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Global stabilization and asymptotic tracking for a class of nonlinear systems by means of time-varying feedback

I. Karafyllis and J. Tsinias*,†

Department of Mathematics, National Technical University of Athens, Zografou Campus 15780, Athens, Greece

SUMMARY

A simple backstepping design scheme is proposed and sufficient conditions for non-uniform in time global stabilization for parameterized systems by means of time-varying feedback are established. Our methodology is applicable to a special class of systems that in general cannot be stabilized by static feedback and includes non-holonomic systems in chained form. For this class of systems the main results on feedback stabilization enable us to derive sufficient conditions for the solvability of the tracking problem. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: global feedback stabilization; asymptotic tracking; time-varying feedback

1. INTRODUCTION

The purpose of this paper is to explore the global feedback stabilization problem for a class of systems whose dynamics contain time-varying unknown parameters. For such systems, sufficient conditions for the existence of time-varying feedback stabilizers, exhibiting non-uniform in time global asymptotic stability at the equilibrium, are derived. Our approach generalizes the backstepping design scheme presented in Reference [1]. The main results on feedback stabilization are used to derive sufficient conditions for the tracking problem for a class of non-holonomic systems. Our work constitutes continuation of References [1, 2] dealing with non-uniform in time global asymptotic stabilization of time-varying systems.

The paper is organized as follows. In Section 2 we provide some definitions and preliminary results that play a key role in proving the main results of the paper. In Section 3 a generalization of the main result in Reference [1] is established for triangular parameterized systems of the form:

$$\begin{aligned}
&\dot{x}_i = f_i(t,\theta,x_1,\ldots,x_i) + \eta_i d_i(t) x_{i+1}, \quad i=1,\ldots,n\\
&u := x_{n+1}\\
&x := (x_1,\ldots,x_n)\in\Re^n, \quad \theta\in\Re^l, \quad \eta := (\eta_1,\ldots,\eta_n)\in\Re^n
\end{aligned} \tag{1}$$

*Correspondence to: Dr. J. Tsinias, Department of Mathematics, Faculty of Applied Sciences, National Technical University, Zografou Campus 15780, Athens, Greece
† E-mail: jtsin@central.ntua.gr

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where (θ, η) are unknown parameters and u is the control input. Particularly, Corollary 3.2 provides sufficient conditions for the existence of a linear time-varying feedback

$$u = k(t)x \tag{2}$$

where $k : \Re^+ \to \Re^{1\times n}$ is of class C^1, exhibiting non-uniform in time global asymptotic stability of the origin for the closed-loop system (1) with (2).

In Section 4 we apply Corollary 3.2 for the solvability of the stabilization problem as well as for the tracking problem for a certain class of two input systems. Particularly, in Section 4.1 we focus our attention on the solvability of the global feedback stabilization problem for two input systems of the form:

$$\dot{z} = Az + bu_1 \tag{3a}$$

$$\dot{x}_i = F_i(\theta, z, u_1, x_1, \ldots, x_i) + G_i(z, u_1)x_{i+1}, \quad i = 1, \ldots, n$$

$$u_2 := x_{n+1} \tag{3b}$$

$$x := (x_1, \ldots, x_n), (z, x, \theta) \in \Re^m \times \Re^n \times \Re^k, \quad (u_1, u_2) \in \Re^2$$

In Proposition 4.1 we establish that, under certain hypotheses, global stabilization for (3) is achieved by means of a smooth time-varying feedback

$$(u_1, u_2) = (U_1(t, z, x), U_2(t, x)) \tag{4}$$

The above class of systems includes the case:

$$\dot{z} = u_1$$

$$\dot{x}_i = u_1(x_{i-1} + f_i(x_1, \ldots, x_i)), \quad 1 \leqslant i \leqslant n-1$$

$$\dot{x}_n = f_n(x_1, \ldots, x_n) + u_2$$

$$(u_1, u_2) \in \Re^2 \tag{5}$$

The stabilization problem as well as the tracking problem for the above class of systems (5), especially when $f_i \equiv 0$ for $i = 1, \ldots, n$ (non-holonomic chained-form case), has attracted the interest of many researchers (see References [3–15]). In Section 4.2 (Proposition 4.5) we use the result of Corollary 3.2 in order to derive sufficient conditions for the solvability of the tracking problem for the case (5).

2. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper we use the notation $\mathcal{N}$ to denote the class of continuous (C^0), non-decreasing, non-negative functions defined on $\Re^+ = [0, +\infty)$. We say that a C^0 function $a : (\Re^+)^2 \to \Re^+$ is of class $\mathcal{NN}$, if for each $s \geqslant 0$ both $a(\cdot, s)$ and $a(s, \cdot)$ are of class $\mathcal{N}$. The notation $\mathcal{NL}$ is used to denote the class of C^0, non-negative functions $a : \Re^+ \times \Re^+ \to \Re^+$, with the property that for each $s \geqslant 0$ the function $a(\cdot, s)$ is of class $\mathcal{N}$ with $\lim_{t\to+\infty} a(s, t) = 0$. Likewise, the notation $\mathcal{NNL}$ is used to denote the class of C^0, non-negative functions $a : \Re^+ \times \Re^+ \times \Re^+ \to \Re^+$, with the property that for each $r, s \geqslant 0$, the functions $a(\cdot, r, s)$ and $a(r, \cdot, s)$ are of class $\mathcal{N}$ with $\lim_{t\to+\infty} a(r, s, t) = 0$. By $L^\infty(A)(L^\infty_{loc}(A))$ we denote the class of measurable (locally) essentially bounded functions $\theta : A \to \Re^l$ and we adopt the notation $\|\theta\| = \mathrm{ess}_{t\in A} \sup |\theta(t)|$, where $|\ |$ is the usual Euclidean norm.

We also introduce certain classes of functions, which are involved in the statements of our main results.

Definition 2.1
We denote by $\mathbf{S}$ the set of C^0 functions $a : \Re^+ \to \Re$ such that for any real constant r it holds that

$$\int_0^{+\infty} (a(\tau) + r)\, \mathrm{d}\tau = +\infty \tag{6}$$

and by $\mathbf{SL}$ the set of C^0 functions $a : \Re^+ \times \Re^+ \to \Re$ satisfying the following properties:

(I) For each $s \geqslant 0$ the mapping $a(\cdot, s)$ is of class $\mathbf{S}$.
(II) For each $t \geqslant 0$, the mapping $a(t, \cdot)$ is non-increasing.

For example the functions t and $\exp\{t\}$ are of class $\mathbf{S}$ and $a(t,s) = \exp\{t\} - s$, is of class $\mathbf{SL}$.

Definition 2.2
We denote by $\mathbf{L}$ the class of C^1 functions $d : \Re^+ \to \Re$ for which the following properties hold:

(I) There exist constants $c \geqslant 0$ and $K > 0$, such that

$$|d(t)| + |\dot{d}(t)| \leqslant K \exp\{ct\}, \quad \forall t \geqslant 0 \tag{7a}$$

($\dot{d}$ denotes the derivative of d);
(II) A pair of constants $p \geqslant 1$ and $r \geqslant 0$ can be found such that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |d(\tau)|^p \exp\{r\tau\}\, \mathrm{d}\tau = +\infty \tag{7b}$$

Example 2.3
The output $y(t, x_0) = Hx(t, x_0) = H \exp\{At\}x_0$ of an autonomous linear system $\dot{x} = Ax$, $x \in \Re^n$ with $x(0, x_0) = x_0 \in \Re^n$ and $(H, A) \in \Re^{1\times n} \times \Re^{n\times n}$ being an observable pair of constant matrices, belongs to $\mathbf{L}$, when $x_0 \neq 0$. Indeed, (7a) holds for $c = |A| := \sup\{|Ax|/|x|,\ x \neq 0\}$ and $K := |H|(1 + |A|)|x_0|$. Moreover, since the pair (H, A) is observable, it follows that for any $\delta > 0$ there exists a constant $\varepsilon > 0$ such that

$$\int_0^{\delta} y^2(t, x_0)\, \mathrm{d}t \geqslant \varepsilon |x_0|^2 > 0, \quad \text{for any } x_0 \neq 0 \tag{8}$$

Notice that $\int_t^{t+\delta} y^2(\tau, x_0)\, \mathrm{d}\tau = x_0^{\mathrm{T}} \exp\{A^{\mathrm{T}} t\}[\int_0^{\delta} \exp\{A^{\mathrm{T}}\tau\} H^{\mathrm{T}} H \exp\{A\tau\}\, \mathrm{d}\tau] \exp\{At\}x_0$, which by virtue of (8) implies $\int_t^{t+\delta} y^2(\tau, x_0)\, \mathrm{d}s \geqslant \varepsilon \exp\{-2|A|t\}|x_0|^2$ for $t \geqslant 0$ and thus we get for any positive integer N:

$$\int_0^{N\delta} y^2(\tau, x_0) \exp\{(2|A| + 1)\tau\}\, \mathrm{d}\tau \geqslant \varepsilon |x_0|^2 \sum_{n=0}^{N-1} \exp\{n\delta\} \tag{9}$$

Let $t \geqslant \delta$ and let N be an integer with $N\delta \leqslant t < (N+1)\delta$. Consequently, by (9)

$$\frac{1}{t}\int_0^t y^2(\tau, x_0)\exp\{(2|A|+1)\tau\}\,\mathrm{d}\tau \geqslant \frac{\varepsilon|x_0|^2\exp\{-\delta\}}{(N+1)\delta}\sum_{n=1}^{N}\exp\{n\delta\} \tag{10}$$

which yields $\lim_{t\to\infty} 1/t\int_0^t y^2(\tau, x_0)\exp\{(2|A|+1)\tau\}\,\mathrm{d}\tau = +\infty$, thus (7b) is satisfied with $p=2$ and $r = 2|A|+1$.

The following lemma provides some elementary properties of the class **L**. Its proof is given in Appendix A.

Lemma 2.4

Suppose that $d(\cdot)$ belongs to **L**. Then for every constant $\mu \geqslant 0$ there exist a C^∞ function $E: \Re^+ \times \Re \to \Re$, a function $\rho_\mu \in \mathcal{N}$ and constants $\sigma \geqslant 0$, $R > 0$ such that:

$$d(\cdot)E(\cdot, d(\cdot)) \in \mathbf{S} \tag{11a}$$

$$\int_0^{+\infty} \max\{M|d(t)|\exp\{\mu t\} - d(t)E(t, d(t)), 0\}\,\mathrm{d}t \leqslant \rho_\mu(M), \quad \forall M \geqslant 0 \tag{11b}$$

$$d(t)E(t, d(t)) \geqslant 0, \quad \forall t \geqslant 0 \tag{11c}$$

$$|E(t,s)| \leqslant R\exp\{\sigma t\}, \quad \forall t \geqslant 0, \ s \in \Re \tag{11d}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}E(t, d(t))\right| \leqslant R(1 + |d(t)| + |\dot{d}(t)|)\exp\{\sigma t\}, \quad \forall t \geqslant 0 \tag{11e}$$

Consider the system

$$\dot{x} = f(t,x), \quad x \in \Re^n, \ t \geqslant 0 \tag{12}$$

where $f: \Re^+ \times \Re^n \to \Re^n$ is measurable in t and locally Lipschitz in x, satisfying $f(t,0) = 0$, for all $t \geqslant 0$. Let us denote its solution initiated from x_0 at time t_0 by $x(t)$. We say that $0 \in \Re^n$ is *globally asymptotically stable* (*GAS*) with respect to (12), if for any initial (t_0, x_0) $x(\cdot)$ is defined for all $t \geqslant t_0$ and the following conditions hold:

(P1) For any $\varepsilon > 0$ and $T \geqslant 0$, it holds that $\sup\{|x(t)|;\ t \geqslant t_0, |x_0| \leqslant \varepsilon,\ t_0 \in [0,T]\} < +\infty$ and there exists a $\delta = \delta(\varepsilon, T) > 0$, such that:

$$|x_0| \leqslant \delta, \ t_0 \in [0,T] \Rightarrow \sup_{t \geqslant t_0} |x(t)| \leqslant \varepsilon \quad \text{(Stability)} \tag{13a}$$

(P2) For any $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists a $\tau = \tau(\varepsilon, T, R) \geqslant 0$, such that:

$$|x_0| \leqslant R, \ t_0 \in [0,T] \Rightarrow \sup_{t \geqslant t_0 + \tau} |x(t)| \leqslant \varepsilon \quad \text{(Attractivity)} \tag{13b}$$

We note that the above type of non-uniform in time asymptotic stability has been recently used in Reference [1] for the problem of feedback stabilization and further analysed in References [16, 17] where Lyapunov characterizations of robust GAS are established.

It is known (see for instance Reference [18]) that (equi)attractivity implies stability, non-uniform with respect to initial time t_0. The following lemma is a slight modification of this fact and for completeness its proof is given in Appendix A.

