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Iasson Karafyllis [a]; Costas Kravaris [b]
[a] Department of Environmental Engineering, Technical University of Crete, Chania, Greece [b] Department of Chemical Engineering, University of Patras, Patras, Greece

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Robust global stabilisability by means of sampled-data control with positive sampling rate

Iasson Karafyllis[a] and Costas Kravaris[b*]

[a]*Department of Environmental Engineering, Technical University of Crete, Chania, Greece;* [b]*Department of Chemical Engineering, University of Patras, Patras, Greece*

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This work proposes a notion of robust reachability of one set from another set under constant control. This notion is used to construct a control strategy, involving sequential set-to-set reachability, which guarantees robust global stabilisation of non-linear sampled data systems with positive sampling rate. Sufficient conditions for robust reachability of one set from another under constant control are also provided. The proposed method is illustrated through a number of examples, including the study of the sampled-data stabilisation problem of the chemostat.

Keywords: sampled-data control; uniform robust global asymptotic stability

1. Introduction

Given the finite-dimensional continuous-time system:

$$\begin{aligned} &\dot{x}(t) = f(d(t), x(t), u(t)) \\ &d(t) \in D \subseteq \Re^l, \quad x(t) \in \Re^n, \quad u(t) \in U \subseteq \Re^m \end{aligned} \tag{1}$$

where the vector field $f: D \times \Re^n \times U \to \Re^n$ is continuous, $u(t)$ represents the control input and $d(t)$ unknown disturbances or model uncertainty. Consider now a state feedback law $u = k(x)$ to be applied to system (1) in discrete time, under zero-order hold with sampling period h:

$$\begin{aligned} &u(t) = k(x(\tau_i)) \text{ on the interval } [\tau_i, \tau_i + h), \\ &\qquad i = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

The resulting closed-loop system is the following hybrid system:

$$\begin{aligned} &\dot{x}(t) = f(d(t), x(t), k(x(\tau_i))), \quad t \in [\tau_i, \tau_{i+1}) \\ &\tau_{i+1} = \tau_i + h, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

and the question is how to select the state feedback function $k(x)$ for desirable stability characteristics of (3).

There is a large body of literature concerning the above very important and very challenging problem of designing sampled-data feedback stabilisers. In particular, the following lines of attack have been pursued to derive stability results (see also the detailed discussion in review article Monaco and Normand-Cyrot (2001)):

- making use of numerical approximations of the solution of the open-loop system e.g. in the work of Nesic, Teel and others (Nesic, Teel and Kokotovic 1999; Nesic, Teel and Sontag 1999; Nesic and Teel 2001; Laila, Nesic and Teel 2002; Nesic and Angeli 2002; Nesic and Laila 2002; Grune and Nesic 2003; Zaccarian, Teel and Nesic 2003; Kellett, Shim and Teel 2004; Nesic and Teel 2004; Laila and Astolfi 2005; Nesic and Grune 2005). The results obtained in this way lead to a systematic procedure for the construction of practical, semi-global feedback stabilisers and provide a list of possible reasons that explain the occasional failure of sampled-data control mechanisms. Recent research takes into account performance and robustness issues as well (Grune and Nesic 2003; Kellet et al. 2004; Khalil 2004; Nesic and Teel 2004).
- exploiting special characteristics of the system such as homogeneity (Grune 1999, 2000), global Lipschitz conditions (Herrmann, Spurgeon and Edwards 1999) or linear structure with uncertainties (Bernstein and Hollot 1989).
- making use of Linear Matrix Inequalities (Ye, Michel and Hou 1998; Mancilla-Aguilar, Garcia and Troparevsky 2000; Hu and Michel 2000a, b), Lyapunov inequalities (Carnevale, Teel and Nesic 2007) or small-gain theorems (Karafyllis and Jiang 2007) in the context of hybrid systems.
- considering the closed-loop system as a discrete-time system (see for instance Monaco and Normand-Cyrot (1988);

*Corresponding author. Email: kravaris@chemeng.upatras.gr

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Artstein and Weiss (2005), as well as Monaco and Normand-Cyrot (1988) which establishes a unified representation for sampled-data systems and discrete-time systems with analytic dynamics). Recent work has established results that characterise the inter-sample behaviour of the solutions based on the behaviour of the solution of the discrete-time system (Nesic et al. 1999).

- considering the closed-loop system under zero order-hold as a time-delay system. This approach was recently explored in the context of linear systems theory (Fridman, Seuret and Richard 2004; Fridman, Shaked and Suplin 2005) and non-linear systems theory (Karafyllis and Kravaris 2007).

It should be mentioned here that the possibility of using state-dependent sampling period $h(x) > 0$ has recently emerged in the literature (see the classical works in Clarke, Ledyaev, Sontag and Subbotin (1997); Sontag (1999) and the links with sampled-data stabilisability given in Grune (1999) in connection with the study of asymptotic controllability of non-linear systems. Notice that the main results in Clarke et al. (1997); Sontag (1999) (and Theorem 9.3.1 in Grune (1999)) lead to semi-global practical stabilisation for the case of sampling schedules with positive sampling rate.

It is important to point out that the above very important research results do not provide conditions for *global* asymptotic stability for *general* non-linear sampled-data systems (usually only semi-global practical stability properties are established or global stability for limited classes of systems; exceptions are the articles Carnevale et al. (2007); Karafyllis and Jiang (2007); Karafyllis and Kravaris (2007).

The goal of the present work is the development of a design methodology to guarantee robust global asymptotic stability for system (3), where the sampled-data feedback is applied with zero-order hold and positive sampling rate. Our proposed approach has been motivated by the following considerations:

(i) When a continuous-time controller is designed for the continuous-time system (1) and subsequently is implemented under sample-and-hold discretisation, the sampled-data control system does not inherit the properties of the continuous-time control system. For example, global closed-loop asymptotic stability in continuous time will *not*, in general, be preserved under the emulation controller: closed-loop stability will, in general, become local.

(ii) If the foregoing emulation design *or any other* sampled-data controller design guarantees a domain of attraction that is too small in size for a particular application, the immediate question that arises concerns the possibility of extending the control strategy for the purpose of enlargement of the domain of attraction. When the system's initial condition is outside the guaranteed domain of attraction under a given controller, is it possible to find a strategy that can bring the system inside?

To be able to address the latter question, some intuitive considerations would be helpful, before a mathematical formulation is developed. In order to drive the system's state to the given target set, the simplest choice of control input that could be tried is constant control. If constant control cannot take the system inside the target set, it will still be able to take it somewhere else. From there, another constant value of the control input can be tried out and, if it still does not hit the target, still another constant control input, etc., until the target set is reached.

The intuitive idea of using different feedback laws in different regions of the state space has appeared recently in the literature. For example, in Behrens and Wirth (2001) the authors exploit null asymptotic controllability of the system and its linearisation in order to obtain a piecewise constant patchy feedback (applied continuously; not under zero-order hold) that brings all Caratheodory (not Filippov) solutions into a feedback invariant neighbourhood of the origin and a sampled-data feedback (applied in the feedback invariant neighbourhood of the origin) which guarantees local exponential stability. Reachability properties of the control system were also exploited in Section 12.1 (Feedback Stabilisation of Regular Systems) of the book Colonius and Kliemann (2000; pp. 434–449) in order to construct measurable feedback laws which are applied continuously (not under zero-order hold) and guarantee global practical stabilisation (see also the references of Colonius and Kliemann (2000)).

