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A vector small-gain theorem for general non-linear control systems

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A new small-gain theorem is presented for general non-linear control systems and can be viewed as unification of previously developed non-linear small-gain theorems for systems described by ordinary differential equations, retarded functional differential equations and hybrid models. The novelty of this research work is that vector Lyapunov functions and functionals are utilized to derive various input-to-output stability and input-to-state stability results. It is shown that the proposed approach is extendible to several important classes of control systems such as large-scale complex systems, non-linear sampled-data systems and non-linear time-delay systems. An application to a biochemical circuit model illustrates the generality and power of the proposed vector small-gain theorem.

Keywords: input-to-output stability; feedback systems; small-gain theorem; vector Lyapunov functions and functionals.

1. Introduction

The small-gain theorem has been widely recognized as an important tool for robustness analysis and robust controller design within the control systems community. For instance, classical small-gain theorems, Desoer & Vidyasagar (1975) and Zames (1966) have played a crucial role for linear robust control of uncertain systems subject to dynamic uncertainties (Zhou *et al.*, 1996). As introduced in the framework of classical small-gain, an essential condition for input–output stability of a feedback system is that the loop gain is less than one. This condition relying upon on the concept of linear finite gain was first relaxed by Hill (1991) and then Mareels & Hill (1992) using the notions of monotone gain and non-linear operators. Quickly after the birth of the notion of input-to-state stability (ISS) originally introduced by Sontag (1989), a non-linear, generalized small-gain theorem was developed in Jiang *et al.* (1994). This non-linear ISS small-gain theorem differs from classical small-gain theorems and the non-linear small-gain theorem of Hill (1991) and Mareels & Hill (1992) in several aspects. One of them is that both internal and external stability properties are discussed in a single framework, while only input–output stability is addressed in previous small-gain theorems. As demonstrated in

Jiang *et al.* (1994) and the subsequent work of many others, non-linear small-gain has led to new solutions to several challenging problems in robust non-linear control, such as stabilization by partial-state and output feedback, robust adaptive tracking and non-linear observers. More interestingly, this perspective of non-linear small gain can find useful applications in monotone systems, an important class of systems in mathematical biology (see Angeli *et al.*, 2004; Angeli & Astolfi, 2007; Enciso & Sontag, 2006). Further extensions of this tool to the cases of non-uniform in time stability, discrete-time systems and Lyapunov characterizations are pursued by several authors independently; see, for instance, Grune (2002), Ito & Jiang (2009), Jiang *et al.* (1996, 2004), Jiang & Mareels (1997), Karafyllis & Tsinias (2004), Karafyllis (2004), Karafyllis & Jiang (2007), Sontag & Ingalls (2002) and Teel (1996).

This paper takes a step further to broaden the applicability and generality of non-linear small gain results by removing essential restrictions in previous small-gain theorems. To better position the novelty and contributions of our present work with respect to numerous variants of non-linear small-gain theorems, some highlights are given below.

- A common feature of the earlier non-linear small-gain theorems is that the semi-group property is required implicitly or explicitly for the solutions of the feedback system in question, whether the feedback system is described by ordinary differential equations (ODEs) or takes the form of hybrid and switched systems. We will adopt a weak semi-group property which is much more relaxed than the semi-group property (see Karafyllis, 2007a,b). As shown in our recent work (Karafyllis & Jiang, 2007), the weak semi-group property allows studying a wide class of non-linear feedback systems such as hybrid and switched systems. As compared with Karafyllis & Jiang (2007) where only two interconnected systems are considered, here we will develop a unifying framework which allows us to study large-scale systems composed of multiple interacting subsystems. To address this goal, additional novel tools will be proposed.

- The new small-gain theorem obtained in this paper, i.e. Theorem 3.1 in Section 3, is a generalization of several previously developed non-linear small-gain theorems. In particular, through examples and detailed analysis, we show that Theorem 3.1 can recover as special cases several newly introduced small-gain theorems for large-scale complex systems (Dashkovskiy *et al.*, 2007, 2010; Jiang & Wang, 2008; Karafyllis *et al.*, 2008a; Teel, 2005). In addition, it is shown that uniform and non-uniform input-to-output stability (IOS) and ISS stability properties can be studied for various important classes of non-linear dynamical controlled systems. Furthermore, explicit formulae are provided for the gain function of the composite system.

- A nice feature of the new small-gain theorem is that we allow non-zero diagonal gains for each interacting system, while all previous non-linear small-gain theorems assume zero diagonal gain (with the exception of Sontag & Ingalls, 2002, which is a scalar small-gain theorem). This generality is important for studying non-linear uncertain and time-delay systems. Indeed, this is one of the cornerstones for our unified framework in considering both delay-free systems and time-delay non-linear systems.

- The new theorem leads to vector Lyapunov function (or functional) characterizations for various stability properties (Theorems 4.1, 4.4 and 4.8). Because of that we coin our new small-gain theorem presented in this paper 'vector small gain'. The advantage of vector Lyapunov function versus single Lyapunov function in non-linear stability analysis has been well documented in past literature (Lakshmikantham *et al.*, 1991; Michel & Miller, 1977). Recent work in Karafyllis *et al.* (2008a) and Karafyllis & Kravaris (2009) provides further evidence on the usefulness of vector Lyapunov functions for the case of ISS stability. Another feature, which is not frequently recognised, is that vector Lyapunov functions can handle large-scale systems more easily than single Lyapunov functions. Indeed, this feature is illustrated by many examples in the paper. For example, Examples 5.1 and 5.2 deal with large-scale time-delay systems. One cannot exclude the possibility of finding an appropriate Lyapunov–Krasovskii functional or a Razumikhin function that can be used for the proof of stability in Examples 5.1 and 5.2. However, the reader can see how easily stability conditions are obtained in both examples by using very simple functions.

The rest of the paper is organized as follows. In Section 2, we provide certain useful results on monotone discrete-time systems. The results contained in this section are used extensively in subsequent sections. Section 3 of the paper provides a brief review of the system-theoretic framework introduced in Karafyllis (2007a,b) and Karafyllis & Jiang (2007), and states the main result (Theorem 3.1). In Section 4, sufficient Lyapunov-like conditions for the verification of the hypotheses of Theorem 3.1 are presented for three types of systems: (i) Systems described by ODEs, (ii) Systems described by retarded functional differential equations (RFDEs) and (iii) sampled-data systems. The results contained in Section 4 are exploited in Section 5, where examples and applications of the vector small-gain methodology are given. The proofs of the main results of Section 4 are given in Section 6. Finally, the conclusions of the paper are provided in Section 7. The proofs of Proposition 2.7 and Theorem 3.1 are given in the Appendix.

Notations. Throughout this paper, we adopt the following notations:

- We denote by K^+ the class of positive, continuous functions defined on $\Re_+ := \{x \in \Re\colon x \geqslant 0\}$. We say that a function $\rho\colon \Re_+ \to \Re_+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. By K, we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho\colon \Re_+ \to \Re_+$ is of class K_∞ if $\lim_{s\to+\infty} \rho(s) = +\infty$. By KL, we denote the set of all continuous functions $\sigma = \sigma(s,t)\colon \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geqslant 0$ the mapping $\sigma(\cdot, t)$ is of class K; (ii) for each $s \geqslant 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s,t) = 0$.
- By $\|\|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. By $||$, we denote the Euclidean norm of $\Re^n$. Let $U \subseteq \mathcal{X}$ with $0 \in U$. By $B_U[0,r] := \{u \in U;\ \|u\|_{\mathcal{X}} \leqslant r\}$, we denote the intersection of $U \subseteq \mathcal{X}$ with the closed ball of radius $r \geqslant 0$, centered at $0 \in U$. If $U \subseteq \Re^n$ then $\mathrm{int}(U)$ denotes the interior of the set $U \subseteq \Re^n$.
- x' denotes the transpose of x.
- $\Re^n_+ := (\Re_+)^n = \{(x_1, \ldots, x_n)' \in \Re^n\colon x_1 \geqslant 0, \ldots, x_n \geqslant 0\}.\{e_i\}_{i=1}^n$ denotes the standard basis of $\Re^n$. Z_+ denotes the set of non-negative integers.
- Let $x, y \in \Re^n$. We say that $x \leqslant y$ if and only if $(y - x) \in \Re^n_+$. We say that a function $\rho\colon \Re^n_+ \to \Re^+$ is of class $\mathcal{N}_n$, if ρ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leqslant \rho(y)$ for all $x, y \in \Re^n_+$ with $x \leqslant y$.
- For $t \geqslant t_0 \geqslant 0$, let $[t_0, t] \ni \tau \to V(\tau) = (V_1(\tau), \ldots, V_n(\tau))' \in \Re^n$ be a bounded map. We define $[V]_{[t_0,t]} := \left(\sup_{\tau\in[t_0,t]} V_1(\tau), \ldots, \sup_{\tau\in[t_0,t]} V_n(\tau)\right)$. For a measurable and essentially bounded function, $x : [a,b] \to \Re^n$, $\operatorname{ess\,sup}_{t\in[a,b]} |x(t)|$ denotes the essential supremum of $|x(\cdot)|$.

- We say that $\Gamma\colon \Re^n_+ \to \Re^m_+$ is non-decreasing if $\Gamma(x) \leqslant \Gamma(y)$ for all $x, y \in \Re^n_+$ with $x \leqslant y$. For an integer $k \geqslant 1$, we define $\Gamma^{(k)}(x) = \underbrace{\Gamma \circ \Gamma \circ \ldots \circ \Gamma}_{k \text{ times}}(x)$, when $m = n$.
- We define $\mathbf{1} = (1, 1, \ldots, 1)' \in \Re^n$. If $u, v \in \Re$ and $u \leqslant v$ then $\mathbf{1}u \leqslant \mathbf{1}v$.
- Let U be a subset of a normed linear space $\mathcal{U}$, with $0 \in U$. By $\mathcal{M}(U)$, we denote the set of all locally bounded functions $u\colon \Re_+ \to U$. By $0 \in \mathcal{M}(U)$, we denote the identically zero input. If $U \subseteq \Re^n$ then M_U denotes the space of measurable, locally bounded functions $u\colon \Re_+ \to U$.
- Let $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces. We denote by $C^0(A; B)$ the class of continuous mappings $f\colon A$. For $x \in C^0([-r, 0]; \Re^n)$, we define $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$. We will use the convention $C^0([0, 0]; \Re^n) = \Re^n$ and if $x \in C^0([0, 0]; \Re^n) = \Re^n$ we have $\|x\|_r = |x|$.

2. Global asymptotic stability for monotone discrete-time systems

The purpose of this section is to introduce some preliminary, technical results which will play an instrumental role in the development of our main results in next sections. Some of these basic results are not new, as compared with Dashkovskiy *et al.* (2007, 2006, 2010), Teel (2005) and Ruffer (2007, 2010) and are reproduced here to make our work self-contained.

Consider the discrete-time system

$$x_{k+1} = \Gamma(x_k), \quad x_k \in \Re^n_+, \tag{2.1}$$

where $\Gamma\colon \Re^n_+ \to \Re^n_+$ is a non-decreasing map with $\Gamma(0) = 0$. For the study of the above system, we adopt the standard stability notions for discrete-time systems (see, for instance, Jiang & Wang, 2002, 2001, and references therein). More specifically, we say that $0 \in \Re^n_+$ is a globally asymptotically stable (GAS) equilibrium point for (2.1) if $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$ for all $x \in \Re^n_+$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x| \leqslant \delta$, $x \in \Re^n_+$ implies $|\Gamma^{(k)}(x)| \leqslant \varepsilon$ for all $k \geqslant 1$. Next a necessary condition for the global asymptotic stability property and a technical result that guarantees convergence to zero are provided. The following results are closely related to Corollary 2.1.2 and Lemma 2.2.4 in Ruffer (2007) and, for completeness, are reproduced below.

PROPOSITION 2.1 If $0 \in \Re^n$ is GAS, then the following implication holds:

$$\Gamma(x) \geqslant x \Rightarrow x = 0. \tag{2.2}$$

LEMMA 2.2 Let $\Gamma\colon \Re^n_+ \to \Re^n_+$ be a continuous, non-decreasing map satisfying (2.2) with $\Gamma(0) = 0$. If the inequality $\Gamma(x) \leqslant x$ holds for some $x \in \Re^n_+$, then $\lim_{k\to\infty} \Gamma^{(k)}(y) = 0$ for all $y \in \Re^n_+$ with $y \leqslant x$.

We now present an algebraic operator on $\Re^n$ that turns out to be useful for the study of discrete monotone systems.

DEFINITION 2.3. Let $x = (x_1, \ldots, x_n)' \in \Re^n$ and $y = (y_1, \ldots, y_n)' \in \Re^n$. We define $z = \mathrm{MAX}\{x, y\}$, where $z = (z_1, \ldots, z_n)' \in \Re^n$ satisfies $z_i = \max\{x_i, y_i\}$ for $i = 1, \ldots, n$. Similarly, for $u_1, \ldots, u_m \in \Re^n$, $z = \mathrm{MAX}\{u_1, \ldots, u_m\}$ is a vector $z = (z_1, \ldots, z_n)' \in \Re^n$ with $z_i = \max\{u_{1i}, \ldots, u_{mi}\}$, $i = 1, \ldots, n$.

REMARK 2.4. The MAX operator possesses some interesting properties. For example, if $x \leqslant y$, then $y = \mathrm{MAX}\{x, y\}$. If $z \leqslant \mathrm{MAX}\{x, y\}$ and $y \leqslant \mathrm{MAX}\{w, v\}$, then $z \leqslant \mathrm{MAX}\{x, w, v\}$. Also $\mathrm{MAX}\{x, y, w\} = \mathrm{MAX}\{x, \mathrm{MAX}\{y, w\}\}$. If $x \leqslant z$ and $y \leqslant z$, then $\mathrm{MAX}\{x, y\} \leqslant z$. In general, we have $\Gamma(\mathrm{MAX}\{x, y\}) \geqslant \mathrm{MAX}\{\Gamma(x), \Gamma(y)\}$ for any non-decreasing map $\Gamma\colon \Re^n_+ \to \Re^n_+$ and for every $x, y \in \Re^n_+$.

The MAX operator becomes a very useful tool for the study of the following special class of monotone vector fields.

DEFINITION 2.5. We say that $\Gamma\colon \Re^n_+ \to \Re^n_+$ is MAX-preserving if $\Gamma\colon \Re^n_+ \to \Re^n_+$ is non-decreasing and for every $x, y \in \Re^n_+$, the following equality holds:

$$\Gamma(\mathrm{MAX}\{x, y\}) = \mathrm{MAX}\{\Gamma(x), \Gamma(y)\} \tag{2.3}$$

The above defined MAX-preserving maps enjoy the following important property.

PROPOSITION 2.6 $\Gamma\colon \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving if and only if there exist non-decreasing functions $\gamma_{i,j}\colon \Re_+ \to \Re_+$, $i, j = 1, \ldots, n$ with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$ for all $x \in \Re^n_+$, $i = 1, \ldots, n$.

Proof. Define $\gamma_{i,j}(s) := \Gamma_i(se_j)$ for all $s \geqslant 0$. Let $x \in \Re^n_+$, i.e., $x = x_1e_1 + \cdots + x_ne_n$ with $x_i \geqslant 0$, $i = 1, \ldots, n$. Note that $x = \mathrm{MAX}\{x_1e_1, \ldots, x_ne_n\}$ and consequently $\Gamma(x) = \mathrm{MAX}\{\Gamma(x_1e_1), \ldots, \Gamma(x_ne_n)\}$. Therefore, $\Gamma_i(x) = \max\{\Gamma_i(x_1e_1), \ldots, \Gamma_i(x_ne_n)\} = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$. The converse statement is a direct consequence of the definition $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$. □

Next, necessary and sufficient conditions are provided for GAS of (2.1) for the case of a continuous MAX-preserving map.

PROPOSITION 2.7 Suppose that $\Gamma\colon \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving and there exist functions $\gamma_{i,j} \in N_1$, $i, j = 1, \ldots, n$ with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$, $i = 1, \ldots, n$. The following statements are equivalent:

(i) $0 \in \Re^n$ is GAS for (2.1).

(ii) The following cyclic small-gain conditions hold:

$$\gamma_{i,i}(s) < s \quad \forall s > 0, \qquad i = 1, \ldots, n \tag{2.4a}$$

and if $n > 1$ then for each $r = 2, \ldots, n$ it holds that

$$(\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \ldots \circ \gamma_{i_r,i_1})(s) < s \quad \forall s > 0 \tag{2.4b}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$.

(iii) The following implication holds: $\Gamma(x) \geqslant x \ \Rightarrow \ x = 0$.

(iv) (iii) holds and for each $k \geqslant 1$ and $x \in \Re^n_+$ it holds that $\Gamma^{(k)}(x) \leqslant Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$.

It should be noted that the equivalence between statements (i), (ii) and (iii) of Proposition 2.7 is implied by Lemma 2.3.14 and Theorem 2.2.8 in Ruffer (2007) or Theorem 6.4 in Ruffer (2010). Here, the new result is the equivalence of statement (iv) with the statements (i), (ii) and (iii). The proof of Proposition 2.7 is provided in the Appendix.

REMARK 2.8. Note that $Q: \Re^n_+ \to \Re^n_+$ is a continuous, MAX-preserving map with $Q(0) = 0$ and $Q(a) \geqslant a$ for all $a \in \Re^n_+$. Moreover, $\Gamma(Q(x)) \leqslant Q(x)$ for all $x \in \Re^n_+$. Indeed, in order to prove $\Gamma(Q(x)) \leqslant Q(x)$ for all $x \in \Re^n_+$, note that since $\Gamma: \Re^n_+ \to \Re^n_+$ is MAX-preserving, we get $\Gamma(Q(x)) = \mathrm{MAX}\{\Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x), \Gamma^{(n)}(x)\}$. Since $\Gamma^{(k)}(x) \leqslant Q(x)$ holds for all integers $k \geqslant 1$, we obtain $\Gamma(Q(x)) \leqslant Q(x)$.

