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STABILIZATION BY MEANS OF APPROXIMATE PREDICTORS FOR SYSTEMS WITH DELAYED INPUT*

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Abstract. Sufficient conditions for global stabilization of nonlinear systems with delayed input by means of approximate predictors are presented. An approximate predictor is a mapping which approximates the exact values of the stabilizing input for the corresponding system with no delay. A systematic procedure for the construction of approximate predictors is provided for globally Lipschitz systems. The resulting stabilizing feedback can be implemented by means of a dynamic distributed delay feedback law. Illustrative examples show the efficiency of the proposed control strategy.

Key words. feedback stabilization, small-gain theorem, time-delay systems

AMS subject classifications. 93B52, 93C23, 93D15, 93C10

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1. Introduction.

In the present work, we consider the stabilization problem for nonlinear systems with input delays of the form

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t-r)),\\ &x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n,\ u(t) \in U, \end{aligned} \tag{1.1}$$

where $r \geq 0$ is a constant and $U \subseteq \Re^m$ is a closed convex set with $0 \in U$. More specifically, we want to address the feedback design problem for system (1.1) based on the knowledge of a feedback stabilizer $u = k(x)$ for system (1.1) with no delay, i.e., (1.1) with $r = 0$, or

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t)),\\ &x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n,\ u(t) \in U. \end{aligned} \tag{1.2}$$

In the literature there are different ways of attacking this problem as follows:

1. One way is to apply the feedback law $u(t) = k(x(t))$ and obtain conditions which guarantee stability for the closed-loop system. Such conditions can be obtained by using Lyapunov, Razumikhin, or small-gain arguments as in [10, 13]. Usually the obtained conditions are delay dependent and impose restrictive upper bounds for the delay.

2. A second way is to modify the feedback law $u(t) = k(x(t))$ by applying a predictor, i.e., a mapping $p(t)$ which guarantees that $p(t-r) = x(t)$. The predictor feedback $u(t) = k(p(t))$ will guarantee that $u(t-r) = k(x(t))$, and thus we will obtain the stability properties of the closed-loop system (1.2) with $u = k(x)$. This idea is classical in linear systems (e.g., the Smith predictor; see [11] and the references in [5, 6]) and was extended recently to nonlinear feedforward systems in [6] by Krstic (see also [7, 15]).

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3. Finally, another way is to exploit certain characteristics of the system in order to obtain a modified feedback law $\tilde{k}(x)$ such that the application of the modified feedback law $u(t) = \tilde{k}(x(t))$ will ensure stability for the corresponding closed-loop system (see [2, 8, 9]).

In the present work, we apply the "predictor approach," and we obtain results which guarantee global asymptotic stability for the closed-loop system for arbitrary large values of the delay r. The proofs of the main results of the present work rely heavily on the recent vector small-gain theorem given in [3]. Our results will extend the results obtained in [5, 6] in several ways as follows:

1. We will show that approximate predictor schemes can be utilized under appropriate assumptions for the nondelayed system (1.2),
2. we will propose implementation schemes for the (approximate or not) predictor-based feedback, and
3. we will propose explicit approximate predictor schemes for globally Lipschitz systems of the form (1.1), which are not necessarily feedforward systems.

Particularly, we will show that our main results (Theorems 2.4 and 2.6 and Corollaries 3.4 and 3.6) can be applied to nonlinear triangular systems of the form

$$\begin{aligned} &\dot{x}_i(t) = f_i(x_1(t), \ldots, x_i(t)) + x_{i+1}(t), \quad i = 1, \ldots, n-1, \\ &\dot{x}_n(t) = f_n(x(t)) + u(t-r), \\ &x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n, \quad u(t) \in \Re, \end{aligned} \tag{1.3}$$

where $f_i \in C^2(\Re^i;\Re)$ $(i = 1, \ldots, n)$ are globally Lipschitz functions.

The structure of the paper is as follows. Section 2 contains results which show that stabilization of (1.1) can be achieved by means of approximate predictors. Section 3 is devoted to the presentation of a systematic construction methodology of approximate predictors for globally Lipschitz systems. In section 4 a simple example is provided, which illustrates the use of approximate predictor schemes. A detailed comparison is made to other stabilization schemes, and simulations are provided. Finally, in section 5 we present the concluding remarks of the work. The appendix contains the proofs of the main results.

Notation. Throughout this paper we adopt the following notation:

(i) For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm, by x' its transpose, and by $|A| := \sup\{\, |Ax|\,;\ x \in \Re^n,\ |x| = 1 \,\}$ the induced norm of a matrix $A \in \Re^{m\times n}$.

(ii) $\Re^+$ denotes the set of nonnegative real numbers.

(iii) A continuous function $\sigma : \Re^+ \times \Re^+ \to \Re^+$ is said to be a KL function (or to belong to the KL class of functions) if the following conditions hold: (i) For each fixed $t \geq 0$ the function $s \to \sigma(s,t)$ is nondecreasing with $\sigma(0,t) = 0$, and (ii) for each fixed $s \geq 0$ the function $t \to \sigma(s,t)$ is nonincreasing with $\lim_{t\to+\infty} \sigma(s,t) = 0$.

(iv) By $C^j(A)$ $(C^j(A\,;\,\Omega))$, where $j \geq 0$ is a nonnegative integer, we denote the class of functions (taking values in Ω) which have continuous derivatives of order j on A.

(v) Let $x : [a-r, b) \to \Re^n$ with $b > a \geq 0$ and $r \geq 0$. By $T_r(t)x$ we denote the "history" of x from $t-r$ to t; i.e., $T_r(t)x : [-r, 0] \to \Re^n$, for $t \in [a,b)$, is the function defined by $(T_r(t)x)(\theta) := x(t+\theta)$ for $\theta \in [-r, 0]$.

(vi) Let $I \subseteq \Re^+ := [0, +\infty)$ be an interval. By $\mathrm{L}^\infty(I;U)$ $(\mathrm{L}^\infty_{loc}(I;U))$ we denote the space of measurable and (locally) essentially bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$. For $x \in \mathrm{L}^\infty([-r,0];\Re^n)$ we define

$\|x\|_r := \sup_{\theta\in[-r,0]} |x(\theta)|$. We will also use the notation M_U for the space of measurable and locally essentially bounded functions $u : \Re^+ \to U$.

(vii) A continuous mapping $f : C^0([-r,0];\Re^n)\times U \to \Re^k$, where $U \subseteq \Re^m$, is said to be completely Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$ if for every bounded set $S \subset C^0([-r,0];\Re^n)\times U$ there exists $L \geq 0$ such that $|f(x,u) - f(y,v)| \leq L\ \|x-y\|_r + L\,|u-v|$ for all $(x,u) \in S$, $(y,v) \in S$.

(viii) Let $U \subseteq \Re^m$ be a closed nonempty convex set. For every $w \in \Re^m$, $Pr_U(w)$ denotes the projection of w on U.

We assume throughout the paper that the mapping $f : \Re^n \times U \to \Re^n$ is locally Lipschitz. Consequently, for every $(x_0,u) \in \Re^n \times M_U$, system (1.2) admits a unique local solution with initial condition $x(0) = x_0 \in \Re^n$ and corresponding to input $u \in M_U$.

2. Stabilization by means of approximate predictors. We start by presenting the assumptions for system (1.2). We say that a system of the form (1.2) is forward complete if for every $x_0 \in \Re^n$, $u \in M_U$ the solution $x(t)$ of (1.2) with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in M_U$ exists for all $t \geq 0$. Using the semigroup property it is clear that (1.2) is forward complete if and only if there exists $r > 0$ such that for every $x_0 \in \Re^n$, $u \in M_U$ the solution $x(t)$ of (1.2) with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in M_U$ exists for all $t \in [0,r]$. Our first assumption concerning system (1.2) is the following.

(H1) *System* (1.2) *is forward complete.*

Assumption (H1) is also a necessary condition for the global stabilization of system (1.1) with $r > 0$: indeed, the solution of (1.1) for $t \in [0,r]$ must exist for every initial condition $x(0) = x_0 \in \Re^n$ and arbitrary measurable and essentially bounded input $u : [-r,0] \to U$. Therefore it follows that (1.2) must necessarily be a forward complete system.

Let $\phi(t,x_0;u)$ denote the solution of (1.2) at time $t \geq 0$ with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in M_U$. The reader should notice that the solution $x(t)$ of (1.1) with initial condition $x(0) = x_0 \in \Re^n$ and corresponding to input $u \in C^0([-r,+\infty);U)$ satisfies

$$x(t) = \phi(t,x_0;\delta_{-r}u) \quad \text{and} \quad x(t+\tau) = \phi(t,x(\tau);\delta_{\tau-r}u) \quad \forall t,\tau \geq 0, \tag{2.1}$$

where $\delta_\theta : \mathrm{L}^\infty_{loc}([-r,+\infty);U) \to M_U$ is the shift operator defined by

$$(\delta_\theta u)(t) := u(t+\theta) \quad \text{for} \quad t \geq 0, \qquad \theta \geq -r. \tag{2.2}$$

We assume next that (1.2) is globally stabilizable.

(H2) *There exists $k \in C^1(\Re^n;U)$ with $k(0) = 0$ such that $0 \in \Re^n$ is globally asymptotically stable for system* (1.2) *with $u = k(x)$; i.e., there exists a function $\sigma \in KL$ such that for every $x_0 \in \Re^n$ the solution $x(t)$ of* (1.2) *with $u = k(x)$ and initial condition $x(0) = x_0 \in \Re^n$ satisfies the following inequality*:

$$|x(t)| \leq \sigma(|x_0|, t) \quad \forall t \geq 0. \tag{2.3}$$

Theorem 2.1 (see Krstic [6]). *Consider system* (1.1) *under hypotheses* (H1) *and* (H2). *Then the feedback law $u(t) = k(\phi(r,x(t);\delta_{t-r}u))$ globally asymptotically stabilizes system* (1.1)*; i.e., there exists $\tilde{\sigma} \in KL$ such that for every $(x_0,u_0) \in \Re^n \times \mathrm{L}^\infty([-r,0];U)$ the solution $(x(t),u(t))$ of* (1.2) *with $u(t) = k(\phi(r,x(t);\delta_{t-r}u))$ and initial condition $x(0) = x_0 \in \Re^n$, $T_r(0)u = u_0$ satisfies the following inequality:*

$$|x(t)| + |u(t)| \leq \tilde{\sigma}(|x_0| + \|u_0\|_r, t) \quad \forall t \geq 0. \tag{2.4}$$

Remark 2.2. It is clear that the implementation of the feedback law $u(t) = k(\phi(r, x(t); \delta_{t-r}u))$ involves the solution of an integral equation. The integral equation $u(t) = k(\phi(r, x(t); \delta_{t-r}u))$ may be transformed into a differential equation under certain regularity assumptions or can be given implicitly by the solution of a system of first order hyperbolic partial differential equations (see [6]). For the practical implementation of the feedback law the knowledge of the mapping $x \to \phi(r, x; u)$ is crucial.

We will next address the problem of the knowledge of the mapping $x \to \phi(r, x; u)$ and the implementation of the feedback law $u(t) = k(\phi(r, x(t); \delta_{t-r}u))$ by means of an approximate predictor scheme. Our hypotheses concerning systems (1.1) and (1.2) follow.