The present work will provide a mathematical formulation of the foregoing intuitive idea of sequential reachability from one region of state space to another, ultimately reaching the target attractor. The goal will be to develop and prove conditions under which this intuitive idea will lead to robust global asymptotic stability for the closed-loop system (Theorem 3.1). In this direction, a new notion of reachability of one set from another under constant control will be proposed (Definition 2.4) and subsequently, this notion will be utilised to establish the main stability results that involve a chain of reachable sets. Simple sufficient conditions to test reachability of one set from another will also be derived (Lemma 2.7 and Lemma 2.9). Finally, the proposed method will be applied to a number of illustrative examples. Example 4.1 will study the simplified Moore–Greitzer model of a jet engine with no stall, which was recently studied in Nesic and Grune (2005). Example 4.2 considers an important class of one-dimensional control systems, where a globally stabilising non-linear sampled-data feedback law is constructed by means of Theorem 3.1. The important class of bilinear systems will be studied in Example 4.3, where the results of Theorem 3.1 lead to a concrete sampled-data stabilisation algorithm by means of bounded feedback. The problem of sampled-data stabilisation of the chemostat will be solved in Example 4.4. The resulting control algorithm will not rely on any monotonicity assumption on the specific growth rate (Smith and Waltman 1995; Karafyllis, Kravaris, Syrou and Lyberatos 2008) and is directly applicable for practical implementation.

Whenever applicable, the proposed control method has very desirable features, including that

- it guarantees *global* asymptotic stability for the closed-loop system,
- it guarantees *robustness* to perturbations of the sampling schedule,
- it provides means to *determine the maximum allowable sampling period*,
- is *not limited* to special cases where the solution map is available,
- is *not limited* to special cases where the non-linear term is homogeneous or globally Lipschitz.

No other existing method can guarantee all of the above at the same time.

On the other hand, even though the proposed methodology is conceptually very simple, its application to a specific control problem requires further work, based on knowledge of the dynamics of the open-loop system under constant input (not necessarily the solution map), in order to come up with a concrete control strategy in a specific application. This will be illustrated in the examples of §4, including the two important engineering applications of a jet engine system and the chemostat.

Notations: Throughout this article we adopt the following notations:

- For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose.
- We say that a non-decreasing continuous function $\gamma : \Re^+ \to \Re^+$ is of class N if $\gamma(0) = 0$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. For the definitions of the classes K and K_∞, see Khalil (1996; p. 135). By KL we denote the set of all continuous functions $\sigma = \sigma(s, t)$: $\Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class K; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s, t) = 0$.
- Let $D \subseteq \Re^l$ be a non-empty set. By $\mathsf{L}^\infty_{\mathrm{loc}}(\Re^+; D)$ we denote the class of all Lebesgue measurable and locally bounded mappings $d : \Re^+ \to D$. Notice that members of $\mathsf{L}^\infty_{\mathrm{loc}}(\Re^+; D)$ are functions $d : \Re^+ \to D$ which are defined pointwise and not equivalent classes of functions.
- Let $A \subseteq \Re^n$ be a non-empty set. For every $\varepsilon > 0$ we define the ε-neighbourhood of A by $\mathsf{N}(A, \varepsilon) := \{y \in \Re^n : \mathrm{dist}(y, A) < \varepsilon\}$, where $\mathrm{dist}(y, A) = \inf\{|y - x| : x \in A\}$.
- By $C^j(A)$ ($C^j(A; \Omega)$), where $j \geq 0$ is a non-negative integer, $A \subseteq \Re^n$, we denote the class of functions (taking values in $\Omega \subseteq \Re^m$) that have continuous derivatives of order j on A.
- For every scalar continuously differentiable function $V : \Re^n \to \Re$, $\nabla V(x)$ denotes the gradient of V at $x \in \Re^n$, i.e. $\nabla V(x) = ((\partial V/\partial x_1)(x), \ldots, (\partial V/\partial x_n)(x))$. We say that a function $V : \Re^n \to \Re^+$ is positive definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$. We say that a continuous function $V : \Re^n \to \Re^+$ is radially unbounded if the following property holds: 'for every $M > 0$ the set $\{x \in \Re^n : V(x) \leq M\}$ is compact'.

2. Main assumptions and notions for sampled-data systems

In the present work we study control systems of the form (1) under the following hypotheses:

(H1) $f(d, x, u)$ is continuous with respect to $(d, x, u) \in D \times \Re^n \times U$ and such that for every bounded $S \subset \Re^n \times U$ there exists constant $L \geq 0$ such that

$$(x - y)'(f(d, x, u) - f(d, y, u)) \leq L|x - y|^2 \quad \forall (x, u, d) \in S \times D, \quad \forall (y, u, d) \in S \times D \tag{4}$$

Hypothesis (H1) is a standard continuity hypothesis and condition (4) is often used in the literature instead of the usual local Lipschitz hypothesis for various purposes and is called a 'one-sided Lipschitz condition' (see, for example, Stuart and Humphries (1998; p. 416) and Fillipov (1988; p. 106)). Notice that the 'one-sided Lipschitz condition' is weaker than the hypothesis of local Lipschitz continuity of the vector field $f(d, x, u)$ with respect to $x \in \Re^n$. It is clear that hypothesis (H1) guarantees that for every $(x_0, d, u) \in \Re^n \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; U)$, there exists a unique solution $x(t)$ of (1) with initial condition $x(0) = x_0$ corresponding to inputs $(d, u) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; U)$.

(H2) There exist a function $a \in K_\infty$ such that

$$|f(d, x, u)| \le a(|x| + |u|), \quad \forall (u, d, x) \in U \times D \times \Re^n \tag{5}$$

Hypothesis (H2) guarantees that $0 \in \Re^n$ is an equilibrium point for (1) and is automatically satisfied if $D \subset \Re^l$ is compact and $f(d, 0, 0) = 0$ for all $d \in D$.

We next provide a definition of robust global stabilisability of (1) by means of bounded sampled-data control with positive sampling rate, which will be used in subsequent developments.

Definition 2.1: We say that the equilibrium point $0 \in \Re^n$ of (1) under hypotheses (H1 and H2) is *robustly globally stabilisable by means of sampled-data control with positive sampling rate*, if there exists a locally bounded mapping $k : \Re^n \to U \subseteq \Re^m$ with $k(0) = 0$ (the feedback function), a function $\kappa \in K_\infty$ with $|f(d, z, k(x))| \le \kappa(|z| + |x|)$ for all $(d, z, x) \in D \times \Re^n \times \Re^n$, a constant $h > 0$ (the maximum allowable sampling period) and a function $\sigma \in KL$ such that the following estimate holds for all $(x_0, d, \tilde{d}) \in \Re^n \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ and $t \ge 0$:

$$|x(t)| \le \sigma(|x_0|, t) \tag{6}$$

where $x(t)$ denotes the solution of the system:

$$\begin{aligned} &\dot{x}(t) = f(d(t), x(t), k(x(\tau_i))), \quad t \in [\tau_i, \tau_{i+1}) \\ &\tau_0 = 0, \quad \tau_{i+1} = \tau_i + h\exp(-\tilde{d}(\tau_i)), \quad i = 0, 1, \ldots \end{aligned} \tag{7}$$

with initial condition $x(0) = x_0$.

We say that the equilibrium point $0 \in \Re^n$ of (1) under hypotheses (H1 and H2) is *robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate*, if the feedback function $k : \Re^n \to U \subseteq \Re^m$ is bounded.

