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Stability and control of nonlinear systems described by retarded functional equations: a review of recent results

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This paper reports on recent results in a series of the work of the authors on the stability and nonlinear control for general dynamical systems described by retarded functional differential and difference equations. Both internal and external stability properties are studied. The corresponding Lyapunov and Razuminkhin characterizations for input-to-state and input-to-output stabilities are proposed. Necessary and sufficient Lyapunov-like conditions are derived for robust nonlinear stabilization. In particular, an explicit controller design procedure is developed for a new class of nonlinear time-delay systems. Lastly, sufficient assumptions, including a small-gain condition, are presented for guaranteeing the input-to-output stability of coupled systems comprised of retarded functional differential and difference equations.

stability, nonlinear control, retarded functional equations, Lyapunov functions, Lyapunov functionals

1 Introduction

The field of nonlinear systems analysis and synthesis continues to attract attention of many researchers due to its strong relevance to several branches of sciences and engineering. Stability is one of the most fundamental problems in natural systems and man-made control engineering systems. Over the last three decades, great efforts and progress have been made in stability and control for finite-dimensional nonlinear dynamical systems with external inputs. Recently there has been a paradigm shift to infinite-dimensional nonlinear control systems such as those systems described by partial differential equations (PDEs) and systems with time delays. Again, these studies on infinite-dimensional systems are motivated by addressing several real-world applications. For example, time delay is a very common phenomenon in both natural and engineering systems. Examples of time delay are found in cell-to-cell communication, wireless and wired communication systems, multi-vehicle coordination and control, and power networks.

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In this work, we will place our attention on time-delay nonlinear systems described by retarded functional differential equations (RFDEs). The primary objective of this research initiative is to develop tools for the stability and stabilization of a wide class of nonlinear dynamical systems described by the following RFDEs:

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, u(t), d(t)) \ , \ t \geqslant t_0, \\ &Y(t) = H(t, T_r(t)x), \\ &x(t) \in \Re^n \ , \ Y(t) \in \mathcal{Y}, \ d(t) \in D \ , \ u(t) \in U, \end{aligned} \tag{1.1}$$

where $D \subseteq \Re^l$ is a non-empty set, $U \subseteq \Re^m$ is a non-empty set with $0 \in U$, $\mathcal{Y}$ is a normed linear space, $r > 0$ is a constant, $T_r(t)x := x(t+\theta); \theta \in [-r, 0]$ and the mappings $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$, $H : \Re^+ \times C^0([-r, 0]; \Re^n) \to \mathcal{Y}$ satisfy $f(t, 0, 0, d) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \Re^+ \times D$. We denote by $x(t)$ with $t \geqslant t_0$ the solution of the initial-value problem (1.1) with initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \Re^n)$.

A major advantage of allowing the output to take values in abstract normed linear spaces is that using the framework of the case (1.1) we are in a position to consider

• outputs with no delay, e.g. $Y(t) = h(t, x(t))$ with $\mathcal{Y} = \Re^k$;

• outputs with discrete or distributed delay, e.g. $Y(t) = h(x(t), x(t-r))$ or $Y(t) = \int_{t-r}^{t} h(t, \theta, x(\theta))d\theta$ with $\mathcal{Y} = \Re^k$;

• functional outputs with memory, e.g. $Y(t) = h(t, \theta, x(t+\theta)); \theta \in [-r, 0]$ or the identity output $Y(t) = T_r(t)x = x(t+\theta); \theta \in [-r, 0]$ with $\mathcal{Y} = C^0([-r, 0]; \Re^k)$.

For RFDE systems in the above form (1.1), we will study both internal stability and external stability properties and characterizations using Lyapunov functionals and Razumikhin functions. Here, internal stability refers mainly to robust stability with respect to disturbance input d and when control input u is absent, while external stability is closely related to input-to-state and input-to-output stability notions which have been extensively studied in the literature of finite-dimensional nonlinear control systems. See section 2 for the details. To add to the generality of our research work, we will consider both uniform and non-uniform in time stability, and present Lyapunov characterizations in section 3 and the Razumikhin method in section 4. In section 5, we study the robust stabilization problem for nonlinear systems described by RFDEs (1.1). Like the context of finite-dimensional systems, we are able to derive necessary and sufficient Lyapunov-like conditions for robust nonlinear stabilization. More interestingly, it is shown that our work at the conceptual level can also lead to an explicit stabilization algorithm for a new class of time-delay nonlinear systems with a lower-triangular structure.

In addition to the efforts on the stability and control of general nonlinear systems of the form (1.1), we will also investigate the stability issue for coupled systems consisting of retarded functional differential and difference equations:

$$\begin{aligned} \dot{x}_1(t) = f_1(t, d(t), &T_{r_1}(t)x_1, \\ &T_{r_2-\tau(t)}(t-\tau(t))x_2, u(t)), \end{aligned} \tag{1.2a}$$

$$\begin{aligned} x_2(t) = f_2(t, d(t), &T_{r_1}(t)x_1, \\ &T_{r_2-\tau(t)}(t-\tau(t))x_2, u(t)), \end{aligned} \tag{1.2b}$$

$$Y = H(t, T_{r_1}(t)x_1, T_{r_2}(t)x_2, u(t)) \in \mathcal{Y}, \tag{1.2c}$$

with $x_1(t) \in \Re^{n_1}$, $x_2(t) \in \Re^{n_2}$, $d(t) \in D$, $u(t) \in U$, $t \geqslant 0$, where $D \subseteq \Re^l$ is a non-empty set, $U \subseteq \Re^m$ is a non-empty set with $0 \in U$, $r_1 \geqslant 0$, $r_2 > 0$, $f_i : \cup_{t \geqslant 0}\{t\} \times D \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty([-r_2+\tau(t), 0]; \Re^{n_2}) \times U \to \Re^{n_i}$, $i = 1, 2$, $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty([-r_2, 0]; \Re^{n_2}) \times U \to \mathcal{Y}$ (again, $\mathcal{Y}$ is a normed linear space) are locally bounded mappings with $f_i(t, d, 0, 0, 0) = 0$ for $i = 1, 2$, $H(t, 0, 0, 0) = 0$ for all $(t, d) \in \Re^+ \times D$.

As demonstrated in the previous work of several authors independently, this class of coupled systems appears in the context of neutral RFDEs as well as quasilinear hyperbolic partial differential equations (see refs. [1–4]).

Here, in section 6 we give minimal regularity hypotheses so that coupled systems of the form (1.2) can be considered as control systems with a well-defined maximal existence time for their unique solution. Moreover, although continuity of the solutions with respect to the initial data will not be guaranteed, we show that our hypotheses guarantee appropriate continuity properties for the solution $x \equiv 0$. Furthermore, we will derive new stability criteria for the internal (Lyapunov) and external (input-to-output) stability of the coupled system (1.2). Interconnected systems composed of ordinary differential equations have been studied by several authors from the perspective of nonlinear small-gain[5]. Interestingly, we show that under a similar nonlinear small-gain condition, the coupled system (1.2) will remain input-to-output stable and input-to-state stable in the sense as defined in section 7.

Finally, we would like to point out that much more needs to be accomplished in this field of nonlinear control for infinite-dimensional systems described by RFDEs (1.1) and (1.2). It is expected that the tools and stability results developed in this paper will be useful for solving control problems related to stabilization and robust control of various classes of time-delay nonlinear systems.

Notations. Throughout this paper we adopt the following notations:

• Let $A \subseteq \Re^n$ be a set. By $C^0(A;\Omega)$, we denote the class of continuous functions on A, which take values in Ω. By $C^k(A;\Omega)$, where $k \geqslant 1$ is an integer, we denote the class of differentiable functions on A with continuous derivatives up to order k, which take values in Ω. By $C^\infty(A;\Omega)$, we denote the class of differentiable functions on A having continuous derivatives of all orders, which take values in Ω, i.e., $C^\infty(A;\Omega) = \bigcap_{k\geqslant 1} C^k(A;\Omega)$.

• By $\| \ \|_{\mathcal{Y}}$, we denote the norm of the normed linear space $\mathcal{Y}$. $\Re^+$ denotes the set of non-negative real numbers.

• For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. For a bounded function $x : [-r, 0] \to \Re^n$ we define $\|x\|_r := \sup_{\theta\in[-r,0]} |x(\theta)|$.

• A continuous mapping $A \times B \ni (z, x) \to k(z, x) \in \Re^m$, where $B \subseteq \mathcal{X}$, $A \subseteq \mathcal{Y}$ and $\mathcal{X}, \mathcal{Y}$ are normed linear spaces, is called completely locally Lipschitz with respect to $x \in B$ if for every closed and bounded set $S \subseteq A \times B$ it holds that

$$\sup\left\{\frac{|k(z,x)-k(z,y)|}{\|x-y\|_{\mathcal{X}}} : (z,x) \in S,\right.$$
$$\left.(z,y) \in S, x \neq y\right\} < +\infty.$$

If the normed linear spaces $\mathcal{X}, \mathcal{Y}$ are finite-dimensional, then we simply say that the continuous mapping $A \times B \ni (z, x) \to k(z, x) \in \Re^m$ is locally Lipschitz with respect to $x \in B$ if for every compact set $S \subseteq A \times B$ it holds that

$$\sup\left\{\frac{|k(z,x)-k(z,y)|}{|x-y|} : (z,x) \in S,\right.$$
$$\left.(z,y) \in S, x \neq y\right\} < +\infty.$$

• We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. We say that a function $\rho : \Re^+ \to \Re^+$ is of class $\mathcal{N}$, if ρ is non-decreasing with $\rho(0) = 0$. By K we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho : \Re^+ \to \Re^+$ is of class K_∞ if $\lim_{s\to+\infty} \rho(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s, t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geqslant 0$ the mapping $\sigma(\,\cdot\,, t)$ is of class K; (ii) for each $s \geqslant 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s, t) = 0$. $\mathcal{E}$ denotes the class of non-negative C^0 functions $\mu : \Re^+ \to \Re^+$, for which it holds: $\int_0^{+\infty} \mu(t)dt < +\infty$ and $\lim_{t\to+\infty} \mu(t) = 0$.

• Let $I \subseteq \Re$ be an interval. By $L^\infty(I;\Omega)$ (resp. $L^\infty_{\mathrm{loc}}(I;\Omega)$), we denote the class of measurable and (resp. locally) bounded functions on I, which take values in $\Omega \subseteq \Re^n$. If $\Omega \subseteq \Re^n$ is a subspace of $\Re^n$, $L^\infty(I;\Omega)$ is a normed linear space with norm $\sup_{t\in I} |x(t)|$, for $x \in L^\infty(I;\Omega)$.

• Let $D \subseteq \Re^l$ be a non-empty set. By M_D we denote the class of all Lebesgue measurable and locally essentially bounded mappings $d : \Re^+ \to D$. By $\tilde{M}_D$ we denote the class of all right-continuous mappings $d : \Re^+ \to D$, with the property that there exists a countable set $A_d \subset \Re^+$ which is either finite or $A_d = \{t_k^d; k = 1, \ldots, \infty\}$ with $t_{k+1}^d > t_k^d > 0$ for all $k = 1, 2, \ldots$ and $\lim t_k^d = +\infty$, such that the mapping $t \in \Re^+ \backslash A_d \to d(t) \in D$ is continuous.

• Let $U \subseteq \Re^m$ be a non-empty set with $0 \in U$. By $B_U[0, r] := \{u \in U; |u| \leqslant r\}$ we denote the intersection of the closed sphere of radius $r \geqslant 0$, centered at $0 \in U$ with $U \subseteq \Re^m$.

• Let $x : [a-r, b) \to \Re^n$ be a continuous mapping with $b > a > -\infty$ and $r > 0$. By $T_r(t)x$ we denote the "r-history" of x at time $t \in [a, b)$, i.e., $T_r(t)x := x(t+\theta); \theta \in [-r, 0]$.

2 Main assumptions and preliminaries for systems described by RFDEs

In this section, we provide background material needed for the study of systems described by RFDEs of the form (1.1). Although the results of this section are technical, they play a fundamental role in the proofs of the main results of the present work.

2.1 Main assumptions for systems described by RFDEs

Concerning systems of the form (1.1) the following hypotheses will be valid throughout the text:

(S1) The mapping $(x, u, d) \to f(t, x, u, d)$ is continuous for each fixed $t \geqslant 0$, such that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]; \Re^n) \times U$, there exists a constant $L \geqslant 0$ such that

$$(x(0)-y(0))'(f(t,x,u,d)-f(t,y,u,d)) \leqslant L \max_{\tau\in[-r,0]} |x(\tau)-y(\tau)|^2 = L\|x-y\|_r^2 \quad \forall\, t \in I, \forall (x,u,y,u) \in S \times S, \forall\, d \in D.$$

Hypothesis (S1) is equivalent to the existence of a continuous function $L : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geqslant 0$ the mappings $L(t, \cdot)$ and $L(\cdot, t)$ are non-decreasing, and that

$$(x(0)-y(0))'(f(t,x,u,d)-f(t,y,u,d)) \leqslant L(t, \|x\|_r + \|y\|_r + |u|)\|x-y\|_r^2 \quad \forall (t,x,y,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^n) \times D \times U. \tag{2.1}$$

(S2) For every bounded $\Omega \subset \Re^+ \times C^0([-r, 0]; \Re^n) \times U$ the image set $f(\Omega \times D) \subset \Re^n$ is bounded.

(S3) There exists a countable set $A \subset \Re^+$, which is either finite or $A = \{t_k; k = 1, \ldots, \infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1, 2, \ldots$ and $\lim t_k = +\infty$, such that mapping $(t, x, u, d) \in (\Re^+ \backslash A) \times C^0([-r, 0]; \Re^n) \times U \times D \to f(t, x, u, d)$ is continuous. Moreover, for each fixed $(t_0, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D$, we have $\lim_{t\to t_0^+} f(t, x, u, d) = f(t_0, x, u, d)$.