Lemma 2.5
Consider system (12) and assume that there exists a function $\Delta \in \mathcal{N}\mathcal{N}\mathcal{L}$, such that for all $(t_0, x_0) \in \Re^+ \times \Re^n$, the solution $x(t)$ of (12) initiated from x_0 at time t_0 satisfies:

$$|x(t)| \leqslant \Delta(|x_0|, t_0, t), \quad \forall t \geqslant t_0 \tag{14}$$

Then zero $0 \in \Re^n$ is GAS.

Finally, we consider the following class of systems whose dynamics contains a pair of time-varying uncertainties (θ, η):

$$\begin{aligned} &\dot{x} = f(t, \theta, \eta, x) \\ &x \in \Re^n, \;\; \theta \in \Re^l, \;\; \eta \in \Re^m, \quad t \geqslant 0 \end{aligned} \tag{15}$$

where f is measurable in t, continuous with respect to (θ, η) and locally Lipschitz with respect to x, with $f(t, \theta, \eta, 0) = 0$ for all $(t, \theta, \eta) \in \Re^+ \times \Re^l \times \Re^m$. The set of admissible uncertainties $(\theta(\cdot), \eta(\cdot))$ is the space $L^\infty(\Re^+) \times L^\infty_{loc}(\Re^+)$.

Let $\Pi_t \subseteq \Re^l \times \Re^m$ be a closed time-varying set, such that its projection on $\Re^l$ along $\Re^m$ coincides with the whole space $\Re^l$. We say that $0 \in \Re^n$ is Π_t-GAS for (15) if for every (t_0, x_0) and input

$$(\theta(\cdot), \eta(\cdot)) \in L^\infty([t_0, +\infty)) \times L^\infty_{loc}([t_0, +\infty)); \;\; (\theta(t), \eta(t)) \in \Pi_t, \quad \forall t \geqslant t_0 \tag{16}$$

the corresponding solution $x(t)$ of (15) initiated from x_0 at time t_0 is defined for all $t \geqslant t_0$ and further satisfies:

(P1) For every $\varepsilon > 0$ and $T \geqslant 0$, it holds that $\sup\{|x(t)|\,;\, t \geqslant t_0, |x_0| \leqslant \varepsilon, t_0 \in [0, T], (\theta(\cdot), \eta(\cdot))$ *satisfying* (16)$\} < +\infty$ and for every pair $(\theta(\cdot), \eta(\cdot))$ satisfying (16), there exists a $\delta := \delta(\varepsilon, T) > 0$, such that (13a) holds.

(P2) For every $\varepsilon > 0$, $T \geqslant 0$, $R \geqslant 0$ and for every pair $(\theta(\cdot), \eta(\cdot))$ satisfying (16), there exists a $\tau := \tau(\varepsilon, T, R) \geqslant 0$, such that (13b) holds.

The following lemma provides a Lyapunov-like criterium for Π_t-global asymptotic stability of the origin for systems of the form (15). Its proof is straightforward and is given in Appendix A.

Lemma 2.6
Consider the parameterized system (15) and assume that there exist a C^1 function $V : \Re^+ \times \Re^n \to \Re^+$, a function $a : (\Re^+)^2 \to \Re$ of class **SL**, and constants $\sigma \geqslant 0$, $K_1, K_2 > 0$, such that for every $(t; x, \theta, \eta) \in \Re^+ \times \Re^n \times \Pi_t$ it holds that

$$K_1 \exp\{-\sigma t\}|x|^2 \leqslant V(t, x) \leqslant K_2|x|^2 \tag{17a}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, \theta, \eta, x) \leqslant -a(t, |\theta|)V(t, x) \tag{17b}$$

Then zero $0 \in \Re^n$ is Π_t-GAS. Particularly, there exists a function $D \in \mathcal{N}\mathcal{L}$ in such a way that:

$$\lim_{t\to+\infty} \exp\{\varepsilon t\} D(s,t) = 0, \quad \forall \varepsilon \geqslant 0, \;\; s \geqslant 0 \tag{18a}$$

$$|x(t)| \leqslant D(\|\theta\|, t) \exp\left\{ \frac{1}{2} \int_0^{t_0} |a(s, \|\theta\|)| \, \mathrm{d}s \right\} |x_0|, \quad \forall t \geqslant t_0 \tag{18b}$$

where $x(t)$ denotes the solution of (15) initiated from x_0 at time t_0 and corresponding to a pair of uncertainties $(\theta(\cdot), \eta(\cdot)) \in L^\infty([t_0, +\infty)) \times L^\infty_{loc}([t_0, +\infty))$, with $(\theta(t), \eta(t)) \in \Pi_t$ for all $t \geqslant t_0$.

3. PARAMETERIZED SYSTEMS

In order to derive sufficient conditions for the problem of global feedback stabilization for case (1), we first need the following lemma that provides a simple backstepping design scheme and generalizes the backstepping technique introduced in Lemma 2.1 in Reference [1]. The result of Lemma 3.1 is inductively used in Corollary 3.2, where sufficient conditions for global stabilization of parameterized systems (1) by means of a linear time-varying feedback are established.

Lemma 3.1

Consider the single-input parameterized system

$$\dot{x} = f(t, \theta, \eta, x) + g(t, \theta, \eta) y \tag{19a}$$

$$\dot{y} = h(t, \theta, \eta, x, y) + \hat{\eta} d(t) u \tag{19b}$$

$$(x, y) \in \Re^n \times \Re, \;\; (\theta, \eta, \hat{\eta}) \in \Re^l \times \Re^m \times \Re$$

where $\theta(\cdot)$, $\eta(\cdot)$, $\hat{\eta}(\cdot)$ are time-varying measurable mappings, f, g, h are measurable in t, continuous with respect to (θ, η) and locally Lipschitz with respect to rest variables and satisfy $f(t, \theta, \eta, 0) = 0$, $h(t, \theta, \eta, 0, 0) = 0$, for every $(t, \theta, \eta) \in \Re^+ \times \Re^l \times \Re^m$, and suppose that d belongs to $\mathbf{L} \cap C^j(\Re^+)$ (for some integer $j \geqslant 1$). Furthermore, assume that there exists a non-empty time-varying set $\Pi_t \subseteq \Re^l \times \Re^m$ of the form

$$\Pi_t := \{(\theta, \eta) \in \Re^l \times \Re^m : \zeta_{i,1}(t, \theta) \leqslant \eta_i \leqslant \zeta_{i,2}(t, \theta), \;\; i = 1, \ldots, m\} \tag{19c}$$

for certain C^0 functions, $\zeta_{i,1}, \zeta_{i,2}$, in such a way that:

A1. There exist a C^1 function $V : \Re^+ \times \Re^n \to \Re^+$, a C^j mapping $k : \Re^+ \to \Re^{1\times n}$, a function $a : \Re^+ \times \Re^+ \to \Re$ of class **SL**, constants $K_i > 0$, $i = 1, 2, 3$, and $\sigma \geqslant 0$, such that the following hold for all $(t; x; \theta, \eta) \in \Re^+ \times \Re^n \times \Pi_t$:

$$K_1 \exp\{-\sigma t\} |x|^2 \leqslant V(t, x) \leqslant K_2 |x|^2 \tag{20a}$$

$$\left| \frac{\partial V}{\partial x}(t, x) \right| \leqslant K_3 |x| \tag{20b}$$

$$\dot{V}|_{(19a),y=k(t)x} := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)(f(t,\theta,\eta,x) + g(t,\theta,\eta)k(t)x)$$
$$\leqslant -a(t,|\theta|)V(t,x) \tag{20c}$$

A2. There exist a function ρ of class $\mathcal{N}$ and constants $K > 0$, $r \geqslant 0$, such that for every $(t;\theta,\eta) \in \Re^+ \times \Pi_t$ we have:

$$|k(t)| + |\dot{k}(t)| \leqslant K \exp\{rt\} \tag{21a}$$

$$|g(t,\theta,\eta)| + \sup_{x\neq 0} \frac{|f(t,\theta,\eta,x)|}{|x|} \leqslant \rho(|\theta|) \exp\{rt\}|d(t)| \tag{21b}$$

$$|a(t,s)| \leqslant \rho(s) \exp\{rt\} \quad \text{and} \quad a(t,s) \leqslant \rho(s) \exp\{rt\}|d(t)|, \;\; \forall s \geqslant 0 \tag{21c}$$

$$\sup_{(x,y)\neq 0} \frac{|h(t,\theta,\eta,x,y)|}{|(x,y)|} \leqslant \rho(|\theta|) \exp\{rt\}|d(t)| \tag{21d}$$

Then

- For every pair of constants $R > 0$ and $\beta \geqslant 0$ and for every function $\zeta \in \mathcal{N}$ with $\zeta(0) \geqslant R$, there exist a C^1 function $W : \Re^+ \times \Re^{n+1} \to \Re^+$, a C^j mapping $\bar{k} : \Re^+ \to \Re^{1\times(n+1)}$, functions $\bar{a} \in$ **SL** and $\bar{\rho} \in \mathcal{N}$, positive constants $\bar{K}, \bar{K}_i > 0$ $i = 1,2,3$, $\bar{\sigma}$, $\bar{r}$, and a non-empty time-varying set $\bar{\Pi}_t \subseteq \Re^l \times \Re^{m+1}$ of the form:
$$\bar{\Pi}_t := \Pi_t \times \{\hat{\eta} \in \Re : R \exp\{-\beta t\} \leqslant \hat{\eta} \leqslant \zeta(|\theta|) \exp\{\beta t\}\} \tag{22a}$$
such that all properties (20), (21) are satisfied with $\bar{\eta} := (\eta,\hat{\eta})$, $\bar{x} := (x,y)$, $\bar{K}$, $\bar{K}_i$ $(i = 1,2,3)$, $\bar{\sigma}, \bar{r}, \bar{a}, \bar{k}, \bar{\rho}, \bar{\Pi}_t, W$,
$$\bar{f}(t,\theta,\bar{\eta},\bar{x}) := \begin{pmatrix} f(t,\theta,\eta,x) + g(t,\theta,\eta)y \\ h(t,\theta,\eta,x,y) \end{pmatrix} \quad \text{and} \quad \bar{g}(t,\theta,\bar{\eta}) := \begin{pmatrix} 0 \\ \hat{\eta}d(t) \end{pmatrix} \tag{22b}$$
instead of η, x, K, K_i $(i = 1,2,3)$, $\sigma, r, a, k, \rho, \Pi_t, V, f$, and g, respectively.
- It turns out that the origin of the closed-loop system (19) with $u = \bar{k}(t)\bar{x}$ is $\bar{\Pi}_t$-GAS. Particularly, there exists a function $D \in \mathcal{NL}$ satisfying (18a) and in such a way that the solution $\bar{x}(t)$ of the closed-loop system (19) with $u = \bar{k}(t)\bar{x}$ satisfies the estimation (18b) with $a := \bar{a}$, for every pair $(\theta(\cdot),\bar{\eta}(\cdot)) \in L^\infty([t_0,+\infty)) \times L^\infty_{loc}([t_0,+\infty))$, with $(\theta(t),\bar{\eta}(t)) \in \bar{\Pi}_t$ for all $t \geqslant t_0$.