The next proposition is a novel useful technical result, which will be used in the following section.

PROPOSITION 2.9 Suppose that $\Gamma: \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving and there exist functions $\gamma_{i,j} \in N_1, i, j = 1, \ldots, n$ with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j), i = 1, \ldots, n$. Moreover, suppose that the small-gain conditions (2.4a,b) hold and that $x \leqslant \mathrm{MAX}\{a, \Gamma(x)\}$ for certain $x, a \in \Re^n_+$. Then $x \leqslant Q(a)$, where $Q(a) = \mathrm{MAX}\{a, \Gamma(a), \Gamma^{(2)}(a), \ldots, \Gamma^{(n-1)}(a)\}$.

Proof. Suppose that $x \leqslant \mathrm{MAX}\{a, \Gamma(x)\}$. Then $\Gamma(x) \leqslant \mathrm{MAX}\{\Gamma(a), \Gamma^{(2)}(x)\}$ and $x \leqslant \mathrm{MAX}\{a, \Gamma(a), \Gamma^{(2)}(x)\}$. By an induction argument $x \leqslant \mathrm{MAX}\{a, \Gamma(a), \ldots, \Gamma^{(k)}(a), \Gamma^{(k+1)}(x)\}$ for all $k \geqslant 1$. It follows from statement (iv) of Proposition 2.7 that $x \leqslant \mathrm{MAX}\{Q(a), \Gamma^{(k+1)}(x)\}$ for all $k \geqslant 1$. Since $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$, we obtain $x \leqslant Q(a)$. □

3. A vector small-gain theorem for a wide class of systems

3.1 *Review of the system-theoretic framework*

To make our work self-contained, we first introduce some basic notions keyed to the system-theoretic framework presented in Karafyllis (2007a,b) and Karafyllis & Jiang (2007). As shown previously, this system-theoretic framework allows us to study a wide class of dynamic systems described by ODEs, RFDEs, and hybrid or impulsive equations.

The notion of a control system-definition 2.1 in Karafyllis & Jiang (2007): A control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs consists of

(i) *a set U (control set) which is a subset of a normed linear space $\mathcal{U}$ with $0 \in U$ and a set $M_U \subseteq \mathcal{M}(U)$ (allowable control inputs) which contains at least the identically zero input $0 \in \mathcal{M}(U)$,*

(ii) *a set D(disturbance set) and a set $M_D \subseteq \mathcal{M}(D)$, which is called the 'set of allowable disturbances',*

(iii) *a pair of normed linear spaces $\mathcal{X}, \mathcal{Y}$ called the 'state space' and the 'output space', respectively,*

(iv) *a continuous map $H: \Re_+ \times \mathcal{X} \times U \to \mathcal{Y}$ that maps bounded sets of $\Re_+ \times \mathcal{X} \times \mathcal{U}$ into bounded sets of $\mathcal{Y}$, called the 'output map',*

(v) *a set-valued map $\Re_+ \times \mathcal{X} \times M_U \times M_D \ni (t_0, x_0, u, d) \to \pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$, with $t_0 \in \pi(t_0, x_0, u, d)$ for all $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$, called the set of 'sampling times'*

(vi) *and the map $\phi: A_\phi \to \mathcal{X}$ where $A_\phi \subseteq \Re_+ \times \Re_+ \times \mathcal{X} \times M_U \times M_D$, called the 'transition map', which has the following properties:*

(1) **Existence:** *For each $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.*

(2) **Identity property:** *For each $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.*

(3) **Causality:** *For each* $(t, t_0, x_0, u, d) \in A_\phi$ *with* $t > t_0$ *and for each* $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ *with* $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ *for all* $\tau \in [t_0, t]$, *it holds that* $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ *with* $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.

(4) **Weak semi-group property:** *There exists a constant* $r > 0$, *such that for each* $t \geqslant t_0$ *with* $(t, t_0, x_0, u, d) \in A_\phi$:

(a) $(\tau, t_0, x_0, u, d) \in A_\phi$ *for all* $\tau \in [t_0, t]$,

(b) $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ *for all* $\tau \in [t_0, t] \cap \pi(t_0, x_0, u, d)$,

(c) *if* $(t + r, t_0, x_0, u, d) \in A_\phi$, *then it holds that* $\pi(t_0, x_0, u, d) \cap [t, t + r] \neq \varnothing$.

(d) *for all* $\tau \in \pi(t_0, x_0, u, d)$ *with* $(\tau, t_0, x_0, u, d) \in A_\phi$ *we have* $\pi(\tau, \phi(\tau, t_0, x_0, u, d), u, d) = \pi(t_0, x_0, u, d) \cap [\tau, +\infty)$.

The BIC and RFC properties-definition 2.4 in Karafyllis & Jiang (2007): *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with outputs. We say that system* Σ

(i) *has the 'boundedness-implies-continuation' (BIC) property if for each* $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$, *there exists a maximal existence time, i.e., there exists* $t_{\max} := t_{\max}(t_0, x_0, u, d) \in (t_0, +\infty]$, *such that* $A_\phi = \cup_{(t_0,x_0,u,d)\in\Re_+\times\mathcal{X}\times M_U\times M_D}[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\}$. *In addition, if* $t_{\max} < +\infty$, *then for every* $M > 0$ *there exists* $t \in [t_0, t_{\max})$ *with* $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$.

(ii) *is robustly forward complete (RFC) from the input* $u \in M_U$ *if it has the BIC property and for every* $r \geqslant 0$, $T \geqslant 0$, *it holds that*

$\sup\{\|\phi(t_0 + s, t_0, x_0, u, d)\|_{\mathcal{X}}; u \in \mathcal{M}(B_U[0, r]) \cap M_U, s \in [0, T], \|x_0\|_{\mathcal{X}} \leqslant r, t_0 \in [0, T], d \in M_D\} < +\infty$.

The notion of a robust equilibrium point-definition 2.5 in Karafyllis & Jiang (2007): *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *and suppose that* $H(t, 0, 0) = 0$ *for all* $t \geqslant 0$. *We say that* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$ *for* Σ *if*

(i) *for every* $(t, t_0, d) \in \Re_+ \times \Re_+ \times M_D$ *with* $t \geqslant t_0$ *it holds that* $\phi(t, t_0, 0, 0, d) = 0$.

(ii) *for every* $\varepsilon > 0, T, h \in \Re_+$ *there exists* $\delta := \delta(\varepsilon, T, h) > 0$ *such that for all* $(t_0, x, u) \in [0, T] \times \mathcal{X} \times M_U$, $\tau \in [t_0, t_0 + h]$ *with* $\|x\|_{\mathcal{X}} + \sup_{t\geqslant 0} \|u(t)\|_{\mathcal{U}} < \delta$ *it holds that* $(\tau, t_0, x, u, d) \in A_\phi$ *for all* $d \in M_D$ *and*

$$\sup\{\|\phi(\tau, t_0, x, u, d)\|_{\mathcal{X}}; d \in M_D, \tau \in [t_0, t_0 + h], t_0 \in [0, T]\} < \varepsilon.$$

Next, we present the IOS property for the class of systems described previously (see also Jiang *et al.*, 1994; Sontag & Wang, 1995, 1996, 1999, for finite-dimensional, time-invariant dynamic systems).

The notions of IOS, UIOS, ISS and UISS-definition 2.14 in Karafyllis & Jiang (2007): *Consider a control system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ *with outputs and the BIC property and for which* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$. *Suppose that* Σ *is RFC from the input* $u \in M_U$. *If there exist functions* $\sigma \in KL, \beta \in K^+, \gamma \in \mathcal{N}_1$ *such that the following estimate holds for all* $u \in M_U$, $(t_0, x_0, d) \in \Re_+ \times \mathcal{X} \times M_D$ *and* $t \geqslant t_0$:

$$\|H(t, \phi(t, t_0, x_0, u, d), u(t))\|_{\mathcal{Y}} \leqslant \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) + \sup_{t_0 \leqslant \tau \leqslant t} \gamma(\|u(\tau)\|_{\mathcal{U}}),$$

then we say that Σ *satisfies the IOS property from the input* $u \in M_U$ *with gain* $\gamma \in \mathcal{N}$. *Moreover, if* $\beta \in K^+$ *may be chosen as* $\beta(t) \equiv 1$, *then we say that* Σ *satisfies the uniform input-to-output stability ((U)IOS) property from the input* $u \in M_U$ *with gain* $\gamma \in \mathcal{N}_1$.

For the special case of the identity output mapping, i.e., $H(t, x, u) := x$, the UIOS property from the input $u \in M_U$ is called (uniform) input-to-state stability ((U) ISS) property from the input $u \in M_U$. When $U = \{0\}$ (the no-input case) and Σ satisfies the (U)IOS property, then we say that Σ satisfies the (uniform) robust global asymptotic output stability (RGAOS) property. When $U = \{0\}$ (the no-input case) and Σ satisfies the (Uniform) ISS property, then we say that Σ satisfies the (Uniform) robust global asymptotic stability (RGAS) property.

Other equivalent definitions of the ISS property, originally introduced by Sontag (1989), are available in the literature (see Grune, 2002; Praly & Wang, 1996).

3.2 *A new small-gain theorem*

We consider an abstract control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that there exist mappings $V_i\colon \Re_+ \times X \times U \to \Re_+$ $(i = 1, \ldots, n)$, $L\colon \Re_+ \times X \to \Re_+$ with $L(t, 0) = 0$, $V_i(t, 0, 0) = 0$ for all $t \geqslant 0 (i = 1, \ldots, n)$ and a MAX-preserving continuous map $\Gamma\colon \Re^n_+ \to \Re^n_+$ with $\Gamma(0) = 0$ such that the following hypotheses hold:

Hypothesis (H1) (the 'IOS-like' inequalities): *There exist functions $\sigma \in KL$, $\nu, c \in K^+$, $\zeta, a, p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$, such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ the mappings $t \to L(t) = L(t, \phi(t, t_0, x_0, u, d))$ and $t \to V(t) = \big(V_1(t, \phi(t, t_0, x_0, u, d), u(t)), \ldots, V_n\big(t, \phi(t, t_0, x_0, u, d), u(t)\big)\big)'$ are locally bounded on $[t_0, t_{\max})$ and the following estimates hold for all $t \in [t_0, t_{\max})$:*

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_0), t - t_0), \Gamma([V]_{[t_0,t]}), \mathbf{1}\zeta([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]})\right\}, \tag{3.1}$$

$$L(t) \leqslant \max\{\nu(t - t_0), c(t_0), a(\|x_0\|_{\mathcal{X}}), p([V]_{[t_0,t]}), p^u([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]})\}, \tag{3.2}$$

where $t_{\max}$ is the maximal existence time of the transition map of Σ.

Hypothesis (H2) (the small-gain conditions): *The small-gain conditions (2.4a,b) hold.*

Hypothesis (H3) (Bounds for the norm of the state): *There exist functions $b \in \mathcal{N}_1$, $g \in \mathcal{N}_n$, $\mu, \beta, \kappa \in K^+$ such that the following inequalities hold for all $(t, x, u) \in \Re_+ \times \mathcal{X} \times U$:*

$$\mu(t)\|x\|_{\mathcal{X}} \leqslant b(L(t, x) + g(V(t, x, u)) + \kappa(t)) \quad \text{and} \quad L(t, x) \leqslant b(\beta(t)\|x\|_{\mathcal{X}}), \tag{3.3}$$

where $V(t, x, u) = (V_1(t, x, u), \ldots, V_n(t, x, u))'$.

Discussion of Hypotheses (H1), (H2) and (H3): In general, the functional $L\colon \Re_+ \times \mathcal{X} \to \Re_+$, that appears at the right-hand side of inequality (3.1), is related to $\|x\|_{\mathcal{X}}$. This is achieved by means of Hypothesis (H3). Hypothesis (H1) is the hypothesis made in every small-gain result: it deals with the 'IOS-like' inequalities, which are to be used and be combined in order to prove the desired estimates. Note that since we are using a family of n functionals, the 'IOS-like' inequalities are given for each functional separately: this is why (3.1) expresses n 'IOS-like' inequalities (in vector notation). Inequality (3.2) guarantees (in conjunction with Hypothesis (H3)) that the norm of the state remains bounded in bounded time intervals as long as the values of the n functionals $V_i(t, \phi(t, t_0, x_0, u, d), u(t))$ $(i = 1, \ldots, n)$ remain bounded. The need for two types of inequalities in order to prove the IOS property by means of small-gain arguments was first shown in Sontag & Ingalls (2002).
For future reference, $V(t, x, u) = (V_1(t, x, u), \ldots, V_n(t, x, u))'$ is called the vector Lyapunov function for the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$. Now, we are ready to state our new small-gain theorem for a general dynamic system described in Section 3.1.

THEOREM 3.1 Consider system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ and suppose that there exist maps $V_i : \Re_+ \times \mathcal{X} \times U \to \Re^+$, with $V_i(t, 0, 0) = 0$ for all $t \geqslant 0 (i = 1, \ldots, n)$ and a MAX-preserving continuous map $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(0) = 0$ such that Hypotheses (H1–H3) hold. Then system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is RFC from the input $u \in M_U$ and there exist functions $\tilde{\sigma} \in KL$ and $\tilde{\beta} \in K^+$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ the following estimate holds for all $t \geqslant t_0$:

$$V(t) \leqslant \mathbf{1}\tilde{\sigma}(\tilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) + G([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}), \tag{3.4}$$

where

$$G(s) = \mathrm{MAX}\{Q(\mathbf{1}\sigma(p^u(s), 0)), Q(\mathbf{1}\sigma(p(Q(\mathbf{1}\zeta(s))), 0)), Q(\mathbf{1}\zeta(s))\} \tag{3.5}$$

and $Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$. Moreover, if $\beta, c \in K^+$ are bounded then $\tilde{\beta} \in K^+$ is bounded. Finally, if in addition to (H1–H3), the following hypothesis holds:

Hypothesis (H4) (bound for the norm of the output): *There exists $q \in \mathcal{N}_n$ such that the following inequality holds for all $(t, x, u) \in \Re_+ \times \mathcal{X} \times U$:*

$$\|H(t, x, u)\|_{\mathcal{Y}} \leqslant q(V(t, x, u)), \tag{3.6}$$

where $V(t, x, u) = (V_1(t, x, u), \ldots, V_n(t, x, u))'$.

then system Σ satisfies the IOS property from the input $u \in M_U$ with gain $\gamma(s) := q(G(s))$. Moreover, if $\beta, c \in K^+$ are bounded, then system Σ satisfies the UIOS property from the input $u \in M_U$ with gain $\gamma(s) := q(G(s))$.

REMARK 3.2. Note that for the control input-free case, i.e., $u \equiv 0$, Theorem 3.1 implies (Uniform) RGAOS for the corresponding system. Moreover, if there exists $M \geqslant 1$ such that $\sigma(s, 0) = Ms$ for all $s \geqslant 0$, then the functions $G_i \in \mathcal{N}_1$ $(i = 1, \ldots, n)$ with $G(s) = (G_1(s), \ldots, G_n(s))'$ are given by

$$G_i(s) := \varphi_i(\max\{Mp^u(s), Mp(\varphi_1(\zeta(s)), \ldots, \varphi_n(\zeta(s))), \zeta(s)\}), \quad i = 1, \ldots, n,$$

where

$$\varphi_i(s) = \max\{s, \max_{k=1,\ldots,n-1} \max\{(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \ldots \circ \gamma_{j_{k-1},j_k})(s) : (j_1, \ldots, j_k) \in \{1, \ldots, n\}^k\}\},$$
$$i = 1, \ldots, n$$

and $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, n$ are the functions with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$, $i = 1, \ldots, n$ and $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$. The reader should note that if Hypothesis (ii) of Proposition 2.7 holds for the functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, n$, then for each $i = 1, \ldots, n$, we have either $\varphi_i(s) = s$ or there exists an index set $(i, j_1, \ldots, j_k) \in \{1, \ldots, n\}^{k+1}$ with no repeated index such that $\varphi_i(s) = (\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \ldots \circ \gamma_{j_{k-1},j_k})(s)$.

The following elementary example demonstrates that Theorem 3.1 can be applied to the time-varying case of non-uniform in time stability.