(S4) For every $\varepsilon > 0$, $t \in \Re^+$, there exists $\delta := \delta(\varepsilon, t) > 0$, such that

$$\sup\{|f(\tau,x,u,d)| : \tau \in \Re^+, d \in D, u \in U, |\tau-t| + \|x\|_r + |u| < \delta\} < \varepsilon.$$

(S5) The mapping $u \to f(t, x, u, d)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]; \Re^n) \times U$, there exists a constant $L_U \geqslant 0$ such that

$$|f(t,x,u,d)-f(t,x,v,d)| \leqslant L_U|u-v|, \quad \forall t \in I, \forall (x,u,x,v) \in S \times S, \forall d \in D.$$

Hypothesis (S5) is equivalent to the existence of a continuous function $L_U : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geqslant 0$ the mappings $L_U(t, \cdot)$ and $L_U(\cdot, t)$ are non-decreasing, with the following property:

$$|f(t,x,u,d)-f(t,x,v,d)| \leqslant L_U(t, \|x\|_r + |u| + |v|)|u-v| \quad \forall (t,x,d,u,v) \in \Re^+ \times C^0([-r,0];\Re^n) \times D \times U \times U. \tag{2.2}$$

(S6) U is a positive cone, i.e., for all $u \in U$ and $\lambda \geqslant 0$ it follows that $(\lambda u) \in U$.

(S7) The mapping $H(t, x)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]; \Re^n)$, there exists a constant $L_H \geqslant 0$ such that

$$\|H(t,x)-H(\tau,y)\|_{\mathcal{Y}} \leqslant L_H(|t-\tau| + \|x-y\|_r), \quad \forall (t,\tau) \in I \times I, \forall (x,y) \in S \times S. \tag{2.3}$$

Hypothesis (S7) is equivalent to the existence of a continuous function $L_H : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geqslant 0$ the mappings $L_H(t, \cdot)$ and $L_H(\cdot, t)$ are non-decreasing, satisfying

$$\|H(t,x)-H(\tau,y)\|_{\mathcal{Y}} \leqslant L_H(t+\tau, \|x\|_r + \|y\|_r)(|t-\tau| + \|x-y\|_r), \quad \forall (t,\tau) \in \Re^+ \times \Re^+, \quad \forall (x,y) \in C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^n). \tag{2.4}$$

Using hypotheses (S1–7) above, Theorem 2.1 in ref. [1] (and its extension given in paragraph 2.6 of the same book) and Theorem 3.2 in ref. [1], we may conclude that for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D \times M_U$ there exists $t_{\max} \in (t_0, +\infty]$, such that the unique solution $x(t)$ of (1.1) is defined on $[t_0 - r, t_{\max})$ and cannot be further continued. Moreover, if $t_{\max} < +\infty$ then we must necessarily have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$. We denote by $T_r(t)x$ the "r-history" of the unique solution of (1.1) with initial condition $T_r(t_0)x = x_0$ corresponding to $(d, u) \in M_D \times M_U$. Moreover, the following inequality holds for every pair $T(\cdot)x : [t_0, t_{\max}^x) \to C^0([-r, 0]; \Re^n)$, $T(\cdot)y : [t_0, t_{\max}^y) \to C^0([-r, 0]; \Re^n)$ of solutions of (1.1) with initial conditions $T_r(t_0)x = x_0$, $T_r(t_0)y = y_0$, corresponding to the same $(d, u) \in M_D \times M_U$ and for all $t \in [t_0, t_1)$ with $t_1 = \min\{t_{\max}^x; t_{\max}^y\}$:

$$
\begin{aligned}
&\|T_r(t)x - T_r(t)y\|_r \\
&\qquad \leqslant \|x_0 - y_0\|_r \exp(L(t, a(t))(t - t_0)), \\
&\|H(t, T_r(t)x) - H(t, T_r(t)y)\|_{\mathcal{Y}} \\
&\qquad \leqslant L_H(t, a(t))\|x_0 - y_0\|_r \exp(L(t, a(t))(t - t_0)), \\
&a(t) = \sup_{\tau \in [t_0, t]} (\|T_r(\tau)x\|_r + \|T_r(\tau)y\|_r) \\
&\qquad + \sup_{\tau \in [t_0, t]} |u(\tau)|.
\end{aligned} \tag{2.5}
$$

Some comments are needed for hypotheses (S1–7). Indeed, hypotheses (S1–7) are the minimal regularity assumptions that guarantee uniqueness of solutions, Lipschitz continuity of the solutions with respect to the initial state (i.e., inequality (2.5)) and the so-called Boundedness-Implies-Continuation property (see refs. [6–9]). Based on these minimal regularity assumptions we are in a position to develop the tools for the study of stability for systems of the form (11).

2.2 Important notions for systems described by RFDEs

An important property for systems of the form (1.1) is robust forward completeness (RFC) from an external input (see refs. [6–9]). This property will be used extensively in the following sections of the present work.

Definition 2.1. We say that (1.1) under hypotheses (S1–7) is robustly forward complete (RFC) from the input $u \in M_U$ if for every $s \geqslant 0$, $T \geqslant 0$, it holds that

$$
\begin{aligned}
&\sup\{\|T_r(t_0 + \xi)x\|_r ; u \in M_{B_U[0,s]}, \xi \in [0, T], \\
&\qquad \|T_r(t_0)x\|_r \leqslant s, t_0 \in [0, T], d \in M_D\} < +\infty.
\end{aligned}
$$

When $U = \{0\}$ we simply say that (1.1) under hypotheses (S1–7) is simply called robustly forward complete (RFC).

In order to study the asymptotic properties of the solutions of systems of the form (1.1), we will use Lyapunov functionals and functions. Therefore, certain notions and properties concerning functionals are needed. Let $x \in C^0([-r, 0]; \Re^n)$ and $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re$. By $E_h(x; v)$, where $0 \leqslant h < r$ and $v \in \Re^n$ we denote the following operator

$$
E_h(x; v) := \begin{cases} x(0) + (\theta + h)v, & -h < \theta \leqslant 0, \\ x(\theta + h), & -r \leqslant \theta \leqslant -h, \end{cases} \tag{2.6}
$$

and we define

$$
V^0(t, x; v) := \limsup_{\substack{h \to 0^+ \\ y \to 0 \\ y \in C^0([-r,0];\Re^n)}} \frac{V(t + h, E_h(x; v) + hy) - V(t, x)}{h}. \tag{2.7}
$$

The class of functionals which are "almost Lipschitz on bounded sets" is introduced in refs. [10,11] and is used extensively in the present work. For reasons of completeness we recall the definition here.

Definition 2.2. We say that a continuous functional $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$, is "almost Lipschitz on bounded sets", if there exist non-decreasing functions $M : \Re^+ \to \Re^+$, $P : \Re^+ \to \Re^+$, $G : \Re^+ \to [1, +\infty)$ such that for all $R \geqslant 0$, the following properties hold:

(P1) For every

$$
x, y \in \{\xi \in C^0([-r, 0]; \Re^n); \|\xi\|_r \leqslant R\},
$$

it holds that

$$
|V(t, y) - V(t, x)| \leqslant M(R)\|y - x\|_r, \forall t \in [0, R].
$$

(P2) For every absolutely continuous function $x : [-r, 0] \to \Re^n$ with $\|x\|_r \leqslant R$ and essentially bounded derivative, it holds that

$$
|V(t + h, x) - V(t, x)| \leqslant hP(R)\left(1 + \sup_{-r \leqslant \tau \leqslant 0} |\dot{x}(\tau)|\right)
$$

for all $t \in [0, R]$ and

$$
0 \leqslant h \leqslant 1/G\left(R + \sup_{-r \leqslant \tau \leqslant 0} |\dot{x}(\tau)|\right).
$$

If the continuous functional $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$ is "almost Lipschitz on bounded sets", then the derivative $V^0(t,x;v)$ defined by (2.7) is simplified in the following way:

$$V^0(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h, E_h(x;v)) - V(t,x)}{h}.$$

The following definition introduces an important relation between output mappings. The equivalence relation, defined next, will be used extensively in the following sections of the present work (see also refs. [10,11]).

Definition 2.3. Suppose that there exists a continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ with $h(t,0) = 0$ for all $t \geqslant -r$ and functions $a_1, a_2 \in K_\infty$ such that $a_1(|h(t,x(0))|) \leqslant \|H(t,x)\|_{\mathcal{Y}} \leqslant a_2(\sup_{\theta\in[-r,0]} |h(t+\theta, x(\theta))|)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$. Then we say that $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping h.

For example the identity output mapping $H(t,x) = x \in C^0([-r,0];\Re^n)$ is equivalent to finite-dimensional mapping $h(t,x) = x \in \Re^n$.

2.3 Useful technical results

The following lemma presents some elementary properties of the generalized derivative given above.

Lemma 2.4. Consider $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re$ and let $x \in C^0([t_0 - r, t_{\max});\Re^n)$ be a solution of (1.1) under hypotheses (S1–7) corresponding to certain $(d,u) \in M_D \times M_U$, where $t_{\max} \in (t_0,+\infty]$ is the maximal existence time of the solution. Then it holds

$$\limsup_{h\to 0^+} h^{-1}(V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \leqslant V^0(t, T_r(t)x; D^+x(t)), \text{ a.e. on } [t_0,t_{\max}), \quad (2.8)$$

where $D^+x(t) = \lim_{h\to 0^+} h^{-1}(x(t+h) - x(t))$. Moreover, if $(d,u) \in \tilde{M}_D \times \tilde{M}_U$ then (2.8) holds for all $t \in [t_0, t_{\max})$.

The following results are direct extensions of the similar results in ref. [12]. More specifically, the proof of Lemma 2.6 utilizes inequality (2.5), which guarantees continuity of the solution with respect to the initial conditions.

Lemma 2.5. Let $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re$ be a functional which is almost Lipschitz on bounded sets and let $x \in C^0([t_0 - r, t_{\max});\Re^n)$ be a solution of (1.1) under hypotheses (S1–7) corresponding to certain $(d,u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$, where $t_{\max} \in (t_0,+\infty]$ is the maximal existence time of the solution. Then for every $T \in (t_0, t_{\max})$, the mapping $[t_0,T] \ni t \to V(t, T_r(t)x)$ is absolutely continuous.

Lemma 2.6. Suppose that there exist mappings $\beta_1 : \Re^+ \times C^0([-r,0];\Re^n) \to \Re$, $\beta_2 : \Re^+ \times \Re^+ \times C^0([-r,0];\Re^n) \times \Theta \to \Re$, where $\Theta \subseteq M_D \times M_U$, with the following properties:

(i) for each $(t,t_0,d,u) \in \Re^+ \times \Re^+ \times \Theta$, the mappings $x \to \beta_1(t,x)$, $x \to \beta_2(t,t_0,x,d,u)$ are continuous;

(ii) there exists a continuous function $M : \Re^+ \times \Re^+ \to \Re^+$ such that

$$\sup\{\beta_2(t_0+\xi, t_0, x_0, d, u) : \sup_{t\geqslant 0} |u(\tau)| \leqslant s,$$
$$\xi \in [0,T], x_0 \in C^0([-r,0];\Re^n), \|x_0\|_r \leqslant s,$$
$$t_0 \in [0,T], (d,u) \in \Theta\} \leqslant M(T,s);$$

(iii) for every $(t_0,x_0,d,u) \in \Re^+ \times C^1([-r,0];\Re^n) \times \Theta$ the solution $x(t)$ of (1.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in \Theta$ satisfies

$$\beta_1(t, T_r(t)x) \leqslant \beta_2(t,t_0,x_0,d,u), \forall t \geqslant t_0. \quad (2.9)$$

Moreover, suppose that one of the following properties holds:

(iv) We have

$$\sup\{\|T_r(t_0+\xi)x\|_r : \sup_{t\geqslant 0} |u(\tau)| \leqslant s, \xi \in [0,T],$$
$$x_0 \in C^0([-r,0];\Re^n), \|x_0\|_r \leqslant s,$$
$$t_0 \in [0,T], (d,u) \in \Theta\} < +\infty.$$

(v) There exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $a(\mu(t)|x(0)|) \leqslant \beta_1(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Then, for every $(t_0,x_0,d,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times \Theta$, the solution $x(t)$ of (1.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in \Theta$ exists for all $t \geqslant t_0$ and satisfies (2.9).

3 Stability notions and their Lyapunov characterizations

In this section, we introduce the reader to the notion of non-uniform and uniform weighted input-to-output stability (IOS) for systems described by

RFDEs and we provide Lyapunov characterizations of these properties for such systems. Notice that the notion of IOS is an "external stability" property since it is applied to systems which operate under the effect of external non-vanishing perturbations.

Definition 3.1. We say that (1.1) under hypotheses (S1–7) satisfies the weighted input-to-output stability property (WIOS) from the input $u \in M_U$ with gain $\gamma \in K$ and weight $\delta \in K^+$, if (1.1) is robustly forward complete (RFC) from the input $u \in M_U$ and there exist functions $\sigma \in KL$, $\beta \in K^+$, such that for all $(d,u) \in M_D \times M_U$, $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$ satisfies the following estimate for all $t \geqslant t_0$:

$$\|H(t,T_r(t)x)\|_{\mathcal{Y}} \leqslant \max\{\sigma(\beta(t_0)\|x_0\|_r, t-t_0), \sup_{t_0\leqslant\tau\leqslant t}\gamma(\delta(\tau)|u(\tau)|\}. \tag{3.1}$$

Moreover,

(i) if $\beta(t) \equiv 1$, then we say that (1.1) satisfies the uniform weighted input-to-output stability property (UWIOS) from the input $u \in M_U$ with gain $\gamma \in K$ and weight $\delta \in K^+$;

(ii) if $\delta(t) \equiv 1$, then we say that (1.1) satisfies the input-to-output stability property (IOS) from the input $u \in M_U$ with gain $\gamma \in K$;

(iii) if $\beta(t) = \delta(t) \equiv 1$, then we say that (1.1) satisfies the uniform input-to-output stability property (UIOS) from the input $u \in M_U$ with gain $\gamma \in K$;

(iv) if $\|x\|_r \leqslant \|H(t,x)\|_{\mathcal{Y}}$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, then we say that (1.1) satisfies the weighted input-to-state stability property (WISS) from the input $u \in M_U$ with gain $\gamma \in K$ and weight $\delta \in K^+$;

(v) if $\|x\|_r \leqslant \|H(t,x)\|_{\mathcal{Y}}$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\beta(t) \equiv 1$, then we say that (1.1) satisfies the uniform weighted input-to-state stability property (UWISS) from the input $u \in M_U$ with gain $\gamma \in K$ and weight $\delta \in K^+$;

(vi) if $\|x\|_r \leqslant \|H(t,x)\|_{\mathcal{Y}}$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\delta(t) \equiv 1$, then we say that (1.1) satisfies the input-to-state stability property (ISS) from the input $u \in M_U$ with gain $\gamma \in K$;

(vii) if $\|x\|_r \leqslant \|H(t,x)\|_{\mathcal{Y}}$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\beta(t) = \delta(t) \equiv 1$, then we say that (1.1) satisfies the uniform input-to-state stability property (UISS) from the input $u \in M_U$ with gain $\gamma \in K$.