Proof
For simplicity we adopt the notation $\bar{x} = (x,y)$. Let R, β be arbitrary positive constants and define

$$W(t,\bar{x}) := V(t,x) + \delta(t)(y - k(t)x)^2 \tag{23}$$

$$\bar{k}(t) := -\frac{1}{2R} \exp\{\beta t\} E(t,d(t))(-k(t),1) \tag{24}$$

$$\delta(t) := \frac{2K_1 \exp\{-\sigma t\}}{1 + 2K^2 \exp\{2rt\}} \tag{25}$$

$$\mu := 4r + 2\sigma + c \tag{26}$$

where the constants c; K_1, σ; K, r are defined in (7a), (20) and (21), respectively and $E(\cdot,\cdot)$ is any C^∞ function which satisfies (11a)–(11e) with μ as defined by (26) and for the specific $d(\cdot)$ above. By (20a), (21a), (23) and by using the elementary inequalities $|x| \leqslant |\bar{x}|$, $|y| \leqslant |\bar{x}|$, $\frac{1}{2}|y|^2 - |k(t)x|^2 \leqslant (y - k(t)x)^2 \leqslant |y|^2 + |k(t)x|^2 + 2|y||k(t)x|$, we find

$$\begin{aligned} &K_1 \exp\{-\sigma t\}|x|^2 + \tfrac{1}{2}\delta(t)|y|^2 - \delta(t)K^2 \exp\{2rt\}|x|^2 \\ &\leqslant W(t,\bar{x}) \leqslant (K_2 + \delta(t)(1+K)^2 \exp\{2rt\})|\bar{x}|^2 \end{aligned} \tag{27a}$$

and by taking into account (25) we estimate:

$$\begin{aligned} \frac{K_1}{1+2K^2} \exp\{-(\sigma + 2r)t\} \leqslant \frac{K_1 \exp\{-\sigma t\}}{1 + 2K^2 \exp\{2rt\}} &= K_1 \exp\{\sigma t\} \\ &\quad - K^2 \delta(t) \exp\{2rt\} \end{aligned} \tag{27b}$$

$$\frac{2K_1}{1+2K^2} \exp\{-(\sigma + 2r)t\} \leqslant \delta(t) \leqslant \frac{K_1}{K^2} \exp\{-2rt\} \tag{27c}$$

and consequently (27a)–(27c) it follows that

$$\bar{K}_1 \exp\{-\bar{\sigma} t\}|\bar{x}|^2 \leqslant W(t,\bar{x}) \leqslant \bar{K}_2 |\bar{x}|^2 \tag{28}$$

with $\bar{\sigma} := 2r + \sigma$ and for certain constants $\bar{K}_1, \bar{K}_2 > 0$. By definition (23)

$$\frac{\partial W}{\partial \bar{x}}(t,\bar{x}) = \left(\frac{\partial V}{\partial x}(t,x) - 2\delta(t)(y - k(t)x)k(t), 2\delta(t)(y - k(t)x) \right)$$

and, using (20b) and (21a), we can also easily estimate:

$$\left| \frac{\partial W}{\partial \bar{x}}(t,\bar{x}) \right| \leqslant K_3 |\bar{x}| + 2\delta(t)(1+K)^2 \exp\{2rt\}|\bar{x}|$$

and thus by (27c):

$$\left| \frac{\partial W}{\partial \bar{x}}(t,\bar{x}) \right| \leqslant \bar{K}_3 |\bar{x}| \tag{29}$$

for certain $\bar{K}_3 > 0$. Next we evaluate the time derivative $\dot{W} := \frac{\mathrm{d}}{\mathrm{d}t} W(t,\bar{x}(t))|_{(19),u=\bar{k}(t)\bar{x}}$ of W along the trajectories of (19) with

$$u = \bar{k}(t)\bar{x} = -\frac{1}{2R} \exp\{\beta t\} E(t, d(t))(y - k(t)x) \quad \text{and} \quad (\theta, \eta, \hat{\eta}) \in \bar{\Pi}_t$$

First, by using (19), (21d), (22a), (23) and (24) we find:

$$\begin{aligned} \frac{\partial W}{\partial t}(t,\bar{x}) &= \frac{\partial V}{\partial t}(t,x) + \dot{\delta}(t)(y - k(t)x)^2 - 2\delta(t)(y - k(t)x)\dot{k}(t)x \\ &\leqslant \frac{\partial V}{\partial t}(t,x) + |\dot{\delta}(t)||y - k(t)x|^2 + 2\delta(t)|\dot{k}(t)||x||y - k(t)x| \end{aligned} \tag{30a}$$

$$\begin{aligned}
&\frac{\partial W}{\partial \bar{x}}(t,\bar{x})(\bar{f}(t,\theta,\bar{\eta},\bar{x})+\bar{g}(t,\theta,\bar{\eta})\bar{k}(t)\bar{x}) \\
&\quad\leqslant \frac{\partial V}{\partial x}(t,x)(f(t,\theta,\eta,x)+g(t,\theta,\eta)k(t)x) \\
&\qquad + \left|\frac{\partial V}{\partial x}(t,x)g(t,\theta,\eta)\right||y-k(t)x| - \delta(t)d(t)E(t,d(t))(y-k(t)x)^2 \\
&\qquad + 2\delta(t)|h(t,\theta,\eta,x,y)||y-k(t)x| + 2\delta(t)|k(t)||f(t,\theta,\eta,x)||y-k(t)x| \\
&\qquad + 2\delta(t)|k(t)||g(t,\theta,\eta)|(y-k(t)x)^2 \\
&\qquad + 2\delta(t)|k(t)|^2|g(t,\theta,\eta)||x||y-k(t)x| \qquad (30b)
\end{aligned}$$

$$|h(t,\theta,\eta,x,y)|\leqslant\rho(|\theta|)|d(t)|\exp\{rt\}((1+|k(t)|)|x|+|y-k(t)x|) \qquad (30c)$$

It turns out from (20b)–(20c), (21a)–(21b) and (30a)–(30c) that

$$\dot{W}\leqslant I_1(t,x,y,|\theta|)+I_2(t,x,y,|\theta|)+I_3(t,x,y) \qquad (31a)$$

$$\begin{aligned}
I_1(t,x,y,|\theta|) &:= \left(\frac{|\dot{\delta}(t)|}{\delta(t)}+2\rho(|\theta|)|d(t)|\exp\{rt\}(1+|k(t)|)\right)\delta(t)(y-k(t)x)^2 \\
&\quad - a(t,|\theta|)V(t,x) - d(t)E(t,d(t))\delta(t)(y-k(t)x)^2 \qquad (31b)
\end{aligned}$$

$$I_2(t,x,y,|\theta|) := \rho(|\theta|)|d(t)|\exp\{rt\}(K_3+2\delta(t)(1+|k(t)|)^2)|x||y-k(t)x| \qquad (31c)$$

$$I_3(t,x,y) =: 2\delta(t)|\dot{k}(t)||x||y-k(t)x| \qquad (31d)$$

Taking into account (25) and combining (21a) and (26) we get $\exp\{rt\}(1+|k(t)|)\leqslant(1+K)\exp\{\mu t\}$ and $|\dot{\delta}(t)|/\delta(t)\leqslant\sigma+2r$. Similarly, by (23) we have $\delta(t)(y-k(t)x)^2\leqslant W(t,\bar{x})$ and thus by taking into account definition (31b) of $I_1(\cdot)$, we estimate:

$$\begin{aligned}
I_1(t,x,y,|\theta|) &\leqslant -a(t,|\theta|)V(t,x)+2(1+K)\rho(|\theta|)|d(t)|\exp\{\mu t\}\delta(t)(y-k(t)x)^2 \\
&\quad + (\sigma+2r)W(t,\bar{x}) - d(t)E(t,d(t))\delta(t)(y-k(t)x)^2 \qquad (32)
\end{aligned}$$

By completing the squares in (31c) and (31d) and using (20a), (21a), (23), (25), (27c) and (7a) we also get:

$$I_2(t,x,y,|\theta|)\leqslant C_1\rho(|\theta|)(V(t,x)+|d(t)|\exp\{\mu t\}\delta(t)(y-k(t)x)^2) \qquad (33)$$

$$I_3(t,x,y)\leqslant C_2W(t,\bar{x}) \qquad (34)$$

for certain $C_1, C_2>0$. By (21c), (23), (26), (31a), (32)–(34) we conclude that

$$\dot{W}\leqslant -\bar{a}(t,|\theta|)W(t,\bar{x}) \qquad (35)$$

$$\bar{a}(t,s) := a(t,s)-M(s)-\max\{0, M(s)\exp\{\mu t\}|d(t)|-d(t)E(t,d(t))\} \qquad (36)$$

where $M(s) := M_0\rho(s)+\sigma+2r+C_2$, for certain constant $M_0>0$ is a function of class $\mathcal{N}$.

We claim that $\bar{a}$ is of class **SL**. Indeed, definition (36) asserts that $\bar{a}:\Re^+\times\Re^+\to\Re$ is a C^0 function and, since $M(\cdot)$ is of class $\mathcal{N}$, for each fixed $t\geqslant 0$ the mapping $\bar{a}(t,\cdot)$ is non-increasing.

Furthermore, by virtue of (11b) of Lemma 2.4 and the fact that a is of class **SL**, it follows that for each fixed $s \geqslant 0$ the mapping $\bar{a}(\cdot, s)$ belongs to **S**, since for every real constant r it holds:

$$\int_0^t (\bar{a}(\tau, s) + r)\,\mathrm{d}\tau \geqslant \int_0^t (a(\tau, s) + r - M(s))\,\mathrm{d}\tau - \rho_\mu(M(s))$$

and thus $\int_0^{+\infty} (\bar{a}(\tau, s) + r)\,\mathrm{d}\tau = +\infty$. Consequently, $\bar{a}$ is of class **SL**.

The latter in conjunction with (28), (35) asserts by virtue of Lemma 2.6 that the origin of the closed-loop system (19) with $u = \bar{k}(t)\bar{x}$ is $\bar{\Pi}_t$-GAS. Particularly, there exists a function $D \in \mathcal{NL}$ satisfying (18a) and in such a way that the solution $\bar{x}(t)$ of the closed-loop system (19) with $u = \bar{k}(t)\bar{x}$ satisfies the estimation (18b) with $a := \bar{a}$. Finally, the desired analogues of inequalities (21) for the dynamics $\bar{f}$, $\bar{g}$ and the feedback $\bar{k}$ are straightforward consequences of (7a), (11d), (11e), (21) and (22). □

As a consequence of Lemma 3.1 we obtain the following result, which provides sufficient conditions for the stabilization of a time-varying system that contains unknown parameters.

Corollary 3.2

Consider the system (1) where each f_i is measurable in t, continuous in θ and locally Lipschitz with respect to rest variables satisfying $f_i(t, \theta, 0, \ldots, 0) = 0$ for all $(t, \theta) \in \Re^+ \times \Re^l$ and each d_i belongs to $\mathbf{L} \cap C^j(\Re^+)$ $(j \geqslant 1)$. Furthermore, suppose that there exist a function $\rho \in \mathcal{N}$ and constants $K > 0$, $r \geqslant 0$ in such a way that

$$\sup_{(x_1,\ldots,x_1)\neq(0,\ldots,0)} \frac{|f_i(t, \theta, x_1, \ldots, x_i)|}{|(x_1, \ldots, x_i)|} \leqslant \rho(|\theta|)\exp\{rt\}|d_i(t)|,$$

$$\forall (t, \theta, x) \in \Re^+ \times \Re^l \times \Re^n \quad (i = 1, \ldots, n) \tag{37}$$

$$|d_{i-1}(t)| \leqslant K|d_i(t)|\exp\{rt\}, \quad \forall t \geqslant 0 \quad (i = 2, \ldots, n) \tag{38}$$

Then

- For every pair of constants $R > 0$ and $\beta \geqslant 0$ and for every function $\zeta \in \mathcal{N}$ with $\zeta(0) \geqslant R$, there exist a C^j mapping $k : \Re^+ \to \Re^{1\times n}$, a positive definite C^1 time-varying matrix $P(t) \in \Re^{n\times n}$, functions $a \in \mathbf{SL}$, $\bar{\rho} \in \mathcal{N}$ and positive constants $\sigma, \bar{r}, K_i$ $(i = 1, 2, 3)$ such that, if we define:

$$\Pi_t = \{(\theta; \eta) \in \Re^l \times \Re^n : R\exp\{-\beta t\} \leqslant \eta_i \leqslant \zeta(|\theta|)\exp\{\beta t\}, i = 1, \ldots, n\} \tag{39}$$

  the following hold for all $(t; x; \theta, \eta) \in \Re^+ \times \Re^n \times \Pi_t$:

$$K_1 \exp\{-\sigma t\}|x|^2 \leqslant x^{\mathrm{T}} P(t) x \leqslant K_2|x|^2 \tag{40}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{T}} P(t) x\right|_{(1), u = k(t)x} \leqslant -a(t, |\theta|) x^{\mathrm{T}} P(t) x \tag{41}$$

$$|k(t)| + |\dot{k}(t)| \leqslant K_3 \exp\{\bar{r}t\} \tag{42}$$

$$|a(t, s)| \leqslant \bar{\rho}(s)\exp\{\bar{r}t\} \quad \text{and} \quad a(t, s) \leqslant \bar{\rho}(s)\exp\{\bar{r}t\}|d_n(t)|, \quad \forall s \geqslant 0 \tag{43}$$

- It turns out that the origin of the closed-loop systems (1) with (2) is Π_t-GAS. Particularly, there exists a function $D \in \mathcal{NL}$, in such a way that (18a) is satisfied and in such a way that the solution $x(t)$ of the closed-loop system (1) with $u = k(t)x$ satisfies the estimation (18b) with $a(\cdot)$ as defined in (41) and (43), for every pair $(\theta(\cdot), \eta(\cdot)) \in L^\infty([t_0, +\infty)) \times L^\infty_{loc}([t_0, +\infty))$, with $(\theta(t), \eta(t)) \in \Pi_t$ for all $t \geqslant t_0$.

Proof
The proof will be made by induction.