(S1) *$f : \Re^n \times U \to \Re^n$ is locally Lipschitz with respect to $x \in \Re^n$ and there exists a constant $L \geq 0$ such that*

$$x' f(x, u) \leq L\, |x|^2 + L\, |u|^2 \quad \forall x \in \Re^n \quad \forall u \in U.$$

Hypothesis (S1) is a growth condition which guarantees hypothesis (H1). Indeed, by utilizing the function $V(x) = \frac{1}{2}|x|^2$, hypothesis (S1) implies that the derivative of $V(\phi(t, x_0; u))$ satisfies $\frac{d}{dt}V(\phi(t, x_0; u)) \leq 2LV(\phi(t, x_0; u)) + L\,|u(t)|^2$ for every $(x_0, u) \in \Re^n \times M_U$ and for almost all $t \geq 0$ for which $\phi(t, x_0; u)$ exists. Direct integration of the previous differential inequality implies

$$|\phi(t, x_0; u)| \leq \exp(Lt) \left(|x_0| + \sup_{0 \leq s \leq t} |u(s)| \right). \tag{2.5}$$

A standard contradiction argument in conjunction with (2.5) guarantees that (2.5) holds for every $(x_0, u) \in \Re^n \times M_U$ and for all $t \geq 0$.

(S2) *There exists $k \in C^1(\Re^n; U)$ being locally Lipschitz with $k(0) = 0$ such that system (1.2) with $u = Pr_U(k(x) + v)$ is input-to-state stable from the input $v \in \Re^m$ with linear gain function; i.e., there exist a function $\sigma \in KL$ and a constant $\gamma \geq 0$ such that for every $x_0 \in \Re^n$, $v \in M_{\Re^m}$, $t \geq 0$ the solution $x(t)$ of (1.2) with $u = Pr_U(k(x) + v)$, initial condition $x(0) = x_0$ corresponding to input $v \in M_{\Re^m}$ satisfies the following inequality for all $t \geq 0$:*

$$|x(t)| \leq \max\left\{ \sigma\left(|x_0|, t\right), \gamma \sup_{0 \leq \tau \leq t} |v(\tau)| \right\}. \tag{2.6}$$

Moreover, there exists a constant $R \geq 0$ such that

$$|k(x)| \leq R\,|x|\, \forall x \in \Re^n. \tag{2.7}$$

Hypothesis (S2) is a more demanding hypothesis than (H2). The notion of input-to-state stability used here is the notion introduced by Sontag in [12].

Finally, we proceed with our last assumption concerning systems (1.1) and (1.2).

(S3) *There exist constants $a_1, a_2 \geq 0$, $G \geq 0$ and completely Lipschitz mappings $p : \Re^n \times C^0([-r, 0]; U) \to U$, $g : \Re^n \times C^0([-r, 0]; U) \to \Re^m$ satisfying the following inequalities for all $(x, u) \in \Re^n \times C^0([-r, 0]; U)$:*

$$|k(\phi(r, x; \delta_{-r}u)) - p(x, u)| \leq \max\{ a_1\,|x|, a_2\,\|u\|_r \}, \tag{2.8}$$

$$|g(x, u)| + |p(x, u)| \leq G\,|x| + G\,\|u\|_r. \tag{2.9}$$

Moreover, for every $(x,u) \in \Re^n \times C^0([-r,+\infty);U)$ *the solution* $x(t)$ *of* (1.1) *with initial condition* $x(0) = x_0$ *corresponding to* $u \in C^0([-r,+\infty);U)$ *satisfies* $\frac{d}{dt}p(x(t),T_r(t)u) = g(x(t),T_r(t)u)$ *for all* $t \geq 0$ *for which the solution exists.*

Remark 2.3. Hypothesis (S3) introduces the mapping $p : \Re^n \times C^0([-r,0];U) \to U$, which approximates the stabilizing mapping $k(\phi(r,x;\delta_{-r}u))$. Indeed, the reader should notice that by virtue of (2.1) and (2.8) the solution $x(t)$ of (1.1) with initial condition $x(0) = x_0$ corresponding to $u \in C^0([-r,+\infty);U)$ satisfies

$$\begin{aligned} &|k(\phi(r,x(t);\delta_{t-r}u)) - p(x(t),T_r(t)u)| \\ &\leq \max\{\, a_1\,|x(t)|\,,\, a_2\,\|T_r(t)u\|_r\,\} \quad \forall t \geq 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &|k(x(t)) - p(x(t-r),T_r(t-r)u)| \\ &\leq \max\{\, a_1\,|x(t-r)|\,,\, a_2\,\|T_r(t-r)u\|_r\,\} \quad \forall t \geq r, \end{aligned} \tag{2.11}$$

$$\frac{d}{dt}p(x(t),T_r(t)u) = g(x(t),T_r(t)u) \quad \forall t \geq 0. \tag{2.12}$$

Because of inequalities (2.10), (2.11), we will call the mapping $p : \Re^n \times C^0([-r,0];U) \to U$ an “approximate predictor” for (1.2). The constants $a_1, a_2 \geq 0$ determine how well the approximate predictor approximates the exact predictor scheme $k(\phi(r,x;\delta_{-r}u))$. For $a_1 = a_2 = 0$, we obtain the exact predictor; i.e., it holds that $k(\phi(r,x;\delta_{-r}u)) = p(x,u)$. Notice that for the exact predictor scheme, identity (2.12) holds with $g(x,u) = \nabla k(\phi(r,x;\delta_{-r}u))f(\phi(r,x;\delta_{-r}u),u(0))$.

We are now ready to state our first main result. Its proof can be found in the appendix.

THEOREM 2.4. *Consider systems* (1.1) *and* (1.2) *under hypotheses* (S1)–(S3) *and further assume that*

$$\gamma\, a_1 < 1, \qquad a_2(1+\gamma\, R) < 1. \tag{2.13}$$

Then for every $\mu > 0$ *there exists* $\tilde{\sigma} \in KL$ *such that for every* $(x_0,w_0) \in \Re^n \times C^0([-r,0];\Re^m)$ *the solution* $(x(t),w(t))$ *of* (1.1) *with*

$$u(t) = Pr_U(w(t)), \tag{2.14}$$

$$\dot{w}(t) = g(x(t),T_r(t)u) - \mu(w(t) - p(x(t),T_r(t)u)), \tag{2.15}$$

with initial conditions $x(0) = x_0$, $T_r(0)w = w_0$, *satisfies the estimate*

$$|x(t)| + |w(t)| \leq \tilde{\sigma}\,(\,|x_0| + \|w_0\|_r\,,\, t\,) \quad \forall t \geq 0; \tag{2.16}$$

i.e., the dynamic feedback law (2.14), (2.15) *achieves global stabilization of system* (1.1). *Moreover, if there exist constants* $M, \omega > 0$ *such that the estimate*

$$|x(t)| \leq \max\left\{ M\exp(-\omega\, t)\,|x_0|\,,\, \gamma \sup_{0\leq\tau\leq t} \exp\left(-\omega(t-\tau)\right)|v(\tau)| \right\} \quad \forall t \geq 0, \tag{2.17}$$

holds instead of (2.6), *then for every* $\mu > 0$ *there exist* $\tilde{M}, \tilde{\omega} > 0$ *such that for every* $(x_0,w_0) \in \Re^n \times C^0([-r,0];\Re^m)$ *the solution* $(x(t),w(t))$ *of* (1.1), (2.14), (2.15), *with* *initial conditions* $x(0) = x_0$, $T_r(0)w = w_0$, *satisfies estimate* (2.16) *with* $\tilde{\sigma}(s,t) := \tilde{M}\exp(-\tilde{\omega}\,t)\,s$*; i.e., the dynamic feedback law* (2.14), (2.15) *achieves global exponential stabilization of system* (1.1).

Remark 2.5. Theorem 2.4 shows that approximate predictors can be used for the stabilization of system (1.1) provided that the approximation is sufficiently accurate. Moreover, Theorem 2.4 shows that the stabilizing feedback can be implemented as a dynamic feedback law; there is no need to solve integral equations. In general the stabilizing feedback law will involve distributed delays. To see this, notice that the classical Smith-like predictor (see [11]) for the linear system $f(x,u) = Ax + Bu$ with $k(x) = k'x$, $U = \Re^m$, where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $k \in \Re^{n\times m}$ are constant matrices, and $A + Bk'$ is Hurwitz, satisfies hypotheses (S1)–(S3) with $a_1 = a_2 = 0$, and

$$\phi(r,x;u) := \exp(Ar)x + \int_0^r \exp(A(r-s))Bu(s)ds,$$

$$p(x,u) := k'\exp(Ar)x + \int_0^r k'\exp(A(r-s))Bu(s-r)ds,$$

$$g(x,u) := k'\exp(Ar)Ax + \int_0^r k'A\exp(A(r-s))Bu(s-r)ds + k'Bu(0).$$

Consequently, Theorem 2.4 guarantees that the closed-loop system with the dynamic distributed-delay feedback (2.14), (2.15), i.e., the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t-r),\\ \dot{u}(t) &= k'\exp(Ar)\,(A+\mu\,I_n)\,x(t) + \int_0^r k'(A+\mu\,I_n)\exp(A(r-s))Bu(t+s-r)ds\\ &\qquad + (k'B-\mu\,I_m)\,u(t),\end{aligned}$$

where $I_n \in \Re^{n\times n}$, $I_m \in \Re^{m\times m}$ denote the identity matrices, is exponentially stable for all $\mu > 0$. Another important case where Theorem 2.4 is directly applicable is the case where hypothesis (S2) holds for certain $k \in C^2(\Re^n;U)$ satisfying

$$|\nabla k(x)| \le R \quad \forall x \in \Re^n \tag{2.18}$$

and there exist constants $L_1, L_2 > 0$ such that the locally Lipschitz mapping $f(x,u)$ satisfies the growth condition

$$|f(x,u)| \le L_1\,|x| + L_2\,|u| \quad \forall x \in \Re^n, \quad u \in U. \tag{2.19}$$

Indeed, in this case hypothesis (S1) automatically holds. We can distinguish two important cases as follows where Theorem 2.4 is directly applicable:

1. The case of the exact predictor scheme $p(x,u) = k(\phi(r,x;\delta_{-r}u))$, $g(x,u) = \nabla k(\phi(r,x;\delta_{-r}u))f(\phi(r,x;\delta_{-r}u),u(0))$. Indeed, utilizing (2.18), (2.19), and (2.5) with $L = L_1 + L_2$, it can be shown that hypothesis (S3) holds in this case with $a_1 = a_2 = 0$ and appropriate $G > 0$. Theorem 2.4 implies that the dynamic feedback law

$$u(t) = Pr_U(w(t)), \tag{2.20}$$

$$\begin{aligned}\dot{w}(t) &= \nabla k(\phi(r,x(t);\delta_{-r}T_r(t)u))f(\phi(r,x(t);\delta_{-r}T_r(t)u),u(t))\\ &\qquad -\mu(w(t) - k(\phi(r,x(t);\delta_{-r}T_r(t)u)))\end{aligned} \tag{2.21}$$

achieves global stabilization of system (1.1).