It should be emphasized that for periodic systems estimate (3.1) leads to a simpler estimate. We say that (1.1) under hypotheses (S1–7) is T-periodic, if there exists $T > 0$ such that $f(t+T,x,u,d) = f(t,x,u,d)$ and $H(t+T,x) = H(t,x)$ for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$. Lemmas 2.19 and 2.20 in ref. [9] show that if system (1.1) is T-periodic and satisfies the WIOS property with gain γ and weight δ from the input $u \in M_U$, then system (1.1) satisfies the UWIOS property from the input $u \in M_U$ with gain γ and weight $\tilde{\delta}$, where $\tilde{\delta}(t) := \max\{\delta(s); s \in [0,t]\}$.

Definition 3.2. Consider system (1.1) under hypotheses (S1–7) with $U = \{0\}$. We say that (1.1) is non-uniformly in time robustly globally asymptotically output stable (RGAOS) with disturbances $d \in M_D$ if (1.1) is RFC and the following properties hold:

P1. (1.1) is robustly lagrange output stable, i.e., for every $\varepsilon > 0$, $T \geqslant 0$, it holds that

$$\sup\{\|H(t,T_r(t)x)\|_{\mathcal{Y}}; t \in [t_0,+\infty), \|x_0\|_r \leqslant \varepsilon, t_0 \in [0,T], d \in M_D\} < +\infty$$

(robust Lagrange output stability).

P2. (2.1) is robustly Lyapunov output stable, i.e., for every $\varepsilon > 0$ and $T \geqslant 0$ there exists a $\delta := \delta(\varepsilon, T) > 0$ such that

$$\|x_0\|_r \leqslant \delta, t_0 \in [0,T] \Rightarrow \|H(t,T_r(t)x)\|_{\mathcal{Y}} \leqslant \varepsilon, \quad \forall\ t \geqslant t_0, \forall d \in M_D$$

(robust Lyapunov output stability)

P3. (2.1) satisfies the robust output attractivity property, i.e. for every $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geqslant 0$, such that

$$\|x_0\|_r \leqslant R, t_0 \in [0,T] \Rightarrow \|H(t,T_r(t)x)\|_{\mathcal{Y}} \leqslant \varepsilon, \quad \forall\ t \geqslant t_0 + \tau, \forall d \in M_D.$$

Moreover, if there exists a function $a \in K_\infty$ such that $a(\|x\|_r) \leqslant \|H(t,x)\|_{\mathcal{Y}}$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, then we say that (2.1) is non-uniformly in time robustly globally asymptotically stable (RGAS) with disturbances $d \in M_D$.

We say that (2.1) is non-uniformly in time robustly globally asymptotically output stable (RGAOS) with disturbances $d \in \tilde{M}_D$ if (1.1) is RFC and properties P1–3 above hold with $d \in \tilde{M}_D$ instead of $d \in M_D$.

The next lemma provides an estimate of the output behavior for non-uniformly in time RGAOS systems. It is a direct corollary of Lemma 3.4 in ref. [6].

Lemma 3.3. System (1.1) under hypotheses (S1–7) with $U = \{0\}$ is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$) if and only if system (1.1) is RFC and there exist functions $\sigma \in KL$, $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0) \in \Re^+ \times C^0([-r, 0]; \Re^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geqslant t_0$:

$$\|H(t, T_r(t)x)\|_{\mathcal{Y}} \leqslant \sigma(\beta(t_0)\|x_0\|_r, t - t_0). \quad (3.2)$$

We next provide the definition of uniform robust global asymptotic output stability, in terms of KL functions, which is completely analogous to the finite-dimensional case (see refs. [13–16]). It is clear that such a definition is equivalent to a $\delta - \varepsilon$ definition (analogous to Definition 3.2).

Definition 3.4. Suppose that (1.1) under hypotheses (S1–7) with $U = \{0\}$ is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$) and there exists $\sigma \in KL$ such that estimate (1.1) holds for all $(t_0, x_0) \in \Re^+ \times C^0([-r, 0]; \Re^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geqslant t_0$ with $\beta(t) \equiv 1$. Then we say that (1.1) is uniformly robustly globally asymptotically output stable (URGAOS) with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$).

The following lemma must be compared to Lemma 1.1 in ref. [1]. It shows that for periodic systems RGAOS is equivalent to URGAOS.

Lemma 3.5. Suppose that (1.1) under hypotheses (S1–7) with $U = \{0\}$ is T-periodic. If (1.1) is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$), then (1.1) is URGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$).

We are now in a position to present Lyapunov-like characterizations for non-uniform in time RGAOS and URGAOS.

Theorem 3.6. Consider system (1.1) under hypotheses (S1–7) with $U = \{0\}$. The following statements are equivalent:

(a) (1.1) is non-uniformly in time RGAOS with disturbances $d \in M_D$.

(b) (1.1) is non-uniformly in time RGAOS with disturbances $d \in \tilde{M}_D$.

(c) (1.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)dt = +\infty$, a positive definite locally Lipschitz function $\rho : \Re^+ \to \Re^+$ and a mapping $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that

$$a_1(\|H(t, x)\|_{\mathcal{Y}}) \leqslant V(t, x) \leqslant a_2(\beta(t)\|x\|_r), \quad \forall (t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n) \quad (3.3)$$

$$V^0(t, x; f(t, x, d)) \leqslant -\gamma(t)\rho(V(t, x)), \quad \forall (t, x, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times D. \quad (3.4)$$

(d) (1.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$ and a mapping $V : \Re^+ \times C^0([-r, 0]; \Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that inequalities (3.3), (3.4) hold with $\gamma(t) \equiv 1$ and $\rho(s) := s$.

(e) (1.1) is RFC and there exist a lower semi-continuous mapping $V : \Re^+ \times C^0([-r-\tau, 0]; \Re^n) \to \Re^+$, a constant $\tau \geqslant 0$, functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)dt = +\infty, \mu \in \mathcal{E}$ (see Notations) and a positive definite locally Lipschitz function $\rho : \Re^+ \to \Re^+$, such that the following inequalities hold:

$$a_1(\|H(t, x)\|_{\mathcal{Y}}) \leqslant V(t, x) \leqslant a_2(\beta(t)\|x\|_{r+\tau}), \quad \forall (t, x) \in \Re^+ \times C^0([-r-\tau, 0]; \Re^n), \quad (3.5)$$

$$V^0(t, x; f(t, T_r(0)x, d)) \leqslant -\gamma(t)\rho(V(t, x)) + \gamma(t)\mu\left(\int_0^t \gamma(s)ds\right), \quad \forall (t, d) \in [\tau, +\infty) \times D\ ,\ \forall x \in S(t), \quad (3.6)$$

where the set-valued map $S(t)$ is defined for $t \geqslant \tau$ by $S(t) := \bigcup_{d \in \tilde{M}_D} S(t, d)$ and the set-valued map $S(t, d)$ is defined for $t \geqslant \tau$ and $d \in \tilde{M}_D$ by

$$S(t, d) := \{\, x \in C^0([-r-\tau, 0]; \Re^n);\ x(\theta) = x(-\tau) + \int_{-\tau}^{\theta} f(t+s, T_r(s)x, d(\tau+s))ds, \quad \forall \theta \in [-\tau, 0]\}. \quad (3.7)$$

Moreover,

i) if $H : \Re^+ \times C^0([-r, 0]; \Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous mapping $h : [-r, +\infty) \times \Re^n \to \Re^p$ then inequalities (3.3) in statements (c) and (d) can be replaced by the following inequalities:

$$a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_r),\\ \forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n); \qquad (3.8)$$

ii) if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ then inequalities (3.5) in statement (e) can be replaced by the following inequalities:

$$a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\beta(t)\,\|x\|_{r+\tau}),\\ \forall(t,x) \in \Re^+ \times C^0([-r-\tau,0];\Re^n); \qquad (3.9)$$

iii) if there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $a(\mu(t)|x(0)|) \leqslant V(t,x)+R$ for all $(t,x) \in \Re^+\times C^0([-r,0];\Re^n)$, then the requirement that (1.1) is RFC is not needed in statements (c) and (d) above.

Theorem 3.7. Consider system (1.1) under hypotheses (S1–7) with $U = \{0\}$. The following statements are equivalent:

(a) (1.1) is URGAOS with disturbances $d \in M_D$.

(b) (1.1) is URGAOS with disturbances $d \in \tilde{M}_D$.

(c) (1.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, a positive definite locally Lipschitz function $\rho : \Re^+ \to \Re^+$ and a mapping $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that

$$a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x) \leqslant a_2(\|x\|_r),\\ \forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n), \qquad (3.10)$$

$$V^0(t,x;f(t,x,d)) \leqslant -\rho(V(t,x)),\\ \forall(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D. \quad (3.11)$$

(d) (1.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$ and a mapping $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, such that inequalities (3.10), (3.11) hold with $\rho(s) := s$. Moreover, if system (1.1) is T-periodic, then V is T-periodic (i.e. $V(t+T,x) = V(t,x)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$) and if (1.1) is autonomous then V is independent of t.

(e) (1.1) is RFC and there exist constants $\tau, \beta \geqslant 0$, a lower semi-continuous mapping $V : \Re^+ \times C^0([-r-\tau,0];\Re^n) \to \Re^+$, functions $a_1, a_2 \in K_\infty$ and a positive definite locally Lipschitz function $\rho : \Re^+ \to \Re^+$, such that the following inequalities hold

$$a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x) \leqslant a_2(\|x\|_{r+\tau}),\\ \forall(t,x) \in \Re^+ \times C^0([-r-\tau,0];\Re^n), \quad (3.12)$$

$$V^0(t,x;f(t,T_r(0)x,d)) \leqslant \beta\, V(t,x),\\ \forall(t,x,d) \in \Re^+ \times C^0([-r-\tau,0];\Re^n)\\ \times D, \qquad (3.13a)$$

$$V^0(t,x;f(t,T_r(0)x,d)) \leqslant -\rho(V(t,x)),\\ \forall(t,d) \in [\tau,+\infty) \times D, \forall x \in S(t), \quad (3.13b)$$

where the set-valued map $S(t)$ is defined for $t \geqslant \tau$ by $S(t) := \bigcup_{d\in\tilde{M}_D} S(t,d)$ and the set-valued map $S(t,d)$ is defined for $t \geqslant \tau$ and $d \in \tilde{M}_D$ by (3.7).

Moreover,

i) if $H : \Re^+\times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous T-periodic mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$, then inequalities (3.10) in statements (c) and (d) can be replaced by the following inequalities:

$$a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\|x\|_r),\\ \forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n); \quad (3.14)$$

ii) if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous T-periodic mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ then inequalities (3.12) in statement (e) can be replaced by the following inequalities:

$$a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\|x\|_{r+\tau}),\\ \forall(t,x) \in \Re^+ \times C^0([-r-\tau,0];\Re^n); \ (3.15)$$

iii) if there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that

$$a(\mu(t)|x(0)|) \leqslant V(t,x) + R$$

for all $(t,x) \in \Re^+\times C^0([-r,0];\Re^n)$ then the requirement that (1.1) is RFC is not needed in statements (c) and (d) above.

Remark 3.8. The set-valued map $S(t,d)$ defined by (3.7) can be equivalently described for given $t \geqslant \tau$ and $d \in \tilde{M}_D$ as "the set of all $x \in C^0([-r-\tau,0];\Re^n)$, which are arbitrary on $[-r-\tau,-\tau]$ (i.e., $T_r(-\tau)x$ is arbitrary) and coincide on $[-\tau,0]$ with the unique solution $y(t)$ of $\dot{y}(t) = f(t,T_r(t)y,d(t))$ with initial condition $T_r(t-\tau)y = T_r(-\tau)x$, i.e., $T_\tau(0)x = T_\tau(t)y$ and $x = T_{r+\tau}(t)y$".

Statements (e) of Theorem 3.6 and Theorem 3.7 are important, since they can be used efficiently when some information about the solution of (1.1) is available (e.g., we have analytical expressions for some components of the solution vector). In this case, the Lyapunov differential inequality is required to hold only for all $(t,d) \in [\tau,+\infty) \times D$ and $x \in S(t)$ since the solution of (1.1) initiated from $t_0 \geqslant 0$ and corresponding to input $d \in \tilde{M}_D$ satisfies $T_{r+\tau}(t)x \in S(t,d)$ for all $t \geqslant t_0+\tau$. Moreover, statements (e) of Theorem 3.6 and Theorem 3.7 have an additional advantage: the Lyapunov functional is not required to be almost Lipschitz on bounded sets (lower semi-continuity is sufficient). Consequently, value functionals of optimal control problems can be used for verification of RGAOS (usually value functionals are not continuous).

We are now in a position to state characterizations for the WIOS property for time-varying uncertain systems.

Theorem 3.9. The following statements are equivalent for system (1.1) under hypotheses (S1–7):

(a) System (1.1) is robustly forward complete (RFC) from the input $u \in M_U$ and there exist functions $\sigma \in KL$, $\beta,\phi \in K^+$, $\rho \in K$ such that that for all $(d,u) \in M_D \times M_U$, $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$ satisfies the following estimate for all $t \geqslant t_0$:

$$\begin{aligned}\|H(t,T_r(t)x)\|_{\mathcal{Y}} \leqslant \max\{&\sigma(\beta(t_0)\|x_0\|_r, t-t_0),\\ &\sup_{t_0\leqslant\tau\leqslant t}\sigma(\rho(\phi(\tau)|u(\tau)|), t-\tau)\}.\end{aligned} \tag{3.16}$$

(b) System (1.1) satisfies the WIOS property from the input $u \in M_U$.