Step 1: Consider the one-dimensional subsystem:

$$\dot{x}_1 = f_1(t, \theta, x_1) + \eta_1 d_1(t) x_2 \quad \text{with } x_2 \text{ as input} \tag{44}$$

Define $V(t, x_1) := x_1^2$. Since $d_1 \in \mathbf{L}$, Lemma 2.4 guarantees the existence of a C^∞ function $E : \Re^+ \times \Re \to \Re$, such that (11a)–(11e) hold for $\mu := r$ and $d \equiv d_1$, where the constant r is defined in (37), (38). Let $R > 0$, $\beta \geqslant 0$ be arbitrary constants and let $\zeta \in \mathcal{N}$ with $\zeta(0) \geqslant R$. Define

$$k_1(t) := -\frac{1}{R} \exp\{\beta t\} E(t, d_1(t)) \tag{45}$$

Since (7a) and (11d)–(11e) hold with $d \equiv d_1$, it follows from definition (45) that (42) holds for $k(t) := k_1(t)$ for appropriate constants $K_3, \bar{r} > 0$. We estimate the derivative $\dot{V}$ of $V(t, x_1)$ along the trajectory of (44) with $x_2 = k_1(t)x_1$ assuming that $(\theta(t), \eta_1(t)) \in \Pi_t$ for all $t \geqslant t_0$, where Π_t is the set defined in (39) for $n = 1$, namely:

$$\Pi_t := \{(\theta, \eta_1) \in \Re^l \times \Re : R \exp\{-\beta t\} \leqslant \eta_1 \leqslant \zeta(|\theta|) \exp\{\beta t\}\} \tag{46}$$

We get by using (37):

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} V(t, x_1) \right|_{(44), x_2 = k_1(t) x_1} \leqslant -a_1(t, |\theta|) V(t, x_1) \tag{47a}$$

$$a_1(t, s) := d_1(t) E(t, d_1(t)) - \max\{0, 2\rho(s) \exp\{rt\} |d_1(t)| - d_1(t) E(t, d_1(t))\} \tag{47b}$$

Obviously, by (11a) of Lemma 2.4 the function $d_1(\cdot) E(\cdot, d_1(\cdot))$ is of class $\mathbf{S}$. Exploiting (11b) and definition (47b) we can establish, as precisely made in the proof of Lemma 3.1, that the mapping a_1 is of class $\mathbf{SL}$. This establishes (41) with $x := x_1$, $P(t) := 1$ and $a(\cdot) := a_1(\cdot)$. Inequalities (43) for $a_1(\cdot)$ are consequences of definition (47b) of $a_1(\cdot)$, in conjunction with the fact that (7a) and (11d)–(11e) hold with $d \equiv d_1$. It turns out by Lemma 2.6 that the origin of the closed-loop system (44) with $x_2 = k_1(t)x_1$ is Π_t-GAS. Particularly, there exists a function $D \in \mathcal{NL}$, in such a way that (18b) is satisfied and in such a way that the solution $x_1(t)$ of the closed-loop system (44) with $x_2 = k_1(t)x_1$ satisfies the estimation (18b) with $a(\cdot) := a_1(\cdot)$, for every pair $(\theta(\cdot), \eta(\cdot)) \in L^\infty([t_0, +\infty)) \times L^\infty_{loc}([t_0, +\infty))$, with $(\theta(t), \eta(t)) \in \Pi_t$ for all $t \geqslant t_0$.

Step 2: Next, consider the two-dimensional subsystem:

$$\dot{x}_1 = f_1(t, \theta, x_1) + \eta_1 d_1(t) x_2, \quad \dot{x}_2 = f_2(t, \theta, x_1, x_2) + \eta_2 d_2(t) x_3$$
$$\text{with } x_3 \text{ as input} \tag{48}$$

We claim that Assumptions Al and A2 of Lemma 3.1 are fulfilled for (48). Indeed, as established in Step 1, A1 is fulfilled for $k(t) := k_1(t)$, $V(t, x_1) := x_1^2$, $a(\cdot) := a_1(\cdot)$ and Π_t as defined by (46). Furthermore, (21b) is a consequence of (37), (38) and (46). Indeed, combining (37),(38) and (46) it follows:

$$|\eta_1| |d_1(t)| + \sup_{x_1 \neq 0} \frac{|f_1(t, \theta, x_1)|}{|x_1|} \leqslant \tilde{\rho}(|\theta|) \exp\{\tilde{r} t\} |d_2(t)|, \quad \forall (t; x; \theta, \eta_1) \in \Re^+ \times \Re^n \times \Pi_t$$

for $\tilde{\rho}(s) := (1+K)(\rho(s)+\zeta(s))$, $\tilde{\rho} \in \mathcal{N}$ and $\tilde{r} := 2r+\beta$. Inequalities (21a), (21c) are consequences of (38), (42) and (43) with $k(\cdot) := k_1(\cdot)$ and $a(\cdot) := a_1(\cdot)$. Finally, (21d) is an immediate consequence of our assumption (37).

It follows from Lemma 3.1 that there exist a C^1 positive definite matrix $P : \Re^+ \to \Re^{2\times 2}$, a C^j mapping $k : \Re^+ \to \Re^{1\times 2}$, functions $a \in \mathbf{SL}$ and $\bar{\rho} \in \mathcal{N}$, constants $K_i > 0$ $i = 1, 2, 3$, σ, $\bar{r}$, such that inequalities (40)–(43) are satisfied for all $(t; x_1, x_2; \theta, \eta_1, \eta_2) \in \Re^+ \times \Re^2 \times \Pi_t$, where Π_t is the set defined in (39) for $n = 2$. It turns out by Lemma 2.6 that the origin of the closed-loop system (48) with

$$x_3 = k(t)\big(x_1,\, x_2\big)$$

is Π_t-GAS. Particularly, there exists a function $D \in \mathcal{NL}$, in such a way that (18a) is satisfied and in such a way that the solution $x_3(t)$ of the closed-loop system (48) with

$$x_3 = k(t)\big(x_1,\, x_2\big)$$

satisfies the estimation (18b).

The proof is completed by inductive use of Lemma 3.1. $\square$

Example 3.3
Consider the linear time-varying system:

$$\dot{x}_i = d(t)x_{i+1} \ (i = 1, \ldots, n-1); \quad \dot{x}_n = u \tag{49}$$

where $d(t) := H\exp\{At\}z_0$, $z_0 \neq 0$ is the output of an autonomous linear observable system $\dot{z} = Az$, $y = Hz$. The system has the structure of (1) with $d_i(t) \equiv d(t)$, $f_i \equiv 0$ and $\eta_t(t) \equiv 1$ for $i = 1, \ldots, n$. Notice that, by virtue of Example 2.3, $d \in \mathbf{L} \cap C^\infty(\Re^+)$. It turns out from Corollary 3.2 that there exists a C^∞ mapping $k : \Re^+ \to \Re^{1\times n}$, such that $0 \in \Re^n$ is GAS with respect to the closed-loop system (49) with $u = k(t)x$.

4. APPLICATIONS

The results of Section 3 are used to derive sufficient conditions for global stabilization by means of time-varying feedback for systems of form (3), as well as for the solution of the tracking problem for the case of system (5).

4.1. Stabilization by smooth time-varying state feedback

We focus our attention to the case of systems (3) that in general cannot be stabilized by a C^0 static (time-invariant) feedback. We show that, under certain hypotheses, there exists a C^∞ time-varying feedback exhibiting non-uniform in time global asymptotic stability of (3) at zero.

Proposition 4.1
Consider system (3), where each F_i is continuous in θ, locally Lipschitz with respect to rest variables and satisfies $F_i(\theta, z, u_1, 0, \ldots, 0) = 0$ for all $(\theta, z, u_1) \in \Re^k \times \Re^m \times \Re$ and each G_i, has

the form:

$$G_i(z,u_1) := q_{i1} + q_{i2}(cz + c_0u_1) \tag{50}$$

for certain $c \in \Re^{1\times m}$ and constants $c_0 \in \Re$, $q_{i1}, Q_{i2} \geqslant 0$ in such a way that

$$q_{i1} + q_{i2} > 0 \quad (i = 1,\dots,n) \tag{51}$$

Moreover, we make the following assumptions:

A1. The pair (A,b) is stabilizable.

A2. There exists a function $\rho \in \mathcal{N}$ such that:

$$\sup_{(x_1,\dots,x_i)\neq 0} \frac{|F_i(\theta,z,u_1,x_1,\dots,x_i)|}{|(x_1,\dots,x_i)|} \leqslant \rho(|(\theta;z,u_1)|)$$

$$\forall(\theta,z,u_1) \in \Re^k \times \Re^m \times \Re \;\; (i = 1,\dots,n) \tag{52}$$

A3. The following condition holds:

$$|c_0| + \sum_{i=0}^{m-1} |cA^ib| > 0 \tag{53}$$

Under the previous assumptions, there exists a pair of C^∞ functions $U_1(t,z,x)$ and $U_2(t,x)$, with $U_1(\cdot,0,0) = U_2(\cdot,0) = 0$, in such a way that for any $\theta(\cdot) \in L^\infty(\Re^+)$, the origin $0 \in \Re^m \times \Re^n$ for the closed-loop system (3) with (4) is GAS. Particularly, for any $\theta(\cdot) \in L^\infty(\Re^+)$ the corresponding solution $(z(t),x(t))$ of the closed-loop system satisfies:

$$\lim_{t\to+\infty} \exp\{\varepsilon t\}|(z(t),x(t))| = 0 \tag{54}$$

for $\varepsilon > 0$ small enough.

Proof

Using A1 and applying a linear change of co-ordinates, subsystem (3a) is written as

$$\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \begin{pmatrix}b_1\\ 0\end{pmatrix}u_1$$

$$z_1 \in \Re^{m_1}, \;\; z_2 \in \Re^{m-m_1} \tag{55}$$

where A_{11}, A_{12}, A_{22} are constant real matrices, in such a way that

$$\operatorname{rank}\{b_1, A_{11}b_1, \dots, A_{11}^{m_1-1}b_1\} = m_1 \tag{56}$$

and A_{22} is Hurwitz, namely:

$$|\exp\{A_{22}(t-t_0)\}| \leqslant M_1 \exp\{-l(t-t_0)\}, \quad \forall t \geqslant t_0 \tag{57}$$

for some constants M_1, $l > 0$. In the above co-ordinates each term G_t takes the form:

$$G_i(z,u_1) = q_{i1} + q_{i2}(c_1z_1 + c_2z_2 + c_0u_1), \quad i = 1,\dots,n \tag{58}$$

for appropriate vectors $c_1 \in \Re^{1\times m_1}$, $c_2 \in R^{1\times(m-m_1)}$. Notice that (53) implies that

$$|c_0| + |c_1| > 0 \tag{59}$$

Without any loss of generality we may assume in the sequel that the pair (A,b) has the canonical form (55) and simultaneously (56)–(59) are fulfilled. The proof is separated into three parts.

First, in Part I we construct the first component $U_1(t,z,x)$ of the desired feedback stabilizer (4) associated with appropriate mappings $p:\Re^+ \to \Re^+$, $T_0 \in \mathcal{N}\mathcal{N}$ and $U_0:\Re^+ \times \Re^m \to \Re$, such that

$$U_1(t,z,x) = U_0(t,z), \quad \text{for } |x| \geqslant \exp\{-p(t)\} \tag{60}$$

and in such a way that the solution $z(t)$ of the closed-loop system (55) with $u_1 = U_0(t,z)$ satisfies:

$$G_i(z(t), U_0(t,z(t))) \geqslant q\exp\{-\beta t\}, \quad i = 1,\ldots,n \tag{61a}$$

$$\text{for } t \geqslant T_0(t_0, |z(t_0)|) \tag{61b}$$

for some appropriate constants $q, \beta > 0$, where G_i's are defined by (58). Part II is devoted to the construction of the second component $U_2(t,x)$ of the desired feedback (4), by employing the results of Section 3. To be more precise, instead of subsystem (3b), we first deal with the parameterized system

$$\dot{x}_i = F_i(\tilde{\theta}, x_1,\ldots,x_i) + \eta_i x_{i+1}, \quad i = 1,\ldots,n; \;\; u_2 := x_{n-1} \tag{62a}$$

$$\tilde{\theta} := (\theta, z, u_1) \tag{62b}$$

that obviously has the same structure with system (1). It turns out by Corollary 3.2 that there exists a feedback law $u_2 = U_2(t,x)$, such that $0 \in \Re^n$ is Π_t-GAS for the closed-loop system (62) with (4), where

$$\Pi_t := \{(\tilde{\theta},\eta) \in \Re^{k+m+1} \times \Re^n, \; q\exp\{-\beta t\} \leqslant \eta_1 \leqslant \zeta(|\tilde{\theta}|)\exp\{\beta t\}, \; i = 1,\ldots,n\} \tag{63a}$$

$$\zeta(s) := (1+s)(|c_0| + |c| + 1)\left(1 + \sum_{i=1}^{n}(q_{i1} + q_{i2})\right) \tag{63b}$$

$$\eta := (\eta_1,\ldots,\eta_n) \tag{63c}$$

Finally, in Part III of the proof we establish that $0 \in \Re^m \times \Re^n$ is GAS for the closed-loop system (3) with (4), by taking into account the analysis made in Parts I and II.