EXAMPLE 3.3. Consider the large-scale system described by the following time-varying ODEs:

$$\dot{x}_i(t) = -ix_i(t) + d_i(t)\exp(t)x_{i+1}(t), \quad i = 1, \ldots, n-1,$$
$$\dot{x}_n(t) = -nx_n(t) + d_n(t)\exp(-nt)x_1(t) + \upsilon(t), \tag{3.7}$$

where $x_i(t) \in \Re (i = 1, \ldots, n)$, $\upsilon\colon \Re_+ \to \Re$ is measurable and locally essentially bounded, $d_i : \Re_+ \to [-a_i, a_i]$ $(i = 1, \ldots, n)$ are measurable functions and $a_i \geqslant 0$ $(i = 1, \ldots, n)$ are constants. Using the variations of constants formula for each one of the states of (3.7), we obtain for the solution $x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n$ with initial condition $x(t_0) = (x_1(t_0), \ldots, x_n(t_0))' \in \Re^n$ corresponding to arbitrary measurable and locally essentially bounded input $\upsilon\colon \Re_+ \to \Re$ and arbitrary measurable functions $d_i\colon \Re_+ \to [-a_i, a_i]$ $(i = 1, \ldots, n)$:

$$x_i(t) = x_i(t_0)\exp(-i(t-t_0)) + \int_{t_0}^{t} \exp(-i(t-\tau))d_i(\tau)\exp(\tau)x_{i+1}(\tau)\mathrm{d}\tau, \; i = 1, \ldots, n-1, \quad (3.8)$$

$$x_n(t) = x_n(t_0)\exp(-n(t-t_0)) + \int_{t_0}^{t} \exp(-n(t-\tau))d_n(\tau)\exp(-(n-1)\tau)x_1(\tau)\mathrm{d}\tau + \int_{t_0}^{t} \exp(-n(t-\tau))\upsilon(\tau)\mathrm{d}\tau \quad (3.9)$$

for all $t \geqslant t_0$. Defining $V_i(t, x) := \exp((i-1)t)|x_i|$ $(i = 1, \ldots, n)$ and $u(t) := \exp((n-1)t)\upsilon(t)$, we obtain from (3.8) and (3.9) for all $t \geqslant t_0$ and $\varepsilon > 0$:

$$V_i(t) \leqslant \max\left\{(1+\varepsilon)V_i(t_0)\exp(-(t-t_0)), a_i(1+\varepsilon^{-1}) \sup_{t_0 \leqslant \tau \leqslant t} V_{i+1}(\tau)\right\}, \; i = 1, \ldots, n-1, \quad (3.10)$$

$$V_n(t) \leqslant \max\left\{2(1+\varepsilon)V_n(t_0)\exp(-(t-t_0)), a_n(1+\varepsilon^{-1}) \sup_{t_0 \leqslant \tau \leqslant t} V_1(\tau), \times 2(1+\varepsilon) \sup_{t_0 \leqslant \tau \leqslant t} |u(\tau)|\right\}. \quad (3.11)$$

Define $L(t, x) := \max_{i=1,\ldots,n} V_i(t, x)$. Using (3.10) and (3.11), it can be directly shown that Hypothesis (H1) holds with $\sigma(s, t) := 2(1+\varepsilon)s\exp(-t)$, $p^u(s) = \zeta(s) := 2(1+\varepsilon)s$, $\Gamma\colon \Re_+^n \to \Re_+^n$ with $\Gamma_i(x) := a_i(1+\varepsilon^{-1})x_{i+1}$ for $i = 1, \ldots, n-1$ and $\Gamma_n(x) := a_n(1+\varepsilon^{-1})x_1$, $p \in \mathcal{N}_n$ with $p(x_1, \ldots, x_n) := (1+\varepsilon^{-1})\max\{a_n x_1, \max_{i=1,\ldots,n-1}(a_i x_{i+1})\}$, $\nu(t) \equiv 1$, $c(t) := 2(1+\varepsilon)\exp(2(n-1)t)$ and $a(s) := s^2$. Hypothesis (H3) holds as well with $\mu(t) = \kappa(t) \equiv 1$, $g \equiv 0$, $b(s) := s\sqrt{n}$ and $\beta(t) := \exp((n-1)t)$. Finally, Hypothesis (H2) holds if and only if

$$\prod_{i=1}^{n} a_i < 1. \quad (3.12)$$

Indeed, the only possible cycle that can be formed by the gain functions is the cycle $(\gamma_{1,2} \circ \gamma_{2,3} \circ \ldots \circ \gamma_{n,1})(s)$, In this case, the small-gain condition gives $(1+\varepsilon^{-1})^n \prod_{i=1}^{n} a_i < 1$. Consequently, if (3.12) holds then there exists $\varepsilon > 0$ such that inequality $(1+\varepsilon^{-1})^n \prod_{i=1}^{n} a_i < 1$ holds.

It follows from Theorem 3.1 and Remark 3.2 that under Hypothesis (3.12), for every $\varepsilon > 0$ with $(1+\varepsilon^{-1})^n \prod_{i=1}^{n} a_i < 1$, there exist functions $\tilde{\sigma} \in KL$ and $\tilde{\beta} \in K^+$ such that for every $x(t_0) = (x_1(t_0), \ldots, x_n(t_0))' \in \Re^n$ and for every measurable and locally essentially bounded functions $d_i\colon \Re_+ \to [-a_i, a_i]$ $(i = 1, \ldots, n)$, $\upsilon\colon \Re_+ \to \Re$, the solution $x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n$ with initial condition $x(t_0) = (x_1(t_0), \ldots, x_n(t_0))' \in \Re^n$ corresponding to $\upsilon\colon \Re_+ \to \Re$ and $d_i\colon \Re_+ \to [-a_i, a_i]$ $(i = 1, \ldots, n)$ satisfies for all $t \geqslant t_0$:

$$\exp((i-1)t)|x_i(t)| \leqslant \tilde{\sigma}(\tilde{\beta}(t_0)|x(t_0)|, t - t_0) + G_i \sup_{t_0 \leqslant \tau \leqslant t} (\exp((n-1)\tau)|\upsilon(\tau)|),$$
$$i = 1, \ldots, n-1, \tag{3.13}$$

where

$$G_i := 4(1+\varepsilon)^2 R \max\left\{1, \max_{k=1,\ldots,n-1} (1+\varepsilon^{-1})^k \prod_{j=0}^{k-1} a_{i+j}\right\},$$

$$R := \max\left\{1, (1+\varepsilon^{-1}) \max_{l=1,\ldots,n}\left(a_l \max\left\{1, \max_{k=1,\ldots,n-1}(1+\varepsilon^{-1})^k \prod_{j=0}^{k-1} a_{l+1+j}\right\}\right)\right\}.$$

In the above formulas, we have used the convention $a_{n+i} = a_i$ for $i = 1, \ldots, n$. Note that inequalities (3.13) can be used for further analysis (for example, if system (3.7) is interconnected with another system). Finally, borrowing the terminology from Karafyllis & Jiang (2007), inequalities (3.13) imply that system (3.7) satisfies the weighted ISS property, as defined in Karafyllis & Jiang (2007).

We finish this section with an important remark.

REMARK 3.4. At this point, the reader may form the intuitive notion that (at least for the ISS case where $H(t, x, u) \equiv x \in \mathcal{X}$) each functional $V_i\colon \Re_+ \times \mathcal{X} \times U \to \Re_+$ $(i = 1, \ldots, n)$ is some kind of measure of a portion of state $x_i \in \mathcal{X}_i$ $(i = 1, \ldots, n)$, where $x = (x_1, \ldots, x_n) \in \mathcal{X} := \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. While this is certainly true for many interesting cases (see Example 3.3 above and Examples 5.1 and 5.2 below), it is not true in general. The following examples show that

- it may be necessary to consider functionals, which are mappings of the state (see Example 3.5 below),
- it may be necessary to consider more than one functionals as measures of a portion of state (see Example 3.6 below).

EXAMPLE 3.5. Consider the input-free linear planar system described by ODEs:

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -b^2 x_1 - 2b x_2,$$
$$x = (x_1, x_2) \in \Re^2, \tag{3.14}$$

where $b > 0$. It is very difficult to prove global asymptotic stability of $0 \in \Re^2$ for (3.14) by means of small-gain arguments (using Theorem 3.1 or any other small-gain result in the literature) and using functions $V_1(x_1)$ and $V_2(x_2)$ each one depending on only one component of the state vector. This happens because the x_1-subsystem, i.e., $\dot{x}_1 = x_2$ does not satisfy the ISS property with respect to the input $x_2 \in \Re$. On the other hand, by using the functions $V_1(x_1) := x_1^2$ and $V_2(x_1, x_2) := (x_2 + b x_1)^2$, one can show that the hypotheses of Corollary 4.2 below hold (and consequently the hypotheses of Theorem 3.1; see the proof of Corollary 4.2) and that $0 \in \Re^2$ is uniformly GAS. The example shows that it may be necessary to consider functions $V_i\colon \Re_+ \times \mathcal{X} \times U \to \Re_+$ $(i = 1, \ldots, n)$, which are mappings of the state.

EXAMPLE 3.6. Consider the input-free planar system described by ODEs:

$$\begin{aligned}\dot{x}_1 &= -ax_1 + \max(0, x_2),\\ \dot{x}_2 &= x_1 - \max(r_1 x_2, r_2 x_2),\\ x &= (x_1, x_2) \in \Re^2,\end{aligned} \tag{3.15}$$

where $a, r_1, r_2 > 0$. If the functions $V_i(x_i) := x_i^2$, $i = 1, 2$ are used, then we can show that $0 \in \Re^2$ for (3.15) is uniformly GAS provided that

$$1 < a \min(r_1, r_2).$$

The proof can be achieved by means of Corollary 4.2 below. One the other hand, if the functions $V_1(x_1) := x_1^2$, $V_2(x_2) := (\max(0, x_2))^2$, $V_3(x_2) := (\min(0, x_2))^2$ are used, then we can show that $0 \in \Re^2$ for (3.15) is uniformly GAS provided that

$$1 < a \max(r_1, r_2).$$

Again, the proof can be achieved by means of Corollary 4.2 below. The example shows that it may be necessary to consider more than one functionals as measures of one component of the state.

4. Vector Lyapunov functions and functionals

In this section, we provide sufficient Lyapunov-like conditions for the verification of Theorem 3.1 for three types of systems: (i) systems described by ODEs, (ii) systems described by RFDEs and (iii) sampled-data systems. Note that since families of Lyapunov functions (or functionals) are employed, the obtained results constitute conditions for vector Lyapunov functions (or functionals) for the (U)IOS property.

4.1 *Systems of ODEs*

We consider systems described by ODEs of the form:

$$\begin{aligned}&\dot{x} = f(t, x, u, d), \quad Y = H(t, x),\\ &x \in \Re^n, \quad Y \in \Re^N, \quad u \in U, \quad d \in D, \quad t \geqslant 0,\end{aligned} \tag{4.1}$$

where $D \subseteq \Re^l$, $U \subseteq \Re^m$ with $0 \in U$ and $f\colon \Re_+ \times \Re^n \times U \times D \to \Re^n$, $H\colon \Re_+ \times \Re^n \to \Re^N$ are continuous mappings with $H(t, 0) = 0$, $f(t, 0, 0, d) = 0$ for all $(t, d) \in \Re_+ \times D$ that satisfy the following hypotheses:

(A1) There exists a symmetric positive-definite matrix $P \in \Re^{n\times n}$ such that for every bounded $I \subseteq \Re_+$ and for every bounded $S \subset \Re^n \times U$, there exists a constant $L \geqslant 0$ satisfying the following inequality:

$$\begin{aligned}&(x - y)' P(f(t, x, u, d) - f(t, y, u, d)) \leqslant L|x - y|^2\\ &\forall t \in I, \quad \forall (x, u, y, u) \in S \times S, \ \forall d \in D.\end{aligned}$$

(A2) There exist $a \in K_\infty$, $\gamma \in K^+$ such that $|f(t,x,u,d)| \leqslant \gamma(t)a(|x|+|u|)$ for all $(t,x,u,d) \in \Re_+ \times \Re^n \times U \times D$.

(A3) There exist functions $V_i \in C^1(\Re_+ \times \Re^n; \Re_+)(i=1,\dots,k)$, $W \in C^1(\Re_+ \times \Re^n; \Re_+)$, $a_1, a_2, a_3, a_4 \in K_\infty$ $\mu, \beta, \kappa \in K^+$, $\zeta \in \mathcal{N}_1$, $g \in \mathcal{N}_k$,$\gamma_{i,j} \in \mathcal{N}_1$, $p_i \in \mathcal{N}_1$, $i, j = 1,\dots,k$, a family of positive-definite functions $\rho_i \in C^0(\Re_+; \Re_+)(i=1,\dots,k)$ and a constant $\lambda \in (0,1)$ such that the following inequalities hold for all $(t,x,u) \in \Re_+ \times \Re^n \times U$:

$$a_1(|H(t,x)|) \leqslant \max_{i=1,\dots,k} V_i(t,x) \leqslant a_2(\beta(t)|x|), \tag{4.2}$$

$$a_3(\mu(t)|x|) - g(V_1(t,x),\dots,V_k(t,x)) - \kappa(t) \leqslant W(t,x) \leqslant a_4(\beta(t)|x|) \tag{4.3}$$

$$\begin{aligned} \sup\left\{\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x) f(t,x,u,d) \colon d \in D\right\} &\leqslant -W(t,x) \\ &+ \lambda \max\{\zeta(|u|), \max\nolimits_{j=1,\dots,k} p_j(V_j(t,x))\} \end{aligned} \tag{4.4}$$

and for every $i=1,\dots,k$ and $(t,x,u) \in \Re_+ \times \Re^n \times U$ the following implication holds:

$$\begin{aligned} \text{‘If} \max\left\{\zeta(|u|), \max_{j=1,\dots,k} \gamma_{i,j}(V_j(t,x))\right\} \leqslant V_i(t,x) \text{ then } &\frac{\partial V_i}{\partial t}(t,x) + \sup_{d \in D} \frac{\partial V_i}{\partial x}(t,x) f(t,x,u,d) \\ &\leqslant -\rho_i(V_i(t,x))\text{’}. \end{aligned} \tag{4.5}$$

Our main result concerning systems of the form (4.1) is the following result which provides sufficient conditions for Theorem 3.1 to hold.

THEOREM 4.1 (VECTOR LYAPUNOV FUNCTION CHARACTERIZATION OF THE IOS PROPERTY). Consider system (4.1) under Hypotheses (A1–A3). If the small-gain conditions (2.4a,b) hold, then system (4.1) satisfies the IOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$, where

$$\theta(s) := \max_{i=1,\dots,k} \varphi_i\left(\max\left\{\max_{l=1,\dots,k} \max_{j=1,\dots,k} \gamma_{l,j}\left(\varphi_j(\zeta(s))\right), \max_{j=1,\dots,k} p_j\left(\varphi_j(\zeta(s))\right), \zeta(s)\right\}\right) \tag{4.6}$$

and

$$\begin{aligned} \varphi_i(s) := \max\left\{s, \max_{l=1,\dots,k-1} \max\{(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \dots \circ \gamma_{j_{l-1},j_l})(s) : (j_1,\dots,j_l) \in \{1,\dots,n\}^l\}\right\}, \\ i = 1,\dots,k \end{aligned} \tag{4.7}$$

Moreover, if $\beta \in K^+$ is bounded, then system (4.1) with output $Y = H(t,x)$ satisfies the UIOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$.

Remark on Theorem 4.1: The proof of Theorem 4.1 (see Section 6) shows that inequalities (4.2) and (4.5) are used for the derivation of inequalities (3.1) and (3.6), while inequalities (4.3) and (4.4) are used for the derivation of inequalities (3.2) and (3.3). Hypotheses (A1) and (A2) are regularity hypotheses that guarantee uniqueness of solutions and continuity of the solutions with respect to initial data for system (4.1).

For the ISS case where $H(t,x) = x$, one can set $W(t,x) \equiv 0$ in Theorem 4.1 to arrive at a corollary on the vector Lyapunov function characterization of the ISS property.

COROLLARY 4.2 (VECTOR LYAPUNOV FUNCTION CHARACTERIZATION OF THE ISS PROPERTY). Consider system (4.1) under hypotheses (A1) and (A2) and suppose that there exists a family of functions $V_i \in C^1(\Re_+ \times \Re^n; \Re_+)(i = 1, \ldots, k)$, functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$ and a family of positive-definite functions $\rho_i \in C^0(\Re_+; \Re_+)(i = 1, \ldots, k)$, such that

$$a_1(|x|) \leqslant \max_{i=1,\ldots,k} V_i(t,x) \leqslant a_2(\beta(t)|x|), \quad \forall (t,x) \in \Re_+ \times \Re^n \tag{4.8}$$

and implication (4.5) holds for every $i = 1, \ldots, k$ and $(t, x, u) \in \Re_+ \times \Re^n \times U$. If, additionally, the small-gain conditions (2.4a,b) hold, then system (4.1) satisfies the ISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$, where $\theta \in \mathcal{N}_1$ is defined by (4.6) and (4.7). Moreover, if $\beta \in K^+$ is bounded, then system (4.1) satisfies the UISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$.