(c) There exists a locally Lipschitz function $\theta \in K_\infty$, functions $\phi,\mu \in K^+$ such that the following system is non-uniformly in time RGAOS with disturbances $(d',d) \in \tilde{M}_\Delta$:

$$\begin{aligned}&\dot{x}(t) = f\left(t, T_r(t)x, \frac{\theta(\|T_r(t)x\|_r)}{\phi(t)}d'(t), d(t)\right),\\ &Y(t) = \tilde{H}(t,T_r(t)x),\end{aligned} \tag{3.17}$$

where $\Delta := B_U[0,1] \times D$,

$$\tilde{H}(t,x) := (H(t,x),\mu(t)x) \in \mathcal{Y} \times C^0([-r,0];\Re^n).$$

(d) There exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1,a_2,a_3 of class K_∞, β,δ,μ of class K^+ such that

$$\begin{aligned}&a_1(\|H(t,x)\|_{\mathcal{Y}} + \mu(t)\|x\|_r)\\ &\quad\leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_r),\\ &\qquad f5.2\forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n),\end{aligned} \tag{3.18}$$

$$\begin{aligned}&V^0(t,x;f(t,x,u,d)) \leqslant -V(t,x) + a_3(\delta(t)|u|),\\ &\forall(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D.\end{aligned} \tag{3.19}$$

(e) System (1.1) is RFC from the input $u \in M_U$ and there exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1,a_2,ζ of class K_∞, β,δ of class K^+ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that

$$\begin{aligned}&a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_r),\\ &\qquad\forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n),\end{aligned} \tag{3.20}$$

$$V^0(t,x;f(t,x,u,d)) \leqslant -\rho(V(t,x)) \tag{3.21}$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$ with $\zeta(\delta(t)|u|) \leqslant V(t,x)$.

(f) System (1.1) is RFC from the input $u \in M_U$ and system (1.1) with $u \equiv 0$ is non-uniformly in time RGAOS with disturbances $d \in M_D$.

Moreover,

i) if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ then inequality (3.20) in the above statement (e) can be replaced by the following inequality:

$$\begin{aligned}&a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_r),\\ &\qquad\forall(t,x) \in \Re^+ \times C^0([-r,0];\Re^n);\end{aligned} \tag{3.22}$$

ii) if there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $a(\mu(t)|x(0)|) \leqslant \|H(t,x)\|_{\mathcal{Y}} + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ then the requirement that (1.1) is RFC from the input $u \in M_U$ is not needed in statement (a) above;

iii) if there exist functions $p \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $p(\mu(t)|x(0)|) \leqslant V(t,x)+R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ then the requirement that (1.1) is RFC from the input $u \in M_U$ is not needed in statement (e) above.

In order to obtain characterizations of the UIOS property, we need an extra hypothesis for system (1.1).

(S8) There exists a constant $R \geqslant 0$ and a function $a \in K_\infty$ such that the inequality $\|x\|_r \leqslant a(\|H(t,x)\|_{\mathcal{Y}}) + R$ holds for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Hypothesis (S8) holds for the important case of the output map $H(t,x) := d(x(\theta),\Gamma); \theta \in [-r,0]$, where $\Gamma \subset \Re^n$ is a compact set which contains $0 \in \Re^n$ and $d(x,\Gamma)$ denotes the distance of the point $x \in \Re^n$ from the set $\Gamma \subset \Re^n$. Notice that it is not required that $\Gamma \subset \Re^n$ is positively invariant for (1.1) with $u \equiv 0$. Hypothesis (S8) allows us to provide characterizations for the UIOS property for periodic uncertain systems.

Theorem 3.10. Suppose that system (1.1) under hypotheses (S1–8) is T-periodic. The following statements are equivalent:

(a) There exist functions $\sigma \in KL$, $\rho \in K_\infty$ such that for all $(d,u) \in M_D \times M_U$, $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$, satisfies the following estimate for all $t \geqslant t_0$:

$$\|H(t,T_r(t)x)\|_{\mathcal{Y}} \leqslant \max\{\sigma(\|x_0\|_r, t-t_0), \sup_{t_0 \leqslant \tau \leqslant t} \sigma(\rho(|u(\tau)|), t-\tau)\}. \tag{3.23}$$

(b) System (1.1) satisfies the UIOS property.

(c) There exists a locally Lipschitz function $\theta \in K_\infty$ such that $0 \in C^0([-r,0];\Re^n)$ is URGAOS with disturbances $(d',d) \in \tilde{M}_\Delta$ for the system

$$\begin{aligned} &\dot{x}(t) = f(t,T_r(t)x,\theta(\|H(t,T_r(t)x)\|_{\mathcal{Y}})d'(t),d(t)),\\ &Y(t) = H(t,T_r(t)x), \end{aligned} \tag{3.24}$$

where $\Delta := B_U[0,1] \times D$.

(d) There exists a T-periodic Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1, a_2, a_3 of class K_∞ such that

$$a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x) \leqslant a_2(\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n), \tag{3.25}$$

$$V^0(t,x;f(t,x,u,d)) \leqslant -V(t,x) + a_3(|u|), \quad \forall (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D. \tag{3.26}$$

(e) There exists a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1, a_2, ζ of class K_∞ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that

$$a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x) \leqslant a_2(\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n), \tag{3.27}$$

$$V^0(t,x;f(t,x,u,d)) \leqslant -\rho(V(t,x)), \tag{3.28}$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$ with $\zeta(|u|) \leqslant V(t,x)$.

Finally, if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous T-periodic mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$, then inequalities (3.25), (3.27) in the above statements (d) and (e), respectively, can be replaced by the following inequality:

$$a_1(|h(t,x(0))|) \leqslant V(t,x) \leqslant a_2(\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n). \tag{3.29}$$

Remark 3.11. A statement like (e) of Theorem 3.10 was extensively used as a tool of proving the UISS property for autonomous time-delay systems in ref. [17]. Moreover, Sontag and Wang[15,16] formulated IOS for continuous-time finite-dimensional systems using an estimate of the form (3.1) with $\beta(t) \equiv \delta(t) \equiv 1$. On the other hand, estimates of the form (3.23) ("fading memory estimates") were first used by Praly and Wang[18] for the formulation of exp-ISS, and by Grune[19,20] for the formulation of input-to-state dynamical stability (ISDS) with $H(t,x) = x$, $\beta(t) \equiv \delta(t) \equiv 1$, which was proved to be qualitatively equivalent to (3.1) for finite-dimensional continuous-time systems.

The following theorem provides sufficient Lyapunov-like conditions for the (U)WIOS property. The proofs of implications (e)⇒(a) of Theorem 3.9 and (e)⇒(a) of Theorem 3.10 are based on the result of Theorem 3.12, which gives quantitative estimates of the solutions of (1.1) under hypotheses (S1–7). The gain functions and the weights of the WIOS property can be determined explicitly in terms of the functions involved in the assumptions of Theorem 3.12.

Theorem 3.12. Consider system (1.1) under hypotheses (S1–7) and suppose that there exists a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a, ζ of class K_∞, β, δ of class K^+ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that

$$V(t,x) \leqslant a(\beta(t)\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n), \quad (3.30)$$

$$V^0(t,x;f(t,x,u,d)) \leqslant -\rho(V(t,x)), \quad (3.31)$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$ with $\zeta(\delta(t)|u|) \leqslant V(t,x)$.

Moreover, suppose that one of the following holds:

a) system (1.1) is RFC from the input $u \in M_U$;

b) there exist functions $p \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $p(\mu(t)|x(0)|) \leqslant V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$.

Then, system (1.1) is RFC from the input $u \in M_U$ and there exists a function $\sigma \in KL$ with $\sigma(s,0) = s$ for all $s \geqslant 0$, such that for all $(d,u) \in M_D \times M_U$, $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$, satisfies the following estimate for all $t \geqslant t_0$:

$$V(t,T_r(t)x) \leqslant \max\{\sigma(a(\beta(t_0)\|x_0\|_r), t-t_0), \sup_{t_0 \leqslant \tau \leqslant t} \sigma(\zeta(\delta(\tau)|u(\tau)|), t-\tau)\}. \quad (3.32)$$

Finally,

(i) if there exists a function a_1 of class K_∞ such that $a_1(\|H(t,x)\|_{\mathcal{Y}}) \leqslant V(t,x)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, then system (1.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ. Moreover, if in addition it holds that $\beta(t) \equiv 1$, then system (1.1) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ;

(ii) if $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ and there exist functions a_1, a_2 of class K_∞ such that $a_1(|h(t,x(0))|) \leqslant V(t,x)$, $\|H(t,x)\|_{\mathcal{Y}} \leqslant a_2(\sup_{\theta\in[-r,0]}|h(t+\theta,x(\theta))|)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, then system (1.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_2(a_1^{-1}(\zeta(s)))$ and weight δ.

4 Razumikhin method for WIOS

Let $V : [-r,+\infty) \times \Re^n \to \Re$ be a locally Lipschitz mapping and let $(t,x,v) \in \Re^+ \times \Re^n \times \Re^n$. We define

$$D^+V(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h,x+hv) - V(t,x)}{h}. \quad (4.1)$$

The following proposition provides conditions in terms of Razumikhin functions for the (U)WIOS property. Its proof follows closely the methodology introduced in ref. [21].

Proposition 4.1. Consider system (1.1) under hypotheses (S1–7) and suppose that $H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$. Moreover, suppose that there exists a locally Lipschitz function $V : [-r,+\infty) \times \Re^n \to \Re^+$, functions a_1, a_2, a, ζ of class K_∞ with $a(s) < s$ for all $s > 0$, functions β, δ of class K^+ and a locally Lipschitz positive definite function ρ such that

$$a_1(|h(t-r,x)|) \leqslant V(t-r,x) \leqslant a_2(\beta(t)|x|), \quad \forall (t,x) \in \Re^+ \times \Re^n, \quad (4.2)$$

$$D^+V(t,x(0);f(t,x,u,d)) \leqslant -\rho(V(t,x(0))), \quad (4.3)$$

for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$ with

$$\max\left\{\zeta(\delta(t)|u|), a\left(\sup_{\theta\in[-r,0]} V(t+\theta,x(\theta))\right)\right\} \leqslant V(t,x(0)).$$

Finally, suppose that one of the following holds:

(i) system (1.1) is RFC from the input $u \in M_U$;

(ii) there exist functions $p \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $p(\mu(t)|x|) \leqslant V(t-r,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$.

Let $a_3 \in K_\infty$ be the function with the property $\|H(t,x)\|_{\mathcal{Y}} \leqslant a_3(\sup_{\theta\in[-r,0]}|h(t+\theta,x(\theta))|)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$. Then, system (1.1) satisfies the WIOS property with gain $\gamma(s) := a_3(a_1^{-1}(\zeta(s)))$ and weight δ. Moreover, if β is bounded, then, system (1.1) satisfies the UWIOS property from the input $u \in M_U$. Finally, if β, δ are bounded, then system (1.1) satisfies the UIOS property from the input $u \in M_U$.

The following corollary extends the classical Razumikhin theorem to systems with disturbances as well as to the case of output asymptotic stability.

Corollary 4.2. Consider system (1.1) under hypotheses (S1–7) with $U = \{0\}$ and suppose that

$H : \Re^+ \times C^0([-r,0];\Re^n) \to \mathcal{Y}$ is equivalent to the finite-dimensional mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$. Moreover, suppose that there exists a locally Lipschitz function $V : [-r,+\infty) \times \Re^n \to \Re^+$, functions a_1, a_2, a of class K_∞, with $a(s) < s$ for all $s > 0$, function β of class K^+ and a locally Lipschitz positive definite function ρ such that

$$a_1(|h(t-r,x)|) \leqslant V(t-r,x) \leqslant a_2(\beta(t)|x|),\quad \forall (t,x) \in \Re^+ \times \Re^n, \tag{4.4}$$

and

$$D^+V(t,x(0);f(t,x,0,d)) \leqslant -\rho(V(t,x(0))),$$

for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$ with

$$a(\sup_{\theta\in[-r,0]} V(t+\theta,x(\theta))) \leqslant V(t,x(0)). \tag{4.5}$$

Finally, suppose that one of the following holds:

(i) system (1.1) with $u \equiv 0$ is RFC;

(ii) there exist functions $\zeta \in K_\infty$, $\mu \in K^+$ and a constant $R \geqslant 0$ such that $\zeta(\mu(t)|x|) \leqslant V(t-r,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$.

Then, system (1.1) with $U = \{0\}$ is non-uniformly in time RGAOS. Moreover, if β is bounded, then system (1.1) with $U = \{0\}$ is URGAOS.

5 Necessary and sufficient conditions for robust global stabilization

In this section, we consider control systems of the form (1.1) under hypotheses (S1–7) and under the additional hypothesis:

(S0) The set $D \subset \Re^l$ is compact and $U \subseteq \Re^m$ is a closed convex set.

We next give the definition of the output robust control Lyapunov functional for system (1.1).

Definition 5.1. We say that (1.1) under hypotheses (S0–7) admits an output robust control Lyapunov functional (ORCLF) if there exists an (almost Lipschitz on bounded sets) functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ (called the output control Lyapunov functional), which satisfies the following properties:

(i) There exist functions $a_1, a_2 \in K_\infty$, $\beta, \mu \in K^+$ such that the following inequality holds for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$

$$\max\{a_1(\|H(t,x)\|_{\mathcal{Y}}), a_1(\mu(t)\|x\|_r)\} \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_r). \tag{5.1}$$

(ii) There exist a function $\Psi : \Re^+ \times \Re^p \times U \to \Re \bigcup \{+\infty\}$ with $\Psi(t,0,0) = 0$ for all $t \geqslant 0$ such that for each $u \in U$ the mapping $(t,\varphi) \to \Psi(t,\varphi,u)$ is upper semi-continuous, a function $q \in \mathcal{E}$, a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to \Phi(t,x) \in \Re^p$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $\Phi(t,0) = 0$ for all $t \geqslant 0$ and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$ such that the following inequality holds:

$$\inf_{u\in U} \Psi(t,\varphi,u) \leqslant q(t),\quad \forall t \geqslant 0, \forall \varphi = (\varphi_1,\ldots,\varphi_p)' \in \Re^p. \tag{5.2}$$

Moreover, for every finite set $\{u_1,u_2,\ldots,u_N\} \subset U$ and for every $\lambda_i \in [0,1] (i = 1,\ldots,N)$ with $\sum_{i=1}^N \lambda_i = 1$, it holds that

$$\sup_{d\in D} V^0\left(t,x;f\left(t,d,x,\sum_{i=1}^N \lambda_i u_i\right)\right) \leqslant -\rho(V(t,x)) + \max\{\Psi(t,\Phi(t,x),u_i), i = 1,\ldots,N\},\quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n). \tag{5.3}$$

If, in addition to the above, there exist $a \in K_\infty$, $\gamma \in K^+$ such that for every $(t,\varphi) \in \Re^+ \times \Re^p$ there exists $u \in U$ with $|u| \leqslant a(\gamma(t)|\varphi|)$, satisfying

$$\Psi(t,\varphi,u) \leqslant q(t), \tag{5.4}$$

then, we say that $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ satisfies the "small-control" property.

For the case when $H(t,x) \equiv x \in C^0([-r,0];\Re^n)$, we simply call $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ a state robust control Lyapunov functional (SRCLF).