Part I: Construction of U_1.

Notice first that (56) guarantees the existence of a constant vector $f \in \Re^{1\times m_1}$, in such a way that, by denoting

$$A_f := A_{11} + b_1 f \tag{64}$$

it holds:

$$|\exp\{A_f(t-t_0)\}| \leqslant M_2\exp\{-2l(t-t_0)\}, \quad \forall t \geqslant t_0 \tag{65}$$

for certain constant $M_2 > 0$. Define

$$c_f := c_1 + c_0 f \tag{66}$$

and let β be a constant with

$$0 < \beta \leqslant \frac{1}{2} \tag{67a}$$

$$c_0 - c_f(A_f + \beta I)^{-1} b_1 \neq 0 \tag{67b}$$

(A detailed establishment of the existence of β, satisfying (67a), (67b) is given in Appendix A).

By (67b) we may define

$$\lambda := \frac{1}{c_0 - c_f(A_f + \beta I)^{-1} b_1} \tag{68}$$

and let

$$U_0(t,z) := f z_1 + \lambda \exp\{-\beta t\} \tag{69}$$

Notice that the z_1 component of the solution of the closed-loop system (55) with $u_1 = U_0(t,z)$, satisfies:

$$\begin{aligned} z_1(t) = {} & \exp\{A_f(t-t_0)\} z_1(t_0) + \int_{t_0}^{t} \exp\{A_f(t-\tau)\} A_{12} z_2(\tau)\, \mathrm{d}\tau \\ & + \lambda \exp\{-\beta t_0\}(A_f + \beta I)^{-1} \exp\{A_f(t-t_0)\} b_1 \\ & - \lambda \exp\{-\beta t\}(A_f + \beta I)^{-1} b_1 \end{aligned} \tag{70}$$

hence, we evaluate for $i = 1, \ldots, n$:

$$G_i(z(t), U_0(t, z(t))) = q_{i1} + \lambda q_{i2}(c_0 - c_f(A_f + \beta I)^{-1} b_1) \exp\{-\beta t\} + q_{i2}\xi(t) \tag{71}$$

where

$$\begin{aligned} \xi(t) := {} & c_f \exp\{A_f(t-t_0)\} z_1(t_0) + \lambda \exp\{-\beta t_0\} c_f (A_f + \beta I)^{-1} \exp\{A_f(t-t_0)\} b_i \\ & + \left(c_f \int_{t_0}^{t} \exp\{A_f(t-\tau)\} A_{12} \exp\{A_{22}(\tau - t_0)\}\, \mathrm{d}\tau \right. \\ & \left. + c_2 \exp\{A_{22}(t-t_0)\} \right) z_2(t_0) \end{aligned} \tag{72}$$

By (57), (65) and (72) we estimate

$$|\xi(t)| \leqslant M_3(1 + |z(t_0)|) \exp\{-l(t-t_0)\}, \quad \forall t \geqslant t_0 \tag{73}$$

for some constant $M_3 > 0$. From (67a), (68), (71) and (73) it follows that (61) holds, where

$$q := \frac{1}{2} \min_{i=1,\ldots,n} (q_{i1} + q_{i2})$$

which according to hypothesis (51) is strictly positive and

$$T_0(t,s) := 2t + \frac{2}{l} \log(M_4(1+s)) \tag{74}$$

for certain constant $M_4 > 1$.

We are now in a position to build the desired map $U_1(\cdot)$. Consider a non-decreasing C^∞ function $\phi : \Re \to \Re^+$ with

$$\phi(0) = 0; \quad \phi(s) = 1 \quad \text{for } s \geqslant 1 \tag{75}$$

and define:

$$U_1(t,z,x) := fz_1 + \lambda\phi(|x|^2 \exp\{2p(t)\})\exp\{-\beta t\} \tag{76a}$$

$$p(t) := \exp\{\exp\{t\}\} \tag{76b}$$

where λ is given by (68). Obviously U_1 is C^∞, and (60) is an immediate consequence of (75) and definitions (69) and (76a) of U_0 and U_1, respectively. Moreover, (76b) guarantees the existence of a function $\delta \in \mathcal{N}\mathcal{N}$, such that

$$\sup_{t\geqslant 0}(s\exp\{\mu t\} - p(t)) \leqslant \delta(\mu, s), \quad \forall \mu, s \geqslant 0 \tag{77}$$

(for example we may take $\delta(\mu,s) := \exp\{2\mu^2\}s(1+s)^\mu$). Estimation (77) above is used in Part III for the stability analysis of the closed-loop system.

Part II: *Construction of* U_2.

In order to define $U_2(\cdot)$, we consider the parameterized system (62). By A2 all assumptions of Corollary 3.2 hold for (62) with $d_1(t) = \cdots = d_n(t) \equiv 1 \in \mathbf{L} \cap C^\infty(\Re^+)$ and therefore there exists a C^∞ mapping $k : \Re^+ \to \Re^{1\times m}$, functions $a \in \mathbf{SL}$ and $\bar{\rho} \in \mathcal{N}$, and constants $\bar{r}, K > 0$, such that the following hold for all $(t,s) \in (\Re^+)^2$:

$$|k(t)| + |\dot{k}(t)| \leqslant K\exp\{\bar{r}t\} \tag{78a}$$

$$|a(t,s)| \leqslant \bar{\rho}(s)\exp\{\bar{r}t\} \tag{78b}$$

Moreover, Corollary 3.2 asserts the existence of a function $D \in \mathcal{N}\mathcal{L}$, in such a way that the following estimate holds:

$$\lim_{t\to+\infty} \exp\{\varepsilon t\}D(s,t) = 0 \quad \text{for all } \varepsilon, s \geqslant 0 \tag{79a}$$

$$|x(t)| \leqslant D(\|\tilde{\theta}\|, t)\exp\left\{\frac{1}{2}\int_0^{t_0} |a(\tau, \|\tilde{\theta}\|)|\,\mathrm{d}\tau\right\}|x_0|, \quad \forall t \geqslant t_0 \tag{79b}$$

where $x(t)$ denotes the solution of the closed-loop system (62) with $u_2 = k(t)x$, corresponding to a pair of inputs $(\tilde{\theta}(\cdot), \eta(\cdot)) \in L^\infty([t_0,+\infty)) \times L^\infty_{loc}([t_0,+\infty))$ with $(\tilde{\theta}(t), \eta(t)) \in \Pi_t$ for all $t \geqslant t_0$, initiated from x_0 at time t_0, where $\tilde{\theta}$, Π_t and η are defined by (62b), (63a) and (63c), respectively. The desired feedback $U_2(\cdot)$ is defined as:

$$U_2(t,x) := k(t)x \tag{80}$$

Part III: Stability analysis for the closed-loop system.

We claim that for any $\theta(\cdot) \in L^\infty(\Re^+)$ the origin $0 \in \Re^m \times \Re^n$ is GAS with respect to the closed-loop system (3) with (4), where $U_1(\cdot)$ and $U_2(\cdot)$ are defined by (76) and (80), respectively, namely, with respect to

$$\dot{z} = Az + bU_1(t,z,x) \tag{81a}$$

$$\dot{x}_i = F_i(\theta, z, U_1(t,z,x), x_1, \ldots, x_i) + G_i(z, U_1(t,z,x))x_{i+1}, \quad i = 1, \ldots, n$$

$$x_{n+1} = U_2(t,x) \tag{81b}$$

Fact I
The solution $(z(t), x(t))$ of (81) initiated from (z_0, x_0) at time t_0 and corresponding to some $\theta(\cdot) \in L^\infty(\mathfrak{R}^+)$ satisfies:

$$|z(t)| + |U_1(t, z(t), x(t))| \leqslant C_1 \exp\{-\beta(t - t_0)\}(|z_0| + 1), \quad \forall t \geqslant t_0 \tag{82}$$

$$\exp\{-\psi(|z_0| + \|\theta\|) \exp\{\bar{r}t\}\}|x_0| \leqslant |x(t)| \leqslant \exp\{\psi(|z_0| + \|\theta\|) \exp\{\bar{r}t\}\}|x_0|, \ \forall t \geqslant t_0 \tag{83}$$

for certain $C_1 > 0$ and $\psi \in \mathcal{N}$, where $\beta, \bar{r}$ are the constants involved in (67) and (78a), (78b), respectively.

Indeed, (82) is a straightforward consequence of (57), (64), (65), (67a), (75), (76a) and application of Gronwall's inequality in (81a). By virtue of (50), (52), (63b) and (78a), we have:

$$\begin{aligned}|\dot{x}(t)| \leqslant C_2(&\rho(|(\theta(t), z(t), U_1(t, z(t), x(t)))|) \\ &+ \zeta(|(z(t), U_1(t, z(t), x(t)))|)) \exp\{\bar{r}t\}|x(t)|\end{aligned}$$

for certain $C_2 > 0$. The desired inequality (83) is a consequence of the inequality above. (82) and application of Gronwall's inequality in (81b). For completeness, we note that (83) is valid by taking

$$\psi(s) := C_3(\rho((1 + C_1)(1 + s)) + \zeta((1 + C_1)(1 + s))) \tag{84}$$

for appropriate constant $C_3 > 0$.

Fact II
There exist mappings $T : \mathfrak{R}^+ \times (\mathfrak{R}^n \setminus \{0\}) \times \mathfrak{R}^m \times L^\infty(\mathfrak{R}^+) \to \mathfrak{R}^+$, $\tilde{\psi} \in \mathcal{N}$ and $\tilde{D} \in \mathcal{NL}$ with

$$\lim_{t \to +\infty} \exp\{\varepsilon t\} \tilde{D}(s, t) = 0 \quad \forall \varepsilon, s \geqslant 0 \tag{85}$$

such that the solution $x(t)$ of subsystem (81b), corresponding to $\theta(\cdot) \in L^\infty(\mathfrak{R}^+)$ satisfies the estimate:

$$\begin{aligned}&|x(t)| \leqslant \tilde{D}(\|\theta\| + |z_0|, t) \exp\{\tilde{\psi}(\|\theta\| + |z_0|) \exp\{\bar{r}T\}\}|x_0| \\ &\text{for all } t \geqslant T := T(t_0, x_0, z_0; \theta(\cdot)), \ x(t_0) = x_0 \neq 0 \text{ and } t_0 \geqslant 0\end{aligned} \tag{86}$$

In order to establish (86), notice first that the left-hand side inequality of (83) and definition (76b) of $p(\cdot)$ yield:

$$|x(t)| \exp\{p(t)\} \to +\infty \quad \text{as } t \to +\infty \quad \text{for any initial } x_0 \neq 0 \tag{87}$$

By taking (87) into account we define a map $T_1 : \mathfrak{R}^+ \times (\mathfrak{R}^n \setminus \{0\}) \times \mathfrak{R}^m \times L^\infty(\mathfrak{R}^+) \to \mathfrak{R}^+$, which satisfies

$$T_1 := T_1(t_0, x_0, z_0, \theta(\cdot)) = \min\{t \geqslant t_0 : |x(s)| \exp\{p(s)\} \geqslant 1, \ \forall s \geqslant t\} \tag{88}$$

where $p(\cdot)$ is defined in (76b). Obviously, the mapping $T_1(\cdot)$ is well-defined and, according to definition (88), it holds $|x(T_1)| = \exp\{-p(T_1)\}$, for the case $T_1 > t_0$. Moreover, we have by (60) and (88):

$$U_1(t, z(t), x(t)) = U_0(t, z(t)), \quad \forall t \geqslant T_1 \tag{89}$$

Define:

$$T = T(t_0, x_0, z_0, \theta(\cdot)) := T_0(T_1(t_0, x_0, z_0, \theta(\cdot)), C_1(|z_0| + 1)) \tag{90}$$

where the map T_0 is defined in (74) and C_1 is the constant involved in (82). Notice that (74), (82) and (88) imply:

$$T(t_0, x_0, z_0, \theta(\cdot)) \geqslant T_0(T_1, |z(T_1)|) \geqslant t_0 \tag{91}$$

We also define:

$$\tilde{\psi}(s) := \frac{1}{2\bar{r}}\bar{\rho}((1 + C_1)(1 + s)) + \psi(s) \tag{92a}$$

$$\tilde{D}(s, t) := D((1 + C_1)(1 + s), t) \tag{92b}$$

where $\bar{r}$, $\bar{\rho}$, D and ψ are defined in (78a), (78b), (79a), (79b) and (84), respectively. Obviously, since $\bar{\rho}$, $\psi \in \mathcal{N}$ and $D \in \mathcal{NL}$ and satisfies (79a), we have that $\tilde{\psi}$ and $\tilde{D}$ are functions of class $\mathcal{N}$ and $\mathcal{NL}$, respectively and satisfy (85).