Comparison of Theorem 4.1 and Corollary 4.2 with existing results: The reader should compare the result of Corollary 4.2 with Theorem 3.4 in Karafyllis *et al.* (2008a). It is clear that Theorem 3.4 in Karafyllis *et al.* (2008a) is a special case of Corollary 4.2 with $\gamma_{i,j}(s) = a(s)$ for all $i, j = 1, \ldots, k$, where $a \in \mathcal{N}_1$ with $a(s) < s$ for $s > 0$. Alternative vector Lyapunov characterizations are based on the main result in Dashkovskiy *et al.* (2007) (e.g., Theorem 3.6 in Karafyllis *et al.*, 2008a) or on the cyclic small-gain condition in Teel (2005) (see for example Theorem 2 in Jiang & Wang, 2008). In order to demonstrate the applicability of our results to large-scale interconnected systems, consider the case

$$\dot{x}_i = f_i(d, x, u), \quad i = 1, \ldots, k,$$

$$x = (x_1', \ldots, x_k')' \in \Re^N, \quad d \in D, u \in U,$$

where $x_i \in \Re^{n_i}$, $i = 1, \ldots, k$, $N = n_1 + \cdots + n_k$, $D \subset \Re^l$ is a non-empty compact set, $U \subseteq \Re^m$ is a non-empty set with $0 \in U$, $f_i \colon D \times \Re^N \times U \to \Re$, $i = 1, \ldots, k$ are locally Lipschitz mappings with $f_i(d, 0, 0) = 0$ for all $d \in D$, $i = 1, \ldots, k$. We assume that the UISS property holds for each subsystem $\dot{x}_i = f_i(d, x, u)$ with input $(u, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(i = 1, \ldots, k)$. Let $V_i \in C^1(\Re_+ \times \Re^{n_i}; \Re_+)(i = 1, \ldots, k)$ be ISS-Lyapunov functions for each of the subsystems, i.e. positive definite and radially unbounded functions for which the following inequalities hold for $i = 1, \ldots, k$:

$$\sup_{d \in D} \Bigg\{ \nabla V_i(x_i) f_i(d, x, u) \colon u \in U, x = (x_1', \ldots, x_k')' \in \Re^N,$$
$$\times \max \left\{ \zeta(|u|), \max_{j=1,\ldots,k} \gamma_{i,j}(V_j(x_j)) \right\} \leqslant V_i(x_i) \Bigg\} \leqslant -\rho_i\,(V_i(x_i)) \quad \forall\, x_i \neq 0$$

for certain functions $\zeta \in \mathcal{N}_1$,$\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$ and certain positive-definite functions $\rho_i \in C^0(\Re_+; \Re_+)$ $(i = 1, \ldots, k)$. Working with the Lyapunov-like functions $V_i \in C^1(\Re_+ \times \Re^{n_i}; \Re_+)$ $(i = 1, \ldots, k)$ and exploiting Corollary 4.2, we can guarantee that the UISS property holds for the above system if the small-gain conditions (2.4a,b) hold. It should be clear that the functions $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$ are the actual gain functions, i.e., the following inequalities hold for all $i = 1, \ldots, k$,

$t \geqslant 0$, $x(0) \in \Re^N$ and $u \in M_U$:

$$V_i(x_i(t)) \leqslant \max\left\{\sigma_i(V_i(x_i(0)), t), \zeta\left(\sup_{0\leqslant\tau\leqslant t}|u(\tau)|\right), \max_{j=1,\ldots,k}\gamma_{i,j}\left(\sup_{0\leqslant\tau\leqslant t}V_j(x_j(\tau))\right)\right\}$$

for certain $\sigma_i \in KL$ $(i = 1, \ldots, k)$. Since $\gamma_{i,i}(s) \equiv 0$ for $i = 1, \ldots, k$, then the above inequalities are nothing else but the inequalities of the max-formulation of the UISS property for each subsystem $\dot{x}_i = f_i(d, x, u)$ with input $(u, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(i = 1, \ldots, k)$ and $V_i(x_i)$ replacing $|x_i|$ for $i = 1, \ldots, k$.

4.2 *Systems described by RFDEs*

Let $D \subseteq \Re^l$ be a non-empty set, $U \subseteq \Re^m$ a non-empty set with $0 \in U$ and $\mathcal{Y}$ a normed linear space. We denote by $x(t)$ the unique solution of the initial-value problem:

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, u(t), d(t)),\\ &Y(t) = H(t, T_r(t)x),\\ &x(t) \in \Re^n, \quad Y(t) \in Y, \quad d(t) \in D, \quad u(t) \in U \end{aligned} \tag{4.9}$$

with initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \Re^n)$, where $r > 0$ is a constant, $T_r(t)x := x(t+\theta); \theta \in [-r, 0]$ and the mappings $f\colon \Re_+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$, $H\colon \Re_+ \times C^0([-r, 0]; \Re^n) \to \mathcal{Y}$ satisfy $f(t, 0, 0, d) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \Re_+ \times D$.

The following hypotheses will be imposed on systems of the form (4.9):

(S1) The mapping $(x, u, d) \to f(t, x, u, d)$ is continuous for each fixed $t \geqslant 0$ and there exists a symmetric positive-definite matrix $P \in \Re^{n\times n}$ with the property that for every bounded $I \subseteq \Re_+$ and for every bounded $S \subset C^0([-r, 0]; \Re^n) \times U$, there exists a constant $L \geqslant 0$ such that

$$(x(0) - y(0))'P(f(t, x, u, d) - f(t, y, u, d)) \leqslant L \max_{\tau\in[-r,0]}|x(\tau) - y(\tau)|^2 = L\|x - y\|_r^2$$
$$\forall t \in I, \forall (x, u, y, u) \in S \times S, \forall d \in D.$$

(S2) There exist $a \in K_\infty$, $\gamma \in K^+$ such that $|f(t, x, u, d)| \leqslant \gamma(t)a(\|x\|_r + |u|)$ for all $(t, x, u, d) \in \Re_+ \times C^0([-r, 0]; \Re^n) \times U \times D$.

(S3) There exists a countable set $A \subset \Re_+$, which is either finite or $A = \{t_k; k = 1, \ldots, \infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1, 2, \ldots$, and $\lim t_k = +\infty$, such that the mapping $(t, x, u, d) \in (\Re_+ \backslash A) \times C^0([-r, 0]; \Re^n) \times U \times D \to f(t, x, u, d)$ is continuous. Moreover, for each fixed $(t_0, x, u, d) \in \Re_+ \times C^0([-r, 0]; \Re^n) \times U \times D$, we have $\lim_{t\to t_0^+} f(t, x, u, d) = f(t_0, x, u, d)$.

(S4) The mapping $H\colon \Re_+ \times C^0([-r, 0]; \Re^n) \to \mathcal{Y}$ is continuous.

Hypotheses (S1–S3) are regularity hypotheses, which guarantee uniqueness of solutions and continuity of the solutions with respect to initial data for system (4.9). Hypothesis (S4) is a standard technical hypothesis for control systems described by RFDEs.

The class of functionals which are ‘almost Lipschitz on bounded sets’ was introduced in Karafyllis *et al.* (2008b,c) and is used extensively in the present work. For the sake of completeness, we recall here the definition in Karafyllis *et al.* (2008b,c).

DEFINITION 4.3. We say that a continuous functional $V\colon [-a,+\infty)\times C^0([-r,0];\Re^n)\to\Re_+$, $r>0$, $a\geqslant 0$ is 'almost Lipschitz on bounded sets', if there exist non-decreasing functions $M\colon \Re_+\to\Re_+$, $P\colon \Re_+\to\Re_+$, $G\colon \Re_+\to[1,+\infty)$ such that for all $R\geqslant 0$, the following properties hold:

(P1) For every $x,y\in\{x\in C^0([-r,0];\Re^n);\ \|x\|_r\leqslant R\}$, it holds that

$$|V(t,y)-V(t,x)|\leqslant M(R)\|y-x\|_r \quad \forall t\in[-a,R].$$

(P2) For every absolutely continuous function $x\colon[-r,0]\to\Re^n$ with $\|x\|_r\leqslant R$ and essentially bounded derivative, it holds that

$$|V(t+h,x)-V(t,x)|\leqslant hP(R)\left(1+\operatorname{ess\,sup}_{-r\leqslant\tau\leqslant 0}|\dot{x}(\tau)|\right) \quad \text{for all } t\in[-a,R] \quad \text{and}$$

$$0\leqslant h\leqslant \frac{1}{G\left(R+\operatorname{ess\,sup}_{-r\leqslant\tau\leqslant 0}|\dot{x}(\tau)|\right)}.$$

For the case $r=0$, we say that a continuous functional $V\colon[-a,+\infty)\times C^0([-r,0];\Re^n)\to\Re_+$, is 'almost Lipschitz on bounded sets', if $V\colon[-a,+\infty)\times\Re^n\to\Re_+$ is locally Lipschitz (note that by convention $C^0([-r,0];\Re^n)=\Re^n$), i.e. for every compact $S\subset[-a,+\infty)\times\Re^n$, there exists $L\geqslant 0$ such that $|V(t,x)-V(\tau,y)|\leqslant L|t-\tau|+L|x-y|$ for all $(t,x)\in S$, $(\tau,y)\in S$.

If the continuous functional $V\colon[-a,+\infty)\times C^0([-r,0];\Re^n)\to\Re_+$, is 'almost Lipschitz on bounded sets', then we can define the derivative $V^0(t,x;v)$ in the following way (see also Karafyllis *et al.*, 2008b,c) for $(t,x,v)\in\Re_+\times C^0([-r,0];\Re^n)\times\Re^n$:

$$V^0(t,x;v):=\limsup_{h\to 0^+}\frac{V(t+h,E_h(x;v))-V(t,x)}{h},$$

where $E_h(x;v)$ with $0\leqslant h<r$ denotes the following operator:

$$E_h(x;v):=\begin{cases} x(0)+(\theta+h)v & \text{for} \quad -h<\theta\leqslant 0,\\ x(\theta+h) & \text{for} \quad -r\leqslant\theta\leqslant -h.\end{cases} \tag{4.10a}$$

Particularly, for the case $r=0$, we define

$$E_h(x;v):=x(0)+hv. \tag{4.10b}$$

The following theorem provides sufficient Lyapunov-like conditions for the (U)IOS property. The gain functions of the IOS property can be determined 'explicitly' in terms of the functions involved in the assumptions of the theorem.

THEOREM 4.4 Consider system (4.9) under hypotheses (S1–S4) and suppose that there exist almost Lipschitz on bounded sets functionals $Q_i\colon[-r+r_i,+\infty)\times C^0([-r_i,0];\Re^n)\to\Re_+$ with $0\leqslant r_i\leqslant r (i=1,\dots,k)$, $Q_0\colon[-r+r_0,+\infty)\times C^0([-r_0,0];\Re^n)\to\Re_+$ with $0\leqslant r_0\leqslant r$, functions $a_1,a_2,a_3,a_4\in K_\infty$, $\mu,\beta,\kappa\in K^+$, $\zeta\in\mathcal{N}_1$, $g\in\mathcal{N}_k$, $\gamma_{i,j}\in\mathcal{N}_1$, $p_i\in\mathcal{N}_1$, $i,j=1,\dots,k$, positive-definite functions $\rho_i\in C^0(\Re_+;\Re_+)(i=1,\dots,k)$ and a constant $\lambda\in(0,1)$ such that for all

$(t, x, u) \in \Re_+ \times C^0([-r, 0]; \Re^n) \times U$, the following inequalities hold:

$$a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant \max_{i=1,\dots,k} V_i(t,x) \leqslant a_2(\beta(t)\|x\|_r), \tag{4.11}$$

$$a_3(\mu(t)\|x\|_r) - g(V_1(t,x),\dots,V_k(t,x)) - \kappa(t) \leqslant W(t,x) \leqslant a_4(\beta(t)\|x\|_r), \tag{4.12}$$

$$\sup_{d\in D} Q_0^0(t, T_{r_0}(0)x;\ f(t,x,u,d)) \leqslant -Q_0(t, T_{r_0}(0)x) + \lambda \max\left\{\zeta(|u|), \max_{j=1,\dots,k} p_j(V_j(t,x))\right\}, \tag{4.13}$$

where

$$V_i(t,x) := \sup_{\theta\in[-r+r_i,0]} Q_i(t+\theta, T_{r_i}(\theta)x), \quad i = 1,\dots,k,$$
$$W(t,x) := \sup_{\theta\in[-r+r_0,0]} Q_0(t+\theta, T_{r_0}(\theta)x) \tag{4.14}$$

and for every $i = 1,\dots,k$ and $(t,x,u) \in \Re_+ \times C^0([-r,0];\Re^n) \times U$, the following implication holds:

$$\text{'If } \max\left\{\zeta(|u|), \max_{j=1,\dots,k} \gamma_{i,j}(V_j(t,x))\right\} \leqslant Q_i(t, T_{r_i}(0)x) \text{ then } \sup_{d\in D} Q_i^0(t, T_{r_i}(0)x;\ f(t,x,u,d))$$
$$\leqslant -\rho_i(Q_i(t, T_{r_i}(0)x))\text{'} \tag{4.15}$$

Finally, suppose that the small-gain conditions (2.4a,b) hold.

Then system (4.9) satisfies the IOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$, where $\theta \in \mathcal{N}_1$ is defined by (4.6) and (4.7). Moreover, if $\beta \in K^+$ is bounded, then system (4.9) satisfies the UIOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$.

When $H(t,x) = x$, setting $Q_0(t,x) \equiv 0$ in Theorem 4.4 leads to a result on the ISS of system (4.9).

COROLLARY 4.5 Consider system (4.9) under hypotheses (S1–S4) and suppose that there exists a family of almost Lipschitz on bounded sets functionals $Q_i\colon [-r+r_i, +\infty) \times C^0([-r_i, 0]; \Re^n) \to \Re_+$ with $0 \leqslant r_i \leqslant r (i = 1,\dots,k)$, functions $a_1, a_2 \in K_\infty, \beta \in K^+, \zeta \in \mathcal{N}_1, \gamma_{i,j} \in \mathcal{N}_1, i, j = 1,\dots,k$, and a family of positive-definite functions $\rho_i \in C^0(\Re_+;\Re_+)(i = 1,\dots,k)$, such that for all $(t,x,u) \in \Re_+ \times C^0([-r,0];\Re^n) \times U$, the following inequality holds:

$$a_1(\|x\|_r) \leqslant \max_{i=1,\dots,k} V_i(t,x) \leqslant a_2(\beta(t)\|x\|_r), \tag{4.16}$$

where

$$V_i(t,x) := \sup_{\theta\in[-r+r_i,0]} Q_i(t+\theta, T_{r_i}(\theta)x), \quad i = 1,\dots,k \tag{4.17}$$

and implication (4.15) holds for every $i = 1,\dots,k$ and $(t,x,u) \in \Re_+ \times C^0([-r,0];\Re^n) \times U$. If, additionally, the small-gain conditions (2.4a,b) hold, then system (4.9) satisfies the ISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$, where $\theta \in \mathcal{N}_1$ is defined by (4.6) and (4.7). Moreover, if $\beta \in K^+$ is bounded, then system (4.9) satisfies the UISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$.

REMARK 4.6. It is of interest to note that some of the functionals $Q_i\colon [-r + r_i, +\infty) \times C^0([-r_i, 0]; \Re^n) \to \Re^+$ in Theorem 4.4 and Corollary 4.5 are allowed to be functions (case of $r_i = 0$). This reminds the case of Razumikhin functions, which are used frequently for the proof of stability properties of systems described by RFDEs (see Karafyllis *et al.*, 2008c; Mazenc & Niculescu, 2001; Niculescu, 2001; Teel, 1998). Consequently, Theorem 4.4 and Corollary 4.5 allow the flexibility of using Lyapunov-like functionals with Razumikhin-like functions in order to prove desired stability properties.

REMARK 4.7. It should be clear that the convention $C^0([0, 0]; \Re^n) = \Re^n$ allows Theorem 4.4 and Corollary 4.5 to be used in the case of systems described by ODEs (case of $r = 0$). In this case, Theorem 4.4 and Corollary 4.5 are generalizations of Theorem 4.1 and Corollary 4.2: the use of locally Lipschitz functions is allowed and discontinuities of the right-hand side of the differential equations with respect to time are allowed. Moreover, in the time-delay case (case of $r > 0$), the diagonal gain functions $\gamma_{i,i}$ for $i = 1, \ldots, k$ play a significant role (see Example 5.1 below) if Razumikhin-like functions are used.

4.3 *Sampled-data systems*

We consider switched systems, described in the following way: given a pair of sets $D \subseteq \Re^l$, $U \subseteq \Re^m$ with $0 \in U$, a positive function $h\colon \Re^n \times U \to (0, r]$, which is bounded by a certain constant $r > 0$ and a pair of vector fields $f\colon \Re^n \times \Re^n \times D \times U \times U \to \Re^n$, $H\colon \Re^n \to \Re^k$, we consider the switched system that produces for each $(t_0, x_0) \in \Re_+ \times \Re^n$ and for each triplet of measurable and locally bounded inputs $d\colon \Re_+ \to D$, $\tilde{d}\colon \Re_+ \to \Re_+$, $u\colon \Re_+ \to U$, the piecewise absolutely continuous function $t \to x(t) \in \Re^n$, via the following algorithm:

Step i:

(1) Given τ_i and $x(\tau_i)$, calculate τ_{i+1} using the equation $\tau_{i+1} = \tau_i + \exp(-\tilde{d}(\tau_i))h(x(\tau_i), u(\tau_i))$,
(2) Compute the state trajectory $x(t)$, $t \in [\tau_i, \tau_{i+1})$ as the solution of the differential equation $\dot{x}(t) = f(x(t), x(\tau_i), d(t), u(t), u(\tau_i))$,
(3) Calculate $x(\tau_{i+1})$ using the equation $x(\tau_{i+1}) = \lim_{t \to \tau_{i+1}^-} x(t)$.

For $i = 0$, we take $\tau_0 = t_0$ and $x(\tau_0) = x_0$ (initial condition). Schematically, we write

$$
\begin{aligned}
&\dot{x}(t) = f(x(t), x(\tau_i), d(t), u(t), u(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}),\\
&\tau_0 = t_0, \quad \tau_{i+1} = \tau_i + \exp(-\tilde{d}(\tau_i))h(x(\tau_i), u(\tau_i)), \quad i = 0, 1, \ldots,\\
&Y(t) = H(x(t))
\end{aligned}
\tag{4.18}
$$

with initial condition $x(t_0) = x_0$. Switched systems of the form (4.18) are called 'sampled-data' systems (see also Nesic, *et al.*, 2009; Tabuada, 2007, for the case of state-dependent sampling period).