2. The no-predictor case, i.e., the case where $p(x,u) = k(x)$ and $g(x,u) = \nabla k(x) f(x,u(-r))$. The reader should notice that in this case there is no prediction and the stabilizing feedback law for (1.2) is used without any modification. Utilizing (2.18) and (2.19) we obtain for all $\varepsilon, \lambda > 0$, $(x,u) \in \Re^n \times C^0([-r,0];U)$

$$\sup_{0 \le s \le r} |\phi(s,x;\delta_{-r}u)| \le \exp\left(\frac{2+\varepsilon}{2} L_1 r\right) |x| + \frac{L_2}{\varepsilon L_1} \sqrt{\frac{\varepsilon}{2+\varepsilon}} \sqrt{\exp\left((2+\varepsilon)L_1 r\right) - 1}\ \|u\|_r$$

and

$$\begin{aligned}
&|k(\phi(r,x;\delta_{-r}u)) - p(x,u)| \le R\,|\phi(r,x;\delta_{-r}u) - x| \le R \left|\int_0^r f(\phi(s,x;\delta_{-r}u), u(s-r))ds\right| \\
&\le R \int_0^r |f(\phi(s,x;\delta_{-r}u), u(s-r))|\, ds \le R \int_0^r (L_1 |\phi(s,x;\delta_{-r}u)| + L_2 |u(s-r)|)\, ds \\
&\le R L_1 r \exp\left(\frac{2+\varepsilon}{2} L_1 r\right) |x| + R L_2 r \left(\frac{1}{\varepsilon}\sqrt{\frac{\varepsilon}{2+\varepsilon}} \sqrt{\exp\left((2+\varepsilon)L_1 r\right) - 1} + 1\right) \|u\|_r \\
&\le Rr \max \left\{ \begin{array}{l} L_1(1+\lambda) \exp\left(\frac{2+\varepsilon}{2} L_1 r\right) |x| \\ L_2\left(1+\frac{1}{\lambda}\right) \left(\frac{1}{\varepsilon}\sqrt{\frac{\varepsilon}{2+\varepsilon}} \sqrt{\exp((2+\varepsilon)L_1 r) - 1} + 1\right) \|u\|_r \end{array} \right\}.
\end{aligned}$$

It follows that hypothesis (S3) holds. By virtue of Theorem 2.4, the dynamic feedback law

$$u(t) = Pr_U(w(t)), \tag{2.22}$$

$$\dot{w}(t) = \nabla k(x(t)) f(x(t), u(t-r)) - \mu(w(t) - k(x(t))) \tag{2.23}$$

achieves global stabilization of system (1.1) provided that the delay $r > 0$ is sufficiently small. More specifically, the above inequalities show that global stabilization of system (1.1) is achieved provided that there exists $\varepsilon > 0$ such that

$$(1+\gamma R) R r L_2 \left(\frac{1}{\varepsilon}\sqrt{\frac{\varepsilon}{2+\varepsilon}} \sqrt{\exp\left((2+\varepsilon)L_1 r\right) - 1} + 1\right) + R r L_1 \gamma \exp\left(\frac{2+\varepsilon}{2} L_1 r\right) < 1,$$

where $\gamma \ge 0$ is the constant involved in hypothesis (S2).

We finish this section by providing an additional result on approximate predictors. Since the formulae for the mappings $p : \Re^n \times C^0([-r,0];U) \to U$ and $g : \Re^n \times C^0([-r,0];U) \to \Re^m$ involved in hypothesis (S3) are usually complicated (see the next section), the following result helps to simplify the formulae at the cost of an additional approximation. The proof of Theorem 2.6 can be found in the appendix.

THEOREM 2.6. *Consider systems* (1.1) *and* (1.2) *under hypotheses* (S1)–(S2) *and further assume that the following hypothesis holds.*

(S4) *There exist constants* $a_1, a_2 \ge 0$, $G \ge 0$ *and completely Lipschitz mappings* $p : \Re^n \times C^0([-r,0];U) \to U$, $g : \Re^n \times C^0([-r,0];U) \to \Re^m$ *satisfying the following inequalities for all* $(x,u) \in \Re^n \times C^0([-r,0];U)$*:*

$$\max \left\{ \begin{array}{l} |k(\phi(r,x;\delta_{-r}u)) - p(x,u)| \\ |\nabla k(\phi(r,x;\delta_{-r}u)) f(\phi(r,x;\delta_{-r}u), u(0)) - g(x,u)| \end{array} \right\} \le \max \left\{ \begin{array}{l} a_1 |x| \\ a_2 \|u\|_r \end{array} \right\} \tag{2.24}$$

$$|g(x,u)| + |p(x,u)| \le G|x| + G\|u\|_r, \tag{2.25}$$

$$\gamma a_1 < 1, \qquad a_2(1+\gamma R) < 1. \tag{2.26}$$

Then for every $\mu > 0$ satisfying

$$\gamma\, a_1 \left(1 + \frac{1}{\mu}\right) < 1, \qquad a_2 \left(1 + \frac{1}{\mu}\right)(1 + \gamma\, R) < 1 \tag{2.27}$$

there exists $\tilde{\sigma} \in KL$ such that for every $(x_0, w_0) \in \Re^n \times C^0([-r, 0]; \Re^m)$ the solution $(x(t), w(t))$ of (1.1) *with* (2.14), (2.15) *and initial condition $x(0) = x_0$, $T_r(0)w = w_0$ satisfies estimate* (2.16), *i.e., the dynamic feedback law* (2.14), (2.15) *achieves global stabilization of system* (1.1). *Moreover, if there exist constants $M, \omega > 0$ such that* (2.17) *holds instead of* (2.6), *then for every $\mu > 0$ satisfying* (2.27) *there exist $\tilde{M}, \tilde{\omega} > 0$ such that for every $(x_0, w_0) \in \Re^n \times C^0([-r, 0]; \Re^m)$ the solution $(x(t), w(t))$ of* (1.1), (2.14), (2.15), *with initial conditions $x(0) = x_0$, $T_r(0)w = w_0$, satisfies estimate* (2.16) *with $\tilde{\sigma}(s, t) := \tilde{M} \exp(-\tilde{\omega}\, t)\, s$; i.e., the dynamic feedback law* (2.14), (2.15) *achieves global exponential stabilization of system* (1.1).

3. Approximate predictors for globally Lipschitz nonlinear systems. In this section we show how we can construct approximate predictors for globally Lipschitz systems, i.e., systems for which there exists a constant $L \geq 0$ satisfying

$$|f(x, u) - f(y, u)| \leq L\, |x - y| \quad \forall x, y \in \Re^n, \quad \forall u \in U, \tag{3.1a}$$

$$|f(x, u)| \leq L\, |x| + L\, |u| \quad \forall x \in \Re^n, \quad \forall u \in U. \tag{3.1b}$$

Particularly, we will show that the solution map for system (1.2) under (3.1a), (3.1b) can be approximated by successive approximations.

Let $u \in L^\infty([0, T]; U)$ be arbitrary, and define the operator $P_{T,u} : C^0([0, T]; \Re^n) \to C^0([0, T]; \Re^n)$ by

$$(P_{T,u}x)(t) = x(0) + \int_0^t f(x(\tau), u(\tau))d\tau \quad \text{for}\, t \in [0, T]; \tag{3.2}$$

and we define for every integer $l \geq 1$

$$P^l_{T,u} = \underbrace{P_{T,u} \dots P_{T,u}}_{l \text{ times}}.$$

The following facts hold for the operator $P_{T,u} : C^0([0, T]; \Re^n) \to C^0([0, T]; \Re^n)$.

Fact I.

$$\begin{aligned} &\max_{0 \leq t \leq T} |(P_{T,u}x)(t) - (P_{T,u}y)(t)| \\ &\leq |x(0) - y(0)| + LT \max_{0 \leq \tau \leq T} |x(\tau) - y(\tau)| \quad \forall x, y \in C^0([0, T]; \Re^n). \end{aligned} \tag{3.3}$$

The above fact is a direct consequence of (3.1a).

Fact II. *For all $x \in C^0([0, T]; \Re^n)$ and for every integer $l \geq 1$ the following implication holds:*

$$\begin{aligned} &\textit{If } LT < 1, \textit{then } \max_{0 \leq t \leq T} \left|(P^l_{T,u}x)(t) - \phi(t, x(0); u)\right| \\ &\leq \frac{(LT)^l}{1 - LT} \max_{0 \leq t \leq T} \left| x(0) + \int_0^t f(x(\tau), u(\tau))d\tau - x(t) \right|. \end{aligned} \tag{3.4}$$

The proof of the above fact closely follows the proof of Banach's fixed point theorem. First, we show (by induction and using Fact I) that

$$\max_{0\le t\le T}\left|(P^{l}_{T,u}x)(t)-(P^{l-1}_{T,u}x)(t)\right|\le (LT)^{l-1}\max_{0\le\tau\le T}\left|x(0)+\int_0^t f(x(\tau),u(\tau))d\tau - x(t)\right| \tag{3.5}$$
$$\forall l\ge 2 \text{ and } x\in C^0([0,T];\Re^n).$$

Then we proceed by estimating the quantity $\max_{0\le t\le T}\left|(P^m_{T,u}x)(t)-x(t)\right|$ by using (3.5) and the inequality

$$\begin{aligned}
&\max_{0\le t\le T}\left|(P^m_{T,u}x)(t)-x(t)\right|\\
&\le \max_{0\le t\le T}\left|(P^m_{T,u}x)(t)-(P^{m-1}_{T,u}x)(t)\right|+\cdots+\max_{0\le t\le T}\left|(P^1_{T,u}x)(t)-x(t)\right|\\
&\le \left[(LT)^{m-1}+\cdots+1\right]\max_{0\le\tau\le T}\left|x(0)+\int_0^t f(x(\tau),u(\tau))d\tau-x(t)\right|\\
&=\tfrac{1-(LT)^m}{1-LT}\max_{0\le\tau\le T}\left|x(0)+\int_0^t f(x(\tau),u(\tau))d\tau-x(t)\right|.
\end{aligned}$$

Replacing x in the above inequality with $P^l_{T,u}x$ and using (3.5), we get for all $x\in C^0([0,T];\Re^n)$

$$\begin{aligned}
&\max_{0\le t\le T}\left|(P^{m+l}_{T,u}x)(t)-(P^l_{T,u}x)(t)\right|\le\frac{1-(LT)^m}{1-LT}\max_{0\le t\le T}\left|(P^{l+1}_{T,u}x)(t)-(P^l_{T,u}x)(t)\right|\\
&\le\frac{1-(LT)^m}{1-LT}(LT)^l\max_{0\le\tau\le T}\left|x(0)+\int_0^t f(x(\tau),u(\tau))d\tau-x(t)\right|.
\end{aligned}$$

Finally, we notice that $\lim_{m\to\infty}\max_{0\le t\le T}\left|(P^m_{T,u}x)(t)-\phi(t,x(0);u)\right|=0$, by virtue of Banach's fixed point theorem. By letting $m\to+\infty$ in the above inequality we obtain (3.4).

We next define the operators $G_T:\Re^n\to C^0([0,T];\Re^n)$, $C_T:C^0([0,T];\Re^n)\to\Re^n$, and $Q^l_{T,u}:\Re^n\to\Re^n$ for $l\ge 1$ by

$$(G_Tx_0)(t)=x_0\quad\text{for } t\in[0,T]\quad\text{and } C_Tx=x(T), \tag{3.6}$$

$$Q^l_{T,u}=C_TP^l_{T,u}G_T. \tag{3.7}$$

The following fact holds for the mapping $Q^l_{T,u}:\Re^n\to\Re^n$.

Fact III. *For every $x,y\in\Re^n$ and $u\in L^\infty([0,T];U)$ the following implication holds:*

If $LT<1$, then

$$\left|Q^l_{T,u}x-\phi(T,y;u)\right|\le\frac{(LT)^{l+1}}{1-LT}\left(|x|+\sup_{0\le\tau\le T}|u(\tau)|\right)+\exp(LT)\,|x-y|. \tag{3.8}$$

Proof. Implication (3.4) and definitions (3.6), (3.7) imply that for every $x\in\Re^n$ the following implication holds:

$$\text{If } LT<1,\text{ then }\left|Q^l_{T,u}x-\phi(T,x;u)\right|\le\frac{(LT)^l}{1-LT}\max_{0\le\tau\le T}\left|\int_0^t f(x,u(\tau))d\tau\right|. \tag{3.9}$$

On the other hand, inequality (3.1a) gives for all $x, y \in \Re^n$

(3.10)
$$|\phi(t,y;u)-\phi(t,x;u)| \leq |x-y| + L\int_0^t |\phi((s,y;u)-\phi(s,x;u))|\, ds \quad \forall t \in [0,T].$$

Application of Gronwall's lemma (see [4]) to inequality (3.10) gives for all $x, y \in \Re^n$

$$|\phi(T,y;u)-\phi(T,x;u)| \leq \exp(LT)\,|x-y|\,. \tag{3.11}$$

Implication (3.8) is a direct consequence of implication (3.9), inequality (3.11), and the inequality $\max_{0\leq\tau\leq T}|\int_0^t f(x,u(\tau))d\tau| \leq LT\,|x| + LT\sup_{0\leq\tau\leq T}|u(\tau)|$, which is a consequence of hypothesis (3.1b). □

The reader should notice at this point that inequality (3.8) guarantees that the mapping $Q_{T,u}^l : \Re^n \to \Re^n$ approximates the solution map $\phi(T,x;u)$ if $LT < 1$. Moreover, the approximation error can be tuned to be "small" by allowing $l \geq 1$ to take large values. Finally, the mapping $Q_{T,u}^l : \Re^n \to \Re^n$ is easily computed. The following example illustrates this point.