We now take into account (61), (89), (91) and definitions (50), (62b), (63b) and (90) of G_i, $\tilde{\theta}$, ζ and T, respectively, and estimate

$$q\exp\{-\beta t\} \leqslant G_i(z(t), U_1(t, z(t), x(t))) \leqslant \zeta(|(z(t), U_1(t, z(t), x(t)))|) \leqslant \exp\{\beta t\}\zeta(|\tilde{\theta}(t)|), \quad \forall t \geqslant T \tag{93}$$

Consequently, by (93) and invoking (62b), (63a) we have:

$$(\theta(t), z(t), U_1(t, z(t), x(t)), G(z(t), U_1(t, z(t), x(t)))) \in \Pi_t, \quad \forall t \geqslant T \tag{94}$$

where $G(z, u_1) := (G_1(z, u_1), \ldots, G_n(z, u_1))^{\mathrm{T}}$. The desired (86) is a consequence of (62b), (78b), (79b), (82), right-hand side of (83), (94) and definitions (62b) and (92a), (92b) of $\tilde{\theta}$ and $\tilde{\psi}, \tilde{D}$, respectively; particularly, we have:

$$\begin{aligned}
|x(t)| &\overset{(79b)}{\leqslant} D(\|\tilde{\theta}\|, t)\exp\left\{\frac{1}{2}\int_0^{\mathrm{T}} |a(\tau, \|\tilde{\theta}\|)|\,\mathrm{d}\tau\right\}|x(T)| \\
&\overset{(78b)}{\leqslant} D(\|\tilde{\theta}\|, t)\exp\left\{\frac{1}{2\bar{r}}\bar{\rho}(\|\tilde{\theta}\|)\exp\{\bar{r}t\}\right\}|x(T)| \\
&\overset{(62b),(82),(92b)}{\leqslant} \tilde{D}(\|\theta\| + |z_0|, t)\exp\left\{\frac{1}{2\bar{r}}\bar{\rho}((1 + C_1)(\|\theta\|\right. \\
&\qquad \left. + |z_0| + 1))\exp\{\bar{r}T\}\right\}|x(T)| \\
&\overset{(83),(92a)}{\leqslant} \tilde{D}(\|\theta\| + |z_0|, t)\exp\{\tilde{\psi}(\|\theta\| + |z_0|)\exp\{\bar{r}T\}\}|x_0|
\end{aligned}$$

for any non-zero $x_0 \in \Re^n$ and for all $t \geqslant T := T(t_0, x_0, z_0, \theta(\cdot))$.

As a consequence of Facts I and II we obtain:

Fact III

Estimation (86) is valid for all $t \geqslant t_0$ and every non-zero x_0.

Indeed, from (79b) we have that:

$$1 \leqslant D(s,t)\exp\left\{\frac{1}{2}\int_0^t |a(\tau,s)|\,\mathrm{d}\tau\right\} \quad \text{for all } (t,s) \in (\Re^+)^2$$

which by virtue of (78b) gives:

$$1 \leqslant D(s,t)\exp\left\{\frac{1}{2\bar{r}}\bar{\rho}(s)\exp\{\bar{r}T\}\right\}, \quad \text{for all } 0 \leqslant t \leqslant T,\ s \geqslant 0,\ T \geqslant 0$$

The right-hand side inequality of (83), in conjunction with the previous inequality yield:

$$\begin{aligned} |x(t)| &\leqslant \exp\{\psi(\|\theta\| + |z_0|)\exp\{\bar{r}T\}\}|x_0| \\ &\leqslant D((1+C_1)(\|\theta\| + |z_0| + 1), t)\exp\left\{\left[\frac{1}{2\bar{r}}\bar{\rho}((1+C_1)(\|\theta\| + |z_0| + 1))\right.\right. \\ &\quad \left.\left. + \psi(\|\theta\| + |z_0|)\right]\exp\{\bar{r}T\}\right\}|x_0| \quad \text{for all } t_0 \leqslant t \leqslant T,\ x_0 \in \Re^n,\ T \geqslant t_0 \end{aligned}$$

and this by virtue of (92a), (92b) and Fact II guarantees that estimation (86) is valid for all $t \geqslant t_0$ and $x_0 \neq 0$.

We complete the proof by establishing the following fact.

Fact IV

There exists a function $\Delta \in \mathcal{N}\mathcal{N}\mathcal{L}$ with

$$\lim_{t\to\infty} \exp\{\varepsilon t\}\Delta(r,s,t) = 0, \quad \forall \varepsilon, r, s \geqslant 0 \tag{95}$$

such that the solution $x(t)$ of subsystem (81b), corresponding to $\theta(\cdot) \in L^\infty(\Re^+)$ and initiated from x_0 at time t_0, satisfies the estimate:

$$|x(t)| \leqslant \Delta(\|\theta\| + |(z_0, x_0)|, t_0, t), \quad \forall t \geqslant t_0 \tag{96}$$

In order to prove (96), we first use Fact III and take into account definitions (74), (88) and (90) of the mappings T_0, T_1 and T, respectively. We find $T = 2T_1 + (2/l)\log(M_4(1 + C_1 + C_1|z_0|))$ and, since (86) holds for all $t \geqslant t_0$, we get

$$|x(t)| \leqslant \tilde{D}(\|\theta\| + |z_0|, t)\exp\{\gamma(\|\theta\| + |z_0|)\exp\{2\bar{r}T_1\}\}|x_0|, \ \forall t \geqslant t_0,\ x_0 \neq 0 \tag{97}$$

for certain $\gamma \in \mathcal{N}$ (for example we can take $\gamma(s) := \tilde{\psi}(s)(M_4(1 + C_1 + C_1 s))(2\bar{r}/l))$. We now recall the precise definition (88) of the map $T_1 := T_1(t_0, x_0, z_0, \theta(\cdot))$. As was pointed out, it holds that

$$|x(T_1)| = \exp\{-p(T_1)\}, \quad \text{when } T_1 > t_0,\ x_0 \neq 0$$

and the latter, in conjunction with the left-hand side inequality of (83), yields

$$|x_0| \leqslant \exp\{-p(T_1) + \exp\{\bar{r}T_1\}\psi(\|\theta\| + |z_0|)\}, \ \text{provided that } T_1 > t_0,\ x_0 \neq 0 \tag{98}$$

Using (97), (98) and recalling (77), we find an estimation of the solution $x(\cdot)$ being independent of T_1

$$|x(t)| \leqslant \tilde{D}(\|\theta\| + |z_0|, t) \exp\{\delta(2\bar{r}, \psi(\|\theta\| + |z_0|) + \gamma(\|\theta\| + |z_0|))\}, \quad \forall t \geqslant t_0$$

$$\text{provided that } T_1 = T_1(t_0, x_0, z_0, \theta(\cdot)) > t_0, \ x_0 \neq 0 \tag{99}$$

Combining (97) and (99) for the cases $T_1 = t_0$ and $T_1 > t_0$, respectively, and using the elementary inequalities $|z_0| \leqslant |(z_0, x_0)|$, $|x_0| \leqslant |(z_0, x_0)|$, as well as the fact that $x(t) \equiv 0$ for $x_0 = 0$, it follows that there exists an $\mathcal{N}\mathcal{N}\mathcal{L}$ function Δ, such that both (95) and (96) are fulfilled. For example, we may take:

$$\Delta(s, t_0, t) := \tilde{D}(s, t)[\exp\{\delta(2\bar{r}, \psi(s) + \gamma(s))\} + s \exp\{\gamma(s) \exp\{2\bar{r}t_0\}\}]$$

that obviously is of class $\mathcal{N}\mathcal{N}\mathcal{L}$ and which, by virtue of (85), satisfies (95). The desired GAS and exponential convergence at the origin for the closed-loop system (81) are direct consequences of (82), (95), (96) and Lemma 2.5. □

We use the result of Proposition 4.1 to give an alternative solution for the stabilization problem for a class of two-input systems that cannot be stabilized by means of a C^0 static state feedback, since they do not satisfy Brockett's necessary condition, Reference [19].

Example 4.2

We consider the global feedback stabilization problem for (5) with $f_i \equiv 0$ $(i = 1, \ldots, n)$. There are many contributions in the literature for the solution of this problem using periodic time-varying continuous feedback and periodically updated hybrid open loop/feedback controls (see References [9–15, 20] and references therein). An alternative solution is given here by employing the result of Proposition 4.1. Indeed, system (5) satisfies all hypotheses of Proposition 4.1 and thus there exists a time-varying C^∞ feedback law of form (4), which globally asymptotically stabilizes (5) at zero. Moreover, by virtue of (54) the rate of convergence is exponential. For the case $n = 2$, the analysis made in the proof of Proposition 4.1 asserts that the feedback law

$$U_1(t, z, x) = -2z - \exp\{-t\}\phi(\exp\{2p(t)\}(x_1^2 + x_2^2))$$
$$U_2(t, x) = -\exp\{10t\}(x_2 + \exp\{2t\}x_1)$$

where $p(t)$ is defined by (76b) and $\phi : \Re \to \Re^+$ is any non-decreasing C^∞ function that satisfies (75), exhibits global stabilization for (5) at zero and further (54) holds for all $\varepsilon < 1$.

Example 4.3

Consider the problem of controlling a mobile robot moving on an uneven surface described by the system (see Reference [21]):

$$\ddot{y}_1 = u_1, \quad \ddot{y}_2 = -\theta_1 y_2 + \theta_2 y_3 - v_2(y_1)y_2 - v_3(y_1)y_3 + u_2, \quad \dot{y}_3 = \dot{y}_1 y_2$$

where v_1, v_2 are C^∞ functions and θ_1, θ_2 are treated in Reference [21] as known constants. In Reference [21] a discontinuous time-invariant control law is proposed exhibiting the so-called 'almost exponential stability'. Here we consider θ_1, θ_2 as bounded time-varying unknown parameters and we strengthen the result in Reference [21] by proposing a C^∞ time-varying feedback that guarantees exponential convergence and global asymptotic stability. The system is equivalently written as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u_1 \end{aligned} \qquad \begin{aligned} \dot{x}_1 &= z_2 x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\theta_1 x_2 + \theta_2 x_1 - v_2(z_1)x_2 - v_3(z_1)x_1 + u_2 \end{aligned}$$

Obviously, for the system above all hypotheses of Proposition 4.1 are fulfilled, hence, the system is globally asymptotically stabilized at $0 \in \Re^5$ by means of C^∞ time-varying feedback of form (4). Particularly, we may apply

$$\begin{aligned} U_1(t,z,x) &= -6z_1 - 5z_2 - \exp\{-t\}\phi(\exp\{2p(t)\}(x_1^2 + x_2^2 + x_3^2)) \\ U_2(t,x) &= -\exp\{58t\}(x_3 + \exp\{10t\}x_2 + \exp\{12t\}x_1) \end{aligned}$$

with same $p(\cdot)$ and $\phi(\cdot)$ as those selected in Example 4.2. The proposed feedback exhibits global stabilization at zero and further (54) holds for all $\varepsilon < 1$.

Example 4.4
Likewise, we can handle the problem of controlling the Cartesian position and orientation of a surface vessel with two independent propellers (see Reference [21]):

$$\ddot{y}_1 = u_1, \quad \ddot{y}_2 = u_2, \quad \ddot{y}_3 = (u_1 + c\dot{y}_1)y_2 - c\dot{y}_3$$

Details are left to the reader.

4.2. Application to Tracking Problem

In this section we apply the result of Corollary 3.2 for the asymptotic tracking problem for the case of systems (5). The same problem has been studied in earlier works under the assumption that $f_i \equiv 0,\ i = 1, \ldots, n$ (see References [3–8] and references therein). Our results are based on a different set of hypotheses with those in the previously mentioned papers.