Remark 2.2:

(a) In this work, the closed-loop system (7) will be regarded as a hybrid system, that produces for each $x_0 \in \Re^n$ and for each pair of measurable and locally bounded inputs $d : \Re^+ \to D$, $\tilde{d} : \Re^+ \to \Re^+$, the absolutely continuous function $t \to x(t) \in \Re^n$, produced by the following algorithm:

Step *i*:

(i) Given τ_i, calculate τ_{i+1} using the equation $\tau_{i+1} = \tau_i + h\exp(-\tilde{d}(\tau_i))$,
(ii) Compute the state trajectory $x(t)$, $t \in [\tau_i, \tau_{i+1})$ as the solution of the differential equation $\dot{x}(t) = f(d(t), x(t), k(x(\tau_i)))$,
(iii) Calculate $x(\tau_{i+1})$ using the equation $x(\tau_{i+1}) = \lim_{t \to \tau_{i+1}^-} x(t)$.

Hybrid systems of the form (7) were considered in Karafyllis (2007a, b). Particularly, it was shown that under hypotheses (H1 and H2) and the hypotheses of Definition 2.1, the hybrid system (7) is an autonomous system which satisfies weak semi-group property, the 'Boundedness Implies Continuation' property and for which $0 \in \Re^n$ is a robust equilibrium point for system (7) from the input $\tilde{d} \in L^\infty_{loc}(\Re^+; \Re^+)$ (Karafyllis 2007a). Moreover, the existence of a function $\sigma \in KL$ that satisfies (6) is equivalent to requiring Uniform Robust Global Asymptotic Stability for the closed-loop system (7).

(b) Under hypothesis (H2) and the assumption that $k : \Re^n \to U \subseteq \Re^m$ is a locally bounded mapping with $k(0) = 0$, the assumption that there exists $\kappa \in K_\infty$ with $|f(d, z, k(x))| \le \kappa(|z| + |x|)$ for all $(d, z, x) \in D \times \Re^n \times \Re^n$ is automatically satisfied if the mapping $k : \Re^n \to U \subseteq \Re^m$ is continuous at $x = 0$.

Remark 2.3: The reader should notice that the sampling period is allowed to be time-varying. The factor $\exp(-\tilde{d}(\tau_i)) \le 1$, with $\tilde{d}(t) \ge 0$ some non-negative function of time, is an uncertainty factor in the end-point of the sampling interval. Proving robust global stabilisability of (1) by bounded sampled-data feedback with positive sampling rate will guarantee stability of the closed-loop system (7) for all sampling schedules with $\tau_{i+1} - \tau_i \le h$ (robustness to perturbations of the sampling schedule). To understand the importance of robustness to perturbations of the sampling schedule, consider the following situation. Suppose that hardware limitations restrict the sampling period to be 1*s*. If we manage to design a sampled-data feedback law with $h(x) \equiv r \ge 2s$, then the application of the feedback control will guarantee stability properties for the closed-loop system even if we 'miss measurements' or if we have 'delayed measurements' (e.g., due to improper operation of the sensor).

In such a case robustness to perturbations of the sampling schedule becomes critical. The introduction of the factor $\exp(-\tilde{d}(\tau_i)) \leq 1$ is a mathematical way of introducing perturbations to the sampling schedule; however, it is not unique. Other ways of introducing perturbations of the sampling schedule can be considered.

We next propose a notion of reachability of one set from another set for control systems of the form (1), which is going to be utilised for the construction of sampled-data feedback stabilisers in the following section.

Definition 2.4: Consider system (1) under hypotheses (H1 and H2) and let $r > 0$ be a constant. A set $A \subseteq \Re^n$ is *r-robustly reachable from a set* $\Omega \subseteq \Re^n$ *for system* (1) *with constant control* if there exist $v \in U$, $c \geq 0$ and functions $a : \Re^+ \to \Re^+$ being non-decreasing and $b \in \mathsf{N}$ with the following property:

> **(Q)** For every $x_0 \in \Omega, d \in \mathsf{L}^\infty_{\text{loc}}(\Re^+; D)$, there exists $T(d, x_0) \in [0, c + b(|x_0|)]$ such that the solution of (1) with $u(t) \equiv v$, initial condition $x(0) = x_0$ corresponding to $d \in \mathsf{L}^\infty_{\text{loc}}(\Re^+; D)$ exists for all $t \in [0, T(d, x_0) + r]$ and satisfies:
>
> (i) $|x(t)| \leq a(|x_0|)$, for all $t \in [0, T(d, x_0) + r]$
> (ii) $x(t) \in A$, for all $t \in [T(d, x_0), T(d, x_0) + r]$
> (iii) $x(t) \in \Omega$, for all $t \in [0, T(d, x_0))$

Remark 2.5: It should be emphasised that *r*-robust reachability of a set with constant control is a much stronger property than simple reachability as defined in Sontag (1998, pp. 81–84):

(a) property (Q)-(ii) requires that the solution remains in the reachable set for at least time r for all possible disturbances,
(b) property (Q)-(i) requires that the solution remains uniformly bounded for all possible disturbances and for initial conditions in a specified compact set of the state space,
(c) property (Q) requires that the time needed in order to reach the set $A \subseteq \Re^n$ is uniformly bounded for all possible disturbances and for initial conditions in a specified compact set of the state space.

Notice that if $A \subseteq \Re^n$ is *r*-robustly reachable from $\Omega \subseteq \Re^n$ for system (1) with constant control then every set $B \subseteq \Re^n$ with $A \subseteq B$ is *r*-robustly reachable from $\Omega \subseteq \Re^n$ for system (1) with constant control.

Example 2.6: Consider the simplified Moore–Greitzer model of a jet engine with no stall presented in Krstic, Kanellakopoulos and Kokotovic (1995), described by the planar system:

$$\begin{aligned} \dot{x}_1 &= \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 + x_2 \\ \dot{x}_2 &= u \\ x &= (x_1, x_2)' \in \Re^2, \quad u \in \Re \end{aligned} \tag{8}$$

The sampled-data stabilisability properties of the jet engine system were studied in in Nesic and Grune (2005), where it was shown that system (8) can be practically, semi-globally stabilised by sampled-data control with positive sampling rate. Here we study the perturbed version of the jet engine system, i.e. the system:

$$\begin{aligned} \dot{x}_1 &= d_1(t)x_1 + \frac{3}{2}d_2(t)x_1^2 - \frac{1}{2}x_1^3 + x_2 \\ \dot{x}_2 &= u \\ x &= (x_1, x_2)' \in \Re^2, \quad u \in \Re, \\ d &= (d_1(t), d_2(t)) \in [-1, 1]^2 \end{aligned} \tag{9}$$

In this example, we show that the set $\Omega_2 = \{(x_1, x_2) \in \Re^2 : |x_2| \leq 1\}$ is *r*-robustly reachable from the set $\Omega_3 = \{(x_1, x_2) \in \Re^2 : x_2 \leq -1\}$ and from the set $\Omega_4 = \{(x_1, x_2) \in \Re^2 : x_2 \geq 1\}$ for system (9) with constant control and $r = 1$.