Example 3.1. Consider the case $f(x,u) = a(x) + b(u)$, where $a : \Re^n \to \Re^n$ is a globally Lipschitz vector field with Lipschitz constant $L \geq 0$ and $b : U \to \Re^n$ is a continuous mapping satisfying the linear growth condition $|b(u)| \leq L\,|u|$ for all $u \in U$. In this case inequalities (3.1a), (3.1b) hold. Applying definitions (3.2), (3.6), (3.7), we get for all $x \in \Re^n$ and $u \in \mathrm{L}^\infty([0,T];U)$

$$Q_{T,u}^1 x = x + Ta(x) + \int_0^T b(u(\tau))d\tau, \tag{3.12}$$

$$Q_{T,u}^2 x = x + \int_0^T a\left(x + \tau\, a(x) + \int_0^\tau b(u(s))ds\right)d\tau + \int_0^T b(u(\tau))d\tau. \tag{3.13}$$

Fact III guarantees that if $LT < 1$, then the following inequality will hold:

$$\left|Q_{T,u}^2 x - \phi(T,x;u)\right| \leq \frac{(LT)^3}{1-LT}\left(|x| + \sup_{0\leq\tau\leq T}|u(\tau)|\right). \tag{3.14}$$

The reader should notice that it is easy to generate mappings $Q_{T,u}^l : \Re^n \to \Re^n$ with $l > 2$. This example will be continued.

We next define the mapping $P_{l,q}^u : \Re^n \to \Re^n$ for arbitrary $u \in \mathrm{L}^\infty([0,r];U)$. Let $l, q \geq 1$ be integers and $T = \frac{r}{q}$. We define for all $x \in \Re^n$

$$P_{l,q}^u x = Q_{T,u_q}^l \ldots Q_{T,u_1}^l x, \tag{3.15}$$

where $u_i(s) = u(s+(i-1)T)$, $i = 1, \ldots, q$ for $s \in [0,T]$. Notice that $u_i \in \mathrm{L}^\infty([0,T];U)$ for $i = 1, \ldots, q$. For the operator $P_{l,q}^u : \Re^n \to \Re^n$ we are in position to prove the following proposition. Its proof is provided in the appendix.

PROPOSITION 3.2. *Let l, q be positive integers with $LT < 1$, where $T = \frac{r}{q}$. Suppose that inequalities* (3.1a), (3.1b) *hold. Then there exists a constant $K := K(q) \geq 0$, independent of l, such that for every $u \in \mathrm{L}^\infty([0,r];U)$ and $x \in \Re^n$ the following inequality holds:*

$$\begin{aligned}\left|P_{l,q}^u x - \phi(r,x;u)\right| &\leq K\frac{(LT)^{l+1}}{1-LT}\left(|x| + \sup_{0\leq\tau\leq r}|u(\tau)|\right)\\ &= \frac{K}{q^l}\frac{(Lr)^{l+1}}{q-Lr}\left(|x| + \sup_{0\leq\tau\leq r}|u(\tau)|\right).\end{aligned} \tag{3.16}$$

The reader should again notice that inequality (3.16) guarantees that the mapping $P_{l,q}^{u} : \Re^n \to \Re^n$ approximates the solution map $\phi(r,x;u)$ if $Lr < q$. Again, the approximation error can be guaranteed to be "sufficiently small" by allowing $l \geq 1$ to take large values. Finally, the mapping $P_{l,q}^{u} : \Re^n \to \Re^n$ can be easily computed.

Example 3.3. Consider again the case $f(x,u) = a(x) + b(u)$, where $a : \Re^n \to \Re^n$ is a globally Lipschitz vector field with Lipschitz constant $L \geq 0$ and $b : U \to \Re^n$ is a continuous mapping satisfying the linear growth condition $|b(u)| \leq L\,|u|$ for all $u \in U$. Applying definitions (3.15), (3.12), (3.13), we get for all $x \in \Re^n$ and $u \in \mathrm{L}^\infty([0,T];U)$

$$P_{1,2}^{u}x = x + \frac{r}{2}a(x) + \int_0^r b(u(\tau))d\tau + \frac{r}{2}a\left(x + \frac{r}{2}a(x) + \int_0^{r/2} b(u(\tau))d\tau\right),$$

$$P_{2,2}^{u}x = x_1 + \int_0^{r/2} a\left(x_1 + \tau\, a(x_1) + \int_{r/2}^{\tau + r/2} b(u(s))ds\right)d\tau + \int_{r/2}^{r} b(u(\tau))d\tau,$$

$$\text{where } x_1 = x + \int_0^{r/2} a\left(x + \tau\, a(x) + \int_0^{\tau} b(u(s))ds\right)d\tau + \int_0^{r/2} b(u(\tau))d\tau.$$

Proposition 3.2 guarantees that if $Lr < 2$, then there exists a constant $K \geq 0$ such that the following inequality holds for all $l \geq 1$:

$$\left|P_{l,2}^{u}x - \phi(r,x;u)\right| \leq K\frac{(LT)^{l+1}}{1-LT}\left(|x| + \sup_{0\leq\tau\leq r}|u(\tau)|\right). \tag{3.17}$$

Although formulae for $P_{l,q}^{u} : \Re^n \to \Re^n$ are complicated for large l, q, the values for $P_{l,q}^{u}x$ can be provided through a simple algorithm.

Finally, let $k \in C^1(\Re^n;U)$ be a mapping with $k(0) = 0$ and for which there exists a constant $R \geq 0$ such that

$$|\nabla k(x)| \leq R \quad \forall x \in \Re^n, \tag{3.18}$$

and consider next the mapping $p_{l,q} : \Re^n \times C^0([-r,0];U) \to U$ defined by

$$p_{l,q}(x,u) := k\left(P_{l,q}^{\delta_{-r}u}x\right). \tag{3.19}$$

Notice that inequality (3.1b) guarantees that inequality (2.5) holds with L replaced by $\frac{1+\sqrt{2}}{2}L$. Proposition 3.2 and definition (3.19) in conjunction with inequality (3.18) and inequality (2.5) (with L replaced by $\frac{1+\sqrt{2}}{2}L$) guarantee the existence of a constant $K := K(q) \geq 0$, independent of l, such that for every $(x,u) \in \Re^n \times C^0([-r,0];U)$ and $\varepsilon > 0$ the following inequality holds:

$$|p_{l,q}(x,u) - k(\phi(r,x;\delta_{-r}u))| \leq \max\left\{\begin{array}{c} (1+\varepsilon)RK\frac{(LT)^{l+1}}{1-LT}\,|x|, \\ (1+\varepsilon^{-1})RK\frac{(LT)^{l+1}}{1-LT}\,\|u\|_r \end{array}\right\}, \tag{3.20}$$

$$|p_{l,q}(x,u)| \leq R\left[\exp\left(\frac{1+\sqrt{2}}{2}Lr\right) + K\frac{(LT)^{l+1}}{1-LT}\right](\,|x| + \|u\|_r\,), \tag{3.21}$$

where l, q are positive integers and $T = \frac{r}{q}$ with $Lr < q$. Therefore, the mapping $p_{l,q} : \Re^n \times C^0([-r, 0]; U) \to U$ is a natural candidate to satisfy the requirements of hypothesis (S3) in section 2, i.e., to be an approximate predictor. Indeed, Theorem 2.4 allows us to obtain the following corollary.

COROLLARY 3.4. *Consider systems* (1.1) *and* (1.2) *under hypothesis* (S2) *and further assume inequalities* (3.1a), (3.1b), (3.18) *hold. Let* $p_{l,q} : \Re^n \times C^0([-r, 0]; U) \to U$ *be the mapping defined by* (3.19) *for positive integers* l, q *with* $Lr < q$*. Moreover, assume that* $p_{l,q} : \Re^n \times C^0([-r, 0]; U) \to U$ *is completely Lipschitz and there exists a completely Lipschitz mapping* $g_{l,q} : \Re^n \times C^0([-r, 0]; U) \to \Re^m$ *and a constant* $G \geq 0$ *satisfying the following inequality for all* $(x, u) \in \Re^n \times C^0([-r, 0]; U)$*:*

$$|g_{l,q}(x, u)| \leq G\,|x| + G\,\|u\|_r\,. \tag{3.22}$$

Finally, assume that for every $(x, u) \in \Re^n \times C^0([-r, +\infty); U)$ *the solution* $x(t)$ *of* (1.2) *with initial condition* $x(0) = x_0$ *corresponding to* $u \in C^0([-r, +\infty); U)$ *satisfies* $\frac{d}{dt} p_{l,q}(x(t), T_r(t)u) = g_{l,q}(x(t), T_r(t)u)$ *for all* $t \geq 0$ *for which the solution exists. If* $l \geq 1$ *is sufficiently large, then for every* $\mu > 0$*, there exists* $\tilde{\sigma} \in KL$ *such that for every* $(x_0, w_0) \in \Re^n \times C^0([-r, 0]; \Re^m)$ *the solution* $(x(t), w(t))$ *of* (1.1) *with*

$$u(t) = Pr_U(w(t)), \tag{3.23}$$

$$\dot{w}(t) = g_{l,q}(x(t), T_r(t)u) - \mu\,(w(t) - p_{l,q}(x(t), T_r(t)u)), \tag{3.24}$$

with initial conditions $x(0) = x_0$, $T_r(0)w = w_0$*, satisfies estimate* (2.16)*; i.e., the dynamic feedback law* (3.23), (3.24) *achieves global stabilization of system* (1.1)*. Moreover, if there exist constants* $M, \omega > 0$ *such that estimate* (2.17) *holds instead of* (2.6) *and* $l \geq 1$ *is sufficiently large, then for every* $\mu > 0$ *there exist* $\tilde{M}, \tilde{\omega} > 0$ *such that for every* $(x_0, w_0) \in \Re^n \times C^0([-r, 0]; \Re^m)$ *the solution* $(x(t), w(t))$ *of* (1.1), (3.23), (3.24)*, with initial conditions* $x(0) = x_0$, $T_r(0)w = w_0$*, satisfies estimate* (2.16) *with* $\tilde{\sigma}(s, t) := \tilde{M} \exp(-\tilde{\omega}\, t)\, s$*; i.e., the dynamic feedback law* (3.23), (3.24) *achieves global exponential stabilization of system* (1.1).

Remark 3.5. It should be emphasized that the hypotheses of Corollary 3.4 usually hold if $k \in C^2(\Re^n; U)$ and $f \in C^2(\Re^n \times U; \Re^n)$ with $|\frac{\partial f}{\partial x}(x, u)| + |\frac{\partial f}{\partial u}(x, u)| \leq K$.