It is clear that the notion of the robust control Lyapunov functional is a generalization of the notion of the robust control Lyapunov function for finite-dimensional systems (see ref. [22]). Particularly, in the finite-dimensional case the map Φ is the identity map, while in the affine in the control case

$$\dot{x} = f(t,d,x) + g(t,d,x)u,$$

the map Ψ is simply

$$\Psi(t,x,u) := \frac{\partial V}{\partial t}(t,x) + \sup_{d\in D} \frac{\partial V}{\partial x}(t,x)(f(t,d,x) + g(t,d,x)u).$$

In the infinite-dimensional case, the map Φ plays a crucial role, since it maps the infinite-dimensional space $C^0([-r,0];\Re^n)$ to the finite-dimensional space $\Re^p$, where partition of unity arguments can be used for the construction of the feedback stabilizer (see Figure 1).

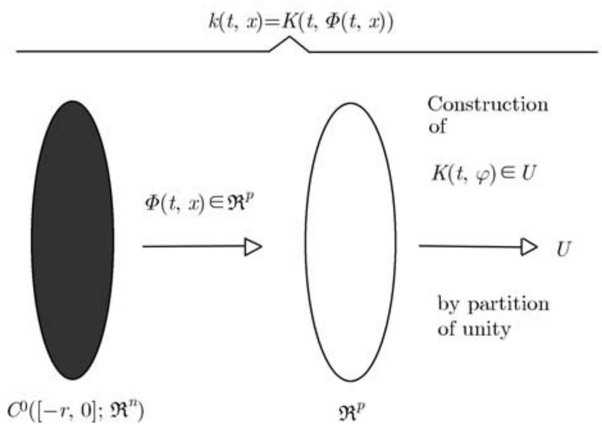

Figure 1 The main idea in the proof of Theorems 5.2 and 5.3.

We are now in a position to state our main results for the infinite-dimensional case (1.1) (see ref. [28]).

Theorem 5.2. Consider system (1.1) under hypotheses (S0–7). The following statements are equivalent:

(a) There exists a continuous mapping

$$\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$$

being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \geqslant 0$, such that the closed-loop system (1.1) with $u = k(t, T_r(t)x)$ is RGAOS.

(b) System (1.1) admits an ORCLF, which satisfies the small control property with $q(t) \equiv 0$.

(c) System (1.1) admits an ORCLF.

Theorem 5.3. Consider system (1.1) under hypotheses (S0–7). The following statements are equivalent:

(a) System (1.1) admits an ORCLF, which satisfies the small-control property and inequalities (5.1), (5.4) with $\beta(t) \equiv 1$, $q(t) \equiv 0$. Moreover, there exist continuous mappings $\eta \in K^+$, $A \ni (t,\varphi) \to K(t,\varphi) \in U$ where $A = \bigcup_{t\geqslant 0}\{t\} \times \{\varphi \in \Re^p : |\varphi| < 4\eta(t)\}$ being locally Lipschitz with respect to φ with $K(t,0) = 0$ for all $t \geqslant 0$ and such that

$$\Psi(t, \Phi(t,x), K(t, \Phi(t,x))) \leqslant 0 \qquad (5.5)$$

for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ with $|\Phi(t,x)| \leqslant 2\eta(t)$, where $\Phi = (\Phi_1,\ldots,\Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ and $\Psi : \Re^+ \times \Re^p \times U \to \Re\bigcup\{+\infty\}$ are the mappings involved in property (ii) of Definition 5.1.

(b) There exists a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \geqslant 0$, such that the closed-loop system (1.1) with $u = k(t, T_r(t)x)$ is URGAOS.

Remark 5.4. From the proof of Theorem 5.3 it becomes apparent that if statement (a) of Theorem 5.3 is strengthened so that the ORCLF V, the mappings $\Phi = (\Phi_1,\ldots,\Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$, Ψ involved in property (ii) of Definition 5.1 and the mapping $K : A \to U$ are time-independent, then the continuous mapping k, whose existence is guaranteed by statement (b) of Theorem 5.3, is time-invariant.

We next show that our generalized RCLF methodology is more than an existence-type result, but can yield constructive design tools for an enlarged class of nonlinear control systems. To this end, we will study in detail a class of triangular time-delay nonlinear systems described by RFDEs, i.e.,

$$\begin{aligned}
\dot{x}_i(t) &= f_i(t, d(t), T_r(t)x_1, \ldots, T_r(t)x_i) \\
&\quad + g_i(t, d(t), T_r(t)x_1, \ldots, T_r(t)x_i)x_{i+1}(t), \\
\dot{x}_n(t) &= f_n(t, d(t), T_r(t)x) + g_n(t, d(t), T_r(t)x)u(t), \\
x(t) &= (x_1(t), \ldots, x_n(t)) \in \Re^n, \qquad (5.6)
\end{aligned}$$

where $d(t) \in D, u(t) \in \Re, t \geqslant 0, i = 1, \ldots, n-1$.

Autonomous and disturbance-free systems of the form (5.6) have been studied in refs. [23–26]. In the present work it is shown that the construction of a stabilizing feedback law for (5.6) proceeds in parallel with the construction of a state robust control Lyapunov functional. Moreover, sufficient conditions for the existence and design of a stabilizing feedback law $u(t) = k(x(t))$, which is independent of the delay, are given below.

Our main result concerning triangular time-delay control systems of the form (5.6) is stated next. It must be compared to Theorem 5.1 in ref. [22], which deals with the triangular finite-dimensional case.

Theorem 5.5. Consider system (5.6), where $r > 0$, $D \subset \Re^l$ is a compact set, the mappings $f_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$, $g_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$ $(i = 1,\dots,n)$ are continuous with $f_i(t,d,0) = 0$ for all $(t,d) \in \Re^+ \times D$ and each $g_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$ $(i = 1,\dots,n)$ is completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^i)$. Suppose that there exists a function $\varphi \in C^\infty(\Re^+;(0,+\infty))$ being non-decreasing, such that for every $i = 1,\dots,n$, it holds that

$$\frac{1}{\varphi(\|x\|_r)} \leqslant g_i(t,d,x) \leqslant \varphi(\|x\|_r), \quad \forall (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^i) \times D. \quad (5.7)$$

Moreover, suppose that for every $i = 1,\dots,n$, it holds that

$$\sup\left\{\frac{|f_i(t,d,x) - f_i(t,d,y)|}{\|x-y\|_r} : (t,d) \in \Re^+ \times D, x \in S, y \in S, x \neq y\right\} < +\infty \quad (5.8)$$

for every bounded $S \subset C^0([-r,0];\Re^i)$.

Then, for every $\sigma > 0$ there exist functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $k_i \in C^\infty(\Re^i;\Re)$ $(i = 1,\dots,n)$ with

$$k_1(\xi_1) := -\mu_1(\xi_1)\xi_1, \quad (5.9a)$$

$$k_j(\xi_1,\dots,\xi_j) := -\mu_j(\xi_1,\dots,\xi_j) \cdot (\xi_j - k_{j-1}(\xi_1,\dots,\xi_{j-1})), \quad (5.9b)$$

for $j = 2,\dots,n$,

such that the following functional

$$V(x) := \max_{\theta\in[-r,0]} \exp(2\sigma\theta)\left(x_1^2(\theta) + \sum_{j=2}^{n} |x_j(\theta) - k_{j-1}(x_1(\theta),\dots,x_{j-1}(\theta))|^2\right) \quad (5.10)$$

is a state robust control Lyapunov functional (SRCLF) for (5.6), which satisfies the "small-control" property. Moreover, the closed-loop system (5.6) with $u(t) = k_n(x(t))$ is URGAS. More specifically, the inequality $V^0(x;v) \leqslant -2\sigma V(x)$ holds for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$ with $v = (f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0),\dots,f_n(t,d,x) + g_n(t,d,x)k_n(x(0)))' \in \Re^n$.

Remark 5.6. The reader should notice that the feedback law $u(t) = k_n(x(t))$ is delay-independent. The proof of Theorem 5.5 shows that the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$ $(i = 1,\dots,n)$ are obtained by a procedure similar to the backstepping procedure used for finite-dimensional triangular control systems. Consequently, as in the finite-dimensional case, the feedback design and the construction of the state robust control Lyapunov functional proceed in parallel. Indeed, inequality (5.8) in conjunction with the fact that $f_i(t,d,0) = 0$ for all $(t,d) \in \Re^+ \times D$ $(i = 1,\dots,n)$ implies the existence of a non-decreasing function $L \in C^\infty(\Re^+;(0,+\infty))$ such that for every $i = 1,\dots,n$, it holds that

$$|f_i(t,d,x)| \leqslant L(\|x\|_r)\|x\|_r, \quad \forall (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^i) \times D. \quad (5.11)$$

Let $\sigma > 0$ be a given number. We next define functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $\gamma_i \in C^\infty(\Re^+;(0,+\infty))$, $b_i \in C^\infty(\Re^+;(0,+\infty))$ $(i = 1,\dots,n)$ using the following algorithm (see ref. [28]).

Algorithm:

Step $i = 1$: We define

$$\mu_1(\xi_1) := \frac{\gamma_1(1+\xi_1^2) + n\sigma}{b_1(1+\xi_1^2)}, \quad (5.12)$$

where

$$\gamma_1(s) := \exp(\sigma r)L(s\exp(\sigma r)) + \varphi(s\exp(\sigma r)), \quad (5.13a)$$

$$b_1(s) := \frac{1}{\varphi(s\exp(\sigma r))}. \quad (5.13b)$$

Step $i \geqslant 2$: Based on the knowledge of the functions $\mu_j \in C^\infty(\Re^j;(0,+\infty))$ $(j = 1,\dots,i-1)$ from previous steps, we can define the function $\mu_i \in C^\infty(\Re^i;(0,+\infty))$. First, for each $1 \leqslant j \leqslant i$, set

$$k_0 \equiv 0, \quad k_1(\xi_1) := -\mu_1(\xi_1)\xi_1, \quad (5.14a)$$

$$k_j(\xi_1,\dots,\xi_j) := -\mu_j(\xi_1,\dots,\xi_j) \cdot (\xi_j - k_{j-1}(\xi_1,\dots,\xi_{j-1})), \quad j = 2,\dots,i-1, \quad (5.14b)$$

$$\gamma_j(s) := \exp(\sigma r)L(s\exp(\sigma r)B_j(s\exp(\sigma r))) \times B_j(s\exp(\sigma r))$$

$$+ \varphi(s\exp(\sigma r)B_j(s\exp(\sigma r))),$$
$$j = 1, \ldots, i. \tag{5.15a}$$
$$b_j(s) := \frac{1}{\varphi(s\exp(\sigma r)B_j(s\exp(\sigma r)))}, \tag{5.15b}$$
where $B_j \in C^\infty(\Re^+; (0, +\infty))(j = 1, \ldots, i)$ are non-decreasing functions that satisfy

$$B_1(s) := 1,$$
$$B_j(s) \geqslant \max\left\{1 + \sum_{l=1}^{j-1} \mu_l(\xi_1, \ldots, \xi_l) : \max_{l=1,\ldots,j} |\xi_l - k_{l-1}(\xi_1, \ldots, \xi_{l-1})| \leqslant s\right\} \tag{5.15c}$$
for all $s \geqslant 0$ and $j \geqslant 2$.

Let $\rho_j \in C^\infty(\Re^+; (0, +\infty))(j = 1, \ldots, i)$ and $\delta_j \in C^\infty(\Re^j; (0, +\infty))(j = 0, \ldots, i-1)$ be functions such that the following inequalities hold:
$$b_j(s') - b_j(s) + s\gamma_j(s) - s'\gamma_j(s') \leqslant (s - s')\rho_j(s), \quad \forall s \geqslant s' \geqslant 0, \tag{5.16}$$
$$\delta_j(\xi_1, \ldots, \xi_j) \geqslant |\nabla k_j(\xi_1, \ldots, \xi_j)|(1 + \mu_1(\xi_1) + \ldots + \mu_j(\xi_1, \ldots, \xi_j)), \quad \forall (\xi_1, \ldots, \xi_j) \in \Re^j. \tag{5.17}$$
Now, let us define
$$\mu_i(\xi_1, \ldots, \xi_i) := b_i^{-1}(p)\Bigg[(n + 1 - i)\sigma + \frac{i-1}{4\sigma}a^2(p, \xi_1, \ldots, \xi_{j-1}) + \gamma_i(p) + c_{i-1}(p)\, \delta_{i-1}(\xi_1, \ldots, \xi_{j-1})\Bigg], \tag{5.18a}$$
where
$$p := \frac{i}{2} + \frac{1}{2}\sum_{j=1}^{i} |\xi_j - k_{j-1}(\xi_1, \ldots, \xi_{j-1})|^2, \tag{5.18b}$$
$c_0 \equiv 0$, $c_j(s) := \sum_{k=1}^{j} \gamma_k(s)$, for $j = 1, \ldots, i$,
$$a(s, \xi_1, \ldots, \xi_{i-1}) := c_{i-1}(s)\delta_{i-1}(\xi_1, \ldots, \xi_{i-1}) + c_i(s) + \left(1 + \sum_{j=1}^{i-1}(s\mu_j(\xi_1, \ldots, \xi_j) + \delta_{j-1}(\xi_1, \ldots, \xi_{j-1}))\right)\sum_{k=1}^{i-1}\rho_k(s). \tag{5.18c}$$

It should be noticed that in every step $i \geqslant 2$ of the above algorithm we only need to compute the functions $\gamma_i(s)$, $b_i(s)$, $B_i(s)$, $\rho_{i-1}(s)$, $\delta_{i-1}(\xi_1, \ldots, \xi_{i-1})$ and $\mu_i(\xi_1, \ldots, \xi_i)$ (the functions $\gamma_j(s)$, $b_j(s)$, $B_j(s)$, $\rho_{j-1}(s)$, $\delta_{j-1}(\xi_1, \ldots, \xi_{j-1})$ and $\mu_j(\xi_1, \ldots, \xi_j)$ for $j = 1, \ldots, i-1$ have been computed in the previous steps).

The proof of Theorem 5.5 is based on the following lemma. The reader should notice that Lemma 5.7 in conjunction with definition (5.10) of the SRCLF for system (5.6) indicate one of the complications frequently encountered in the study of infinite-dimensional systems. Namely, although the differential equations (5.6) are affine in the control input $u \in \Re$, the derivative $V^0(x; v)$, where $v = (f_1(t, d, x_1) + g_1(t, d, x_1)x_2(0), \ldots, f_n(t, d, x) + g_n(t, d, x)u)' \in \Re^n$, $x = (x_1, \ldots, x_n) \in C^0([-r, 0]; \Re^n)$ is not affine in the control input $u \in \Re$.