To prove reachability of Ω_2 from Ω_4, let $v = -1$ and notice that the solution $x(t)$ of (9) with initial condition $x_0 = (x_{10}, x_{20})' \in \Omega_4$ satisfies $x_2(t) = x_{20} - t$ for all $t \geq 0$ such that the solution of (9) exists. Moreover, we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2(t)\right) &= 2d_1(t)x_1^2(t) + 3d_2(t)x_1^3(t) - x_1^4(t) + 2x_1(t)x_2(t) \\ &\leq 8x_1^2(t) + x_2^2(t) \end{aligned}$$

The above differential inequality in conjunction with the fact that $x_2(t) = x_{20} - t$ gives $x_1^2(t) \leq (x_{10}^2 + (1/8) \times \max_{\tau\in[0,t]}(x_{20} - \tau)^2)\exp(8 + 8|x_0|)$ for all $t \in [0, 1 + |x_0|]$. Consequently, the solution of (9) exists for all $t \in [0, 1 + |x_0|]$. It follows that:

$$\left|x(t)\right| \leq \left(|x_0| + \max_{\tau\in[0,t]}|x_{20} - \tau|\right)\exp(4 + 4|x_0|), \quad \text{for all } t \in [0, 1 + |x_0|] \tag{10}$$

Next we show that property (Q) of Definition 2.4 holds with $c := 0$, $b(s) := s \in \mathsf{N}$, $a(s) := 2s\exp(4 + 4s) \in \mathsf{N}$ and $T(d, x_0) = x_{20} - 1$. Indeed, we have $x(t) \in \Omega_2$ for all $t \in [T(d, x_0), T(d, x_0) + 1]$, $x(t) \in \Omega_4$ for all $t \in [0, T(d, x_0)]$, where $T(d, x_0) = x_{20} - 1$. Moreover, we have $T(d, x_0) \leq c + b(|x_0|)$, where $c := 0$ and $b(s) := s \in \mathsf{N}$. Finally, from (10) we also obtain $|x(t)| \leq a(|x_0|)$ for all $t \in [0, T(d, x_0) + 1]$, where $a(s) := 2s\exp(4 + 4s)$.

Similarly, we can prove that the set $\Omega_2 = \{(x_1, x_2) \in \Re^2 : |x_2| \le 1\}$ is r-robustly reachable from the set $\Omega_3 = \{(x_1, x_2) \in \Re^2 : x_2 \le -1\}$ for system (9) with constant control and $r = 1$. Particularly, using the same arguments we can show that property (Q) of Definition 2.4 holds with $v = 1$, $c := 0$, $b(s) := s \in N$, $a(s) := 2s\exp(4 + 4s) \in N$ and $T(d, x_0) = x_{20} + 1$.

The following simple lemma provides sufficient conditions for r-robust reachability of sets with constant control. More specifically, given a positively invariant set $\Omega \subseteq \Re^n$ for system (1), we present conditions for the construction of an appropriate subset $A \subseteq \Omega$, which is r-robustly reachable from $\Omega \subseteq \Re^n$ for system (1) with constant control for every $r > 0$. The following lemma will be used in the examples of the present work.

Lemma 2.7: *Consider system* (1) *under hypotheses* (H1 *and* H2) *and suppose that there exists a set* $\Omega \subseteq \Re^n$, *a continuously differentiable function* $h : \Re^n \to \Re$ *and constants* $v \in U$, $R \ge 0$, $\delta > 0$ *such that* $\{x \in \Omega : h(x) \ge R\} \ne \emptyset$ *and*

$$\sup_{d \in D} \nabla h(x) f(d, x, v) \le -\delta, \quad \text{for all } x \in \Omega \quad \text{with } h(x) \ge R \tag{11}$$

Moreover, suppose that there exist functions $a_1, a_2 \in K_\infty$ *and constants,* $p, G \ge 0$, *such that for every* $x_0 \in \Omega$, $d \in L^\infty_{\text{loc}}(\Re^+; D)$, *the solution of* (1) *with* $u(t) \equiv v$ *and initial condition* $x(0) = x_0$ *exists for all* $t \ge 0$ *and satisfies* $x(t) \in \Omega$, $a_1(|x(t)|) \le (G + a_2(|x_0|))\exp(pt)$ *for all* $t \ge 0$. *Then for every* $r > 0$, *the set* $A := \Omega \cap \{x \in \Re^n : h(x) \le R\}$ *is r-robustly reachable from* $\Omega \subseteq \Re^n$ *for system* (1) *with constant control.*

Proof: Let $r > 0$. Notice that inequality (11) guarantees that the set $A := \Omega \cap \{x \in \Re^n : h(x) \le R\}$ is positively invariant for system (1) with $u(t) \equiv v$. Consequently, if $x_0 \in A$ then $x(t) \in A$ for all $t \ge 0$ and $d \in L^\infty_{\text{loc}}(\Re^+; D)$. Let arbitrary $x_0 \in \Omega$, with $h(x_0) > R$, $d \in L^\infty_{\text{loc}}(\Re^+; D)$ and consider the solution of (1) with $u(t) \equiv v$ and initial condition $x(0) = x_0$. Define the set $\{t \ge 0 : x(t) \notin A\}$. Clearly this set is non-empty (since $0 \in \{t \ge 0 : x(t) \notin A\}$). We next claim that $\sup\{t \ge 0 : x(t) \notin A\} \le \delta^{-1}(h(x_0) - R)$. Suppose that this is not the case. Then there exists $t > \delta^{-1}(h(x_0) - R)$ with $h(x(t)) > R$. Since $A := \Omega \cap \{x \in \Re^n : h(x) \le R\}$ is positively invariant for system (1) with $u(t) \equiv v$, this implies that $h(x(\tau)) > R$ for all $\tau \in [0, t]$. Consequently, it follows from (11) that $(d/d\tau)h(x(\tau)) \le -\delta$, a.e. on $[0, t]$. Thus we obtain $h(x(t)) \le h(x_0) - \delta t$, which combined with the hypothesis $t > \delta^{-1}(h(x_0) - R)$ gives $h(x(t)) \le R$, a contradiction.

Thus, for every $x_0 \in \Omega$, $d \in L^\infty_{\text{loc}}(\Re^+; D)$ there exists time $T(d, x_0) \ge 0$ with $T(d, x_0) \le \delta^{-1}\max\{0, h(x_0) - R\}$ such that $x(t) \in A$, for all $t \in [T(d, x_0), T(d, x_0) + r]$. Furthermore, inequality $T(d, x_0) \le \delta^{-1}\max\{0, h(x_0) - R\}$ implies $T(d, x_0) \le c + b(|x_0|)$, where $b(s) := \delta^{-1}(\max_{|x| \le s}\max\{0, h(x) - R\} - \max\{0, h(0) - R\}) \in N$ and $c := \delta^{-1}\max\{0, h(0) - R\} \ge 0$. By virtue of the hypotheses of the lemma, $|x(t)| \le a(|x_0|)$, for all $t \in [0, T(d, x_0) + r]$, where $a(s) := a_1^{-1}(\exp(p(c + r) + pb(s))(G + a_2(s)))$. Consequently, all requirements of Definition 2.4 hold and the set $A := \Omega \cap \{x \in \Re^n : h(x) \le R\}$ is r-robustly reachable from $\Omega \subseteq \Re^n$ for system (1) with constant control. The proof is complete. □

The following example illustrates how Lemma 2.7 can be used for the establishment of r-robust reachability of sets with constant control.