Proof. By virtue of the assumptions and inequalities (3.20), (3.21), hypotheses (S1)–(S3) hold. Particularly, hypothesis (S2) holds with $a_1 = (1 + \varepsilon) RK \frac{(LT)^{l+1}}{1-LT}$, $a_2 = (1 + \varepsilon^{-1}) RK \frac{(LT)^{l+1}}{1-LT}$, and $T = \frac{r}{q}$ for every $\varepsilon > 0$. It follows that (2.13) holds provided that

$$(\gamma + 1 + \gamma R)\, RK \frac{(LT)^{l+1}}{1 - LT} < 1. \tag{3.25}$$

Since $LT < 1$, the above inequality is satisfied for sufficiently large $l \geq 1$. The conclusion is a consequence of Theorem 2.4. $\square$

When the computation of the mapping $g_{l,q} : \Re^n \times C^0([-r, 0]; U) \to \Re^m$ is difficult (due to high complexity of the formulae), one can use the following corollary (which is based on Theorem 2.6).

COROLLARY 3.6. *Consider systems* (1.1) *and* (1.2) *with* $U = \Re^m$ *and assume that hypothesis* (S2) *holds with a linear feedback* $u = k'x$*, where* $k \in \Re^{n \times m}$*. Further assume that inequalities* (3.1a), (3.1b) *hold. Define* $\Phi_{l,q}(x, u) := P_{l,q}^{\delta_{-r}u} x$ *for positive integers* l, q *with* $Lr < q$*, and assume that* $\Phi_{l,q} : \Re^n \times C^0([-r, 0]; U) \to \Re^n$ *is completely*

Lipschitz. Let $\mu > 0$ be given. If $l \geq 1$ is sufficiently large, then there exists $\tilde{\sigma} \in KL$ such that for every $(x_0, u_0) \in \Re^n \times C^0([-r,0];\Re^m)$ the solution $(x(t), u(t))$ of (1.1) *with*

$$\dot{u}(t) = k' f\left(\Phi_{l,q}(x(t), T_r(t)u), u(t)\right) - \mu(u(t) - k'\Phi_{l,q}(x(t), T_r(t)u)), \tag{3.26}$$

with initial conditions $x(0) = x_0$, $T_r(0)u = u_0$ satisfies estimate (2.16)*; i.e., the dynamic feedback law* (3.26) *achieves global stabilization of system* (1.1)*. Moreover, if there exist constants $M, \omega > 0$ such that estimate* (2.17) *holds instead of* (2.6) *and $l \geq 1$ is sufficiently large, then there exist $\tilde{M}, \tilde{\omega} > 0$ such that for every $(x_0, u_0) \in \Re^n \times C^0([-r,0];\Re^m)$ the solution $(x(t), u(t))$ of* (1.1), (3.26) *with initial condition $x(0) = x_0$, $T_r(0)u = u_0$ satisfies estimate* (2.16) *with $\tilde{\sigma}(s,t) := \tilde{M}\exp(-\tilde{\omega}\, t)\, s$; i.e., the dynamic feedback law* (3.26) *achieves global exponential stabilization of system* (1.1).

Proof. By virtue of the assumptions and inequality (3.16), hypotheses (S1), (S2), and (S4) hold with $p(x,u) := k'\Phi_{l,q}(x,u)$, $g(x,u) := k' f\left(\Phi_{l,q}(x,u), u(0)\right)$ for sufficiently large $l \geq 1$. Particularly, hypothesis (S4) holds with

$$a_1 = (1+\varepsilon)KR\max\{1, L\}\frac{(LT)^{l+1}}{1-LT}, \qquad a_2 = (1+\varepsilon^{-1})RK\max\{1, L\}\frac{(LT)^{l+1}}{1-LT},$$

and $T = \frac{r}{q}$ for every $\varepsilon > 0$, where $R := |k|$. It follows that (2.27) holds provided that

$$(\gamma + 1 + \gamma R)\, RK\max\{1, L\}\left(1 + \frac{1}{\mu}\right)\frac{(LT)^{l+1}}{1-LT} < 1. \tag{3.27}$$

Since $LT < 1$, the above inequality is satisfied for sufficiently large $l \geq 1$. The conclusion is a consequence of Theorem 2.6. ∎

4. Illustrative example. The following example illustrates the use of Corollaries 3.4 and 3.6 for the scalar system

$$\begin{aligned} &\dot{x}(t) = f(x(t)) + u(t-r), \\ &x(t) \in \Re, \qquad u \in \Re, \end{aligned} \tag{4.1}$$

where $f \in C^1(\Re;\Re)$ is a globally Lipschitz function with a locally Lipschitz derivative satisfying $\sup_{x\in\Re}|f'(x)| < +\infty$. The feedback law

$$k(x) := -(\Lambda + \kappa)x, \tag{4.2}$$

where $\kappa > 0$ is a constant and $\Lambda := \sup_{x\in\Re}|f'(x)|$, satisfies hypothesis (S2) with $\gamma = \frac{1+\varepsilon}{\kappa}$, $\sigma(s,t) := s\,(1+\varepsilon^{-1})\exp\left(-\frac{\kappa}{2}t\right)$, and $R = \Lambda + \kappa$, where $\varepsilon > 0$ is arbitrary. This may be shown directly by using the time derivative of the function $V(x) = \frac{1}{2}x^2$ and the growth condition $|f(x)| \leq \Lambda|x|$ for all $x \in \Re$. More specifically, inequality (2.17) holds with $\gamma = \frac{1+\varepsilon}{\kappa\sqrt{1-\varepsilon}}$, $M := 1+\varepsilon^{-1}$, and $\omega := \frac{\kappa\varepsilon}{2}$, where $\varepsilon \in (0,1)$ is arbitrary.

Hypothesis (S2) can be used for the small-gain analysis presented in [13]. Indeed, utilizing the small-gain arguments in [13], it may be shown that the closed-loop system (4.1) with $u(t) = k(x(t))$ will be globally asymptotically stable provided that

$$r < \frac{\kappa}{(\Lambda+\kappa)(2\Lambda+\kappa)}. \tag{4.3}$$

As a case study, we consider the case $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$. For this function we have $f'(x) = \frac{|x|\left(2+x^2\right)}{(1+x^2)\sqrt{1+x^2}}$, which implies that $\Lambda := \sup_{x\in\Re}|f'(x)| = \frac{4\sqrt{2}}{3\sqrt{3}} \approx 1.088662$.

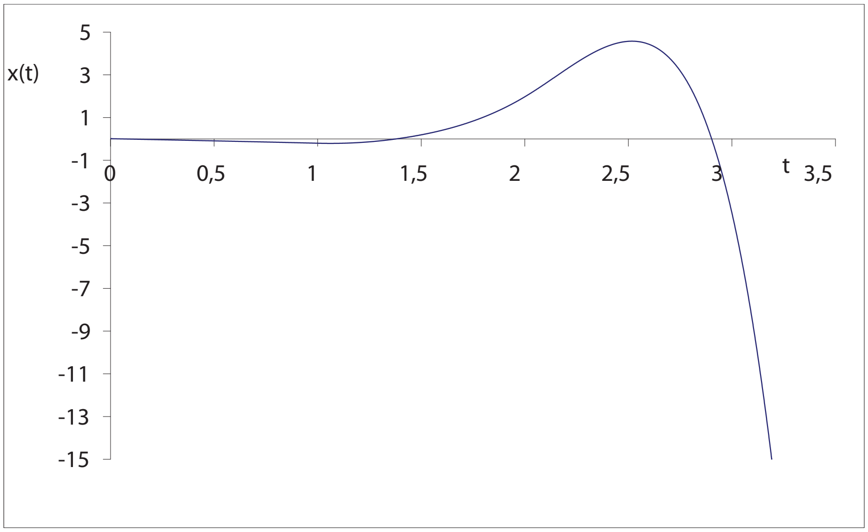

FIG. 1. *Time evolution of the state $x(t)$ for the closed-loop system* (4.1), (4.2) *with* $u(t) = k(x(t))$, $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - \Lambda$, *and* $r = 1$ *with initial condition* $x(\theta) = 0.01$ *for* $\theta \in [-1, 0]$.

It follows that inequality (4.3) is not satisfied for $\kappa = 20 - \Lambda$ and $r > 0.044838$. Simulations confirm that we should not use the "no-prediction" controller $u(t) = k(x(t))$ for large values of the delay. Figure 1 shows the time evolution of the state $x(t)$ for the closed-loop system (4.1), (4.2) with $u(t) = k(x(t))$, $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - L$, and $r = 1$ with initial condition $x(\theta) = 0.01$ for $\theta \in [-1, 0]$. The solution does not approach the equilibrium and obtains very large (in absolute value) values after a short time.

Since the nonlinearity $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$ is considered to be mild in the sense that it is globally Lipschitz and a linear growth condition holds, one may assume that a classical linear Smith-like predictor can be used for the linearized system $\dot{x}(t) = u(t - r)$ using the linear controller (4.2), which stabilizes the nonlinear system when no delays are present (notice that $f'(0) = 0$). The applied input is given by the implicit expression

$$u(t) = -(\Lambda + \kappa)\left(x(t) + \int_{t-r}^{t} u(\tau)d\tau\right) \quad \text{for } t \geq 0. \tag{4.4}$$

Simulations show that local stabilization can be achieved by using the controller (4.4). However, global stability is not achieved by using (4.4). Figure 2 shows the time evolution of the state $x(t)$ for the closed-loop system (4.1), (4.4) with $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - \Lambda$, and $r = 1$ with initial condition $x(\theta) = 1$ for $\theta \in [-1, 0]$ and $u(\theta) = 3$ for $\theta \in [-1, 0)$. It is clear that the solution does not approach the equilibrium.

We conclude that nonlinear feedback design tools must be used in order to achieve global exponential stabilization of system (4.1).

Notice that inequalities (3.1a), (3.1b) hold with

$$L := \max(1, \Lambda) = \max\left(1, \sup_{x \in \Re} |f'(x)|\right).$$

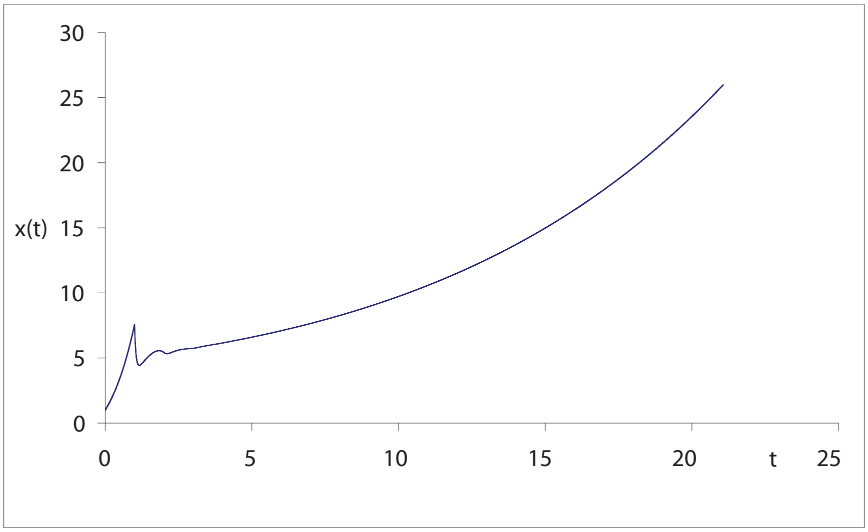

FIG. 2. *Time evolution of the state $x(t)$ for the closed-loop system* (4.1), (4.4) *with* $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - \Lambda$, *and* $r = 1$ *with initial condition* $x(\theta) = 1$ *for* $\theta \in [-1, 0]$ *and* $u(\theta) = 3$ *for* $\theta \in [-1, 0)$.