Lemma 5.7. Let $Q \in C^1(\Re^n; \Re^+)$, $\sigma > 0$ and consider the functional $V : C^0([-r, 0]; \Re^n) \to \Re^+$ defined by
$$V(x) := \max_{\theta \in [-r, 0]} \exp(2\sigma\theta)Q(x(\theta)). \tag{5.19}$$
The functional $V : C^0([-r, 0]; \Re^n) \to \Re^+$ defined by (5.19) is Lipschitz on bounded sets of $C^0([-r, 0]; \Re^n)$ and enjoys the following facts:
$$V^0(x; v) \leqslant -2\sigma V(x) \tag{5.20a}$$
for all $(x, v) \in C^0([-r, 0]; \Re^n) \times \Re^n$ with $Q(x(0)) < V(x)$, and
$$V^0(x; v) \leqslant \max\{-2\sigma V(x), \nabla Q(x(0))v\} \tag{5.20b}$$
for all $(x, v) \in C^0([-r, 0]; \Re^n) \times \Re^n$ with $Q(x(0)) = V(x)$.

6 Coupled systems

In this section, we consider coupled systems of the form (1.2) with initial conditions $x_1(t_0 + \theta) = x_{10}(\theta); \theta \in [-r_1, 0]$ and $x_2(t_0 + \theta) = x_{20}(\theta)$; $\theta \in [-r_2, 0]$ with $x_{10} \in C^0([-r_1, 0]; \Re^{n_1})$, $x_{20} \in L^\infty([-r_2, 0]; \Re^{n_2})$, under the following hypotheses:

(P1) The function $\tau : \Re^+ \to (0, +\infty)$ is continuous with $\sup_{t \geqslant 0} \tau(t) \leqslant r_2$.

(P2) There exist functions $a \in K_\infty$, $\beta \in K^+$ such that
$$|f_i(t, d, x_1, T_{r_2-\tau(t)}(-\tau(t))x_2, u)| \leqslant a(\beta(t)\|T_{r_2-\tau(t)}(-\tau(t))x_2\|_{r_2-\tau(t)}) + a(\beta(t)|u|) + a(\beta(t)\|x_1\|_{r_1})$$

for each $i = 1,2$, and for all $(t,d,x_1,x_2,u) \in \Re^+ \times D \times C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2}) \times U$.

(P3) For every $x_1 \in C^0([-r_1,+\infty);\Re^{n_1})$, $d \in L^\infty_{\rm loc}(\Re^+;D)$, $u \in L^\infty_{\rm loc}(\Re^+;U)$ and $x_2 \in L^\infty_{\rm loc}([-r_2,+\infty);\Re^{n_2})$ the mappings $t \to f_i(t,d(t),T_{r_1}(t)x_1, T_{r_2-\tau(t)}(t-\tau(t))x_2,u(t))$, $i=1,2$ are measurable. Moreover, for each fixed $(t,d,x_2,u) \in \Re^+ \times D \times L^\infty([-r_2,0];\Re^{n_2}) \times U$ the mapping $f_1(t,d,x_1,T_{r_2-\tau(t)}(-\tau(t))x_2,u)$ is continuous with respect to $x_1 \in C^0([-r_1,0];\Re^{n_1})$.

(P4) For every pair of bounded sets $I \subset \Re^+$ and $\Omega \subset C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2}) \times U$, there exists $L := L(I,\Omega) \geqslant 0$ such that

$$\begin{aligned}&(x_1(0)-y_1(0))'(f_1(t,d,x_1,T_{r_2-\tau(t)}(-\tau(t))x_2,u)\\ &\qquad - f_1(t,d,y_1,T_{r_2-\tau(t)}(-\tau(t))x_2,u))\\ &\leqslant L\|x_1-y_1\|^2_{r_1}, \forall (t,d) \in I \times D,\\ &\qquad \forall (x_1,x_2,u) \in \Omega, \forall (y_1,x_2,u) \in \Omega. \qquad (6.1)\end{aligned}$$

(P5) The mapping $H: \Re^+ \times C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2}) \times U \to \mathcal{Y}$ is continuous with $H(t,0,0,0) = 0$ for all $t \geqslant 0$. Moreover, the image set $H(\Omega)$ is bounded for each bounded set $\Omega \subset \Re^+ \times C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2}) \times U$.

The reader should notice that hypothesis (P1) and the fact that $\tau(t) > 0$ for all $t \geqslant 0$, guarantee that eq. (1.2b) is a functional difference equation. It should be pointed out that hypotheses (P1), (P2), (P3) are satisfied if $D \subset \Re^l$ is compact and there exist continuous functions $\tau_i : \Re^+ \to (0,+\infty)$ $(i=1,\ldots,p)$, $\tau: \Re^+ \to (0,+\infty)$ with $\tau(t) \leqslant \tau_1(t) < \tau_2(t) < \ldots < \tau_p(t) \leqslant r_2$ for all $t \geqslant 0$, continuous mappings $g_i : \Re^+ \times D \times C^0([-r_1,0];\Re^{n_1}) \times \Re^{pn_2} \times \Re^k \times U \to \Re^{n_i}$, $i=1,2$, $h: \Re^+ \times [-r_2,0] \times \Re^{n_2} \to \Re^k$ with $g_i(t,d,0,0,0,0) = 0$, $h(t,\theta,0) = 0$ for all $(t,\theta,d) \in \Re^+ \times [-r-T,0] \times D$, such that

$$\begin{aligned}&f_i(t,d,x_1,T_{r_2-\tau(t)}(-\tau(t))x_2,u)\\ &\quad = g_i\Bigg(t,d,x_1,x_2(-\tau_1(t)),x_2(-\tau_2(t)),\ldots,\\ &\qquad x_2(-\tau_p(t)),\int_{-r_2}^{-\tau(t)} h(t,\theta,x_2(\theta))d\theta,u\Bigg),\\ &\qquad i=1,2,\end{aligned}$$

for all $(t,d,x_1,x_2,u) \in \Re^+ \times D \times C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2}) \times U$.

We start with an existence-uniqueness-continuation theorem for the solution of (1.2). We say that a mapping $x: [a,b) \to \Re^n$ with $-\infty < a < b \leqslant +\infty$ is absolutely continuous on $[a,b)$ if for every $c \in (a,b)$ the mapping $x: [a,b) \to \Re^n$ is absolutely continuous on $[a,c]$.

Theorem 6.1. Consider system (1.2) under hypotheses (P1–4). Then for every $t_0 \geqslant 0$, $(x_{10},x_{20}) \in C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2})$, $d \in L^\infty_{\rm loc}(\Re^+;D)$, $u \in L^\infty_{\rm loc}(\Re^+;U)$ there exists $t_{\max} \in (t_0,+\infty]$ and a unique pair of mappings $x_1 \in C^0([t_0-r_1,t_{\max});\Re^{n_1})$, $x_2 \in L^\infty_{\rm loc}([t_0-r_2,t_{\max});\Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, $x_1 \in C^0([t_0-r_1,t_{\max});\Re^{n_1})$ being absolutely continuous on $[t_0,t_{\max})$ such that (1.2a) holds a.e. for $t \in [t_0,t_{\max})$ and (1.2b) holds for all $t \in (t_0,t_{\max})$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0,t_{\max})$ with $\|T_{r_1}(t)x_1\|_{r_1} > M$.

Theorem 6.1 guarantees that $t_{\max} \in (t_0,+\infty]$ is the maximal existence time for the solution of (1.2). The idea behind the proof of Theorem 6.1 is the method of steps, used already in ref. [3].

Remark 6.2. According to Theorem 6.1 above, Definitions 2.1 and 2.4 in ref. [9], system (1.2) under hypotheses (P1–5) is a control system $\Sigma := (\mathcal{X},\mathcal{Y},M_U,M_D,\phi,\pi,H)$ with outputs that satisfies the "Boundedness-Implies-Continuation" property (BIC property in refs. [6–9]) with state space $\mathcal{X} = C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2})$, output space $\mathcal{Y}$, set of allowable control inputs $M_U = L^\infty_{\rm loc}(\Re^+;U)$, set of allowable disturbances $M_D = L^\infty_{\rm loc}(\Re^+;D)$ and set of sampling times $\pi(t_0,x_0,u,d) = [t_0,t_{\max})$, where $t_{\max} > t_0$ is the maximal existence time of the solution. Moreover, if a finite escape time occurs then the component x_1 of the solution of (1.2) must be unbounded (but x_2 may or may not be unbounded).

The following theorem guarantees that $(0,0) \in C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2})$ is a robust equilibrium point from the input u (in the sense of Definition 2.6 in ref. [9]) for system (1.2) under hypotheses (P1–4).

Theorem 6.3. Consider system (1.2) under hypotheses (P1–4). Then for every $\varepsilon > 0$, $T,h \in \Re^+$ there exists $\delta := \delta(\varepsilon,T,h) > 0$ such that for all $(t_0,x_{10},x_{20}) \in [0,T] \times C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2})$, $(u,d) \in L^\infty_{\rm loc}(\Re^+;U) \times L^\infty_{\rm loc}(\Re^+;D)$ with $\|x_{10}\|_{r_1} + \|x_{20}\|_{r_2} +$

$\sup_{t\geqslant 0}|u(t)| < \delta$ there exists $t_{\max} \in (t_0 + h, +\infty]$ and a unique pair of mappings $x_1 \in C^0([t_0-r_1, t_{\max}); \Re^{n_1})$, $x_2 \in L^\infty_{\rm loc}([t_0-r_2, t_{\max}); \Re^{n_2})$ with $T_{r_1}(t_0)x_1 = x_{10}$, $T_{r_2}(t_0)x_2 = x_{20}$, $x_1 \in C^0([t_0 - r_1, t_{\max}); \Re^{n_1})$ being absolutely continuous on $[t_0, t_{\max})$, such that (1.2a) holds a.e. for $t \in [t_0, t_{\max})$, (1.2b) holds for all $t \in (t_0, t_{\max})$ and

$$\sup\{\|T_{r_1}(t)x_1\|_{r_1} + \|T_{r_2}(t)x_2\|_{r_2};\\ t \in [t_0, t_0 + h]\} < \varepsilon. \tag{6.2}$$

Remark 6.4. It should be emphasized that Theorems 6.1 and 6.3 guarantee that all stability results obtained in refs. [6–9] for general control systems with the "Boundedness Implies Continuation" property hold as well for system (1.2) under hypotheses (P1–5). This implication enables us to obtain the stability results of the following section.

Remark 6.5. It is important to notice that Theorems 6.1 and 6.3 can be applied to systems described by FDEs of the form

$$x(t) = f(t, d(t), T_{r-\tau(t)}(t - \tau(t))x, u(t)),\\ x(t) \in \Re^n, d(t) \in D, u(t) \in U, t \geqslant 0, \tag{6.3}$$

where $D \subseteq \Re^l$ is a non-empty set, $U \subseteq \Re^m$ is a non-empty set with $0 \in U$, $r > 0$, $f : \Re^+ \times D \times L^\infty([-r+\tau(t), 0]; \Re^n) \times U \to \Re^n$, under the following hypotheses:

(Q1) The function $\tau : \Re^+ \to (0, +\infty)$ is continuous with $\sup_{t\geqslant 0} \tau(t) \leqslant r$.

(Q2) There exist functions $a \in K_\infty$, $\beta \in K^+$ such that

$$|f(t, d, T_{r-\tau(t)}(-\tau(t))x, u)|\\ \leqslant a(\beta(t)\|T_{r-\tau(t)}(-\tau(t))x\|_{r-\tau(t)}) + a(\beta(t)|u|),$$

for all $(t, d, x, u) \in \Re^+ \times D \times L^\infty([-r, 0]; \Re^n) \times U$.

(Q3) For every $d \in L^\infty_{\rm loc}(\Re^+; D)$, $u \in L^\infty_{\rm loc}(\Re^+; U)$ and $x \in L^\infty_{\rm loc}([-r, +\infty); \Re^n)$ the mapping $t \to f(t, d(t), T_{r-\tau(t)}(t - \tau(t))x, u(t))$ is measurable.

Indeed, system (6.3) can be embedded into the following system described by coupled RFDEs and FDEs:

$$\dot\xi(t) = 0,\\ x(t) = f(t, d(t), T_{r-\tau(t)}(t - \tau(t))x, u(t)),\\ \xi(t) \in \Re, x(t) \in \Re^n, d(t) \in D, u(t) \in U,\\ t \geqslant 0, \tag{6.4}$$

which is a system of the form (1.2) that satisfies hypotheses (P1–4). Consequently, Theorems 6.1 and 6.3 can be applied to system (6.4) and we obtain

Corollary 6.6. Consider system (6.3) under hypotheses (Q1–3). Then for every $t_0 \geqslant 0$, $x_0 \in L^\infty([-r, 0]; \Re^n)$, $(u, d) \in L^\infty_{\rm loc}(\Re^+; U) \times L^\infty_{\rm loc}(\Re^+; D)$ there exists a unique mapping $x \in L^\infty_{\rm loc}([t_0 - r, +\infty); \Re^n)$ with $T_r(t_0)x = x_0$, such that (6.3) holds for all $t > t_0$. Moreover, for every $\varepsilon > 0$, $T, h \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x_0) \in [0, T] \times L^\infty([-r, 0]; \Re^n)$, $(u, d) \in L^\infty_{\rm loc}(\Re^+; U) \times L^\infty_{\rm loc}(\Re^+; D)$ with $\|x_0\|_r + \sup_{t\geqslant 0}|u(t)| < \delta$ the solution $x(t)$ of (6.3) with initial condition $T_r(t_0)x = x_0$, corresponding to inputs $(u, d) \in L^\infty_{\rm loc}(\Re^+; U) \times L^\infty_{\rm loc}(\Re^+; D)$ satisfies $\sup\{\|T_r(t)x\|_r; t \in [t_0, t_0 + h]\} < \varepsilon$.