Example 2.8: Consider again the perturbed jet engine system (9). In this example, we show that for every $r > 0$ the set $A = \{(x_1, x_2) \in \Re^2 : |x_2| \le 1, |x_1| \le 4\}$ is r-robustly reachable from the set $\Omega_2 = \{(x_1, x_2) \in \Re^2 : |x_2| \le 1\}$ for system (9) with constant control. Define $\Omega = \Omega_2$, $h(x) = x_1^2$ and $v = 0$. Notice that the solution $x(t)$ of (9) with initial condition $x_0 = (x_{10}, x_{20})' \in \Omega_2$ satisfies $x_2(t) = x_{20} \in [-1, 1]$ for all $t \ge 0$ such that the solution of (9) exists. Moreover, we have:

$$\frac{d}{dt}\left(x_1^2(t)\right) = 2d_1(t)x_1^2(t) + 3d_2(t)x_1^3(t) - x_1^4(t) + 2x_1(t)x_2(t) \le 8x_1^2(t) + x_2^2(t)$$

The above differential inequality in conjunction with the fact that $x_2(t) = x_{20}$ gives $x_1^2(t) \le (x_{10}^2 + (1/8)x_{20}^2) \times \exp(8t)$ for all $t \ge 0$. Consequently, the solution of (9) exists for all $t \ge 0$ and satisfies $x(t) \in \Omega$ and

$$\left|x(t)\right| \le 2|x_0|\exp(4t), \quad \text{for all } t \ge 0 \tag{12}$$

Moreover, notice that

$$\sup_{d \in [-1,1]^2} 2d_1x_1^2 + 3d_2x_1^3 - x_1^4 + 2x_1x_2 \le -7, \quad \text{for all } x \in \Omega \text{ with } h(x) \ge 16 \tag{13}$$

It follows from (12), (13) that the hypotheses of Lemma 2.7 hold with $a_1(s) := s$, $a_2(s) := 2s$, $p = 4$, $G = 0$, $\delta = 7$ and $R := 16$. Consequently for every $r > 0$ the set $A := \Omega \cap \{x \in \Re^n : h(x) \le 16\}$ is r-robustly reachable from $\Omega \subseteq \Re^n$ for system (9) with constant control. Notice that $A = \{(x_1, x_2) \in \Re^2 : |x_2| \le 1, |x_1| \le 4\}$.

Finally, we end this section with a result that provides links between r-robust reachability of sets with constant control and attractor theory for systems without disturbances. Particularly, we show that for every ε, $r>0$ an ε-neighbourhood of a compact global attractor is r-robustly reachable from $\Re^n$. Consequently, knowledge of the dynamics of a control system under constant input may be used for the construction of r-robustly reachable sets.

Lemma 2.9: *Let $U \subseteq \Re^m$ with $0 \in U$ and consider the control system*:

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t)) \\ &x(t) \in \Re^n, \quad u(t) \in U \end{aligned} \tag{14}$$

where f is a locally Lipschitz vector field with $f(0)=0$. Suppose that there exists $v \in U$ such that the dynamical system (14) *with $u(t) \equiv v$ has a compact global attractor $A \subset \Re^n$. Then for every ε, $r>0$ the ε-neighbourhood of $A \subset \Re^n$, $\mathsf{N}(A,\varepsilon)$ (see notations) is r-robustly reachable from $\Re^n$ for system* (14) *with constant control.*

Proof: Let $x(t,x_0)$ denote the solution of (14) with $u(t)\equiv v$ and initial condition $x(0)=x_0$. Since $A\subset\Re^n$ is a global attractor, for every $\varepsilon, R>0$ there exists $T(\varepsilon,R)\geq 0$ such that the following implication holds (Stuart and Humphries 1998, p. 166):

$$\text{'if } |x_0| \leq R \text{ then } x(t,x_0) \in \mathsf{N}(A,\varepsilon) \quad \text{for all } t \geq T(\varepsilon,R)\text{'} \tag{15}$$

Let $g:\Re^+\to\Re^+$ be defined by $g(s):=T(\varepsilon,k+1)+(s-k)(T(\varepsilon,k+2)-T(\varepsilon,k+1))$ for $s\in[k,k+1)$ and every non-negative integer k. Clearly, $g:\Re^+\to\Re^+$ is continuous with $g(s)\geq \min\{T(\varepsilon,[s]+1),T(\varepsilon,[s]+2)\}$, where $[s]$ is the integer part of $s\geq 0$. Define $c:=g(0)$ and $b(s):=\max\{g(y)-g(0): y\in[0,s]\}$. Clearly, $b\in\mathsf{N}$ with $g(s)\leq c+b(s)$ for all $s\geq 0$.

Let $x_0\in\Re^n$ and consider the solution of (14) with $u(t)\equiv v$ and initial condition $x(0)=x_0$. By virtue of implication (15), there exists $T(x_0)\geq 0$ such that $x(t,x_0)\in\mathsf{N}(A,\varepsilon)$ for all $t\geq T(x_0)$. Moreover, $T(x_0)\leq \min\{T(\varepsilon,[|x_0|]+1),\ T(\varepsilon,[|x_0|]+2)\}$, where $[|x_0|]$ is the integer part of $|x_0|$ and consequently, we obtain $T(x_0)\leq g(|x_0|)\leq c+b(|x_0|)$. Consequently, requirements (ii), (iii) of Property (Q) of Definition 2.4 hold.

We next show that requirement (i) of Property (Q) of Definition 2.4 holds as well for appropriate $a\in\mathsf{N}$. Since $A\subset\Re^n$ is bounded, there exists $M>0$ such that $\mathsf{N}(A,\varepsilon)\subseteq\bar{B}_M$, where $\bar{B}_M$ denotes the closed sphere in $\Re^n$ of radius $M>0$, centred at $0\in\Re^n$. Consequently, by virtue of implication (15), we obtain for all $s\geq 0$:

$$\begin{aligned} &\sup\{|x(t,x_0)|: t\geq 0, |x_0|\leq s\} \\ &\quad\leq \max\big(\sup\{|x(t,x_0)| : t\in[0,T(\varepsilon,s)], |x_0|\leq s\}, \\ &\sup\{|x(t,x_0)|: t\geq T(\varepsilon,s), |x_0|\leq s\}\big) \\ &\quad\leq \max\big(\sup\{|x(t,x_0)|: t\in[0,T(\varepsilon,s)], |x_0|\leq s\}, M\big). \end{aligned}$$

By virtue of continuity of the mapping $\Re^+\times\Re^n\ni(t,x_0)\to|x(t,x_0)|\in\Re^+$ and compactness of the set $\{(t,x_0)\in\Re^+\times\Re^n: t\in[0,T(\varepsilon,s)], |x_0|\leq s\}$, it follows that $\sup\{|x(t,x_0)|: t\in[0,T(\varepsilon,s)], |x_0|\leq s\}<+\infty$. Therefore, for all $s\geq 0$, it holds that $a(s):=\sup\{|x(t,x_0)|: t\geq 0, |x_0|\leq s\}<+\infty$. By definition we have:

$$|x(t,x_0)|\leq a(|x_0|), \quad \text{for all } (t,x_0)\in\Re^+\times\Re^n \tag{16}$$

It follows from (16) that requirement (i) of Property (Q) of Definition 2.4 holds as well. The proof is complete. □

3. Main results

Our main result is presented below. Theorem 3.1 is an existing result for (bounded) sampled-data feedback. The reader should notice that Theorem 3.1 does not guarantee continuity of the sampled-data feedback stabiliser.

Theorem 3.1: *Consider system* (1) *under hypotheses* (H1 *and* H2) *and suppose the following*:

(P1) *There exist a locally bounded mapping $\tilde{k}:\Re^n\to U\subseteq\Re^m$ with $\tilde{k}(0)=0$, a bounded open set $\Theta\subseteq\Re^n$ which contains a neighbourhood of $0\in\Re^n$, a function $\gamma\in K_\infty$ with $|f(d,z,\tilde{k}(x))|\leq\gamma(|z|+|x|)$ for all $(d,z,x)\in D\times\Re^n\times\Re^n$, a constant $\tilde{h}>0$ and a function $\sigma\in KL$ such that the following estimate holds for all $(x_0,d,\bar{d})\in\Theta\times\mathsf{L}^\infty_{\mathrm{loc}}(\Re^+;D)\times\mathsf{L}^\infty_{\mathrm{loc}}(\Re^+;\Re^+)$ and $t\geq 0$*:

$$|x(t)|\leq\sigma(|x_0|,t), \quad x(t)\in\Theta \tag{17}$$

where $x(t)$ denotes the solution of the system:

$$\begin{aligned} &\dot{x}(t)=f\big(d(t),x(t),\tilde{k}(x(\tau_i))\big), \quad t\in[\tau_i,\tau_{i+1}) \\ &\tau_0=0, \quad \tau_{i+1}=\tau_i+\tilde{h}\exp\big(-\bar{d}(\tau_i)\big), \quad i=0,1,\ldots \end{aligned} \tag{18}$$

with initial condition $x(0)=x_0\in\Theta$.