Corollary 3.4 guarantees that the approximate predictors $p_{l,1} : \Re^n \times C^0([-r, 0]; \Re) \to \Re$ (parameterized by the integer $l \geq 1$) will result in stabilizing feedback laws provided that (3.25) holds, i.e., provided that

$$Lr < 1 \text{ and } (1 + \Lambda + 2\kappa)(\Lambda + \kappa)(Lr)^{l+1} < \kappa(1 - Lr). \tag{4.5}$$

Indeed, the proof of Proposition 3.2 shows that the constant $K(q)$ involved in (3.16) and (3.20) satisfies $K = 1$ for $q = 1$. Notice that the hypotheses of Corollary 3.4 are satisfied

1. for $l = 1$ with

$$p_{1,1}(x, u) = -(\Lambda + \kappa)\left(x + rf(x) + \int_0^r u(\tau - r)d\tau\right)$$

$$\text{and } \; g_{1,1}(x, u) = -(\Lambda + \kappa)\,(f(x) + rf'(x)f(x) + rf'(x)u(-r) + u(0))\,;$$

2. for $l = 2$ with

$$p_{2,1}(x, u) = -(\Lambda + \kappa)\left(x + \int_0^r f\left(x + \tau\, f(x) + \int_0^\tau u(s - r)ds\right)d\tau \right.$$
$$\left. + \int_0^r u(\tau - r)d\tau\right)$$

and $g_{2,1}(x, u) = -(\Lambda + \kappa)\,(f(x) + u(0) + I)$, where

$$I = \int_0^r f'\left(x + \tau\, f(x) + \int_0^\tau u(s - r)ds\right)(f(x) + \tau\, f'(x)f(x)$$
$$+ \tau\, f'(x)u(-r) + u(\tau - r))d\tau.$$

Returning to the case study $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $L = \Lambda := \sup_{x\in\Re}|f'(x)| = \frac{4\sqrt{2}}{3\sqrt{3}} \approx 1.088662$, $\kappa = 20 - L$, we notice that system (4.1) will be stabilized by the "no-prediction" feedback $u(t) = k(x(t))$ for $r < 0.044838$, i.e., when (4.3) holds. On the other hand, inequality (4.5) shows that for every $\mu > 0$ the dynamic feedback (3.23), (3.24) with $l = q = 1$ will achieve global exponential stabilization for $r < 0.130922$. Moreover, inequality (4.5) shows that for every $\mu > 0$ the dynamic feedback (3.23), (3.24) with $q = 1$ will achieve global exponential stabilization for $r < r_{\max}(l)$, where $r_{\max}(l) \to L^{-1}$ as $l \to +\infty$. For example, for $l = 2$ global exponential stabilization is achieved for $r < 0.238642$.

Other values for q can be used as well. However, the formulae become complicated for high values for the integers l, q. For $l = 1$, $q = 2$ we obtain the formulae

$$\begin{aligned} p_{1,2}(x,u) = -(\Lambda+\kappa)\Bigg(& x + \frac{r}{2} f(x) + \int_0^r u(\tau - r)d\tau \\ &+ \frac{r}{2} f\left(x + \frac{r}{2} f(x) + \int_0^{r/2} u(\tau - r)d\tau\right)\Bigg), \\ g_{1,2}(x,u) = -(\Lambda+\kappa)\ & \left(f(x) + \frac{r}{2} f'(x) f(x) + \frac{r}{2} f'(x) u(-r) + u(0)\right) \\ &-(\Lambda+\kappa)\frac{r}{2} f'\left(x + \frac{r}{2} f(x) + \int_0^{r/2} u(\tau - r)d\tau\right) \\ \times \left(f(x) + u\left(-\frac{r}{2}\right) + \frac{r}{2} f'(x) f(x) + \frac{r}{2} f'(x) u(-r)\right).& \end{aligned}$$

Exact computation of the constant $K(q)$ involved in (3.16) and (3.20) in conjunction with (3.25) shows that for every $\mu > 0$ the dynamic feedback (3.23), (3.24) with $q = 2$ will achieve global exponential stabilization provided that

$$\begin{aligned} &Lr < 2 \text{ and } (1 + 2\kappa + \Lambda)(\Lambda+\kappa)(Lr)^{l+1} B < 2^l \kappa (2 - Lr) \\ &\text{where } B = 1 + \exp\left(\frac{\sqrt{2}+1}{4} Lr\right) + \exp\left(\frac{Lr}{2}\right) + \frac{1}{2^l}\frac{(Lr)^{l+1}}{2 - Lr}. \end{aligned} \tag{4.6}$$

For the case study $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $L = \Lambda := \sup_{x\in\Re} |f'(x)| = \frac{4\sqrt{2}}{3\sqrt{3}} \approx 1.088662$, $\kappa = 20 - L$, we notice that inequality (4.6) with $l = 1$ holds for $r < 0.151464$. For $r = 1$, inequality (4.6) shows that for every $\mu > 0$ the dynamic feedback (3.23), (3.24) with $q = 2$ will achieve global exponential stabilization for $l \geq 9$.

Corollary 3.6 can be used as well. For the case $l = 2$, $q = 2$ we have

$$\begin{aligned} \Phi_{2,2}(x,u) &= x_1 + \int_0^{r/2} f\left(x_1 + \tau f(x_1) + \int_{r/2}^{\tau + r/2} u(s - r)ds\right) d\tau + \int_{r/2}^r u(\tau - r)d\tau, \\ x_1 &= x + \int_0^{r/2} f\left(x + \tau f(x) + \int_0^\tau u(s - r)ds\right) d\tau + \int_0^{r/2} u(\tau - r)d\tau. \end{aligned}$$

By virtue of Corollary 3.6 and inequality (3.27) the dynamic feedback

$$\dot{u}(t) = -(\Lambda+\kappa)\left(f\left(\Phi_{2,2}(x(t), T_r(t)u)\right) + u(t)\right) - \mu(u(t) + (\Lambda+\kappa)\Phi_{2,2}(x(t), T_r(t)u))$$

will achieve exponential stabilization provided that

$$\begin{aligned} &Lr < 2 \text{ and } (1 + 2\kappa + \Lambda)(\Lambda+\kappa)L(Lr)^3\Gamma < \frac{4\mu}{1+\mu}\kappa(2 - Lr) \\ &\text{where } \Gamma = 1 + \exp\left(\frac{\sqrt{2}+1}{4} Lr\right) + \exp\left(\frac{Lr}{2}\right) + \frac{1}{4}\frac{(Lr)^3}{2 - Lr}. \end{aligned} \tag{4.7}$$

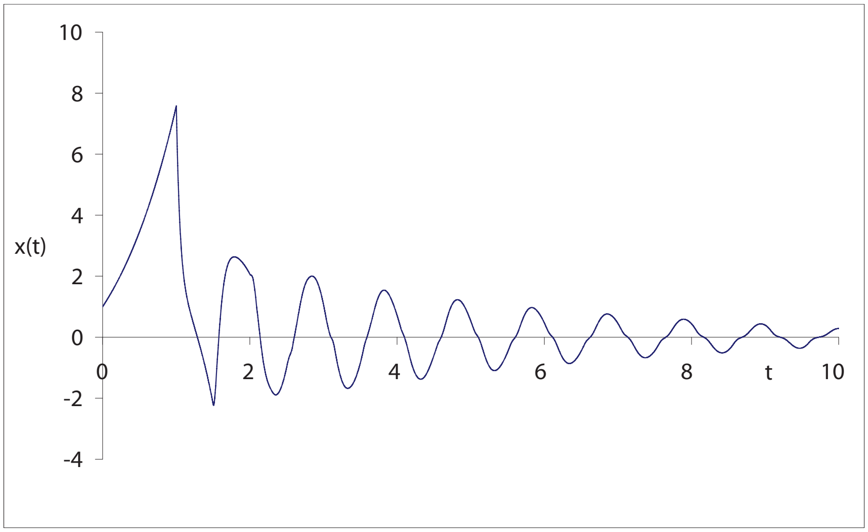

FIG. 3. *Time evolution of the state $x(t)$ for the closed-loop system* (4.1), (3.23), (3.24) *with* $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - \Lambda$, $\mu = 100$, $l = 1$, $q = 2$, *and* $r = 1$ *with initial condition* $x(0) = 1$ *and* $u(\theta) = 3$ *for* $\theta \in [-1, 0]$.

For the case study $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $L = \Lambda := \sup_{x\in\Re} |f'(x)| = \frac{4\sqrt{2}}{3\sqrt{3}} \approx 1.088662$, $\kappa = 20 - L$, we notice that inequality (4.7) with $\mu = 100$ holds for $r < 0.31821$. Therefore, the use of simple predictor formulae allowed a 609% increase in the value of the maximum allowable delay compared to the use of the "no-prediction" feedback $u(t) = k(x(t))$.

In practice, inequalities (4.5), (4.6), and (4.7) are proved to be conservative. For the case study $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $L = \Lambda := \sup_{x\in\Re} |f'(x)| = \frac{4\sqrt{2}}{3\sqrt{3}} \approx 1.088662$, $\kappa = 20 - L$ with $r = 1$, simulations show that low values for the integers l, q can guarantee global exponential stabilization. Figure 3 shows the time evolution of the state $x(t)$ for the closed-loop system (4.1), (3.23), (3.24) with $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - L$, $\mu = 100$, $l = 1$, $q = 2$, and $r = 1$ with initial conditions $x(0) = 1$ and $u(\theta) = 3$ for $\theta \in [-1, 0]$. It is clear that exponential convergence to the equilibrium is achieved. Similar time evolutions are obtained for other initial conditions. The time evolution of the input $u(t)$, shown in Figure 4, demonstrates that after a short transient period, where large values of the input are applied, the input converges to zero asymptotically as expected. Notice that in this case there is no distinction between the applied input $u(t)$ and the controller state $w(t)$, i.e., $w(t) \equiv u(t)$.

The results can be applied in a similar way to triangular systems of the form (1.3), where $f_i \in C^1(\Re^i; \Re)$ $(i = 1, \cdots, n)$ are globally Lipschitz functions with locally Lipschitz derivatives satisfying $|\nabla f_i(x)| \leq L$ for all $x \in \Re^i$ $(i = 1, \cdots, n)$. For such systems there exists a linear feedback for which hypothesis (S2) and inequality (2.17) hold (see [14]).

5. Concluding remarks. In this work, sufficient conditions for global stabilization of nonlinear systems with delayed input by means of "approximate predictors" were presented. The approximate predictor is a notion introduced in the present work and, roughly speaking, is a mapping which approximates the exact values of the

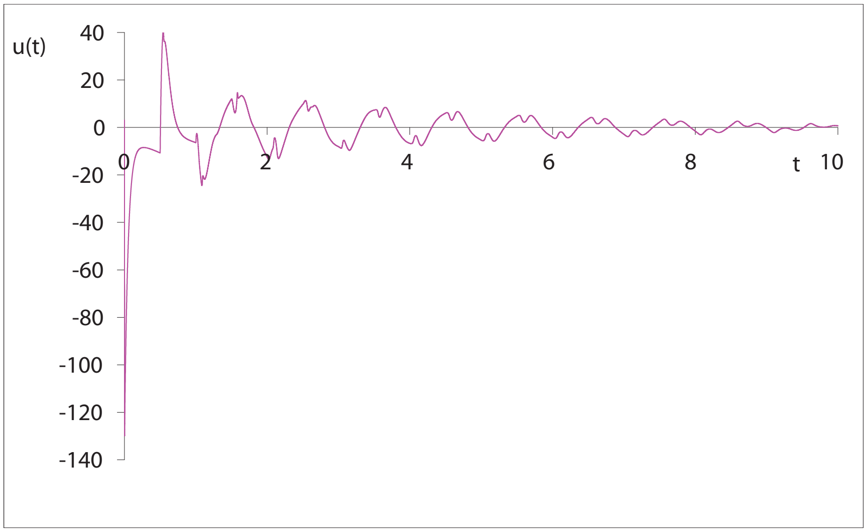

FIG. 4. *Time evolution of the input $u(t)$ for the closed-loop system* (4.1), (3.23), (3.24) *with* $f(x) = sgn(x)\frac{x^2}{\sqrt{1+x^2}}$, $\kappa = 20 - \Lambda$, $\mu = 100$, $l = 1$, $q = 2$, *and* $r = 1$ *with initial condition* $x(0) = 1$ *and* $u(\theta) = 3$ *for* $\theta \in [-1, 0]$.

stabilizing input for the corresponding system with no delay. A systematic procedure for the construction of families of approximate predictors is provided for globally Lipschitz systems: the construction is based on successive approximations on appropriate time intervals. The resulting stabilizing feedback for the system with delayed input can be implemented by means of a dynamic distributed delay feedback law. An illustrative example showed the efficiency of the proposed control strategy for various predictor schemes.