7 Stability results for coupled systems

In this section, we present stability results for a wide class of systems described by coupled RFDEs and FDEs. Particularly, we consider the following class of systems described by coupled RFDEs and FDEs:

$$\dot x_1(t) = f_1(t, d(t), T_{r_1}(t)x_1, u(t),\\ H_2(t, T_{r_2-\tau(t)}(t - \tau(t))x_2)), \tag{7.1a}$$

$$x_2(t) = f_2(t, d(t), T_{r_2-\tau(t)}(t - \tau(t))x_2,\\ u(t), H_1(t, T_{r_1}(t)x_1)), \tag{7.1b}$$

$$x_1(t) \in \Re^{n_1}, x_2(t) \in \Re^{n_2}, u(t) \in U, d(t) \in D, t \geqslant 0,$$

$$Y(t) = H(t, T_{r_1}(t)x_1, T_{r_2}(t)x_2) \in \mathcal{Y}, \tag{7.1c}$$

where $r_1 \geqslant 0$, $r_2 > 0$, $D \subseteq \Re^l$ a non-empty set, $U \subseteq \Re^m$ a non-empty set with $0 \in U$, $\mathcal{Y}$ is a normed linear space, $H_1 : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \to S_1$, $H_2 : \cup_{t\geqslant 0}\{t\} \times L^\infty([-r_2 + \tau(t), 0]; \Re^{n_2}) \to S_2$, $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty([-r_2, 0]; \Re^{n_2}) \to \mathcal{Y}$ are continuous mappings, $S_1 \subseteq \Re^{k_1}$, $S_2 \subseteq \Re^{k_2}$ are sets with $0 \in S_1$, $0 \in S_2$ and the mappings $f_1 : \Re^+ \times D \times C^0([-r_1, 0]; \Re^{n_1}) \times U \times S_2 \to \Re^{n_1}$, $f_2 : \cup_{t\geqslant 0}\{t\} \times D \times L^\infty_{\rm loc}([-r_2 - \tau(t), 0]; \Re^{n_2}) \times U \times S_1 \to \Re^{n_2}$ are locally bounded mappings, which satisfy the following hypotheses:

(R1) The function $\tau : \Re^+ \to (0, +\infty)$ is continuous with $\sup_{t\geqslant 0} \tau(t) \leqslant r_2$.

(R2) For every $v \in L^\infty_{\rm loc}(\Re^+; S_1)$, $d \in L^\infty_{\rm loc}(\Re^+; D)$, $u \in L^\infty_{\rm loc}(\Re^+; U)$ and $x_2 \in L^\infty_{\rm loc}([-r_2, +\infty); \Re^{n_2})$

the mapping $t \to f_2(t, d(t), T_{r_2-\tau(t)}(t-\tau(t))x_2, u(t), v(t))$ is measurable.

(R3) The output map $H_1 : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \to S_1$, is a continuous mapping that maps bounded sets of $\Re^+ \times C^0([-r_1, 0]; \Re^{n_1})$ into bounded sets of $\Re^{k_1}$ with $H_1(t, 0) = 0$ for all $t \geqslant 0$.

(R4) $H_2 : \cup_{t\geqslant 0}\{t\} \times L^\infty([-r_2+\tau(t), 0]; \Re^{n_2}) \to S_2$ is a continuous mapping that maps bounded subsets of $\cup_{t\geqslant 0}\{t\} \times L^\infty([-r_2+\tau(t), 0]; \Re^{n_2})$ into bounded sets of $\Re^{k_2}$ with $H_2(t, 0) = 0$ for all $t \geqslant 0$. Moreover, for every $x_2 \in L^\infty_{\rm loc}([-r_2, +\infty); \Re^{n_2})$ the mapping $t \to H_2(t, T_{r_2-\tau(t)}(t-\tau(t))x_2)$ is measurable.

(R5) There exist functions $\beta \in K^+$, $a \in K_\infty$ such that

$$
\begin{aligned}
&|f_2(t, d, T_{r_2-\tau(t)}(-\tau(t))x_2, u, v)| \\
&\quad\leqslant a\left(\beta(t) \sup_{\theta\in[-r_2,-\tau(t)]} |x_2(\theta)|\right) \\
&\qquad + a(\beta(t)|u|) + a(\beta(t)|v|)
\end{aligned}
$$

for all $(t, x_2, u, v, d) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}) \times U \times S_1 \times D$, and

$$
\begin{aligned}
&|f_1(t, d, x, u, v)| \\
&\quad\leqslant a(\beta(t)\|x\|_{r_1}) + a(\beta(t)|u|) + a(\beta(t)|v|)
\end{aligned}
$$

for all $(t, x, u, v, d) \in \Re^+ \times C^0([-r_1, 0]; \Re^n) \times U \times S_2 \times D$.

(R6) The mapping $(x, u, v, d) \to f_1(t, d, x, u, v)$ is continuous for each fixed $t \geqslant 0$ and such that for every bounded $I \subseteq \Re^+$ and for every bounded $\Omega \subset C^0([-r_1, 0]; \Re^n) \times U \times S_2$, there exists a constant $L \geqslant 0$ such that

$$
\begin{aligned}
&(x(0) - y(0))'(f_1(t, d, x, u, v) - f_1(t, d, y, u, v)) \\
&\quad\leqslant L \max_{\tau\in[-r_1,0]} |x(\tau) - y(\tau)|^2, \\
&\qquad\qquad \forall t \in I, \forall (x, u, v, y, u, v) \in \Omega \times \Omega, \forall d \in D.
\end{aligned}
$$

(R7) There exists a countable set $A \subset \Re^+$, which is either finite or $A = \{t_k; k = 1, \ldots, \infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1, 2, \ldots$ and $\lim t_k = +\infty$, such that mapping $(t, x, u, v, d) \in (\Re^+\backslash A) \times C^0([-r, 0]; \Re^n) \times U \times S_2 \times D \to f_1(t, d, x, u, v)$ is continuous. Moreover, for each fixed $(t_0, x, u, v, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times S_2 \times D$, we have $\lim_{t\to t_0^+} f_1(t, d, x, u, v) = f_1(t_0, d, x, u, v)$.

(R8) The mapping $H : \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty([-r_2, 0]; \Re^{n_2}) \to \mathcal{Y}$ is continuous with $H(t, 0, 0) = 0$ for all $t \geqslant 0$. Moreover, the image set $H(\Omega)$ is bounded for each bounded set $\Omega \subset \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty([-r_2, 0]; \Re^{n_2})$.

By virtue of Lemma 3.2 in ref. [6] and Lemma 1 in ref. [27], it follows that system (7.1) under hypotheses (R1–8) is a system of the form (1.2) which satisfies hypotheses (P1–5). However, it should be emphasized that not every system of the form (1.2) can be expressed in the form (7.1). Next, we consider the following systems

$$
\begin{aligned}
&\dot{x}_1(t) = f_1(t, d(t), T_{r_1}(t)x_1, u(t), v_1(t)), \\
&Y_1(t) = H_1(t, T_{r_1}(t)x_1), \\
&x_1(t) \in \Re^{n_1}, Y_1(t) \in S_1, \\
&(u(t), v_1(t)) \in U \times S_2, d(t) \in D, t \geqslant 0,
\end{aligned}
\tag{7.2}
$$

which is a system described by RFDEs, and the following system described by FDEs:

$$
\begin{aligned}
&x_2(t) = f_2(t, d(t), T_{r_2-\tau(t)}(t-\tau(t))x_2, u(t), v_2(t)), \\
&Y_2(t) = H_2(t, T_{r_2-\tau(t)}(t-\tau(t))x_2), \\
&x_2(t) \in \Re^{n_2}, Y_2(t) \in S_2, (u(t), v_2(t)) \in U \times S_1, \\
&d(t) \in D, t \geqslant 0.
\end{aligned}
\tag{7.3}
$$

The following things can be remarked for systems (7.2) and (7.3).

• The theory of retarded functional differential equations guarantees that under hypotheses (R3–7), for each $(t_0, x_{10}) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1})$ and for each triple of measurable and locally bounded inputs $v_1 \in L^\infty_{loc}(\Re^+; S_2)$, $d \in L^\infty_{loc}(\Re^+; D)$, $u \in L^\infty_{loc}(\Re^+; U)$ there exists a unique absolutely continuous mapping $x_1(t)$ that satisfies a.e. the differential equation (7.2) with initial condition $T_{r_1}(t_0)x_1 = x_{10} \in C^0([-r_1, 0]; \Re^{n_1})$ (see refs. [1, 11]). Moreover, Theorem 3.2 in ref. [1] guarantees that (7.2) is a control system $\Sigma_1 := (C^0([-r_1, 0]; \Re^{n_1}), \Re^{k_1}, M_{U\times S_2}, M_D, \phi, \pi, H_1)$ with outputs that satisfies the Boundedness Implies Continuation property with $M_{U\times S_2}, M_D$ the sets of all measurable and locally bounded mappings $(u, v) : \Re^+ \to U \times S_2$, $d : \Re^+ \to D$, respectively (in the sense described in refs. [6–9]). Furthermore, the classical semigroup property is satisfied for this system, i.e., we have $\pi(t_0, x_0, u, d) = [t_0, t_{\max})$, where $t_{\max} > t_0$ is the maximal existence time of the solution. Finally, hypotheses (R3–7) guarantee that $0 \in C^0([-r_1, 0]; \Re^{n_1})$ is a robust equilibrium point from the input $(u,v) \in M_{U\times S_2}$ for Σ_1.

• Hypotheses (R1–5) guarantee that for each $(t_0, x_{20}) \in \Re^+ \times \mathcal{X}$ with $\mathcal{X} := L^\infty([-r_2, 0]; \Re^{n_2})$ and for each triple $v_2 \in L^\infty_{\mathrm{loc}}(\Re^+; S_1)$, $d \in L^\infty_{\mathrm{loc}}(\Re^+; D)$, $u \in L^\infty_{\mathrm{loc}}(\Re^+; U)$ there exists a unique measurable and locally bounded mapping $x_2(t)$ that satisfies the difference equations (7.3) for all $t > t_0$ with initial condition $x_2(t_0 + \theta) = x_{20}(\theta); \theta \in [-r_2, 0]$. Consequently, (7.3) describes a control system $\Sigma_2 := (\mathcal{X}, \Re^{k_2}, M_{U\times S_1}, M_D, \phi, \pi, H_2)$ with outputs, $M_{U\times S_1}$ being the set of all measurable and locally bounded functions $(u,v) : \Re^+ \to U \times S_1$ and M_D being the set of all measurable and locally bounded functions $d : \Re^+ \to D$ (in the sense described in ref. [9]). Furthermore, Remark 6.5 and Corollary 6.6 show that system (7.3) is robustly forward complete from the input $(u,v) \in M_{U\times S_1}$ and that $0 \in \mathcal{X}$ is a robust equilibrium point from the input $(u,v) \in M_{U\times S_1}$ for system (7.3) in the sense described in ref. [9]. Finally, notice that the classical semigroup property is satisfied for system (7.3), i.e., we have $\pi(t_0, x_{20}, u, d) = [t_0, +\infty)$.

Systems (7.1), (7.2) and (7.3) are related in the following way: it can be said that system (7.1) is the feedback interconnection of subsystems (7.2) and (7.3), in the sense described in ref. [9]. Figure 2 presents schematically the interconnection of subsystems (7.2) and (7.3) that produces the composite system (7.1).

We are now in a position to present our main result, which is a direct consequence of the Small-Gain Theorem presented in ref. [9]. Here we are using the notion of the (U)WIOS property for abstract control systems given in ref. [9].

Theorem 7.1. Consider system (7.1) under hypotheses (R1–8) and assume that

(H1) Subsystem (7.2) satisfies the WIOS property from the inputs v_1 and u. Particularly, there exist functions $\sigma_1 \in KL$, $\beta_1, \mu_1, c_1, \delta_1, \delta_1^u, q_1^u \in K^+$, $\gamma_1, \gamma_1^u, a_1, p_1, p_1^u \in \mathcal{N}$, such that the following estimate holds for all $(t_0, x_{10}, (v_1, u, d)) \in \Re^+ \times C^0([-r_1, 0]; \Re^{n_1}) \times L^\infty_{\mathrm{loc}}(\Re^+; S_2 \times U \times D)$ and $t \geqslant t_0$ for the solution $x_1(t)$ of (7.2) with initial condition $T_{r_1}(t_0)x_1 = x_{10}$ corresponding to inputs $(v_1, u, d) \in L^\infty_{\mathrm{loc}}(\Re^+; S_2 \times U \times D)$:

$$\begin{aligned} |H_1(t, T_{r_1}(t)x_1)| \leqslant\ & \sigma_1(\beta_1(t_0)\|x_{10}\|_{r_1}, t - t_0) \\ & + \sup_{t_0\leqslant\tau\leqslant t} \gamma_1(\delta_1(\tau)|v_1(\tau)|) \\ & + \sup_{t_0\leqslant\tau\leqslant t} \gamma_1^u(\delta_1^u(\tau)|u(\tau)|), \end{aligned} \tag{7.4}$$

$$\begin{aligned} \beta_1(t)\|T_{r_1}(t)x_1\|_{r_1} \leqslant\ & \max\{\mu_1(t - t_0), c_1(t_0), \\ & a_1(\|x_{10}\|_{r_1}), \sup_{t_0\leqslant\tau\leqslant t} p_1(|v_1(\tau)|), \\ & \sup_{t_0\leqslant\tau\leqslant t} p_1^u(q_1^u(\tau)|u(\tau)|)\}. \end{aligned} \tag{7.5}$$

(H2) Subsystem (7.3) satisfies the WIOS property from the inputs v_2 and u. Particularly, there exist functions $\sigma_2 \in KL$, $\beta_2, \mu_2, c_2, \delta_2, \delta_2^u, q_2^u \in K^+$, $\gamma_2, \gamma_2^u, a_2, p_2, p_2^u \in \mathcal{N}$, such that the following estimate holds for all $(t_0, x_{20}, (v_2, u, d)) \in \Re^+ \times L^\infty$

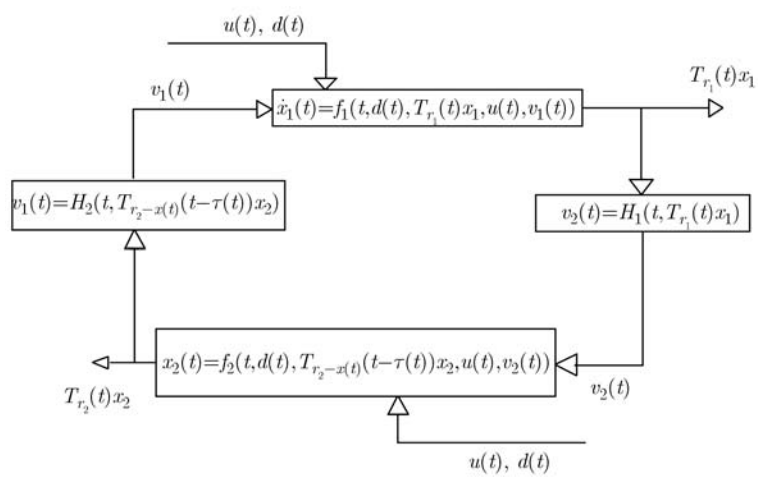

Figure 2 System (7.1) regarded as the feedback interconnection of subsystem (7.2) described by RFDEs and subsystem (7.3) described by FDEs.