Moreover, suppose that one of the following statements hold:

(P2) *There exist sets $\Omega_j\subseteq\Re^n$, $j=1,\ldots,N$ with $\Omega_1=\Theta$, $\cup_{j=1,\ldots,N}\Omega_j=\Re^n$, such that for each $j\in\{2,\ldots,N\}$ the set $\cup_{i=1}^{j-1}\Omega_i$ is r-robustly reachable from $\Omega_j\subseteq\Re^n$ for system* (1) *with constant control.*

(P3) *There exists a sequence of sets $\Omega_j\subseteq\Re^n$, $j=1,2,\ldots$ with $\Omega_1=\Theta,\cup_{j=1}^\infty\Omega_j=\Re^n$, such that for each $j\in\{2,3,\ldots\}$ the set $\cup_{i=1}^{j-1}\Omega_i$*

is r-robustly reachable from $\Omega_j \subseteq \Re^n$ for system (1) *with constant control. Moreover, for each compact $K \subseteq \Re^n$, there exists $N \in \{2,3,\ldots\}$ such that $K \subseteq \cup_{j=1}^{N} \Omega_j$.*

If hypotheses (P1) *and* (P2) *hold, then the equilibrium point $0 \in \Re^n$ of* (1) *under hypotheses* (H1 *and* H2) *is robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate. Moreover, if hypotheses* (P1) *and* (P3) *hold then the equilibrium point $0 \in \Re^n$ of* (1) *under hypotheses* (H1 *and* H2) *is robustly globally stabilisable by means of sampled-data control with positive sampling rate.*

Remark 3.2: *Discussion of hypothesis* (P1). Hypothesis (P1) is a local hypothesis, which guarantees the existence of a sampled-data feedback, which 'works effectively' in the set $\Theta \subseteq \Re^n$. There are many tools in the literature that can be used for the verification of hypothesis (P1) (see for instance Nesic et al. (1999); Nesic and Teel (2004)). It should be emphasised that sampled-data feedback designed by emulation is expected to satisfy hypothesis (P1) for appropriate set $\Theta \subseteq \Re^n$.

Remark 3.3: *Discussion of hypotheses* (P2) *and* (P3). Hypotheses (P2) and (P3) are hypotheses of global nature. All tools presented in previous section can be used in order to show the existence of appropriate sets $\Omega_j \subseteq \Re^n$. It should be emphasised that the role of nonlinearities in the verification of hypotheses (P2) and (P3) is essential (contrary to hypothesis (P1), which as a local hypothesis depends heavily on the linearisation of system (1)). Finally, it should be observed that (P2) and (P3) are very similar in nature, except that (P2) involves a finite chain of sets whereas (P3) a countably infinite chain. (P3) is a weaker assumption than (P2).

The following lemma can be used for the verification of hypothesis (P1). Its proof can be found in the Appendix.

Lemma 3.4: *Consider system* (1) *under hypotheses* (H1 *and* H2) *and suppose that there exists a locally bounded mapping $\tilde{k} : \Re^n \to U \subseteq \Re^m$ with $\tilde{k}(0) = 0$, a function $\gamma \in K_\infty$ with $|f(d,z,\tilde{k}(x))| \le \gamma(|z|+|x|)$ for all $(d,z,x) \in D \times \Re^n \times \Re^n$, a continuous positive function $\rho : \Re^+ \to \Re^+$, a positive definite, continuously differentiable and radially unbounded function $V : \Re^n \to \Re^+$, constants R, $q > 0$, M, $L \ge 0$ such that the following inequalities hold for all $z \in \Theta$, $x \in \Theta$ and $d \in D$:*

$$(z-x)' f(d,z,\tilde{k}(x)) \le L|z-x|^2 + q|x|^2 \tag{19}$$

$$\sup\left\{\nabla V(z) f\left(d,z,\tilde{k}(x)\right) : d \in D, M|z-x| \le |z|\right\} \le -\rho(V(z)) \tag{20}$$

where $\Theta := \{x \in \Re^n : V(x) < R\}$. Let $x(t)$ denote the solution of (18), *initial condition $x(0) = x_0 \in \Re^n$ and corresponding to input $(d,\tilde{d}) \in L^\infty_{\mathrm{loc}}(\Re^+; D) \times L^\infty_{\mathrm{loc}}(\Re^+; \Re^+)$. Then there exists $\sigma \in KL$ such that hypothesis* (P1) *of Theorem* 3.1 *holds with $\tilde{k} : \Re^n \to U \subseteq \Re^m$, $\gamma \in K_\infty$ as above, $\Theta := \{x \in \Re^n : V(x) < R\}$ and for every $\tilde{h} > 0$ satisfying $\tilde{h} < (1/2L)\ln(1 + (L/q) \times ((1/(1+M)))^2)$, for the case $L > 0$ or $\tilde{h} < (1/2q) \times ((1/(1+M)))^2$, for the case $L = 0$.*

When additional regularity properties hold, then the requirements of Lemma 3.4 are simplified. The following lemma is proved in the Appendix and exploits additional regularity properties for the feedback law and the right-hand side of system (1).

Lemma 3.5: *Consider system* (1) *under hypotheses* (H1 and H2) *and suppose that there exists a locally Lipschitz mapping $\tilde{k} : \Re^n \to U \subseteq \Re^m$ with $\tilde{k}(0) = 0$ and a positive definite, radially unbounded function $V \in C^2(\Re^n; \Re^+)$, such that for every pair of compact sets $S \subseteq \Re^n$, $W \subseteq U$ there exist constants $C, K > 0$ satisfying the following inequalities*:

$$\sup_{d \in D} |f(d,z,v) - f(d,x,u)| \le C|z-x| + C|u-v|, \quad \textit{for all } z,x \in S, u,v \in W \tag{21}$$

$$\sup\left\{\nabla V(z) f\left(d,z,\tilde{k}(z)\right) : d \in D\right\} \le -K|z|^2, \quad \textit{for all } z \in S \tag{22}$$

Then for every $R > 0$ there exists a continuous positive function $\rho : \Re^+ \to \Re^+$ and constants $q > 0$, $M, L \ge 0$ such that inequalities (19), (20) *hold with $\Theta := \{x \in \Re^n : V(x) < R\}$. Moreover, there exists $\gamma \in K_\infty$ such that for every $R > 0$ hypothesis* (P1) *of Theorem* 3.1 *holds with $\tilde{k} : \Re^n \to U \subseteq \Re^m$ as above, $\Theta := \{x \in \Re^n : V(x) < R\}$ and appropriate $\sigma \in KL$, $\tilde{h} > 0$.*

The rest of the section is devoted to the proof of Theorem 3.1.