In the present work, we study systems of the form (4.18) under the following hypotheses:

(R1) $f(x, x_0, d, u, u_0)$ is continuous with respect to $(x, d, u) \in \Re^n \times D \times U$ and such that for every bounded $S \subset \Re^n \times \Re^n \times U \times U$ there exists constant $L \geqslant 0$ such that

$$
\begin{aligned}
&(x - y)'(f(x, x_0, d, u, u_0) - f(y, x_0, d, u, u_0)) \leqslant L|x - y|^2\\
&\forall (x, x_0, u, u_0, d) \in S \times D, \forall (y, x_0, u, u_0, d) \in S \times D.
\end{aligned}
$$

(R2) There exists a function $a \in K_\infty$ such that

$$
|f(x, x_0, d, u, u_0)| \leqslant a(|x| + |x_0| + |u| + |u_0|), \quad \forall (u, u_0, d, x, x_0) \in U \times U \times D \times \Re^n \times \Re^n
$$

(R3) $H\colon \Re^n \to \Re^k$ is a continuous map with $H(0) = 0$.

(R4) The function $h\colon \Re^n \times U \to (0, r]$ is a positive, continuous and bounded function.

The following theorem provides sufficient Lyapunov-like conditions for the (U)IOS property. The gain functions of the IOS property can be determined explicitly in terms of the functions involved in the assumptions of the theorem.

THEOREM 4.8 (VECTOR LYAPUNOV FUNCTION CHARACTERIZATION OF UIOS). Consider system (4.18) under Hypotheses (R1–R4) and suppose that there exist non-negative functions $V_i \in C^1(\Re^n; \Re_+)(i = 1, \ldots, k)$, $Q \in C^1(\Re^n; \Re_+)$, $a_1, a_2, a_3, a_4 \in K_\infty$, $\zeta \in \mathcal{N}_1$, $g \in \mathcal{N}_k$, $\gamma_{i,j} \in \mathcal{N}_1$, $p_i \in \mathcal{N}_1$, $i, j = 1, \ldots, k$, constants $\mu, \kappa \geqslant 0$, $\lambda \in (0, 1)$ and positive-definite functions $\rho_i \in C^0(\Re_+; \Re_+)$ $(i = 1, \ldots, k)$, such that the following inequalities hold for all $(x, x_0, u, u_0) \in \Re^n \times \Re^n \times U \times U$:

$$a_1(|H(x)|) \leqslant \max_{i=1,\ldots,k} V_i(x) \leqslant a_2(|x|), \tag{4.19}$$

$$a_3(|x|) - g(V_1(x), \ldots, V_k(x)) - \kappa \leqslant Q(x) \leqslant a_4(|x|) \tag{4.20}$$

$$\sup_{d\in D} \nabla Q(x) f(x, x_0, d, u, u_0) \leqslant \mu Q(x) + \lambda \max\left\{\zeta(|u|), \zeta(|u_0|), \max_{j=1,\ldots,k} p_j(V_j(x)), \max_{j=1,\ldots,k} p_j\left(V_j(x_0)\right)\right\} \tag{4.21}$$

and for every $i = 1, \ldots, k$ and $(x, u, u_0) \in \Re^n \times U \times U$ the following implication holds:

$$\text{'If } \max\left\{\zeta(|u|), \zeta(|u_0|), \max_{j=1,\ldots,k} \gamma_{i,j}(V_j(x))\right\} \leqslant V_i(x) \text{ and } x_0 \in A_i(h(x_0, u_0), x) \text{then } \sup_{d\in D} \nabla V_i(x) f(x, x_0, d, u, u_0) \leqslant -\rho_i(V_i(x))\text{'}, \tag{4.22}$$

where the family of set-valued maps $\Re_+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n (i = 1, \ldots, k)$ is defined by

$$A_i(T, x) = \bigcup_{0\leqslant s\leqslant T} \left\{ \begin{array}{c} x_0 \in \Re^n\colon \exists (d, u) \in M_D \times M_U \text{ with } \phi(s, x_0; d, u) = x, \\ \zeta(|u(t)|) \leqslant V_i(x), \gamma_{i,j}(V_j(\phi(t, x_0; d, u))) \leqslant V_i(x) \\ \text{for all } t \in [0, s] \text{ and } j = 1, \ldots, k \end{array} \right\} \tag{4.23}$$

and $\phi(t, x_0; d, u)$ denotes the solution of $\dot{x}(t) = f(x(t), x_0, d(t), u(t), u(0))$ with initial condition $x(0) = x_0$ corresponding to $(d, u) \in M_D \times M_U$.

Furthermore, if the small-gain conditions (2.4a,b) hold, then system (4.18) satisfies the UIOS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$ and zero gain from the input $\tilde{d} \in M_{\Re_+}$, where $\theta \in \mathcal{N}_1$ is defined by (4.6) and (4.7).

For the ISS case where $H(t, x) = x$, one can set $Q(x) \equiv 0$ in Theorem 4.8 and obtain a result on the vector Lyapunov characterization of UISS.

COROLLARY 4.9 (VECTOR LYAPUNOV FUNCTION CHARACTERIZATION OF UISS:) Consider system (4.18) under Hypotheses (R1–R4) and suppose that there exists a family of functions $V_i \in C^1(\Re^n; \Re_+)(i = 1, \ldots, k)$, functions $a_1, a_2 \in K_\infty$, $\zeta \in \mathcal{N}_1$, $\gamma_{i,j} \in \mathcal{N}_1$, $i, j = 1, \ldots, k$, and a family of positive-definite functions $\rho_i \in C^0(\Re_+;\Re_+)(i = 1, \ldots, k)$, such that the following inequality holds for all $x \in \Re^n$:

$$a_1(|x|) \leqslant \max_{i=1,\ldots,k} V_i(x) \leqslant a_2(|x|) \tag{4.24}$$

and implication (4.22) holds for every $i = 1, \ldots, k$ and $(x, u, u_0) \in \Re^n \times U \times U$, where the family of set-valued maps $\Re_+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n (i = 1, \ldots, k)$ is defined by (4.23) and $\phi(t, x_0; d, u)$ denotes the solution of $\dot{x}(t) = f(x(t), x_0, d(t), u(t), u(0))$ with initial condition $x(0) = x_0$ corresponding to $(d, u) \in M_D \times M_U$.

Under the small-gain conditions (2.4a,b), system (4.18) satisfies the UISS property with gain $\gamma = a_1^{-1} \circ \theta \in \mathcal{N}_1$ from the input $u \in M_U$ and zero gain from the input $\tilde{d} \in M_{\Re_+}$, where $\theta \in \mathcal{N}_1$ is defined by (4.6) and (4.7).

REMARK 4.10. It is worth noting that Theorem 4.1 and Corollary 4.3 in Karafyllis & Kravaris (2009) can be easily derived from Theorem 4.8 and Corollary 4.9 with $\gamma_{i,j}(s) = a(s)$ for all $i, j = 1, \ldots, k$ and $Q(x) \equiv 0$, where $a \in \mathcal{N}_1$ with $a(s) < s$ for $s > 0$. Moreover, it is assumed in Karafyllis & Kravaris (2009) that there exist a constant $R \geqslant 0$ and a function $p \in K_\infty$ such that $|x| \leqslant R + p(|H(x)|)$. In the present work, such a hypothesis is not needed.

The interpretation of the family of set-valued maps $\Re_+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ $(i = 1, \ldots, k)$, defined in (4.23) is the following (the same with Karafyllis & Kravaris, 2009): each $A_i(T, x) \subseteq \Re^n$ is the set of all states $x_0 \in \Re^n$ so that the solution of $\dot{x}(t) = f(x(t), x_0, d(t), u(t), u(0))$ with initial condition $x(0) = x_0$ can be controlled to $x \in \Re^n$ in time s less or equal than T by means of appropriate inputs $(d, u) \in M_D \times M_U$ that satisfy $\zeta(\sup_{t\in[0,s]} |u(t)|) \leqslant V_i(x)$ and such that the trajectory of the solution satisfies the constraint $\max_{j=1,\ldots,k} \sup_{t\in[0,s]} \gamma_{i,j}(V_j(x(t))) \leqslant V_i(x)$. In general, it is very difficult to obtain an accurate description of the set-valued maps $\Re_+ \times \Re^n \ni (T, x) \to A_i(T, x) \subseteq \Re^n$ defined by (4.23). However, for every $g \in C^1(\Re^n;\Re)$, we have

$$A_i(T, x) \subseteq B_i^g(T, x) = \{x_0 \in \Re^n : |g(x_0) - g(x)| \leqslant T b_i^g(x)\} \quad \forall (T, x) \in \Re_+ \times \Re^n,$$

where

$$b_i^g(x) := \max \Big\{ |\nabla g(\xi) f(\xi, x_0, d, u, u_0)| : d \in D, \zeta(\max\{|u|, |u_0|\}) \leqslant V_i(x),$$

$$\max_{j=1,\ldots,k} \gamma_{i,j}(\max\{V_j(\xi), V_j(x_0)\}) \leqslant V_i(x) \Big\} < +\infty$$

and $V_i \in C^1(\Re^n;\Re_+)(i = 1, \ldots, k)$ are the functions involved in hypotheses of Theorem 4.8.

5. Examples and applications

EXAMPLE 5.1. Consider the time-delay system:

$$\dot{x}_i(t) = -a_i x_i(t) + g_i(d(t), T_r(t)x), \quad i = 1, \ldots, n, \tag{5.1}$$

where $d(t) \in D \subseteq \Re^m$, $a_i > 0$ $(i = 1, \ldots, n)$ and $g_i\colon D \times C^0([-r, 0];\Re^n) \to \Re$ $(i = 1, \ldots, n)$ are continuous mappings with

$$\sup_{d\in D} |g_i(d, x)| \leqslant \max_{j=1,\ldots,n} c_{i,j} \|x_j\|_r \tag{5.2}$$

and

$$\sup_{d \in D}[(x_i(0) - y_i(0))(g_i(d,x) - g_i(d,y))] \leqslant L\|x-y\|_r^2, \quad i = 1,\ldots,n$$

for certain constants $L \geqslant 0$, $c_{i,j} \geqslant 0$ $(i,j = 1,\ldots,n)$ and for all $x, y \in C^0([-r,0];\Re^n)$. We next show that $0 \in C^0([-r,0];\Re^n)$ is RGAS for (5.1) if $c_{i,i} < a_i$ for all $i = 1,\ldots,n$ and the following small-gain conditions hold for each $r = 2,\ldots,n$:

$$c_{i_1,i_2}c_{i_2,i_3}\ldots c_{i_r,i_1} < a_{i_1}a_{i_2}\ldots a_{i_r} \tag{5.3}$$

for all $i_j \in \{1,\ldots,n\}$, $i_j \neq i_k$ if $j \neq k$.

First, we note that Hypotheses (S1–S4) hold for system (5.1) under Hypothesis (5.2) with output $H(t,x) := x \in C^0([-r,0];\Re^n)$. Define the family of functions $Q_i(x) = \frac{1}{2}x_i^2(0)$ and $V_i(x) := \sup_{\theta\in[-r,0]} Q_i(x(\theta)) = \frac{1}{2}\|x_i\|_r^2$ $(i = 1,\ldots,n)$ for $x \in C^0([-r,0];\Re^n)$. These mappings satisfy inequality (4.16) and definition (4.17) with $a_1(s) := \frac{1}{2n}s^2$, $a_2(s) := \frac{1}{2}s^2$, $\beta(t) \equiv 1$ and $r_i := 0$, $i = 1,\ldots,n$. Let $\lambda \in (0,1)$ and note that implication (4.15) holds with $\gamma_{i,j}(s) := \frac{c_{i,j}^2}{\lambda^2 a_i^2}s$ and $\rho_i(s) := 2(1-\lambda)a_i s$. Condition (5.3) and the fact that $c_{i,i} < a_i$ for all $i = 1,\ldots,n$ implies that the small-gain conditions (2.4a,b) hold for $\lambda \in (0,1)$ sufficiently close to 1. We conclude from Corollary 4.5 that $0 \in C^0([-r,0];\Re^n)$ is RGAS for (5.1).

It is important to note that the conditions on the diagonal terms cannot be avoided in general if Razumikhin-like functions are used. Such a situation occurs, for example when

$$\dot{x}_1(t) = -a_1x_1(t) + c_{1,1}d_1(t)x_1(t-r) + c_{1,2}d_2(t)x_2(t-r),$$

$$\dot{x}_2(t) = -a_2x_2(t) + c_{2,1}d_3(t)x_1(t),$$

$$d_i(t) \in [-1,1], \quad i = 1,2,3,$$

with $c_{1,1} > 0$, $c_{1,2} \geqslant 0$ and $c_{2,1} \geqslant 0$. In this case, $0 \in C^0([-r,0];\Re^2)$ is RGAS for the above system if $c_{1,1} < a_1$ and $c_{1,2}c_{2,1} < a_1a_2$.

Another thing that should be noted is that system (5.1) includes the case of a system described by ODEs, i.e., the case $g_i(d(t),T_r(t)x) = g_i(d(t),x_1(t),\ldots,x_n(t))$ for $i = 1,\ldots,n$. In order to illustrate the superiority of the results of the present paper compared to the results in Karafyllis *et al.* (2008a), we note that the results in Karafyllis *et al.* (2008a) can show robust global asymptotic stability of $0 \in \Re^n$ for the system (5.1) under Hypothesis (5.2) with $g_i(d(t),T_r(t)x) = g_i(d(t),x_1(t),\ldots,x_n(t))$ for $i = 1,\ldots,n$, provided that the inequalities

$$c_{i,j} < a_i \quad \text{for} \quad i,j = 1,\ldots,n$$

hold. The above inequalities imply directly inequalities (5.3). In order to understand how conservative the results in Karafyllis *et al.* (2008a) are compared to the results of the present paper, we note that the case $c_{1,1} = c_{2,2} = c_{3,3} = c_{2,1} = 0$, $a_1 = a_2 = a_3 = c_{1,3} = c_{3,2} = 1$, $c_{2,3} = c_{3,1} = 1/2$ and $c_{1,2} = 2$ satisfies inequalities (5.3) and does not satisfy the above inequalities.

EXAMPLE 5.2. Consider the following biochemical control circuit model:

$$\begin{aligned} \dot{X}_1(t) &= g(X_n(t-\tau_n)) - a_1X_1(t), \\ \dot{X}_i(t) &= X_{i-1}(t-\tau_{i-1}) - a_iX_i(t), \quad i = 2,\ldots,n, \\ X(t) &= (X_1(t),\ldots,X_n(t))' \in \Re_+^n, \end{aligned} \tag{5.4}$$

where $a_i > 0$, $\tau_i \geqslant 0$ ($i = 1, \ldots, n$) are constants and $g \in C^1(\Re_+; \Re_+)$ is a function with $g(X) > 0$ for all $X > 0$. This model has been studied in Smith (1994) (see pp. 58–60 and 93–94). In this book, it is further assumed that $g \in C^1(\Re_+; \Re_+)$ is bounded and strictly increasing (a typical choice for $g \in C^1(\Re_+; \Re_+)$ is $g(X) = \frac{X^p}{1+X^p}$ with p being a positive integer or $g(X) = \frac{\mu X}{c+X}$ with $\mu, c > 0$). It is shown that if there is one equilibrium point for (5.4), then it attracts all solutions. If there are two equilibrium points, then all solutions are attracted to these points. Here, we study (5.4) under the following assumption:

(H) *There exist* $X_n^* > 0$, $K > 0$ *and* $\lambda \in (0, 1)$ *with* $aX_n^* = g(X_n^*)$ *and such that*

$$\frac{K + X_n^*}{K + X} X \leqslant a^{-1} g(X) \leqslant X_n^* + \lambda |X - X_n^*|, \text{ for all } X \geqslant 0, \tag{5.5}$$

where $a = \prod_{j=1}^n a_j$.

The reader should noted that Hypothesis (H) is automatically satisfied for the case of Monod kinetics, i.e., $g(X) = \frac{\mu X}{c+X}$ with $c > 0$ and $\mu > ac$. Indeed, in this case, inequality (5.5) holds with $K = c$ and $\lambda = \frac{c}{X_n^* + c}$, where $X_n^* = \frac{\mu - ac}{a}$. The case of Monod kinetics is typical for biochemical models (see, for example Smith & Waltman, 1995).

Using small-gain analysis, we are in a position to prove

'Consider system (5.4) under Hypothesis (H) and let $r := \max_{i=1,\ldots,n} \tau_i$. Then for every $X_0 \in C^0([-r, 0]; int(\Re_+^n))$ the solution of (5.4) with initial condition $T_r(0)X = X_0$ satisfies $\lim_{t\to+\infty} X(t) = X^*$, where $X^* = (X_1^*, \ldots, X_n^*)' \in int(\Re_+^n)$ with $\left(\prod_{j=1}^i a_j\right) X_i^* = g(X_n^*)$, for $i = 1, \ldots, n-1$'.

It should be clear that in contrast to the analysis performed in Smith (1994) for (5.4) (based on the monotone dynamical system theory), we do not assume that $g \in C^1(\Re_+; \Re_+)$ is bounded or strictly increasing, Moreover, even if there are two equilibrium points note that (5.5) allows $g(0) = 0$ and therefore $0 \in \Re_+^n$ can be an equilibrium point), we prove almost global convergence to the non-trivial equilibrium.