We assume that each term f_i $(i = 1, \ldots, n)$ vanishes at zero and is globally Lipschitz, namely, there is a positive constant L, such that:

$$\begin{aligned} &|f_i(x_1, \ldots, x_i) - f_i(y_1, \ldots, y_i)| \leqslant L|(x_1 - y_1, \ldots, x_i - y_i)| \\ &\forall (x_1, \ldots, x_i, y_1, \ldots, y_i) \in \Re^l \times \Re^l,\ i = 1, \ldots, n \end{aligned} \tag{100}$$

Consider a reference trajectory $(z_d(t), x_d(t)) = (z_d(t); x_{1d}(t), \ldots, x_{nd}(t)) \in \Re^{n-1},\ t \geqslant 0$ of system (5) namely:

$$\dot{z}_d = u_{1d}; \quad \dot{x}_{id} = u_{1d}(f_i(x_{1d}, \ldots, x_{id}) + x_{(i+1)}d), \quad 1 \leqslant i \leqslant n-1 \quad ; \ \dot{x}_{nd} = f_n(x_d) + u_{2d} \tag{101}$$

for certain reference control inputs u_{1d}, u_{2d} and denote the tracking error as

$$(z_e(t), e(t)) := (z(t) - z_d(t), x(t) - x_d(t))$$

where $(z(t), x(t))$ is any arbitrary solution of (5). Then $(z_e(t), e(t))$ satisfies:

$$\dot{z}_e = v_1 \tag{102a}$$

$$\begin{aligned} \dot{e}_i &= (u_{1d}(t) + v_1)(e_{i-1} + g_i(t, e_1, \dots, e_i)) + v_1(x_{(i+1)d}(t) \\ &\quad + f_i(x_{1d}(t), \dots, x_{id}(t))) \quad 1 \leqslant i \leqslant n-1 \\ \dot{e}_n &= g_n(t, e) + v_2 \end{aligned} \tag{102b}$$

where

$$g_i(t, e_1, \dots, e_i) := f_i(x_{1d}(t) + e_1, \dots, x_{id}(t) + e_i) - f_i(x_{1d}(t), \dots, x_{id}(t)), \quad i = 1, \dots, n \tag{103a}$$

$$v_1 := u_1 - u_{1d}, \quad v_2 := u_2 - u_{2d} \tag{103b}$$

The tracking problem (TP) is said to be *globally solvable* if there exists a pair of time-varying feedback controllers of the form

$$v_1 = U_1(t, z_e, e), \quad v_2 = U_2(t, z_e, e) \tag{104}$$

such that $0 \in \Re^{n+1}$ is GAS for the closed-loop system (102) with (104).

The result of the following proposition is a consequence of Corollary 3.2.

Proposition 4.5

Consider system (101), where each f_i $(i = 1, \dots, n)$ vanishes at zero and satisfies (100). Suppose that:

- $u_{1d}(\cdot)$ is of class $\mathbf{L}$ and $u_{2d}(\cdot)$ is measurable and locally essentially bounded.
- The $x_d(\cdot)$ component of the solution of (101) satisfies

$$|x_d(t)| \leqslant M \exp\{\lambda t\} \tag{105}$$

for some constants $M,\ \lambda \geqslant 0$.

Then there are C^1 mappings k_i $(i = 1, 2)$ such that the time-varying feedback law

$$v_1 = U_1(t, z_e) = k_1(t) z_e, \quad v_2 = U_2(t, e) = k_2(t) e \tag{106}$$

solves the TP globally.

Proof

We establish the existence of C^1 mappings k_i $(i = 1, 2)$ such that $0 \in \Re^{n+1}$ is GAS for the closed-loop system (102) with (106). By taking into account (100), definition (103a) of $g_i(\cdot)$ and our hypothesis $f_i(0, \dots, 0) = 0$, we have:

$$|g_i(t, e_1, \dots, e_i)| \leqslant L|(e_1, \dots, e_i)| \tag{107a}$$

$$|f_i(x_{1d}, \dots, x_{id})| \leqslant L|x_d| \tag{107b}$$

Consider first the subsystem (102b) with zero input v_1:

$$\dot{e}_i = u_{1d}(t) e_{i+1} + u_{1d}(t) g_i(t, e, \dots, e_i), \quad 1 \leqslant i \leqslant n-1; \quad \dot{e}_n = g_n(t, e) + v_2 \tag{108}$$

Because of (107a) and our assumption $u_{1d} \in \mathbf{L}$, system (108) satisfies all hypotheses of Corollary 3.2, hence, there exist a C^1 mapping $k : \Re^+ \to \Re^{1 \times n}$, a C^1 positive definite time-varying matrix

$P(\cdot) \in \Re^{n\times n}$, a function $a \in \mathbf{S}$ and constants $\sigma, \bar{r}, K_i > 0$ $(i = 1, 2, 3)$ such that:

$$K_1 \exp\{-\sigma t\}|e|^2 \leqslant e^{\mathrm{T}} P(t) e \leqslant K_2 |e|^2 \tag{109a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathrm{T}}(t)P(t)e(t)\bigg|_{(108), v_2 = k(t)e} \leqslant -a(t)e^{\mathrm{T}}P(t)e(t) \tag{109b}$$

$$|a(t)| + |k(t)| + |\dot{k}(t)| \leqslant K_3 \exp\{\bar{r}, t\} \tag{109c}$$

We define:

$$k_1(t) := -C_1 \exp\{C_2 t\}, \quad k_2(t) := k(t) \tag{110}$$

where $C_1, C_2 > 0$ are certain constants yet to be specified and $k(\cdot)$ as defined in (109). We are in a position to establish that $0 \in \Re^{n+1}$ is GAS for the closed-loop system (102) with (106), where k_i $(i = 1, 2)$ are given by (110). Notice first that the solution $z_e(\cdot)$ of (102a) with $v_1 = k_1(t)z_e$, where k_1 is given by (110), satisfies

$$z_e(t) = z_e(t_0) \exp\left\{ -C_1 \int_{t_0}^{t} \exp\{C_2 \tau\}\, \mathrm{d}\tau \right\} \tag{111a}$$

Moreover, by (110) and (111a), it holds

$$|v_1(t)| = C_1 |z_e(t_0)| \exp\left\{ C_2 t - C_1 \int_{t_0}^{t} \exp\{C_2 \tau\}\, \mathrm{d}\tau \right\} \tag{111b}$$

We now take into account (105), (107a), (107b), (109a), (109b) and estimate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathrm{T}}(t)P(t)e(t)\bigg|_{(102\mathrm{b}), v_2 = k(t)e} \\
&\leqslant -a(t)e^{\mathrm{T}}(t)P(t)e(t) + 2v_1(t)e^{\mathrm{T}}(t)P(t)\begin{pmatrix} e_2 + g_1(t, e_1) \\ \vdots \\ e_n + g_{n-1}(t, e_1, \ldots, e_{n-1}) \\ 0 \end{pmatrix} \\
&\quad + 2v_1(t)e^{\mathrm{T}}(t)P(t)\begin{pmatrix} x_{2d}(t) + f_1(x_{1d}(t)) \\ \vdots \\ x_{nd}(t) + f_{n-1}(x_{1d}(t) \ldots x_{(n-1)d}(t)) \\ 0 \end{pmatrix} \\
&\leqslant -(a(t) - C|v_1(t)| \exp\{lt\})e^{\mathrm{T}}(t)P(t)e(t) + C\exp\{lt\}|v_1(t)|
\end{aligned} \tag{112}$$

for certain constants $C > 0$ and $l > 0$ (specifically, we may take $l = 2\lambda + \sigma$, where λ, σ are the constants involved in (105), (109a), respectively). It follows from (109a) and (112) that

$$|e(t)| \leqslant \sqrt{\frac{K_2}{K_1}} \exp\left\{ -\frac{1}{2} \int_0^t (a(\tau) - \sigma)\, \mathrm{d}\tau \right\} \exp\left\{ \frac{1}{2} \int_0^{t_0} |a(\tau)|\, \mathrm{d}\tau \right\} D(t, t_0, e(t_0); v_1), \quad \forall t \geqslant t_0 \tag{113}$$

where

$$D(t, t_0, s; v_1) := \exp\left\{ \frac{C}{2} \int_{t_0}^t \exp\{l\tau\}|v_1(\tau)|\, \mathrm{d}\tau \right\} s + \left(C \int_{t_0}^t |v_1(\tau)| \exp\left\{ l\tau + \int_{t_0}^\tau |a(w)|\, \mathrm{d}w + C \int_r^t \exp\{lw\}|v_1(w)|\, \mathrm{d}w \right\} \mathrm{d}\tau \right)^{1/2}$$

It turns out by picking

$$C_1 := 2(l + C_2) + \frac{2K_3}{\bar{r}} C_2 + C_2; \quad C_2 := \bar{r} \tag{114}$$

and taking into account (109c) and (111b) that there exists a function $E(t_0, s)$ of class $\mathcal{N}\mathcal{N}$, such that

$$D(t, t_0, s; v_1) \leqslant E(t_0, s + |z_e(t_0)|), \quad \forall t \geqslant t_0, \; s \geqslant 0 \tag{115}$$

(for example, we may take $E(t,s) := \exp\left\{ \frac{CC_1}{2} \exp\left\{ \frac{C_1}{C_2} \exp\{C_2 t\} \right\} s \right\} \left(s + (CC_1 s)^{1/2} \exp\left\{ \frac{C_1}{2C_2} \exp\{C_2 t\} \right\} \right)$.

Notice that, since $a \in \mathbf{S}$, (113) and (115) guarantee the existence of a function $\Delta \in \mathcal{N}\mathcal{N}\mathcal{L}$ such that

$$|e(t)| \leqslant \Delta(|z_e(t_0)| + |e(t_0)|, t_0, t), \quad \forall t \geqslant t_0 \tag{116}$$

The desired GAS for the closed-loop system (102) with (106) are direct consequences of (111a), (116) and Lemma 2.5. □

Remark 4.6

For the case $f_i \equiv 0$ $(i = 1, \ldots, n)$, (105) can be relaxed by assuming that:

$$\sum_{i=2}^n |x_{id}(t)| \leqslant M \exp\{\lambda t\} \tag{117}$$

Indeed, if each f_i vanishes identically then inequality (112) is a consequence of (117).

Remark 4.7

We make some comparison with earlier existing works for the tracking problem for the case $f_i \equiv 0$ $(i = 1, \ldots, n)$. We first notice that the results in Reference [7] generalize those in References [3–6]. Particularly, in Reference [7, Theorem 1] it is proved that the TP is solvable under the assumptions that $x_{id}(\cdot)$ $(i = 2, \ldots, n)$, $u_{1d}(\cdot)$, $u_{2d}(\cdot)$, $\dot{u}_{1d}(\cdot)$ are bounded over $\Re^+$ and $u_{1d}(\cdot)$ does not converge to zero as $t \to \infty$. Clearly, our assumption concerning x_d is weaker, since it is not required to be bounded. Furthermore, it should be noticed that there are functions of class **L**, which do not meet the requirements of Reference [7, Theorem 1]. A typical example is the function $u_{1d}(t) = \exp\{-t\} \sin t$, which obviously belongs to **L** (since it can be seen as the output response of an appropriate linear system for non-zero initial condition), but does not meet the requirements imposed in Reference [7]. We also mention the recent result [8, Proposition 10.3.1], where it is proved that the TP is solvable, under the assumptions that $u_{1d}(\cdot)$ is continuous, $x_{id}(\cdot)$ $(i = 2, \ldots, n)$ are bounded over $\Re^+$ and there exist positive constants $\delta, \varepsilon_1, \varepsilon_2$ such that for all $t > 0$:

$$\varepsilon_1 I \leqslant \int_t^{t+\delta} w_r(t,\tau) w_r^{\mathrm{T}}(t,\tau)\, \mathrm{d}\tau \leqslant \varepsilon_2 I$$

$$w_r^{\mathrm{T}}(t,\tau) := \left(1, \int_\tau^t u_{1d}(s)\, \mathrm{d}s, \ldots, \left(\int_\tau^t u_{1d}(s)\, \mathrm{d}s\right)^{n-1}\right)$$

Again, our hypothesis concerning x_d is weaker and there are functions of class **L**, which do not meet the requirements of Reference [8, Proposition 10.3.1]; a typical example again is the function $u_{1d}(t) = \exp\{-t\} \sin t$.

5. CONCLUSIONS

In this paper we explored the idea of non-uniform in time global asymptotic stabilization to derive sufficient conditions for the existence of linear time-varying feedback that stabilizes the equilibrium point of an uncertain triangular time-varying system. As applications we considered the problem of robust stabilization for two-input systems that in general cannot be stabilized by C^0 time-invariant feedback, as well as the state feedback tracking problem for a class of systems that includes the non-holonomic chained form case. Our results require a different set of hypotheses than those encountered in the literature and specifically for the tracking problem, we have enriched the class of functions for which the tracking problem is solvable.