Proof of Theorem 3.1: Define recursively the following sets by the following formulae:

$$C_i = \Omega_i \backslash B_{i-1}, \quad B_i = B_{i-1} \cup \Omega_i, \quad i > 1 \tag{21a}$$

with

$$C_1 = \Omega_1 = \Theta, \quad B_1 = \Omega_1 = \Theta \tag{21b}$$

Notice that $B_i = \cup_{k=i,\ldots,i} \Omega_k = \cup_{k=i,\ldots,i} C_k$ for all $i = 1,\ldots,N$. Let $v_i \in U$ be the constant control that guarantees property (Q) of Definition 2.4 for every set Ω_i with $i > 1$. We define:

$$k(x) = v_i \quad \text{if } x \in C_i \text{ with } i > 1 \tag{22a}$$

$$k(x) = \tilde{k}(x) \quad \text{if } x \in C_1 = \Theta \tag{22b}$$

$$h = \min\{\tilde{h}, r\} \tag{22c}$$

If hypothesis (P3) holds then for each compact $K \subseteq \Re^n$, there exists $N \in \{2,3,\ldots\}$ such that $K \subseteq \cup_{j=1}^{N} \Omega_j = \cup_{j=1}^{N} C_j$. Since $\tilde{k}: \Re^n \to U \subseteq \Re^m$ is locally bounded and Θ is bounded, it follows that the mapping $k: \Re^n \to U$ as defined by (22a, b) is locally bounded. Moreover, if hypothesis (P2) holds then it follows that the mapping $k: \Re^n \to U$ as defined by (22a, b) is bounded.

We next claim that there exists a function $\kappa \in K_\infty$ with $|f(d,z,k(x))| \le \kappa(|z|+|x|)$ for all $(d,z,x) \in D \times \Re^n \times \Re^n$ and a function $\sigma \in KL$ such that estimate (6) holds for all $(x_0, d, \tilde{d}) \in \Re^n \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ and $t \ge 0$ for the solution $x(t)$ of (7) with initial condition $x(0) = x_0$ and corresponding to inputs $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$.

Notice that by virtue of hypotheses (P1), (H2) the function $\Re^+ \ni s \to \tilde{\kappa}(s) := \sup\{|f(d,z,k(x))|: |z|+|x| \le s, d \in D\}$ is a non-decreasing, locally bounded function which satisfies $\tilde{\kappa}(s) \le \gamma(s)$ for $s \ge 0$ sufficiently small and $|f(d,z,k(x))| \le \tilde{\kappa}(|z|+|x|)$ for all $(d,z,x) \in D \times \Re^n \times \Re^n$. It turns out that $\tilde{\kappa}$ can be bounded from above by the K_∞ function κ defined by $\kappa(s) := s + (1/s) \times \int_s^{2s} \tilde{\kappa}(w)\mathrm{d}w$ for $s > 0$ and $\kappa(0) = 0$. Consequently, there exists a function $\kappa \in K_\infty$ with $|f(d,z,k(x))| \le \kappa(|z|+|x|)$ for all $(d,z,x) \in D \times \Re^n \times \Re^n$.

In order to show the existence of a function $\sigma \in KL$ such that estimate (6) holds for all $(x_0, d, \tilde{d}) \in \Re^n \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ and $t \ge 0$ for the solution $x(t)$ of (7) with initial condition $x(0) = x_0$ and corresponding to inputs $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$, we need to show the following things:

- for every $s > 0$, it holds that

$$\sup\left\{|x(t)|;\, t \ge 0, |x_0| \le s, (d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)\right\} < +\infty$$

(Robust Lagrange Stability)

- for every $\varepsilon > 0$ there exists a $\delta := \delta(\varepsilon) > 0$ such that:

$$\sup\left\{|x(t)|;\, t \ge 0, |x_0| \le \delta, \left(d, \tilde{d}\right) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)\right\} \le \varepsilon$$

(Robust Lyapunov Stability)

- for every $\varepsilon > 0$ and $s \ge 0$, there exists a $\tau := \tau(\varepsilon, s) \ge 0$, such that:

$$\sup\left\{|x(t)|;\, t \ge \tau, |x_0| \le s, \left(d, \tilde{d}\right) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)\right\} \le \varepsilon.$$

(Uniform Attractivity)

The above properties guarantee the existence of a function $\sigma \in KL$ such that estimate (6) holds for all $(x_0, d, \tilde{d}) \in \Re^n \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ and $t \ge 0$ for the solution $x(t)$ of (7) with initial condition $x(0) = x_0$ and corresponding to inputs $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$. Indeed, we may define $\tilde{\sigma}(s,t) := \sup\{|x(\xi)|;\, \xi \ge t, |x_0| \le s, (d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)\}$ for all s, $t \ge 0$. By defining $\tilde{\sigma}(s,t) := \tilde{\sigma}(s,0)$ for all $s \ge 0$, $t \in [-1, 0)$, the desired $\sigma \in KL$ can be defined by $\sigma(s,t) = s\exp(-t) + (1/s)\int_{t-1}^{t}\int_s^{2s} \times \tilde{\sigma}(\xi, w)\mathrm{d}\xi\,\mathrm{d}w$ for all $t \ge 0$, $s > 0$ and $\sigma(0,t) = 0$ for all $t \ge 0$.

Since the solution of (7) with $x(0) = x_0$ corresponding to inputs $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ coincides with the solution of (18) with same initial condition corresponding to inputs $(d, \bar{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ with $\bar{d}(t) = \tilde{d}(t) + \ln(\tilde{h}/h)$, it follows that Robust Lyapunov Stability is an immediate consequence of hypothesis (P1) (notice that $\Theta \subseteq \Re^n$ contains a neighbourhood of $0 \in \Re^n$). Thus, we are left with the proofs of Robust Lagrange Stability and Uniform Attractivity.

Let $s \ge 0$ and consider the closed ball $\{x \in \Re^n: |x| \le s\}$. By virtue of hypothesis (P3) (or hypothesis (P2)) there exists $N \in \{2,3,\ldots\}$ such that $\{x \in \Re^n: |x| \le s\} \subseteq \cup_{j=1}^{N} \Omega_j$. Let $c_i \ge 0$ and the functions a_i, b_i that guarantee property (Q) of Definition 2.4 for every Ω_i with $i > 1$ and let $a(s) := \max_{i=2,\ldots,N} a_i(s)$, $b(s) := \max_{i=2,\ldots,N} b_i(s)$, $c := \max_{i=2,\ldots,N} c_i$. Robust Lagrange Stability and Uniform Attractivity will be shown with the help of the following fact, which is shown in the Appendix.

Fact: Let $\tilde{d} \in L^\infty_{loc}(\Re^+; \Re^+)$ and $\pi(\tilde{d}) := \{\tau_0, \tau_1, \tau_2, \ldots\}$ (the set of sampling times), where $\tau_0 = 0$ and $\tau_{i+1} = \tau_i + h\exp(-\tilde{d}(\tau_i))$ for $i \ge 0$. If $x(\tau_i) \in C_k$ for certain $k \in \{2,\ldots,N\}$, then for every $d \in L^\infty_{loc}(\Re^+; D)$ there exists $\tau \in \pi(\tilde{d}) \cap [\tau_i, \tau_i + c + b(|x(\tau_i)|) + r]$ and $m \in \{1,\ldots,k-1\}$ such that $x(\tau) \in C_m$. Moreover, $|x(t)| \le a(|x(\tau_i)|)$ for all $t \in [\tau_i, \tau]$.