A typical analysis of the equilibrium points of (5.4) under Hypothesis (H) shows that there exists an equilibrium point $X^* \in int(\Re_+^n)$ satisfying:

$$\left(\prod_{j=1}^i a_j\right) X_i^* = g(X_n^*), i = 1, \ldots, n. \tag{5.6}$$

In order to be able to study solutions of (5.4) evolving in $\text{int}(\Re_+^n)$, we consider the following transformation:

$$X_i = X_i^* \exp(x_i), i = 1, \ldots, n. \tag{5.7}$$

Therefore, system (5.4) under transformation (5.7) is expressed by the following set of differential equations:

$$\dot{x}_1 = a_1 \left(\frac{g(X_n^* \exp(x_n(t - \tau_n)))}{g(X_n^*)} \exp(-x_1(t)) - 1 \right), \tag{5.8a}$$

$$\dot{x}_i(t) = a_i(\exp(x_{i-1}(t - \tau_{i-1}) - x_i(t)) - 1), \quad i = 2, \ldots, n,$$

$$x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n. \tag{5.8b}$$

First, we note that Hypotheses (S1–S4) hold for system (5.8) under Hypothesis (H) with output $H(t,x) := x \in C^0([-r,0];\Re^n)$ and that $0 \in C^0([-r,0];\Re^n)$ is an equilibrium point for (5.8). Define the family of functions $Q_i(x) = \frac{1}{2}x_i^2(0)$ and $V_i(x) := \sup_{\theta\in[-r,0]} Q_i(x(\theta)) = \frac{1}{2}\|x_i\|_r^2$ $(i = 1,\ldots,n)$ for $x \in C^0([-r,0];\Re^n)$. These mappings satisfy inequality (4.16) and definition (4.17) with $a_1(s) := \frac{1}{2n}s^2$, $a_2(s) := \frac{1}{2}s^2$, $\beta(t) \equiv 1$ and $r_i := 0$, $i = 1,\ldots,n$.

We define $\gamma_{1,j}(s) \equiv 0$ for $j \neq n$ and $\gamma_{1,n}(s) := \frac{1}{2}[\log(1+\theta(\exp(\sqrt{2s})-1))]^2$, where $\theta \in (\max\{\frac{b}{b+1},\lambda\},1)$, $\lambda \in (0,1)$ being the constant involved in Hypothesis (H) and $b := \frac{K}{X_n^*}$. Note that

$$Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)}\exp(-x_1(0))-1\right)\right)$$
$$= a_1x_1(0)\left(\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)}\exp(-x_1(0))-1\right).$$

We consider the following cases:

(1) $x_1(0) < 0$. In this case, the left-hand side inequality (5.5) implies that $\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)} \geqslant \frac{b+1}{b+\exp(x_n(-\tau_n))}\exp(x_n(-\tau_n)) \geqslant \frac{b+1}{b+\exp(-|x_n(-\tau_n)|)}\exp(-|x_n(-\tau_n)|)$, where $b := \frac{K}{X_n^*}$. The inequality $\gamma_{1,n}(V_n(x)) \leqslant Q_1(x_1(0))$ implies $\ln(1+\theta(\exp(|x_n(-\tau_n)|)-1)) \leqslant -x_1(0)$, which combined with the previous inequalities gives

$$Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)}\exp(-x_1(0))-1\right)\right)$$
$$\leqslant a_1x_1(0)\frac{(b+1-b\theta^{-1})(\exp(-x_1(0))-1)}{b+1+b\theta^{-1}(\exp(-x_1(0))-1)}. \tag{5.9}$$

(2) $x_1(0) \geqslant 0$. In this case, the right-hand side inequality (5.5) implies that $\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)} \leqslant 1+\lambda|\exp(x_n(-\tau_n))-1| \leqslant 1+\lambda(\exp(|x_n(-\tau_n)|)-1)$. The inequality $\gamma_{1,n}(V_n(x)) \leqslant Q_1(x_1(0))$ implies $\ln(1+\theta(\exp(|x_n(-\tau_n)|)-1)) \leqslant x_1(0)$, which combined with the previous inequalities gives

$$Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)}\exp(-x_1(0))-1\right)\right)$$
$$\leqslant a_1x_1(0)(\lambda\theta^{-1}-1)(1-\exp(-x_1(0))). \tag{5.10}$$

Combining the two cases, we obtain from (5.9) and (5.10) that the following implication holds:

$$\gamma_{1,n}(V_n(x)) \leqslant Q_1(x_1(0)) \Rightarrow Q_1^0\left(x_1(0); a_1\left(\frac{g(X_n^*\exp(x_n(-\tau_n)))}{g(X_n^*)}\exp(-x_1(0))-1\right)\right)$$
$$\leqslant -\rho_1(Q_1(x_1(0))) \tag{5.11}$$

with $\rho_1(s) := a_1\sqrt{2s}\min\left\{\left(1-\lambda\theta^{-1}\right)(1-\exp(-\sqrt{2s})), \frac{(b+1-b\theta^{-1})(\exp(\sqrt{2s})-1)}{b+1+b\theta^{-1}(\exp(\sqrt{2s})-1)}\right\}$.

We next define for $i = 2, \dots, n$, $\gamma_{i,j}(s) \equiv 0$ for $j \neq i-1$ and $\gamma_{i,i-1}(s) := \frac{1}{2}[\log(1 + \mu(\exp(\sqrt{2s}) - 1))]^2$, where $\mu > 1$ is to be selected. Working in a similar way as above, we obtain for $i = 2, \dots, n$:

$$\gamma_{i,i-1}(V_i(x)) \leqslant Q_i(x_i(0)) \Rightarrow Q_i^0(x_i(0); a_i(\exp(x_{i-1}(-\tau_{i-1}) - x_i(0)) - 1)) \\ \leqslant -\rho_i(Q_i(x_i(0))) \tag{5.12}$$

with $\rho_i(s) := (1 - \mu^{-1})a_i\sqrt{2s}\frac{(1-\exp(-\sqrt{2s}))}{1+\mu^{-1}(\exp(\sqrt{2s})-1)}$ for $i = 2, \dots, n$.

Therefore, we conclude from (5.11) and (5.12) that implication (4.15) holds.

Finally, we check the small-gain conditions. Exploiting the previous definitions of the functions $\gamma_{i,j}(s)$, $i, j = 1, \dots, n$, we conclude that the small-gain conditions (2.4a,b) hold if and only if $(\gamma_{n,n-1} \circ \gamma_{n-1,n-2} \circ \dots \circ \gamma_{1,n})(s) < s$ for all $s > 0$. Since

$$(\gamma_{n,n-1} \circ \gamma_{n-1,n-2} \circ \dots \circ \gamma_{1,n})(s) = \frac{1}{2}[\log(1 + \mu^{n-1}\theta(\exp(\sqrt{2s}) - 1))]^2,$$

the small-gain conditions (2.4a,b) hold with $\mu \in \left(1, \theta^{-\frac{1}{n-1}}\right)$. Thus, Corollary 4.5 implies that $0 \in C^0([-r, 0]; \Re^n)$ is uniformly GAS for system (5.8). Taking into account transformation (5.7), this implies that for every $X_0 \in C^0([-r, 0]; \mathrm{int}(\Re^n_+))$, the solution of (5.4) with initial condition $T_r(0)X = X_0$ satisfies $\lim_{t\to+\infty} X(t) = X^*$, where $X^* = (X_1^*, \dots, X_n^*)' \in \mathrm{int}(\Re^n_+)$ with $\left(\prod_{j=1}^{i} a_j\right) X_i^* = g(X_n^*)$, for $i = 1, \dots, n-1$.

Again, we can make a comparison between the results of the present work and the results in Karafyllis *et al.* (2008a) for the delay-free case $\tau_i = 0$, $i = 1, \dots, n$. In this case, the results in Karafyllis *et al.* (2008a) cannot show global asymptotic stability: the gain functions $\gamma_{i,i-1}(s) := \frac{1}{2}[\log(1 + \mu(\exp(\sqrt{2s}) - 1))]^2$ for $i = 2, \dots, n$ and $\mu > 1$ do not satisfy $\gamma_{i,i-1}(s) < s$ for $s > 0$.

6. Proofs of the main results of Section 4

Proof of Theorem 4.1. We want to show that all hypotheses of Theorem 3.1 hold with

$$L(t, x) := \max\left\{W(t, x), \max_{i=1,\dots,k} V_i(t, x)\right\}. \tag{6.1}$$

Note that Hypothesis (H3) of Theorem 3.1 is a direct consequence of inequalities (4.2), (4.3) and definition (6.1). Moreover, Hypothesis (H4) of Theorem 3.1 is a direct consequence of inequality (4.2) with $q(x) := a_1^{-1}\left(\max_{i=1,\dots,k} x_i\right)$ for all $x \in \Re^n_+$.

Consider a solution $x(t)$ of (4.1) corresponding to arbitrary $(u, d) \in M_U \times M_D$ with arbitrary initial condition $x(t_0) = x_0 \in \Re^n$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. Let $V_i(t) = V_i(t, x(t))$, $i = 1, \dots, k$, $W(t) = W(t, x(t))$ absolutely continuous functions on $[t_0, t_{\max})$ and let $L(t) = L(t, x(t))$. Moreover, let $I \subset [t_0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or $\dot{x}(t) \neq f(t, x(t), u(t), d(t))$. By virtue of (4.5), it follows that the following implication holds for $t \in [t_0, t_{\max})\backslash I$ and $i = 1, \dots, k$:

$$V_i(t) \geqslant \max\left\{\zeta(|u(t)|), \max_{j=1,\dots,k} \gamma_{i,j}(V_j(t))\right\} \Rightarrow \quad \dot{V}_i(t) \leqslant -\rho_i(V_i(t)) \tag{6.2}$$

and by virtue of (4.4), we get for $t \in [t_0, t_{\max})\backslash I$:

$$\dot{W}(t) \leqslant -W(t) + \lambda \max\left\{\zeta(|u(t)|),\ \max_{j=1,\dots,k} p_j(V_j(t))\right\}. \tag{6.3}$$

Lemma 3.5 in Karafyllis & Kravaris (2009) in conjunction with (6.2) implies that there exists a family of continuous functions σ_i ($i = 1, \dots, k$) of class KL, with $\sigma_i(s, 0) = s$ for all $s \geqslant 0$ such that for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$, we have

$$V_i(t) \leqslant \max\left\{\begin{array}{l}\sigma_i(V_i(t_0), t - t_0);\ \sup\limits_{t_0 \leqslant \tau \leqslant t} \sigma_i\left(\max\limits_{j=1,\dots,k} \gamma_{i,j}\left(V_j(\tau)\right), t - \tau\right)\\ \sup\limits_{t_0 \leqslant \tau \leqslant t} \sigma_i(\zeta(|u(\tau)|), t - \tau).\end{array}\right\} \tag{6.4}$$

Moreover, inequality (6.3) directly implies that for all $t \in [t_0, t_{\max})$, we have

$$W(t) \leqslant W(t_0) + \lambda \max\left\{\zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right),\ \max_{j=1,\dots,k} p_j\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right)\right\}. \tag{6.5}$$

Let $\sigma(s, t) := \max_{i=1,\dots,k} \sigma_i(s, t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \geqslant 0$. It follows from (6.4), (6.5) and definition (6.1) that for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$, we get

$$V_i(t) \leqslant \max\left\{V_i(t_0),\ \max_{j=1,\dots,k} \gamma_{i,j}\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right), \zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right)\right\}, \tag{6.6}$$

$$V_i(t) \leqslant \max\left\{\sigma(L(t_0), t - t_0),\ \max_{j=1,\dots,k} \gamma_{i,j}\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right), \zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right)\right\}, \tag{6.7}$$

$$W(t) \leqslant \max\left\{\frac{1}{1-\lambda} W(t_0), \zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right),\ \max_{i=1,\dots,k} p_i\left(\sup_{t_0 \leqslant s \leqslant t} V_i(s)\right)\right\}. \tag{6.8}$$

Clearly, inequalities (6.7) show that (3.1) holds with $\Gamma\colon \Re_+^k \to \Re_+^k$, $\Gamma(x) = (\Gamma_1(x), \dots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\dots,k} \gamma_{i,j}(x_j)$ for all $i = 1, \dots, k$ and $x \in \Re_+^n$. Furthermore, Hypothesis (H2) of Theorem 3.1 holds as well.
Define

$$p^u(s) := \zeta(s), \text{ for all } s \geqslant 0 \tag{6.9}$$

$$p(x) := \max\left\{\max_{i=1,\dots,k} \max_{j=1,\dots,k} \gamma_{i,j}(x_j),\ \max_{j=1,\dots,k} p_j(x_j)\right\} \text{ for all } x \in \Re_+^n. \tag{6.10}$$

Combining estimates (6.6), (6.8) and exploiting definitions (6.1), (6.9) and (6.10), we get for all $t \in [t_0, t_{\max})$:

$$L(t) \leqslant \max\left\{\frac{1}{1-\lambda} L(t_0),\ p\left(\sup_{t_0 \leqslant s \leqslant t} V_1(s), \dots, \sup_{t_0 \leqslant s \leqslant t} V_k(s)\right), p^u\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right)\right\}. \tag{6.11}$$

Inequality (3.2) is a direct consequence of (6.11), inequalities (4.2), (4.3) and Corollary 10 in Sontag (1998) with $\nu(t) \equiv 1$, $p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$ as defined by (6.9), (6.10) and appropriate $a \in \mathcal{N}_1$ and $c \in K^+$. The reader should note that if $\beta \in K^+$ is bounded then $c \in K^+$ is bounded as well.

Consequently, all hypotheses of Theorem 3.1 hold with $\sigma(s,t) := \max_{i=1,\ldots,k} \sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0) = s$ for all $s \geqslant 0$. The rest of proof is a consequence of Remark 3.2 in conjunction with definitions (6.9) and (6.10). The proof is complete.

Proof of Theorem 4.4. We want to show that all hypotheses of Theorem 3.1 hold with $L(t,x)$ as defined by (6.1). Note that Hypothesis (H3) of Theorem 3.1 is a direct consequence of inequalities (4.11), (4.12) and definition (6.1). Moreover, Hypothesis (H4) of Theorem 3.1 is a direct consequence of inequality (4.11) with $q(x) := a_1^{-1}(\max_{i=1,\ldots,k} x_i)$ for all $x \in \Re_+^n$.

We next show that Hypotheses (H1) and (H2) of Theorem 3.1 hold as well. The proof consists of two steps:

Step 1. We show that Hypotheses (H1), (H2) of Theorem 3.1 hold for arbitrary $(t_0, u, d) \in \Re_+ \times M_U \times M_D$ and $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$.

Step 2. We show that Hypotheses (H1), (H2) of Theorem 3.1 hold for arbitrary $(t_0, u, d) \in \Re_+ \times M_U \times M_D$ and $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$.

Step 1: Consider the solution $x(t)$ of (4.9) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. By virtue of Lemma A.2 in Karafyllis *et al.* (2008b), and Lemma 2.5 in Karafyllis *et al.* (2008c), we can guarantee that the functions $Q_i(t) = Q_i(t, T_{r_i}(t)x)$, $i = 1,\ldots,k$, $Q_0(t) = Q_0(t, T_{r_0}(t)x)$ are absolutely continuous functions on $[t_0, t_{\max})$. Let $V_i(t) = V_i(t, T_r(t)x) = \sup_{\theta\in[-r+r_i,0]} Q_i(t+\theta)$, $i = 1,\ldots,k$, $W(t) = W(t, T_r(t)x) = \sup_{\theta\in[-r+r_0,0]} Q_0(t+\theta)$ and $L(t) = L(t, T_r(t)x)$ be mappings defined on $[t_0, t_{\max})$. Moreover, let $I \subset [t_0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ or $Q_i(t) (i = 0,\ldots,k)$ is not differentiable or $\dot{x}(t) \neq f(t, T_r(t)x, u(t), d(t))$. By virtue of (4.15) and Lemma 2.4 in Karafyllis *et al.* (2008c), it follows that the following implication holds for $t \in [t_0, t_{\max})\backslash I$ and $i = 1,\ldots,k$:

$$Q_i(t) \geqslant \max\left\{\zeta(|u(t)|), \max_{j=1,\ldots,k} \gamma_{i,j}(V_j(t))\right\} \Rightarrow \quad \dot{Q}_i(t) \leqslant -\rho_i(Q_i(t)) \tag{6.12}$$

and by virtue of (4.13), we have

$$\dot{Q}_0(t) \leqslant -Q_0(t) + \lambda \max\left\{\zeta(|u(t)|), \max_{j=1,\ldots,k} p_j(V_j(t))\right\}. \tag{6.13}$$

Lemma 3.5 in Karafyllis & Kravaris (2009) in conjunction with (6.12) implies that there exists a family of continuous functions $\tilde{\sigma}_i$ $(i = 1,\ldots,k)$ of class KL, with $\tilde{\sigma}_i(s,0) = s$ for all $s \geqslant 0$ such that for all $t \in [t_0, t_{\max})$ and $i = 1,\ldots,k$, we have

$$Q_i(t) \leqslant \max\left\{\begin{array}{l} \tilde{\sigma}_i(Q_i(t_0), t-t_0);\ \sup\limits_{t_0\leqslant\tau\leqslant t} \tilde{\sigma}_i\left(\max\limits_{j=1,\ldots,k} \gamma_{i,j}(V_j(\tau)), t-\tau\right) \\ \sup\limits_{t_0\leqslant\tau\leqslant t} \tilde{\sigma}_i\left(\zeta\left(|u(\tau)|\right), t-\tau\right) \end{array}\right\}. \tag{6.14}$$

Moreover, inequality (6.13) directly implies that for all $t \in [t_0, t_{\max})$, we have

$$Q_0(t) \leqslant Q_0(t_0) + \lambda \max\left\{\zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right), \max_{j=1,\dots,k} p_j\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right)\right\}. \tag{6.15}$$

Using the fact that $\tilde{\sigma}_i(s, 0) = s$ for all $s \geqslant 0$, we obtain from (6.14) for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$:

$$Q_i(t) \leqslant \max\left\{\tilde{\sigma}_i(Q_i(t_0), t - t_0), \max_{j=1,\dots,k} \gamma_{i,j}\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right), \zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right)\right\}. \tag{6.16}$$

Let σ_i $(i = 1, \dots, k)$ be functions of class KL, defined by $\sigma_i(s, t) = s$ for all $s \geqslant 0$, $t \in [0, r]$ and $\sigma_i(s, t) = \tilde{\sigma}_i(s, t - r)$ for all $s \geqslant 0$, $t > r$. Using the fact that $V_i(t) = V_i(t, T_r(t)x) = \sup_{\theta \in [-r+r_i, 0]} Q_i(t + \theta)$, $i = 1, \dots, k$, we obtain from (6.16) for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$:

$$V_i(t) \leqslant \max\left\{\sigma_i(V_i(t_0), t - t_0), \max_{j=1,\dots,k} \gamma_{i,j}\left(\sup_{t_0 \leqslant s \leqslant t} V_j(s)\right), \zeta\left(\sup_{t_0 \leqslant s \leqslant t} |u(s)|\right)\right\}. \tag{6.17}$$

Similarly, using (6.15) and the fact that $W(t) = W(t, T_r(t)x) = \sup_{\theta \in [-r+r_0, 0]} Q_0(t + \theta)$, we conclude that (6.8) holds for all $t \in [t_0, t_{\max})$. Define $\sigma(s, t) := \max_{i=1,\dots,k} \sigma_i(s, t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \geqslant 0$. It follows from (6.17) and definition (6.1) that inequalities (6.6) and (6.7) hold for all $t \in [t_0, t_{\max})$ and $i = 1, \dots, k$. Clearly, inequalities (6.7) show that (3.1) holds with $\Gamma : \Re_+^k \to \Re_+^k$, $\Gamma(x) = (\Gamma_1(x), \dots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\dots,k} \gamma_{i,j}(x_j)$ for all $i = 1, \dots, k$ and $x \in \Re_+^n$. Moreover, Hypothesis (H2) of Theorem 3.1 holds as well. Define $p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$ by (6.9) and (6.10). Combining estimates (6.6), (6.8) and exploiting definitions (6.1), (6.9) and (6.10), we get inequality (6.11) for all $t \in [t_0, t_{\max})$. Inequality (3.2) is a direct consequence of (6.11), inequalities (4.11), (4.12) and Corollary 10 in Sontag (1998) with $\nu(t) \equiv 1$, $p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$ as defined by (6.9), (6.10) and appropriate $a \in \mathcal{N}_1$ and $c \in K^+$. The reader should note that if $\beta \in K^+$ is bounded then $c \in K^+$ is bounded as well.