APPENDIX A

Proof of Lemma 2.4
Let $\phi : \Re \to \Re$ be any C^∞ bounded function that satisfies

$$x\phi(x) \geqslant 0, \quad \forall x \in \Re \quad \text{and} \quad \phi(x) = \mathrm{sgn}(x) \;\; \text{for} \;\; |x| \geqslant 1 \tag{A1}$$

Define

$$E(t,s) := \phi(s \exp\{C_1 t\}) \exp\{C_2 t\} \tag{A2}$$

for some constants $C_1, C_2 > 0$ yet to be chosen. Obviously, by (A1) and (A2) for any selection of $C_1, C_2 > 0$, property (11c) holds. In order to establish (11d) and (11e) let

$$Q := \max_{|s| \leqslant 1} \left(|\phi(s)| + \left| \frac{\mathrm{d}\phi}{\mathrm{d}x}(s) \right| \right)$$

and notice that

$$|E(t,s)| \leqslant Q \exp\{C_2 t\}$$

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} E(t,d(t)) \right| &= \left| C_2 E(t,d(t)) + \exp\{(C_1 + C_2)t\} \frac{\mathrm{d}\phi}{\mathrm{d}x}(d(t)\exp\{C_1 t\})\,(\dot{d}(t) + C_1 d(t)) \right| \\ &\leqslant Q(C_2 + |\dot{d}(t)| + C_1 |d(t)|) \exp\{(C_1 + C_2)t\} \end{aligned}$$

Thus (11d) and (11e) hold with $R := Q(1 + C_1 + C_2)$, $\sigma := C_1 + C_2$. We next show that (11b) is satisfied for appropriate selection of the constants C_1 and C_2. Indeed, let

$$C_1, C_2 \geqslant 1 + \mu \tag{A3}$$

and define

$$I^+ := \{t \geqslant 0 : |d(t)| > \exp\{-C_1 t\}\}; \quad I^- := \Re^+ \setminus I^+ \tag{A4}$$

Notice that by (A1), (A3) and definitions (A2), (A4) of E and I^+, respectively, we have:

$$d(t)E(t,d(t)) = |d(t)| \exp\{C_2 t\}, \quad \forall t \in I^+ \tag{A5}$$

$$M \exp\{\mu t\}|d(t)| - |d(t)| \exp\{C_2 t\} \leqslant 0, \quad \forall t \geqslant M \tag{A6}$$

Consequently, by (A3), (A4), (A5), (A6) and invoking (7a) we get

$$\begin{aligned} &\int_0^{+\infty} \max\{M|d(t)| \exp\{\mu t\} - d(t)E(t,d(t)), 0\}\, \mathrm{d}t \\ &\overset{(A4),(A5)}{\leqslant} \int_{I^+} \max\{M|d(t)| \exp\{\mu t\} - |d(t)| \exp\{C_2 t\}, 0\}\, \mathrm{d}t \\ &\quad + M \int_{I^-} \exp\{(\mu - C_1)t\}\, \mathrm{d}t \\ &\overset{(A3),(A6)}{\leqslant} M \int_{I^+ \cap [0,M]} |d(t)| \exp\{\mu t\}\, \mathrm{d}t + M \leqslant M(K \exp\{(c+\mu)M\} + 1) \end{aligned}$$

Thus (11b) is fulfilled with $\rho_\mu(s) := s(K \exp\{(\sigma + \mu)s\} + 1)$. Finally, we show that (11a) holds, provided that, in addition to (A3), the constants C_1, C_2 satisfy

$$C_1 \geqslant 1 + C_2; \quad C_2 \geqslant r + c(p-1) \tag{A7}$$

where $p \geqslant 1$ and $r \geqslant 0$ are the constants involved in (7b) and c is defined in (7a). Let q be an arbitrary real number and $T \geqslant 0$. By (11c), (A4) and (A5) we obtain:

$$\begin{aligned}\int_0^T (d(t)E(t,d(t))+q)\,\mathrm{d}t &\overset{\text{(11c),(A4)}}{\geqslant} \int_{I^+\cap[0,T]} d(t)E(t,d(t))\,\mathrm{d}t - |q|^T \\ &\overset{\text{(A5)}}{=} \int_{I^+\cap[0,T]} |d(t)|\exp\{C_2 t\}\,\mathrm{d}t - |q|^T \\ &\overset{\text{(A4)}}{=} \int_0^T |d(t)|\exp\{C_2 t\}\mathrm{d}t - |q|T \\ &\quad - \int_{I^-\cap[0,T]} |d(t)|\exp\{C_2 t\}\,\mathrm{d}t \end{aligned} \tag{A8}$$

and by (7a) we get:

$$K^{1-p}\exp\{-c(p-1)t\}|d(t)|^p \leqslant |d(t)| \tag{A9}$$

Inequalities (A7), (A8), (A9) in conjunction with definition (A4) of the set I^-, give:

$$\begin{aligned}\int_0^T (d(t)E(t,d(t))+q)\,\mathrm{d}t &\overset{\text{(A4),(A8),(A9)}}{\geqslant} K^{1-p}\int_0^T |d(t)|^p \exp\{(C_2 - c(p-1))t\}\,\mathrm{d}t \\ &\quad - |q|T - \int_{[0,T]} \exp\{(C_2 - C_1)t\}\,\mathrm{d}t \\ &\overset{\text{(A7)}}{\geqslant} TK^{1-p}\left[\frac{1}{T}\int_0^T |d(t)|^p\exp\{rt\}\,\mathrm{d}t - |q|K^{p-1} - \frac{K^{p-1}}{T}\right]\end{aligned}$$

The latter in conjunction with (7b), imply (11a) as $T \to +\infty$. We conclude that for appropriate selection of C_1 and C_2, the map $E(\cdot)$ as defined by (A2) satisfies all properties of Lemma 2.4. □

Proof of Lemma 2.5

We recall our hypothesis that $\Delta \in \mathcal{NNL}$, which implies that for any $\varepsilon > 0$, $T \geqslant 0$ and $R > 0$ there exists a time $\tau = \tau(\varepsilon, R, T) \geqslant 0$ such that $\Delta(R, T, \xi) \leqslant \varepsilon$, for all $\xi \geqslant \tau$; thus by virtue of (14) it holds:

$$|x(t)| \leqslant \varepsilon, \quad \forall t \geqslant t_0 + \tau, \quad |x_0| \leqslant R, \quad t_0 \in [0,T] \quad \text{(Attractivity)} \tag{A10}$$

In order to establish stability, notice first that, by virtue of (14), we have

$$\sup\{|x(t)|;\ t \geqslant t_0, |x_0| \leqslant \varepsilon, t_0 \in [0,T]\} \leqslant \sup_{t \geqslant 0} \Delta(\varepsilon, T, t) < +\infty \tag{A11}$$

for all $\varepsilon > 0$, $T \geqslant 0$. Consider next the function $L : (\Re^+)^2 \to \Re^+$, defined as

$$L(t,r) := \sup\left\{\frac{|f(\tau,x) - f(\tau,y)|}{|x-y|};\ \tau \in [0,t], |x| \leqslant r, |y| \leqslant r, x \neq y\right\} \tag{A12}$$

This function is well-defined by the fact that the dynamics are locally Lipschitz with respect to $x \in \Re^n$. It turns out from definition (A12) that for each fixed $s \geqslant 0$, the mappings $L(s,\cdot)$ and $L(\cdot,s)$ are non-negative and non-decreasing and the following holds:

$$|f(t,x)| \leqslant L(t,|x|)|x|, \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{A13}$$

From (14) and (A13) we have:

$$|f(t,x(t))| \leqslant M(t,t_0,|x_0|)|x(t)|$$

$$M(t,t_0,s) := L\left(t, \sup_{0 \leqslant r \leqslant t} \Delta(s,t_0,r)\right) \tag{A14}$$

where $x(t)$ is the solution of (12) with $x(t_0) = x_0$. Obviously, for each fixed $(t,t_0,s) \in (\Re^+)^3$, the mappings $M(t,t_0,\cdot)$, $M(\cdot,t_0,s)$ and $M(t,\cdot,s)$ are non-negative and non-decreasing. The latter in conjunction with (12) and (A14) guarantees:

$$|x(t)| \leqslant \exp\left\{\int_{t_0}^{t} M(\tau,t_0,|x_0|)\,\mathrm{d}\tau\right\}|x_0|, \quad \forall t \geqslant t_0 \tag{A15}$$

Since the mappings $M(t,t_0,\cdot)$, $M(t,\cdot,s)$ and $M(\cdot,t_0,s)$ are non-negative and non-decreasing, (A15) implies that for any $\varepsilon > 0$, $T \geqslant 0$ and $R > 0$ it holds:

$$\begin{aligned} &|x(t)| \leqslant \varepsilon, \quad \forall t \in [t_0, t_0 + \tau], \\ &\forall |x_0| \leqslant \delta(\varepsilon,T) := \min\left\{R, \varepsilon \exp\left\{-\int_{T}^{T+\tau} M(s,T,R)\,\mathrm{d}s\right\}\right\}, \ t_0 \in [0,T] \end{aligned} \tag{A16}$$

It follows by using (A10) and (A16) that for any $\varepsilon > 0$ and $T \geqslant 0$, there exists $\delta := \delta(\varepsilon,T) > 0$ such that $|x(t)| \leqslant \varepsilon$ for all $t_0 \in [0,T]$, $t \geqslant t_0$ and $|x_0| \leqslant \delta$. This fact, in conjunction with (A11), establishes stability. □

Proof of Lemma 2.6

By (17b) and the fact that $a(t,r)$ is non-increasing in r, we get:

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t)) f(t,\theta(t),\eta(t),x(t)) \\ &\leqslant -a(t,\|\theta\|)V(t,x(t)), \ \text{a.e.} \quad \text{for } t \geqslant t_0 \end{aligned} \tag{A16}$$

where $x(\cdot)$ denotes the solution of (15) corresponding to a pair of inputs $(\theta(\cdot),\eta(\cdot)) \in L^\infty \times ([t_0,+\infty)) \times L^\infty_{loc}([t_0,+\infty))$ with $(\theta(t),\eta(t)) \in \Pi_t$ for all $t \geqslant t_0$. Using (A16) we obtain:

$$V(t,x(t)) \leqslant V(t_0,x(t_0)) \exp\left\{-\int_{t_0}^{t} a(s,\|\theta\|)\,\mathrm{d}s\right\} \tag{A17}$$

It follows by (17a) and (A17) that (18b) holds with

$$D(s,t) := \sqrt{\frac{K_2}{K_1}} \exp\left\{ -\frac{1}{2} \int_0^t (a(\tau,s) - \sigma)\, \mathrm{d}\tau \right\} \tag{A18}$$

since $a(t,s)$ is non-increasing in s, it follows that for each $t \geqslant 0$ the mapping $D(\cdot,t)$ is non-decreasing. Furthermore, for every $\varepsilon \geqslant 0$, it holds:

$$\exp\{\varepsilon t\} D(s,t) = \sqrt{\frac{K_2}{K_1}} \exp\left\{ -\frac{1}{2} \int_0^t (a(\tau,s) - \sigma - 2\varepsilon)\, \mathrm{d}\tau \right\} \tag{A19}$$

We now recall our assumption that for each fixed $s \geqslant 0$, the map $a(\cdot,s)$ is of class $\mathbf{S}$ and consequently $\lim_{t\to+\infty} \int_0^t (a(\tau,s) - \sigma - 2\varepsilon)\,\mathrm{d}\tau = +\infty$. This in conjunction with (A19) implies (18a). We conclude that D is of class $\mathcal{NL}$. Finally, by (18b) we get:

$$|x(t)| \leqslant \Delta(|x_0|, t_0, t), \quad \forall t \geqslant t_0 \tag{A20}$$

$$\Delta(s,t_0,t) := D(\|\theta\|, t) \exp\left\{ \frac{1}{2} \int_0^{t_0} |a(\tau, \|\theta\|)|\, \mathrm{d}\tau \right\} s \tag{A21}$$

Obviously, (18a) and definition (A21) of Δ, implies that Δ is of class $\mathcal{NNL}$. Therefore, estimation (A20), guarantees by virtue of Lemma 2.5 that $0 \in \Re^n$ is Π_t-GAS. □

Proof of (67)

Notice first by (65) that the matrix $A_f + \beta I$ is invertible for β small enough. In order to establish (67b) suppose the contrary

$$c_0 = c_f (A_f + \beta I)^{-1} b_1, \quad \forall \beta > 0 \text{ close to zero} \tag{A22}$$

We define the complex rational function:

$$g(s) := c_f (A_f - sI)^{-1} b_1 \tag{A23}$$

which is the transfer function of the system:

$$\begin{aligned} \dot{x} &= A_f x + b_1 u \\ y &= -c_f x \end{aligned} \tag{A24}$$

where A_f and c_f are defined in (64) and (66), respectively. It then follows by (A22) and (A23) that $g(\cdot)$ must be equal to c_0 for all real $s < 0$ close to zero, which implies both $c_0 = 0$ and $c_f(A_f - sI)^{-1} b_1 = 0$ for almost all (complex) s. Since $c_0 = 0$, it turns out from (59) and (66) that $c_1 \neq 0$ and $c_1(A_f - sI)^{-1} b_1 = 0$ for almost all s. The latter implies that the pair (A_f, b_1) is uncontrollable and thus by (64) that the pair (A_{11}, b_1) is uncontrollable as well, which contradicts hypothesis (56). □

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