Future research will address the important open problem of applying numerical methods for the construction of approximate predictors. Indeed, the recent work presented in [1] can be an alternative way for constructing approximate predictors for nonlinear systems which are not necessarily globally Lipschitz.

Appendix.

Proof of Theorem 2.4. Since $U \subseteq \Re^m$ is closed and convex with $0 \in U$, the following inequalities will be used repeatedly in the proof:

$$|Pr_U(w)| \leq |w| \text{ and } |Pr_U(w) - Pr_U(v)| \leq |w - v| \quad \forall\, w,\, v \in \Re^m.$$

Let $(x_0, w_0) \in \Re^n \times C^0([-r, 0]; \Re^m)$ be arbitrary. Exploiting hypothesis (S1) and the linear growth condition (2.9), it can be shown (using the functional $V(x, w) = |x|^2 + \sup_{-r \leq s \leq 0} |w(s)|^2$) that the solution of (1.1), (2.14), (2.15), with initial condition $x(0) = x_0$, $T_r(0)w = w_0$, exists for all $t \geq 0$ and satisfies the inequality

$$|x(t)| + \|T_r(t)w\|_r \leq B\exp(\sigma\, t)\ (|x_0| + \|w_0\|_r) \quad \forall t \geq 0 \tag{A.1}$$

for certain constants $B, \sigma > 0$. Differential equation (2.15) and hypothesis (S2) imply that the following inequalities hold for the solution of (1.1), (2.14), (2.15) with initial conditions $x(0) = x_0$, $T_r(0)w = w_0$:

$$|w(t) - p(x(t), T_r(t)u)| \leq \exp(-\mu\, t)\ |w(0) - p(x_0, T_r(0)u)| \quad \forall t \geq 0, \tag{A.2}$$

$$|x(t)| \leq \max\left\{ \sigma(|x_0|, t),\, \gamma \sup_{0\leq s\leq t} |w(s-r) - k(x(s))| \right\} \quad \forall t \geq 0. \tag{A.3}$$

Using (2.5) and (2.7) we obtain for $t \in [0, r)$

$$\begin{aligned}
|w(t-r) - k(x(t))| &\leq |w(t-r)| + R\,|x(t)| \leq \|w_0\|_r + R\,|x(t)| \\
&\leq \|w_0\|_r + R\exp(Lr)\left(|x_0| + \sup_{-r\leq s\leq 0} |u(s)| \right) \\
&\leq (1 + R\exp(Lr))\,\|w_0\|_r + R\exp(Lr)\,|x_0| \\
&\leq (1 + R\exp(Lr))\exp(-\mu(t-r))\,(\,|x_0| + \|w_0\|_r). \qquad \text{(A.4)}
\end{aligned}$$

Using (2.9), (2.11), and (A.2) we obtain for $t \geq r$

(A.5)

$$\begin{aligned}
&|w(t-r) - k(x(t))| \\
&\leq |w(t-r) - p(x(t-r), T_r(t-r)u)| + |p(x(t-r), T_r(t-r)u) - k(x(t))| \\
&\leq \exp(-\mu\,(t-r))\;|w(0) - p(x_0, T_r(0)u)| + \max\{\, a_1\,|x(t-r)|\,,\, a_2\,\|T_r(t-r)u\|_r\,\} \\
&\leq (1+G)\exp(-\mu\,(t-r))\;(\,\|w_0\|_r + |x_0|)\; + \max\{\, a_1\,|x(t-r)|\,,\, a_2\,\|T_r(t-r)u\|_r\,\} \\
&\leq (1+G)\exp(-\mu\,(t-r))\;(\,\|w_0\|_r + |x_0|)\; + \max\{\, a_1\,|x(t-r)|\,,\, a_2\,\|T_r(t-r)w\|_r\,\}
\end{aligned}$$

Combining (A.4) and (A.5) we conclude that there exists a constant $Q > 0$ such that the following inequality holds for all $t \geq 0$ and $\varepsilon > 0$:

$$|w(t-r) - k(x(t))| \leq \max\left\{ \begin{array}{l} Q(1+\varepsilon^{-1})\exp(-\mu\,t)\;(\,\|w_0\|_r + |x_0|) \\ a_1(1+\varepsilon)\sup\limits_{0\leq s\leq t} |x(s)| \\ a_2(1+\varepsilon)\sup\limits_{0\leq s\leq t} |w(s-r)| \end{array} \right\}. \tag{A.6}$$

On the other hand, using (2.7) we conclude that the following inequalities hold for all $t \geq 0$ and $\lambda > 0$:

$$\begin{aligned}
|w(t-r)| &\leq |w(t-r) - k(x(t))| + |k(x(t))| \\
&\leq |w(t-r) - k(x(t))| + R\,|x(t)| \\
&\leq \sup_{0\leq s\leq t} |w(s-r) - k(x(s))| + R\sup_{0\leq s\leq t} |x(s)| \\
&\leq \max\left\{ (1+\lambda)\sup_{0\leq s\leq t} |w(s-r) - k(x(s))|\,,\, (1+\lambda^{-1})R\sup_{0\leq s\leq t} |x(s)| \right\}. \qquad \text{(A.7)}
\end{aligned}$$

Using (A.1), (A.3), (A.6), and (A.7) in conjunction with the vector small-gain theorem in [3], we conclude that there exists $\tilde{\sigma} \in KL$ such that (2.16) holds, provided that there exist $\varepsilon, \lambda > 0$ satisfying the following inequalities:

$$(1+\varepsilon)\gamma a_1 < 1, \quad (1+\varepsilon)a_2(1+\lambda^{-1})\gamma R < 1, \quad a_2(1+\varepsilon)(1+\lambda) < 1.$$

The reader should notice that inequalities (2.13) guarantee the existence $\varepsilon, \lambda > 0$ such that the above inequalities hold.

To finish the proof, consider the case where (2.17) holds for certain constants $M, \omega > 0$. Let $0 < \tilde{\omega} \le \min\{\omega, \mu\}$ sufficiently small such that

$$\begin{aligned} &(1+\varepsilon)\gamma a_1 \exp(\tilde{\omega}\, r) < 1,\\ &(1+\varepsilon)a_2 \exp(\tilde{\omega}\, r)(1+\lambda^{-1})\gamma R < 1,\\ &a_2 \exp(\tilde{\omega}\, r)(1+\varepsilon)(1+\lambda) < 1 \end{aligned} \tag{A.8}$$

for certain $\varepsilon, \lambda > 0$. Again the existence of appropriate $\varepsilon, \lambda > 0$ and sufficiently small $\tilde{\omega} > 0$ is guaranteed by (2.13). Inequality (2.17) gives

$$\exp(\tilde{\omega} t)\,|x(t)| \le \max\left\{ M\,|x_0|\,,\, \gamma \sup_{0\le s\le t} \exp(\tilde{\omega}\, s)\,|w(s-r) - k(x(s))| \right\} \quad \forall t \ge 0. \tag{A.9}$$

Using (2.9), (2.11), and (A.2) we obtain for $t \ge r$ and $\tilde{\omega} \le \mu$

$$\begin{aligned} &\exp(\tilde{\omega}\, t)\,|w(t-r) - k(x(t))|\\ &\quad \le \exp(\tilde{\omega}\, t)\,|w(t-r) - p(x(t-r), T_r(t-r)u)| + I_1\\ &\quad \le \exp(\tilde{\omega}\, t)\exp(-\mu\,(t-r))\;|w(0) - p(x_0, T_r(0)u)| + I_2\\ &\quad \le (1+G)\exp(\mu\, r)\;(\|w_0\|_r + |x_0|)\; + I_2\\ &\quad \le (1+G)\exp(\mu\, r)\;(\|w_0\|_r + |x_0|)\; + I_3\\ &\quad \le (1+G)\exp(\mu\, r)\;(\|w_0\|_r + |x_0|)\; + I_4, \end{aligned}$$

$$\begin{aligned} I_1 &= \exp(\tilde{\omega}\, t)\,|p\,(x(t-r),\, T_r(t-r)u) - k(x(t))|\,,\\ I_2 &= \max\left\{ \begin{array}{l} a_1 \exp(\tilde{\omega}\, t)\,|x(t-r)| \\ a_2 \exp(\tilde{\omega}\, t)\,\|T_r(t-r)u\|_r \end{array} \right\},\\ I_3 &= \max\left\{ \begin{array}{l} a_1 \exp(\tilde{\omega} r)\exp(\tilde{\omega}\,(t-r))\,|x(t-r)| \\ a_2 \exp(\tilde{\omega}\, t) \displaystyle\sup_{-r\le s\le 0} |w(t-r+s)| \end{array} \right\},\\ I_4 &= \max\left\{ \begin{array}{l} a_1 \exp(\tilde{\omega} r)\exp(\tilde{\omega}\,(t-r))\,|x(t-r)| \\ a_2 \exp(2\tilde{\omega}\, r) \displaystyle\sup_{-r\le s\le 0} \exp(\tilde{\omega}(t-r+s))\,|w(t-r+s)| \end{array} \right\}. \end{aligned}$$

Combining (A.4) and the above inequality, we conclude that there exists a constant $\tilde{Q} > 0$ such that the following inequality holds for all $t \ge 0$:

$$\begin{aligned} &\exp(\tilde{\omega} t)\,|w(t-r) - k(x(t))|\\ &\le \max\left\{ \begin{array}{l} \tilde{Q}(1+\varepsilon^{-1})\;(\|w_0\|_r + |x_0|)\,,\; a_1 \exp(\tilde{\omega}\, r)(1+\varepsilon) \displaystyle\sup_{0\le s\le t} \exp(\tilde{\omega} s)\,|x(s)| \\ a_2 \exp(2\tilde{\omega}\, r)(1+\varepsilon) \displaystyle\sup_{0\le s\le t} \exp(\tilde{\omega}(s-r))\,|w(s-r)| \end{array} \right\}. \end{aligned} \tag{A.10}$$

On the other hand, using (2.7) we conclude that the following inequalities hold for all $t \ge 0$:

$$\begin{aligned} &\exp(\tilde{\omega}(t-r))\,|w(t-r)|\\ &\quad \le \exp(\tilde{\omega}(t-r))\,|w(t-r) - k(x(t))| + \exp(\tilde{\omega}(t-r))\,|k(x(t))|\\ &\quad \le |w(t-r) - k(x(t))|\exp(\tilde{\omega}(t-r)) + R\,|x(t)|\exp(\tilde{\omega}(t-r))\\ &\le \exp(-\tilde{\omega}\, r) \sup_{0\le s\le t} |w(s-r) - k(x(s))|\exp(\tilde{\omega}\, s) + R\,\exp(-\tilde{\omega}\, r) \sup_{0\le s\le t} |x(s)|\exp(\tilde{\omega} s)\\ &\le \max\left\{ \begin{array}{l} (1+\lambda)\exp(-\tilde{\omega}\, r) \displaystyle\sup_{0\le s\le t} |w(s-r) - k(x(s))|\exp(\tilde{\omega} s) \\ (1+\lambda^{-1})R\,\exp(-\tilde{\omega}\, r) \displaystyle\sup_{0\le s\le t} |x(s)|\exp(\tilde{\omega} s) \end{array} \right\}. \end{aligned} \tag{A.11}$$