Step 2: Let $(t_0, x_0, u, d) \in \Re_+ \times C^1([-r, 0]; \Re^n) \times M_U \times M_D$. Inequalities (6.7) in conjunction with Proposition 2.9 imply for the solution $x(t)$ of (4.9) corresponding to $(u, d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$ and for all $t \in [t_0, t_{\max})$:

$$V(t) \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\sigma\,(L(t_0), 0)),\, Q\left(\mathbf{1}\zeta\left(\left[\|u(\tau)\|_U\right]_{[t_0,t]}\right)\right)\right\}, \tag{6.18}$$

where $Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \dots, \Gamma^{(n-1)}(x)\}$. Using (4.11), (4.12), (6.1), (6.11) and (6.18), we obtain functions $\rho \in K^+$, $a \in K_\infty$ such that the solution $x(t)$ of (4.9) corresponding to $(u, d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \Re^n)$ and for all $t \in [t_0, t_{\max})$:

$$\|T_r(t)x\|_r \leqslant a\left(\rho(t) + \|x_0\|_r + \sup_{t_0 \leqslant s \leqslant t} |u(s)|\right). \tag{6.19}$$

Lemma 2.6 in Karafyllis *et al.* (2008c) and (6.19) imply that system (4.9) is RFC from the input $u \in M_U$.

We next claim that inequalities (3.1) and (3.2) hold for all $(t_0, x_0, u, d) \in \Re_+ \times C^0([-r, 0]; \Re^n) \times M_U \times M_D$ and $t \geqslant t_0$. The proof will be made by contradiction. Suppose on the contrary that there exists $(t_0, x_0, u, d) \in \Re_+ \times C^0([-r, 0]; \Re^n) \times M_U \times M_D$ and $t_1 > t_0$ such that the solution $x(t)$ of (4.9) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(u,d) \in M_U \times M_D$ satisfies$\beta(t_1,t_0,x_0,d,u) > 0$, where

$$\beta(t,t_0,x_0,d,u) := \max\left\{\begin{array}{l} L(t) - \max\left\{\nu(t-t_0), c(t_0), a\left(\|x_0\|_r\right), p\left([V]_{[t_0,t]}\right)\right. \\ \quad \left. p^u\left([\|u(\tau)\|_U]_{[t_0,t]}\right)\right\} \\ \max_{i=1,\ldots,k}\left\{V_i(t) - \max\left\{\sigma\left(L(t_0), t-t_0\right), \Gamma_i\left([V]_{[t_0,t]}\right),\right.\right. \\ \quad \left.\left. \zeta\left([\|u(\tau)\|_U]_{[t_0,t]}\right)\right\}\right\} \end{array}\right\}.$$

Using continuity of the mappings $x \to L(t,x)$, $x \to V_i(t,x)$ $(i = 1,\ldots,k)$ and continuity of the solution of (4.9) with respect to the initial condition, we can guarantee that the mapping $x_0 \to \beta(t_1,t_0,x_0,d,u)$ is continuous. Using density of $C^1([-r,0];\Re^n)$ in $C^0([-r,0];\Re^n)$, continuity of the mapping $x_0 \to \beta(t_1,t_0,x_0,d,u)$, we conclude that there exists $\tilde{x}_0 \in C^1([-r,0];\Re^n)$ such that

$$|\beta(t_1,t_0,x_0,d,u) - \beta(t_1,t_0,\tilde{x}_0,d,u)| \leqslant \frac{1}{2}\beta(t_1,t_0,x_0,d,u).$$

Thus, we obtain a contradiction.

Consequently, all hypotheses of Theorem 3.1 hold with $\sigma(s,t) := \max_{i=1,\ldots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s,0) = s$ for all $s \geqslant 0$. The rest of proof is a consequence of Remark 3.2 in conjunction with definitions (6.9), (6.10). The proof is complete.

Proof of Theorem 4.8. We want to show that all hypotheses of Theorem 3.1 hold with $L(t,x)$ as defined by (6.1) and

$$W(t,x) := \exp(-(\mu+1)t)Q(x). \tag{6.20}$$

Note that definition (6.20) in conjunction with inequalities (4.20) imply the following inequality for all $(t,x) \in \Re_+ \times \Re^n$:

$$\exp(-(\mu+1)t)a_3(|x|) - g(V_1(x),\ldots,V_k(x)) - \kappa \leqslant W(t,x) \leqslant a_4(|x|).$$

Using Corollary 10 in Sontag (1998), we can find functions $\tilde{a} \in K_\infty$, $\eta \in K^+$ such that $a_3^{-1}(s\exp((\mu+1)t)) \leqslant \frac{1}{\eta(t)}\tilde{a}(s)$ for all $t,s \geqslant 0$. Consequently, we obtain $\exp(-(\mu+1)t)a_3(s) \geqslant \tilde{a}^{-1}(\eta(t)s)$ for all $t,s \geqslant 0$. Note that Hypothesis (H3) of Theorem 3.1 with $\beta(t) \equiv 1$ is a direct consequence of previous inequalities, (4.19) and definitions (6.1) and (6.20). Moreover, Hypothesis (H4) of Theorem 3.1 is a direct consequence of inequality (4.19) with $q(x) := a_1^{-1}(\max_{i=1,\ldots,k} x_i)$ for all $x \in \Re^n_+$.

Consider the solution $x(t)$ of (4.18) under Hypotheses (R1–R4) corresponding to arbitrary $(u,d,\tilde{d}) \in M_U \times M_D \times M_{\Re_+}$ with arbitrary initial condition $x(t_0) = x_0 \in \Re^n$. Note that since system (4.18) is autonomous (see Karafyllis, 2007a), it suffices to consider the case $t_0 = 0$. By virtue of Proposition 2.5 in Karafyllis (2007a), there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. Let $V_i(t) = V_i(x(t))$, $i = 1,\ldots,k$, $W(t) = W(t,x(t))$, $L(t) = L(t,x(t))$ absolutely continuous functions on $[0,t_{\max})$. Moreover, let $\pi := \{\tau_0,\tau_1,\ldots\}$ be the set of sampling times (which may be finite if $t_{\max} < +\infty$) and $p(t) := \max\{\tau \in \pi : \tau \leqslant t\}$, $q(t) := \min\{\tau \in \pi : \tau \geqslant t\}$. Let $I \subset [0,t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or where $\dot{x}(t) \neq f(x(t),x(\tau_i),d(t),u(t),u(\tau_i))$. Clearly, we have $x(t) = \phi(t-p(t),x(p(t));P_t d,P_t u)$ for all $t \in [0,t_{\max})$, where $(P_t u)(s) = u(p(t)+s)$, $(P_t d)(s) = d(p(t)+s)$, $s \geqslant 0$. Next, we show that the following implication holds for $t \in [0,t_{\max})\backslash I$ and $i = 1,\ldots,k$:

$$V_i(t) \geqslant \max\left\{\zeta\left(\sup_{p(t)\leqslant s\leqslant t}|u(s)|\right), \max_{j=1,\ldots,k}\gamma_{i,j}\left(\sup_{p(t)\leqslant s\leqslant t}V_j(s)\right)\right\} \Rightarrow \quad \dot{V}_i(t) \leqslant -\rho_i(V_i(t)). \tag{6.21}$$

In order to prove implication (6.21), let $t \in [0, t_{\max})\backslash I$, $i = 1, \ldots, k$, $\tau = p(t)$ and suppose that $V_i(t) \geqslant \max\left\{\zeta\left(\sup_{p(t)\leqslant s\leqslant t}|u(s)|\right), \max_{j=1,\ldots,k}\gamma_{i,j}\left(\sup_{p(t)\leqslant s\leqslant t}V_j(s)\right)\right\}$. By virtue of the semi-group property for the previous inequality implies that $\zeta(|u(\tau+s)|) = \zeta(|(P_t u)(s)|) \leqslant V_i(x(t)), \gamma_{i,j}(V_j(\phi(s, x(\tau); P_t d, P_t u))) \leqslant V_i(x(t))$ for all $s \in [0, t-\tau]$ and $j = 1, \ldots, k$. In this case, by virtue of definition (4.23) and the fact that $t - \tau \leqslant h(x(\tau), u(\tau))$, it follows that $x(\tau) \in A_i(h(x(\tau), u(\tau)), x(t))$. Since $\dot{x}(t) = f(x(t), x(\tau), d(t), u(t), u(\tau))$, we conclude from (4.22) that $\dot{V}_i(t) \leqslant -\rho_i(V_i(t))$.

Lemma 3.5 in Karafyllis & Kravaris (2009) implies that there exists a family of continuous function σ_i of class $KL (i = 1, \ldots, k)$, with $\sigma_i(s, 0) = s$ for all $s \geqslant 0$ such that for all $t \in [0, t_{\max})$ and $i = 1, \ldots, k$, we have

$$V_i(t) \leqslant \max\left\{\sigma_i(V_i(0), t); \max_{j=1,\ldots,k}\sup_{0\leqslant\tau\leqslant t}\sigma_i\left(\gamma_{i,j}\left(\sup_{p(\tau)\leqslant s\leqslant\tau}V_j(s)\right), t-\tau\right); \right.$$
$$\left.\sup_{0\leqslant\tau\leqslant t}\sigma_i\left(\zeta\left(\sup_{p(\tau)\leqslant s\leqslant\tau}|u(s)|\right), t-\tau\right)\right\}. \tag{6.22}$$

Let $\sigma(s,t) := \max_{i=1,\ldots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \geqslant 0$. Inequalities (3.1) with $\Gamma: \Re_+^k \to \Re_+^k$, $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\ldots,k}\gamma_{i,j}(x_j)$ for all $i = 1, \ldots, k$ and $x \in \Re_+^n$ are direct consequences of the previous definition, estimates (6.22), definition (6.1) and the fact that $\sigma_i(s, 0) = s$ for all $s \geqslant 0$ and $i = 1, \ldots, k$. Moreover, Hypothesis (H2) of Theorem 3.1 holds as well.

Exploiting (4.21) and definition (6.20), we get for $t \in [0, t_{\max})\backslash I$:

$$\dot{W}(t) \leqslant -W(t) + \lambda\max\left\{\zeta\left(\sup_{p(t)\leqslant s\leqslant t}|u(s)|\right), \max_{j=1,\ldots,k}p_j\left(\sup_{p(t)\leqslant s\leqslant t}V_j(s)\right)\right\}. \tag{6.23}$$

Inequality (6.23) directly implies that for all $t \in [0, t_{\max})$, we have

$$W(t) \leqslant \max\left\{\frac{1}{1-\lambda}W(0), \zeta\left(\sup_{0\leqslant s\leqslant t}|u(s)|\right), \max_{i=1,\ldots,k}p_i\left(\sup_{0\leqslant s\leqslant t}V_i(s)\right)\right\}. \tag{6.24}$$

Define $p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$ by (6.9) and (6.10). Combining estimates (6.22) and (6.24) and exploiting definitions (6.9), (6.10) and (6.1), we obtain (6.11) with $t_0 = 0$ for all $t \in [0, t_{\max})$. Inequality (3.2) is a direct consequence of (6.11) with $t_0 = 0$, inequalities (4.19) and (4.20) with $\nu(t) = c(t) \equiv 1$, $p^u \in \mathcal{N}_1$, $p \in \mathcal{N}_n$ as defined by (6.9), (6.10) and appropriate $a \in \mathcal{N}_1$.

Consequently, all hypotheses of Theorem 3.1 hold with $\sigma(s,t) := \max_{i=1,\ldots,k}\sigma_i(s,t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \geqslant 0$. The rest of proof is a consequence of Remark 3.2 in conjunction with definitions (6.9) and (6.10). The proof is complete.

7. Conclusions

A novel small-gain theorem is presented, which leads to vector Lyapunov characterizations of the (uniform and non-uniform) IOS property for various important classes of non-linear control systems. The results presented in this work generalize many recent small-gain results in the literature and allow the explicit computation of the gain function of the overall system. Moreover, since the gain map $\Gamma : \Re_+^n \to \Re_+^n$ is allowed to contain diagonal terms, the obtained results have direct applications to time-delay systems. Examples have demonstrated the effectiveness of the vector small-gain methodology to large-scale time-delay systems, such as those encountered in mathematical biology.

Our future work will be directed at applications of the vector small-gain theorem to the non-linear feedback design issue for various classes of non-linear control systems. Another interesting topic for future research is to study the internal and external stability properties for coupled systems involving integral input-to-state stable (iISS, a weaker notion than ISS; see Sontag, 1998) subsystems from a viewpoint of vector small gain. Some preliminary results are reported upon in Ito & Jiang, 2009 for interconnected systems consisting of two ISS and/or iISS subsystems.

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Appendix-Proofs of Proposition 2.7 and Theorem 3.1

Proof of Proposition 2.7. We prove implications (iv) ⇒ (i), (iii) ⇒ (ii) and (ii) ⇒ (iv) since the implication (i) ⇒ (iii) is a consequence of Proposition 2.1.

(iv) ⇒ (i): Clearly, since $\Gamma^{(k)}(x) \leqslant Q(x)$ for all $k \geqslant 1$, $x \in \Re^n_+$, we have $|\Gamma^{(k)}(x)| \leqslant |Q(x)|$ for all $k \geqslant 1$, $x \in \Re^n_+$ (note that $\Gamma^{(k)}(x) \in \Re^n_+$). Continuity of the mapping $Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$ (which is a direct consequence of continuity of the mapping $\Gamma(x)$) implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x| \leqslant \delta$, $x \in \Re^n_+$ implies $|Q(x)| \leqslant \varepsilon$ (note that $Q(0) = 0$). This implies stability.

Since $\Gamma\colon \Re^n_+ \to \Re^n_+$ is MAX-preserving, we have $\Gamma(Q(x)) = \mathrm{MAX}\{\Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n)}(x)\}$ for all $x \in \Re^n_+$. Moreover, since $\Gamma^{(k)}(x) \leqslant Q(x)$ for all $k \geqslant 1$, $x \in \Re^n_+$, it follows that $\Gamma(Q(x)) \leqslant Q(x)$ for all $x \in \Re^n_+$. Lemma 2.2 in conjunction with the fact that (iii) holds and $x \leqslant Q(x)$ for all $x \in \Re^n_+$ implies that $\lim_{k\to\infty} \Gamma^{(k)}(x) = 0$ for all $x \in \Re^n_+$.

(iii) ⇒ (ii): If there exist $s > 0$ and some integer $i = 1, \ldots, n$ such that $\gamma_{i,i}(s) \geqslant s$, then the non-zero vector $x \in \Re^n_+$ with $x_i = s$ and $x_j = 0$ for $j \neq i$ will violate (iii). Consequently, $\gamma_{i,i}(s) < s$, for all $s > 0$, $i = 1, \ldots, n$.

Next, suppose that $n > 1$. Suppose that there exist some $s > 0$, $r \in \{2, \ldots, n\}$, indices $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, r$ with $i_j \neq i_k$ if $j \neq k$ such that $(\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \ldots \circ \gamma_{i_r,i_1})(s) \geqslant s$. Without loss of generality, we may assume that $i_j = j$, for $j = 1, \ldots, r$ and consequently $(\gamma_{1,2} \circ \gamma_{2,3} \circ \ldots \circ \gamma_{r,1})(s) \geqslant s$. The non-zero vector $x \in \Re^n_+$ with $x_1 = s$, $x_j = (\gamma_{j,j+1} \circ \gamma_{j+1,j+2} \circ \ldots \circ \gamma_{r,1})(s)$ for $j = 2, \ldots, r$ and $x_j = 0$ for $j > r$ satisfies $\Gamma(x) \geqslant x$ and consequently Hypothesis (iii) is violated. Therefore, (ii) must hold.