$([-r_2,0];\Re^{n_2}) \times L^{\infty}_{\rm loc}(\Re^+;S_1 \times U \times D)$ and $t \geqslant t_0$ for the solution $x_2(t)$ of (7.3) with initial condition $T_{r_2}(t_0)x_2 = x_{20}$ corresponding to inputs $(v_2,u,d) \in L^{\infty}_{\rm loc}(\Re^+;S_1 \times U \times D)$:

$$\begin{aligned} &|H_2(t, T_{r_2-\tau(t)}(t-\tau(t))x_2)| \\ &\quad \leqslant \sigma_2(\beta_2(t_0)\|x_{20}\|_{r_2}, t-t_0) \\ &\qquad + \sup_{t_0 \leqslant s \leqslant t} \gamma_2(\delta_2(s)|v_2(s)|) \\ &\qquad + \sup_{t_0 \leqslant s \leqslant t} \gamma_2^u(\delta_2^u(s)|u(s)|), \end{aligned} \tag{7.6}$$

$$\begin{aligned} &\beta_2(t)\|T_{r_2}(t)x_2\|_{r_2} \\ &\quad \leqslant \max\{\mu_2(t-t_0), c_2(t_0), a_2(\|x_{20}\|_{r_2}), \\ &\qquad \sup_{t_0 \leqslant s \leqslant t} p_2(|v_2(s)|), \\ &\qquad \sup_{t_0 \leqslant s \leqslant t} p_2^u(q_2^u(s)|u(s)|)\}. \end{aligned} \tag{7.7}$$

(H3) There exist functions $\rho \in K_\infty$, $a \in \mathcal{N}$ and a constant $M > 0$ such that one of the following conditions holds for all $t, s \geqslant 0$ and $x = (x_1,x_2) \in C^0([-r_1,0];\Re^{n_1}) \times L^\infty([-r_2,0];\Re^{n_2})$:

$$\delta_1(t) \leqslant M, \tag{7.8a}$$

$$g_1\left(\delta_1(t) g_2\left(\max_{\theta \in [0,t]} \delta_2(\theta) s\right)\right) \leqslant s, \tag{7.8b}$$

$$\begin{aligned} &\|H(t,x_1,x_2)\|_{\mathcal{Y}} \leqslant a(|H_1(t,x_1)| \\ &\quad + \gamma_1(\delta_1(t)|H_2(t, T_{r_2-\tau(t)}(-\tau(t))x_2)|)); \end{aligned} \tag{7.8c}$$

or

$$\delta_2(t) \leqslant M, \tag{7.9a}$$

$$g_2\left(\delta_2(t) g_1\left(\max_{\theta \in [0,t]} \delta_1(\theta) s\right)\right) \leqslant s, \tag{7.9b}$$

$$\begin{aligned} &\|H(t,x_1,x_2)\|_{\mathcal{Y}} \\ &\quad \leqslant a(|H_2(t, T_{r_2-\tau(t)}(-\tau(t))x_2)| \\ &\qquad + \gamma_2(\delta_2(t)|H_1(t,x_1)|)), \end{aligned} \tag{7.9c}$$

where $g_i(s) := \gamma_i(s) + \rho(\gamma_i(s))$, $i = 1,2$.

Then, there exists a function $\gamma \in \mathcal{N}$ such that system (7.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$, where

$$\delta(t) := \max\{\delta_1^u(t), \delta_2^u(t), q_1^u(t), q_2^u(t)\}. \tag{7.10}$$

Moreover, if $\beta_1, \beta_2, c_1, c_2, \delta_1, \delta_2 \in K^+$ are bounded, then system (7.1) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma \in \mathcal{N}$ and weight $\delta \in K^+$.

Remark 7.2.

(a) It should be clear that Theorem 7.1 gives sufficient conditions (but not necessary) for the WIOS property for system (7.1). The main advantage of Theorem 7.1 is that the stability properties for system (7.1) can be verified by studying the stability properties of subsystems (7.2) and (7.3), which are simpler systems.

(b) When $\gamma_1 \in \mathcal{N}$ or $\gamma_2 \in \mathcal{N}$ is identically zero, it follows that (7.8b) and (7.9b) are automatically satisfied. On the other hand, if $\gamma_i(s) = K_i s$ for certain constants $K_i \geqslant 0$ $(i = 1,2)$, then hypothesis (7.8b) (or (7.9b)) is satisfied if $K_1 K_2 \sup_{t \geqslant 0}(\delta_1(t) \max_{\tau \in [0,t]} \delta_2(\tau)) < 1$ (or $K_1 K_2 \sup_{t \geqslant 0}(\delta_2(t) \max_{\tau \in [0,t]} \delta_1(\tau)) < 1$).

It is clear that hypothesis (H1) of Theorem 7.1 can be verified by using the results of sections 3 and 4. Next, sufficient Lyapunov-like conditions for hypothesis (H2) of Theorem 7.1 are presented.

Theorem 7.3 (Lyapunov-like sufficient conditions for hypothesis (H2))**.** Consider system (7.3) under hypotheses (R1–5) and suppose that there exist a Lyapunov functional $V : \Re^+ \times L^\infty([-r_2,0];\Re^{n_2}) \to \Re^+$, functions a_2 of class K_∞, functions ζ, ζ^u of class $\mathcal{N}$, functions $\beta, \delta_2, \delta_2^u$ of class K^+ and a continuous positive definite function $\rho : \Re^+ \to \Re^+$ such that

$$\begin{aligned} &V(t,x_2) \leqslant a_2(\beta(t)\|x_2\|_{r_2}), \\ &\qquad \forall (t,x_2) \in \Re^+ \times L^\infty([-r_2,0];\Re^{n_2}), \end{aligned} \tag{7.11}$$

$$\begin{aligned} &V(t+h, G_h(t,x_2;d,u,v_2)) \\ &\quad \leqslant \max\{\sigma(V(t,x_2),h), \\ &\qquad \sup_{t \leqslant \tau \leqslant t+h} \zeta(\delta_2(\tau)|v_2(\tau)|), \\ &\qquad \sup_{t \leqslant \tau \leqslant t+h} \zeta^u(\delta_2^u(\tau)|u(\tau)|)\}, \end{aligned} \tag{7.12}$$

for all $(t,x_2,u,v_2,d) \in \Re^+ \times L^\infty([-r_2,0];\Re^{n_2}) \times M_U \times M_{S_1} \times M_D$ and $h \in (0, g(t)]$.

In the above equation,

$$\begin{aligned} &G_h(t,x_2;d,u,v_2) \\ &\quad = \begin{cases} x_2(h+\theta), & \theta \in [-r_2,-h], \\ \tilde{f}_2(s), & \theta \in (-h,0], \end{cases} \end{aligned} \tag{7.13}$$

where

$$\tilde{f}_2(s) = f_2(s, d(s), T_{r_2-\tau(s)}(-\tau(s))x_2, u(s), v_2(s)); \quad s = t+h+\theta,$$

$$g(t) = \min\left\{1, \min_{t \leqslant s \leqslant t+1} \tau(s)\right\} \tag{7.14}$$

and $\sigma \in KL$ is the function that satisfies

$$\frac{\partial}{\partial t}\sigma(s,t) = -\rho(\sigma(s,t)), \quad \forall\, t, s \geqslant 0, \tag{7.15a}$$

$$\sigma(s,0) = s, \quad \forall s \geqslant 0. \tag{7.15b}$$

Moreover, suppose that there exist functions a_1, p of class K_∞, μ of class K^+ and a constant $R \geqslant 0$ such that one of the following inequalities holds:

$$a_1(|H_2(t, T_{r_2-\tau(t)}(-\tau(t))x_2)|) \leqslant V(t, x_2),$$
$$\forall (t, x_2) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}), \tag{7.16a}$$

or

$$p(\mu(t)|x_2(0)|) \leqslant V(t, x_2) + R,$$
$$\forall (t, x_2) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}). \tag{7.16b}$$

If

• (7.16a) holds then there exists a function $\sigma_2 \in KL$, such that estimate (7.6) holds with $\beta_2(t) \equiv \beta(t)$, $\gamma_2(s) := a_1^{-1}(\zeta(s))$, $\gamma_2^u(s) := a_1^{-1}(\zeta^u(s))$ for all $(t_0, x_{20}, (v_2, u, d)) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}) \times L^\infty_{loc}(\Re^+; S_1 \times U \times D)$ and $t \geqslant t_0$ for the solution $x_2(t)$ of (7.3) with initial condition $T_{r_2}(t_0)x_2 = x_{20}$ corresponding to inputs $(v_2, u, d) \in L^\infty_{loc}(\Re^+; S_1 \times U \times D)$.

• (7.16b) holds and $\delta_2(t) \equiv 1$ then for every $\phi \in K^+$ there exist functions $\mu_2, c_2 \in K^+$, $g_2, p_2, p_2^u \in \mathcal{N}$, such that the following estimate holds for all $t \geqslant t_0$

$$\phi(t)\|T_{r_2}(t)x_2\|_{r_2}$$
$$\leqslant \max\{\mu_2(t-t_0), c_2(t_0), g_2(\|x_{20}\|_{r_2}),$$
$$\sup_{t_0 \leqslant s \leqslant t} p_2(|v_2(s)|), \sup_{t_0 \leqslant s \leqslant t} p_2^u(\delta_2^u(s)|u(s)|)\}, \tag{7.17}$$

for all $(t_0, x_{20}, (v_2, u, d)) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}) \times L^\infty_{loc}(\Re^+; S_1 \times U \times D)$ and $t \geqslant t_0$ for the solution $x_2(t)$ of (7.3) with initial condition $T_{r_2}(t_0)x_2 = x_{20}$ corresponding to inputs $(v_2, u, d) \in L^\infty_{loc}(\Re^+; S_1 \times U \times D)$. Moreover, if $\phi \in K^+$ is bounded and there exists constant $L > 0$ such that

$$\beta(t) + \frac{1}{\mu(t)} \leqslant L, \quad \forall\, t \geqslant 0, \tag{7.18}$$

then the function $c_2 \in K^+$ is bounded.

The following corollary shows how a Lyapunov functional satisfying the assumptions of Theorem 7.3 for system (7.3) can be constructed.

Corollary 7.4 (Lyapunov-like sufficient conditions for hypothesis (H2))**.** Consider system (7.3) under hypotheses (R1–5) and suppose that there exists a function $W : [-r_2, +\infty) \times \Re^{n_2} \to \Re^+$, functions $\tilde{a}_1, \tilde{a}_2, b$ of class K_∞, ζ, ζ^u of class $\mathcal{N}$, $\tilde{\beta}, \delta_2, \delta_2^u$ of class K^+ and a constant $\lambda \in [0, 1)$ such that

$$W(t - r_2, x_2) \leqslant \tilde{a}_2(\tilde{\beta}(t)|x_2|),$$
$$\forall (t, x_2) \in \Re^+ \times \Re^{n_2}, \tag{7.19}$$

$$W(t, f_2(t, d, T_{r_2-\tau(t)}(-\tau(t))x_2, u, v_2))$$
$$\leqslant \max\left\{\lambda \sup_{-r_2 \leqslant \theta \leqslant -\tau(t)} W(t+\theta, x_2(\theta)),\right.$$
$$\left.\zeta(\delta_2(t)|v_2|), \zeta^u(\delta_2^u(t)|u|)\right\}, \tag{7.20}$$

for all

$$(t, x_2, u, v_2, d) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}) \times U \times S_1 \times D.$$

Select constant $\mu > 0$ such that $\lambda \exp(\mu r_2) \leqslant 1$. Define for all $(t, x_2) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2})$ the functional

$$V(t, x_2) = \sup_{-r_2 \leqslant \theta \leqslant 0} \exp(\mu\theta) W(t+\theta, x_2(\theta)). \tag{7.21}$$

Then, the functional

$$V : \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}) \to \Re^+$$

satisfies inequalities (7.11), (7.12) with

$$\beta(\tau) := \max_{t \leqslant s \leqslant t+r_2} \tilde{\beta}(s), a_2(s) := \tilde{a}_2(s),$$
$$\sigma(s,t) = s\exp(-\mu t)$$

and consequently $\sigma \in KL$ is a function satisfying (7.15a,b) with $\rho(s) := \mu s$. Moreover,

• if there exists a function $\tilde{a}_1$ of class K_∞, such that the following inequality holds:

$$\tilde{a}_1(|H_2(t, T_{r_2-\tau(t)}(-\tau(t))x_2)|)$$
$$\leqslant \sup_{\theta \in [-r_2, 0]} W(t+\theta, x_2(\theta)),$$
$$\forall (t, x_2) \in \Re^+ \times L^\infty([-r_2, 0]; \Re^{n_2}), \tag{7.22}$$

then inequality (7.16a) holds with

$$a_1(s) := \exp(-\mu r_2)\tilde{a}_1(s);$$

• if there exist functions p of class K_∞, $\mu \in K^+$ and a constant $R \geqslant 0$ such that the following inequality holds:

$$p(\mu(t)|x_2|) \leqslant W(t, x_2) + R, \forall (t, x_2)$$
$$\in \Re^+ \times \Re^{n_2}, \tag{7.23}$$

then inequality (7.16b) holds.

8 Conclusions

As already remarked in the Introduction, we would like to stress that much more needs to be accomplished in the field of nonlinear control for infinite-dimensional systems described by RFDEs of the form (1.1) or coupled systems of the form (1.2). The results of the present work, and in particular ref. [28], are expected to have numerous applications for mathematical control theory. For example, the characterizations presented in this work can be directly used (exactly as in the finite-dimensional case) in order to:

– develop Lyapunov redesign methodologies which guarantee robustness to disturbance inputs;

– study the solution of tracking control problems where the signal to be tracked is not necessarily bounded with respect to time;

– study the observer existence/design problem for systems described by RFDEs by means of Lyapunov-like conditions (e.g., Observer Lyapunov Function, Lyapunov characterizations of observability/detectability);

– study the stability properties and feedback stabilization problems for systems described by quasi-linear hyperbolic partial differential equations.

Future work will address these problems.

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