Combining (A.10) and (A.11) we obtain for all $t \geq 0$

(A.12)

$$\sup_{0\leq s\leq t} \exp(\tilde{\omega}s)\,|w(s-r)-k(x(s))| \leq \max\left\{\begin{array}{l} \tilde{Q}(1+\varepsilon^{-1})\,(\|w_0\|_r+|x_0|)\,,\ a_1\exp(\tilde{\omega}\,r)(1+\varepsilon)\sup\limits_{0\leq s\leq t}\exp(\tilde{\omega}s)\,|x(s)| \\ a_2\exp(\tilde{\omega}\,r)(1+\varepsilon)(1+\lambda)\sup\limits_{0\leq s\leq t}|w(s-r)-k(x(s))|\exp(\tilde{\omega}s) \\ a_2\exp(\tilde{\omega}\,r)(1+\varepsilon)(1+\lambda^{-1})R\sup\limits_{0\leq s\leq t}|x(s)|\exp(\tilde{\omega}s) \end{array}\right\}.$$

Since $a_2\exp(\tilde{\omega}\,r)(1+\varepsilon)(1+\lambda)<1$ (recall (A.8)), inequality (A.12) is simplified in the following way:

(A.13)

$$\sup_{0\leq s\leq t} \exp(\tilde{\omega}s)\,|w(s-r)-k(x(s))| \leq \max\left\{\begin{array}{l} \tilde{Q}(1+\varepsilon^{-1})\,(\|w_0\|_r+|x_0|) \\ \exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\sup\limits_{0\leq s\leq t}|x(s)|\exp(\tilde{\omega}s) \end{array}\right\}.$$

Inequality (A.13) in conjunction with inequality (A.9) gives for all $t \geq 0$

$$\sup_{0\leq s\leq t} \exp(\tilde{\omega}s)\,|w(s-r)-k(x(s))| \leq \max\left\{\begin{array}{l} \tilde{Q}(1+\varepsilon^{-1})\,(\|w_0\|_r+|x_0|) \\ \exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\,M\,|x_0| \\ \exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\,\gamma\sup\limits_{0\leq s\leq t}\exp(\tilde{\omega}\,s)\,|w(s-r)-k(x(s))| \end{array}\right\}.$$

Since $\exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\gamma<1$ (recall (A.8)), we obtain

$$\sup_{0\leq s\leq t} \exp(\tilde{\omega}s)\,|w(s-r)-k(x(s))| \leq \max\left\{\begin{array}{l} \tilde{Q}(1+\varepsilon^{-1})\,(\|w_0\|_r+|x_0|) \\ \exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\,M\,|x_0| \end{array}\right\}. \tag{A.14}$$

Inequality (A.14) in conjunction with (A.9) gives

$$\sup_{0\leq s\leq t} \exp(\tilde{\omega}s)\,|x(s)| \leq \max\left\{\begin{array}{l} M\,|x_0|\,,\ \gamma\tilde{Q}(1+\varepsilon^{-1})\,(\|w_0\|_r+|x_0|) \\ \gamma\exp(\tilde{\omega}\,r)(1+\varepsilon)\max\left\{a_1, a_2(1+\lambda^{-1})R\right\}\,M\,|x_0| \end{array}\right\} \quad \forall t\geq 0. \tag{A.15}$$

Finally, from (A.14), (A.15), and (A.11) we get

$$\exp(\tilde{\omega}(t-r))\,|w(t-r)| \leq P\,(\|w_0\|_r+|x_0|) \quad \forall t\geq 0 \tag{A.16}$$

for certain appropriate constant $P>0$. Inequalities (A.15) and (A.16) imply that there exist $\tilde{M}, \tilde{\omega}>0$ such that for every $(x_0, w_0)\in\Re^n\times C^0([-r,0];\Re^m)$ the solution $(x(t), w(t))$ of (1.1), (2.14), (2.15) with initial condition $x(0)=x_0$, $T_r(0)w=w_0$, satisfies estimate (2.16) with $\tilde{\sigma}(s,t):=\tilde{M}\exp(-\tilde{\omega}\,t)\,s$. The proof is complete. ∎

Proof of Theorem 2.6. The proof is exactly the same as the proof of Theorem 2.4 except for the estimate for the quantity $|w(t-r)-k(x(t))|$. By virtue of inequality (2.24) and noticing that

$$\frac{d}{dt}k(\phi(r,x(t);\delta_{-r}T_r(t)u)) = \nabla k(\phi(r,x(t);\delta_{-r}T_r(t)u))f(\phi(r,x(t);\delta_{-r}T_r(t)u),u(t)),$$

we obtain

$$\left|\frac{d}{dt}k(\phi(r,x(t);\delta_{-r}T_r(t)u)) - g(x(t),T_r(t)u)\right| \leq \max\left\{\, a_1\,|x(t)|\,,\; a_2\,\|T_r(t)u\|_r\,\right\}.$$

Integrating (2.15) and using (2.24) and the above inequality, we obtain for all $t \geq 0$

$$\begin{aligned} &|w(t)-k(\phi(r,x(t);\delta_{-r}T_r(t)u))| \leq \exp(-\mu\, t)\; |w(0)-k(\phi(r,x(0);\delta_{-r}T_r(0)u))| \\ &\qquad + \left(1+\frac{1}{\mu}\right)\max\left\{ a_1 \sup_{0\leq\tau\leq t}|x(\tau)|\,,\; a_2 \sup_{0\leq\tau\leq t}\|T_r(\tau)u\|_r \right\}. \end{aligned} \tag{A.17}$$

Combining (A.4), (A.17), (2.5), and (2.7) we conclude that there exists a constant $Q>0$ such that the inequality (A.6) holds for all $t\geq 0$ and $\varepsilon>0$ with a_1,a_2 replaced by $a_1(1+\mu^{-1}), a_2(1+\mu^{-1})$. Similar changes are needed for the case of exponential stability. Details are left to the reader. ☐

Proof of Proposition 3.2. For notational convenience we set $a := \frac{(LT)^{l+1}}{1-LT}$. Define the sequence

$$g_i := \left|\underbrace{Q^l_{T,u_i}\dots Q^l_{T,u_1}}_{i \text{ times}} x - \phi(iT,x;u)\right|. \tag{A.18}$$

By virtue of inequality (3.8) and definition (3.15) this sequence satisfies

$$g_1 \leq a\,(\,|x| + \|u\|\,) \textit{ and } g_q = \left|P^u_{l,q}x - \phi(r,x;u)\right|, \tag{A.19}$$

where $\|u\| := \sup_{0\leq\tau\leq r}|u(\tau)|$. Equation (A.19) shows that inequality (3.16) holds with $K=1$ for the case $q=1$.

Next assume that $q\geq 2$. Inequality (3.8) implies the following recursive relation for $i=1,\dots,q-1$:

$$\begin{aligned} g_{i+1} &:= \left|\underbrace{Q^l_{T,u_{i+1}}Q^l_{T,u_i}\dots Q^l_{T,u_1}}_{i+1 \text{ times}} x - \phi((i+1)T,x;u)\right| \\ &= \left|\underbrace{Q^l_{T,u_{i+1}}Q^l_{T,u_i}\dots Q^l_{T,u_1}}_{i+1 \text{ times}} x - \phi(T,\phi(iT,x;u);u_{i+1})\right| \\ &\leq a\left(\left|\underbrace{Q^l_{T,u}\dots Q^l_{T,u}}_{i \text{ times}} x\right| + \|u\|\right) + \exp(LT)\,g_i \\ &\leq (a+\exp(LT))\,g_i + a\,(|\phi(iT,x;u)| + \|u\|\,). \end{aligned} \tag{A.20}$$

Inequality (3.1b) guarantees that inequality (2.5) holds with L replaced by $\frac{1+\sqrt{2}}{2}L$. Consequently, we obtain from (A.20) for $i = 1, \ldots, q-1$

$$g_{i+1} \leq (a + \exp(LT))\, g_i + a\, (\exp(ipLT) + 1)\, (|x| + \|u\|)\,, \tag{A.21}$$

where $p := \frac{1+\sqrt{2}}{2}$. Using (A.21) and (A.19) we can obtain the following estimate for $q \geq 2$:

$$\begin{aligned} &\left|P^u_{l,q}x - \phi(r,x;u)\right| \\ &\leq \left[(a + \exp(LT))^{q-1} + (\exp(pLr) + 1)\,\frac{(a + \exp(LT))^{q-1} - 1}{a + \exp(LT) - 1}\right] a\,(|x| + \|u\|)\,. \end{aligned} \tag{A.22}$$

Since $a = \frac{(LT)^{l+1}}{1-LT} \leq \frac{(LT)^2}{1-LT}$, we obtain from (A.22)

$$\begin{aligned} &\left|P^u_{l,q}x - \phi(r,x;u)\right| \\ &\leq \left[(b + \exp(LT))^{q-1} + (\exp(pLr) + 1)\,\frac{(b + \exp(LT))^{q-1} - 1}{b + \exp(LT) - 1}\right] a\,(|x| + \|u\|)\,, \end{aligned} \tag{A.23}$$

where $b := \frac{(LT)^2}{1-LT}$. It follows that inequality (3.16) holds with $K := (b + \exp(LT))^{q-1} + (\exp(pLr) + 1)\,\frac{(b+\exp(LT))^{q-1}-1}{b+\exp(LT)-1}$ for the case $q \geq 2$. The proof is complete. ∎

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REFERENCES

[1] L. Grüne and P. E. Kloeden, *Higher order numerical schemes for affinely controlled nonlinear systems*, Numer. Math., 89 (2001), pp. 669–690.

[2] I. Karafyllis, *Finite-time global stabilization by means of time-varying distributed delay feedback*, SIAM J. Control Optim., 45 (2006), pp. 320–342.

[3] I. Karafyllis and Z. P. Jiang, *A vector small-gain theorem for general nonlinear control systems*, in Proceedings of the 48th IEEE Conference on Decision and Control (Shanghai, China), IEEE Press, Piscataway, NJ, 2009, pp. 7996–8001; also available online at http://arxiv.org/abs/0904.0755.

[4] H. K. Khalil, *Nonlinear Systems*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1996.

[5] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhäuser Boston, Inc., Boston, MA, 2009.

[6] M. Krstic, *Input delay compensation for forward complete and strict-feedforward nonlinear systems*, IEEE Trans. Automat. Control, 55 (2010), pp. 287–303.

[7] M. Krstic, *Lyapunov stability of linear predictor feedback for time-varying input delay*, IEEE Trans. Automat. Control, 55 (2010), pp. 554–559.

[8] F. Mazenc, S. Mondie, and R. Francisco, *Global asymptotic stabilization of feedforward systems with delay at the input*, IEEE Trans. Automat. Control, 49 (2004), pp. 844–850.

[9] F. Mazenc and P. A. Bliman, *Backstepping design for time-delay nonlinear systems*, IEEE Trans. Automat. Control, 51 (2006), pp. 149–154.

[10] F. Mazenc, M. Malisoff, and Z. Lin, *Further results on input-to-state stability for nonlinear systems with delayed feedbacks*, Automatica J. IFAC, 44 (2008), pp. 2415–2421.

[11] S. I. Niculescu, *Delay Effects on Stability. A Robust Control Approach*, Lecture Notes in Control and Inform. Sci. 269, Springer-Verlag London, Ltd., London, 2001.

[12] E. D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control, 34 (1989), pp. 435–443.
[13] A. R. Teel, *Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem*, IEEE Trans. Automat. Control, 43 (1998), pp. 960–964.
[14] J. Tsinias, *A theorem on global stabilization of nonlinear systems by linear feedback*, Systems Control Lett., 17 (1991), pp. 357–362.
[15] X. Zhang, H. Gao, and C. Zhang, *Global asymptotic stabilization of feedforward nonlinear systems with a delay in the input*, Internat. J. Systems Sci., 37 (2006), pp. 141–148.