(ii) ⇒ (iv) The proof of this implication is a direct consequence of the fact that

$$\Gamma_i^{(k)}(x) = \max\{(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \ldots \circ \gamma_{j_{k-1},j_k})(x_{j_k}) : (j_1, \ldots, j_k) \in \{1, \ldots, n\}^k\}$$

for all $k \geqslant 1$, $x \in \Re^n_+$ and $i = 1, \ldots, n$. Using (ii), it may be shown that $\Gamma^{(n)}(x) \leqslant Q(x) = \mathrm{MAX}\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\}$ for all $x \in \Re^n_+$. Since $\Gamma\colon \Re^n_+ \to \Re^n_+$ is MAX-preserving, we have $\Gamma(Q(x)) = MAX\{\Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n)}(x)\}$ for all $x \in \Re^n_+$. As a result, we obtain $\Gamma(Q(x)) \leqslant Q(x)$ for all $x \in \Re^n_+$. By induction, it follows that $\Gamma^{(k)}(Q(x)) \leqslant Q(x)$ for all $k \geqslant 1$, $x \in \Re^n_+$. Since $x \leqslant Q(x)$, we obtain $\Gamma^{(k)}(x) \leqslant Q(x)$ for all $k \geqslant 1$, $x \in \Re^n_+$.

The fact that implication (iii) holds is shown by contradiction. Suppose that there exists a non-zero $x \in \Re^n_+$ with $\Gamma(x) \geqslant x$. Consequently, for every $i \in \{1, \ldots, n\}$, there exists $p(i) \in \{1, \ldots, n\}$ with $\gamma_{i,p(i)}(x_{p(i)}) \geqslant x_i$. With these inequalities in mind, there is at least one $i \in \{1, \ldots, n\}$ with $x_i > 0$, and a closed cycle $(i, j_1, \ldots, j_r, i)$ such that $(\gamma_{i,j_1} \circ \gamma_{j_1,j_2} \circ \ldots \circ \gamma_{j_r,i})(x_i) \geqslant x_i$, which contradicts (ii). Therefore, the implication (ii) $\Rightarrow$ (iv) holds.

The proof is thus completed.

Proof of Theorem 3.1. The proof consists of two steps:

Step 3. We show that Σ is RFC from the input $u \in M_U$ and that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ the following inequality holds for all $t \geqslant t_0$:

$$V(t) \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\sigma(L(t_0), 0)), Q(\mathbf{1}\zeta([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}))\right\}. \tag{A.1}$$

Therefore by virtue of (A.1), (3.3) and definition (3.5), properties P1 and P2 of Lemma 2.16 in Karafyllis & Jiang (2007) hold for system Σ with $V = V_i$ and $\gamma = G_i$ $(i = 1, \ldots, n)$. Moreover, if $\beta \in K^+$ is bounded then (3.3) implies that properties P1 and P2 of Lemma 2.17 in Karafyllis & Jiang (2007) hold for system Σ with $V = V_i$ and $\gamma = G_i$ $(i = 1, \ldots, n)$.

Proof of Step 1: Let $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$. Inequality (3.1) implies for all $t \in [t_0, t_{\max})$

$$[V]_{[t_0,t]} \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_0), 0), \Gamma([V]_{[t_0,t]}), \mathbf{1}\zeta\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}. \tag{A.2}$$

Proposition 2.9 in conjunction with (A.2) implies (A.1) for all $t \in [t_0, t_{\max})$. It should be emphasized that the small-gain conditions are exploited at this point: Proposition 2.9 assumes that the cyclic small-gain conditions (2.4a,b) hold.

We show next that Σ is RFC from the input $u \in M_U$ by contradiction. Suppose that $t_{\max} < +\infty$. Then by virtue of the BIC property for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$. On the other hand, estimate (A.1) in conjunction with the hypothesis $t_{\max} < +\infty$ shows that there exists $M_1 \geqslant 0$ such that $\sup_{t_0 \leqslant \tau < t_{\max}} |V(\tau)| \leqslant M_1$. The fact that $V(t)$ is bounded in conjunction with estimate (3.2) implies that there exists $M_2 \geqslant 0$ such that $\sup_{t_0 \leqslant \tau < t_{\max}} L(\tau) \leqslant M_2$. It follows from (3.3) and inequality $\mu(t)\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} \leqslant b(L(t) + g(V(t)) + \kappa(t))$ that the transition map of Σ, i.e., $\phi(t, t_0, x_0, u, d)$, is bounded on $[t_0, t_{\max})$ and this contradicts the requirement that for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$. Hence, we must have $t_{\max} = +\infty$ and consequently, (A.1) holds for all $t \geqslant t_0$.

Let arbitrary $r \geqslant 0$, $T \geqslant 0$ and let arbitrary $u \in \mathcal{M}(B_U[0, r]) \cap M_U$, $\|x_0\|_{\mathcal{X}} \leqslant r$, $t_0 \in [0, T]$, $d \in M_D$ be given. Estimate (A.1) shows that there exists $M_1 := M_1(r, T) \geqslant 0$ such that$\sup_{t_0 \leqslant \tau < t_0+T} |V(\tau)| \leqslant M_1 < +\infty$. Consequently, estimate (3.2) implies that there exists $M_2 := M_2(r, T) \geqslant 0$ such that $\sup_{t_0 \leqslant \tau < t_0+T} L(\tau) \leqslant M_2 < +\infty$. It follows from (3.3) and inequality $\mu(t)\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} \leqslant b(L(t) + g(V(t)) + \kappa(t))$ that there exists $M_3 := M_3(r, T) \geqslant 0$ such that $\sup_{t_0 \leqslant \tau < t_0+T} \|\phi(\tau, t_0, x_0, u, d)\|_{\mathcal{X}} \leqslant M_3 < +\infty$. Hence, it holds that

$$\sup\{\|\phi(t_0 + s, t_0, x_0, u, d)\|_{\mathcal{X}}; u \in M(B_U[0, r]) \cap M_U, s \in [0, T], \|x_0\|_{\mathcal{X}} \leqslant r, t_0 \in [0, T], d \in M_D\}.$$

$$< +\infty$$

Therefore, we conclude that Σ is RFC from the input $u \in M_U$.

Step 4. We prove the following claim.

Claim: *For every $\varepsilon > 0, k \in Z_+, R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ the following inequality holds:*

$$V(t) \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(L(t_0), 0))), G\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right\} \quad \text{for all} \quad t \geqslant t_0 + \tau_k. \tag{A.3}$$

Moreover, if $\beta, c \in K^+$ are bounded then for every $\varepsilon > 0, k \in Z_+, R \geqslant 0$, there exists $\tau_k(\varepsilon, R) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $\|x_0\|_{\mathcal{X}} \leqslant R$ inequality (A.3) holds.

Note that Hypothesis (H2), which is equivalent with statement (i) of Proposition 2.7, and inequality (3.3) guarantees the existence of $k(\varepsilon, T, R) \in Z_+$ such that $Q(\mathbf{1}\varepsilon) \geqslant \Gamma^{(l)}(Q(\mathbf{1}\sigma(b(R \max_{0\leqslant t\leqslant T} \beta(t)), 0)))$ for all $l \geqslant k$. If $\beta \in K^+$ is bounded then k is independent of T. Therefore, by virtue of (A.3), property P3 of Lemma 2.16 in Karafyllis & Jiang (2007) holds for system Σ with $V = V_i$ and $\gamma = G_i$ $(i = 1, \ldots, n)$. Moreover, if $\beta, c \in K^+$ are bounded then (A.3) implies that property P3 of Lemma 2.17 in Karafyllis & Jiang (2007) hold for system Σ with $V = V_i$ and $\gamma = G_i$ $(i = 1, \ldots, n)$. The proof of Theorem 3.1 is thus completed with the help of Lemma 2.16 (or Lemma 2.17) in Karafyllis & Jiang (2007).

Proof of Step 2. The proof of the claim will be made by induction on $k \in Z_+$.

First, we show inequality (A.3) for $k = 1$. Let arbitrary $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ be given. Inequality (3.1) in conjunction with inequality (A.1) give for $t \geqslant t_0$:

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_0), t - t_0), \Gamma(Q(\mathbf{1}\sigma(L(t_0), 0))), \Gamma\left(Q\left(\mathbf{1}\gamma\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right)\right),\right.$$
$$\left.\mathbf{1}\gamma\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}. \tag{A.4}$$

Since $\Gamma(Q(x)) \leqslant Q(x)$ and $Q(x) \geqslant x$ for all $x \in \Re^n_+$, inequality (A.4) implies for all $t \geqslant t_0$:

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_0), t - t_0), \Gamma(Q(\mathbf{1}\sigma(L(t_0), 0))), Q\left(\mathbf{1}\zeta\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right)\right\}. \tag{A.5}$$

Similarly inequality (3.2) in conjunction with inequality (A.1) give for $t \geqslant t_0$:

$$L(t) \leqslant \max\left\{\nu(t - t_0), c(t_0), a(\|x_0\|_{\mathcal{X}}), p(Q(\mathbf{1}\sigma(L(t_0), 0))), p\left(Q\left(\mathbf{1}\zeta\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right)\right),\right.$$
$$\left.p^u\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}. \tag{A.6}$$

Note that (3.3) implies $L(t_0) \leqslant b(\beta(t_0)\|x_0\|_{\mathcal{X}}) \leqslant b(R \max_{0\leqslant t\leqslant T} \beta(t))$. Using the properties of the *KL* functions we can guarantee that there exists $\tau_1(\varepsilon, R, T) \geqslant 0$ such that $\sigma(b(R \max_{0\leqslant t\leqslant T} \beta(t)), \tau_1) \leqslant \varepsilon$. Note that if $\beta \in K^+$ is bounded then $\tau_1 \geqslant 0$ is independent of T. Then it follows from (A.5) that we have $V(t) \leqslant \mathrm{MAX}\{\mathbf{1}\varepsilon, \Gamma(Q(\mathbf{1}\sigma(L(t_0), 0))), Q(\mathbf{1}\zeta([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}))\}$ for all $t \geqslant t_0 + \tau_1$. Since $G(s) \geqslant Q(\mathbf{1}\zeta(s))$ for all $s \geqslant 0$ (a consequence of (3.5)) and $Q(\mathbf{1}\varepsilon) \geqslant \mathbf{1}\varepsilon$, we conclude that inequality (A.3) holds for $k = 1$.

Next, suppose that for every $\varepsilon > 0$, $R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ (A.3) holds for some $k \in Z_+$. Let arbitrary $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ be given. Note that the weak semi-group property implies that $\pi(t_0, x_0, u, d) \cap [t_0 + \tau_k, t_0 + \tau_k + r] \neq \emptyset$. Let $t_k \in \pi(t_0, x_0, u, d) \cap [t_0 + \tau_k, t_0 + \tau_k + r]$. Then (3.1) implies

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_k), t - t_k), \Gamma([V]_{[t_k,t]}), \mathbf{1}\zeta\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_k,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_k. \tag{A.7}$$

Moreover, inequality (A.3) gives

$$[V]_{[t_k,t]} \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k)}(Q(\mathbf{1}\sigma(L(t_0), 0))), G\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_k. \tag{A.8}$$

Inequality (A.6) also implies:

$$L(t_k) \leqslant \max\left\{\nu(t_k - t_0), c(t_0), a(R), p(Q(\mathbf{1}\sigma(L(t_0), 0))), p\left(Q\left(\mathbf{1}\zeta\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right)\right),\right.$$
$$\left.\times\ p^u\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t_k]}\right)\right\}. \tag{A.9}$$

Using (A.8) and the fact that $\Gamma(G(s)) \leqslant G(s)$for all $s \geqslant 0$ (a direct consequence of definition (3.5) and the fact that $\Gamma(Q(x)) \leqslant Q(x)$ for all $x \in \Re^n_+$), we obtain

$$\Gamma([V]_{[t_k,t]}) \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(t_0), 0))), G\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_k. \tag{A.10}$$

Inequality (A.10) in conjunction with inequality (A.7), the fact that $G(s) \geqslant Q(\mathbf{1}\zeta(s)) \geqslant \mathbf{1}\zeta(s)$ for all $s \geqslant 0$ and the fact that $t_k \leqslant t_0 + \tau_k + r$ implies:

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(L(t_k), t - t_0 - \tau_k - r), Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(t_0), 0))), G\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right\},$$
$$\text{for all} \quad t \geqslant t_0 + \tau_k + r. \tag{A.11}$$

Inequality (A.9) in conjunction with the fact that $\mathbf{1}\sigma(p^u(s), 0) \leqslant G(s)$, $\mathbf{1}\sigma(p(Q(\mathbf{1}\zeta(s))), 0) \leqslant G(s)$ for all $s \geqslant 0$ and the facts that $t_k \leqslant t_0 + \tau_k + r$, $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ implies that

$$\mathbf{1}\sigma(L(t_k), t - t_0 - \tau_k - r) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(f(\varepsilon, T, R), t - t_0 - \tau_k - r), G\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\},$$
$$\text{for all} \quad t \geqslant t_0 + \tau_k + r, \tag{A.12}$$

where

$$f(\varepsilon, T, R) := \max\left\{\max_{0\leqslant t\leqslant \tau_k(\varepsilon,R,T)+r} \nu(t), \max_{0\leqslant t\leqslant T} c(t), a(R),\right.$$
$$\left.\times\ p\left(Q\left(\mathbf{1}\sigma\left(b\left(R\max_{0\leqslant t\leqslant T}\beta(t)\right), 0\right)\right)\right)\right\}. \tag{A.13}$$

The reader should note that if $\beta, c \in K^+$ are bounded and τ_k is independent of T then f can be chosen to be independent of T as well. Note that by combining (A.11) and (A.12), we get

$$V(t) \leqslant \mathrm{MAX}\left\{\mathbf{1}\sigma(f(\varepsilon, T, R), t - t_0 - \tau_k - r), Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(t_0), 0))),\right.$$
$$\left.G\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_0 + \tau_k + r. \tag{A.14}$$

Clearly, there exists $\tau(\varepsilon, R, T) \geqslant 0$ such that $\sigma(f(\varepsilon, T, R), \tau) \leqslant \varepsilon$. Define

$$\tau_{k+1}(\varepsilon, R, T) = \tau_k(\varepsilon, R, T) + r + \tau(\varepsilon, R, T) \tag{A.15}$$

Again, the reader should notice that if f and τ_k are independent of T then τ_{k+1} is independent of T as well. Since $Q(\mathbf{1}\varepsilon) \geqslant \mathbf{1}\varepsilon$, we obtain from (A.14):

$$V(t) \leqslant \mathrm{MAX}\left\{Q(\mathbf{1}\varepsilon), \Gamma^{(k+1)}(Q(\mathbf{1}\sigma(L(t_0), 0))), G\left(\left[\|u(\tau)\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_0 + \tau_{k+1} \tag{A.16}$$

which shows that (A.3) holds for $k + 1$.

To finish the proof, we assume that Hypothesis (H4) holds. Let $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ be arbitrary and denote $Y(t) = H(t, \phi(t, t_0, x_0, u, d), u(t))$ for $t \geqslant t_0$. Using (3.6), (A.1) and the following fact:

Fact: *If $p \in N_n$ and R: $\Re^n_+ \to \Re^n_+$ is a non-decreasing mapping, then the following inequality holds for all $s, r \in \Re_+$: $p(\mathrm{MAX}\{R(\mathbf{1}s), R(\mathbf{1}r)\}) = \max(p(R(\mathbf{1}s)), p(R(\mathbf{1}r)))$.*

we obtain for all $t \geqslant t_0$:

$$\|Y(t)\|_{\mathcal{Y}} \leqslant \max\left\{q(Q(\mathbf{1}\sigma(L(t_0), 0))), q\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathcal{U}}\right]_{[t_0,t]}\right)\right)\right)\right\}. \tag{A.17}$$

Inequality (A.17) shows that properties P1 and P2 of Lemma 2.16 in Karafyllis & Jiang (2007) hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$ and $\gamma(s) := q(G(s))$. Moreover, if $\beta \in K^+$ is bounded, then (A.17) implies that properties P1 and P2 of Lemma 2.17 in Karafyllis & Jiang (2007) hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$ and $\gamma(s) := q(G(s))$.

Inequality (A.3) in conjunction with Hypothesis (H2), which is equivalent with statement (i) of Proposition 2.7, and inequality (3.3) guarantees the existence of $k := k(\varepsilon, T, R) \in Z_+$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathcal{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathcal{X}} \leqslant R$ the following inequality holds:

$$V(t) \leqslant MAX\left\{Q(\mathbf{1}\varepsilon), G\left([\|u(\tau)\|_{\mathcal{U}}]_{[t_0,t]}\right)\right\}, \quad \text{for all} \quad t \geqslant t_0 + \tau_k \tag{A.18}$$

If $\beta, c \in K^+$ are bounded then k is independent of T. The above Fact in conjunction with (3.6), (A.18) and definition (3.5) of G imply that property P3 of Lemma 2.16 in Karafyllis & Jiang (2007) holds for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$ and $\gamma(s) := q(G(s))$. Moreover, if $\beta, c \in K^+$ are bounded then (A.18) and (3.6) imply that property P3 of Lemma 2.17 in Karafyllis & Jiang (2007) hold for system Σ with $V = \|H(t, x, u)\|_{\mathcal{Y}}$ and $\gamma(s) := q(G(s))$. The proof is complete. □