Since $\{x \in \Re^n: |x| \le s\} \subseteq \cup_{k=1,\ldots,N} C_k$ and $C_1 = \Theta$, the above fact implies that for every $(x_0, d, \tilde{d}) \in \{x \in \Re^n: |x| \le s\} \times L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ the solution $x(t)$ of (7) with initial condition $x(0) = x_0$ and corresponding to (arbitrary) inputs $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$, satisfies $x(\tau) \in \Theta$ for certain $\tau \in \pi(\tilde{d}) \cap [0, Nc + Nb(a^{(N)}(|x_0|)) + Nr]$ and $|x(t)| \le a^{(N)}(|x_0|)$ for all $t \in [0, \tau]$, where $a^{(N)} = \underbrace{a \circ a \circ \cdots \circ a}_{N \text{ times}}$. By virtue of (17), it follows that $|x(t)| \le \sigma(a^{(N)}(|x_0|), t - \tau)$ for all $t \ge \tau$. The properties of the KL functions in conjunction with the previous estimate of the solution imply the Uniform Attractivity property. Moreover, we have $|x(t)| \le \sigma(a^{(N)}(|x_0|), 0)$, for all $t \ge 0$ (Uniform Lagrange Stability). The proof is complete. □

4. Examples and applications

In this section, a number of examples are presented, which illustrate how the main result of the present work (Theorem 3.1) can be used for the construction of robust sampled-data feedback stabilisers.

Example 4.1: We consider again the perturbed jet engine system (9). Here, we intend to prove that the perturbed jet engine system (9) is robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate. The proof will exploit Theorem 3.1.

Consider the function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 5x_1)^2 \tag{23}$$

which is obtained by applying the backstepping procedure to the uncertain control system (9). Notice that the set $A = \{(x_1, x_2) \in \Re^2 : |x_2| \leq 1, |x_1| \leq 4\}$ is a subset of $\Theta = \{x \in \Re^2 : V(x) < (457/2) + \varepsilon\}$ for all $\varepsilon > 0$. Consequently, Examples 2.6 and 2.8 show that the sets $\Omega_4 = \{(x_1, x_2) \in \Re^2 : x_2 \geq 1\}$, $\Omega_3 = \{(x_1, x_2) \in \Re^2 : x_2 \leq -1\}$, $\Omega_2 = \{(x_1, x_2) \in \Re^2 : |x_2| \leq 1\}$, $\Omega_1 = \Theta$, satisfy hypothesis (P2) of Theorem 3.1. We next show that the hypotheses of Lemma 3.5 are fulfilled for the function V defined by (23).

The derivative of V along the trajectories of system (9) is expressed by the following equality for all $(d, x) \in [-1, 1]^2 \times \Re^2$:

$$\begin{aligned} \nabla V(x)\begin{bmatrix} d_1 x_1 + \frac{3}{2}d_2 x_1^2 - \frac{1}{2}x_1^3 + x_2 \\ u \end{bmatrix} &= x_1\left(d_1 x_1 + \frac{3}{2}d_2 x_1^2 - \frac{1}{2}x_1^3 + x_2\right) \\ &\quad + (x_2 + 5x_1) \times \left(u + 5d_1 x_1 + \frac{15}{2}d_2 x_1^2 - \frac{5}{2}x_1^3 + 5x_2\right) \end{aligned}$$

Using the inequalities $(3/2)d_2 x_1^3 \leq (1/4)x_1^4 + (9/4)x_1^2$, $5|x_1||x_2 + 5x_1| \leq (1/4)x_1^2 + 25(x_2 + 5x_1)^2$, $(15/2)x_1^2 \times |x_2 + 5x_1| \leq (1/4)x_1^4 + (225/4)(x_2 + 5x_1)^2$, we obtain the following inequality for all $(d, x) \in [-1, 1]^2 \times \Re^2$:

$$\begin{aligned} \nabla V(x)\begin{bmatrix} d_1 x_1 + \frac{3}{2}d_2 x_1^2 - \frac{1}{2}x_1^3 + x_2 \\ u \end{bmatrix} &\leq -\frac{3}{2}x_1^2 - \frac{11}{4}(x_2 + 5x_1)^2 \\ &\quad + (x_2 + 5x_1)\left(u + 421x_1 - \frac{5}{2}x_1^3 + 89x_2\right) \end{aligned} \tag{24}$$

We define $\tilde{k}(x) := -421x_1 - 89x_2 + (5/2)x_1^3$ and we notice that by virtue of inequality (24), the hypotheses of Lemma 3.5 are fulfilled. Consequently, for every $R > 0$, hypothesis (P1) of Theorem 3.1 holds with $\tilde{k}(x) := -421x_1 - 89x_2 + (5/2)x_1^3$, $\Theta := \{x \in \Re^n : V(x) < R\}$ and appropriate $\sigma \in KL$, $\tilde{h} > 0$.

It follows from Theorem 3.1 that the perturbed jet engine system (9) is robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate. Since Theorem 3.1 is proved constructively, a bounded sampled-data feedback can be suggested. Particularly, following the proof of Theorem 3.1, the following discontinuous feedback law:

$$\begin{aligned} &k(x) = -421x_1 - 89x_2 + \frac{5}{2}x_1^3, \\ &\qquad \text{if } x \in C_1 = \left\{x \in \Re^2 : V(x) < \frac{457}{2} + \varepsilon\right\}, \\ &k(x) = 0, \quad \text{if } x \in C_2 = \left\{(x_1, x_2) \in \Re^2 : |x_2| \leq 1, \right. \\ &\qquad\qquad \left. V(x) \geq \frac{457}{2} + \varepsilon\right\}, \\ &k(x) = 1, \quad \text{if } x \in C_3 = \left\{(x_1, x_2) \in \Re^2 : x_2 < -1, \right. \\ &\qquad\qquad \left. V(x) \geq \frac{457}{2} + \varepsilon\right\}, \\ &k(x) = -1, \quad \text{if } x \in C_1 = \left\{(x_1, x_2) \in \Re^2 : x_2 > 1, \right. \\ &\qquad\qquad \left. V(x) \geq \frac{457}{2} + \varepsilon\right\} \end{aligned}$$

is a robust sampled-data feedback stabiliser for system (9) for all $\varepsilon > 0$. In Figures 1–3 it is shown the evolution of the states for the closed-loop system (9) with

$$\begin{aligned} &u(t) = k(x(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}) \\ &\tau_0 = 0, \quad \tau_{i+1} = \tau_i + h\exp\left(-\tilde{d}(\tau_i)\right), \\ &\qquad i = 0, 1, \ldots \end{aligned} \tag{25}$$

The parameters h, ε were selected to be $h = \varepsilon = 0.001$ and the initial state is $x_1(0) = 10$, $x_2(0) = 2$.

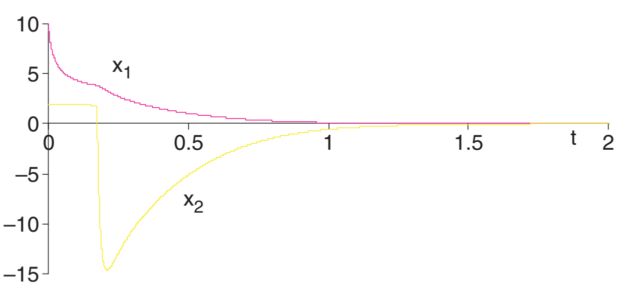

Figure 1. The evolution of the states of the closed-loop system (9) with (25) corresponding to inputs $d_1(t) = \tilde{d}(t) \equiv 0$, $d_2(t) \equiv 1$.

Figures 1–3 show the evolution of the states of the closed-loop system (9) with (25) corresponding to the triplets of inputs $d_1(t) \equiv 0$ $d_2(t) \equiv 1$ $\tilde{d}(t) \equiv 0$, $d_1(t) \equiv 1$, $d_2(t) = \sin(t)$, $\tilde{d}(t) \equiv 0$ and $d_1(t) \equiv 1$, $d_2(t) \equiv 1$, $\tilde{d}(t) = |\sin(t)|$, respectively. It is clear that in all cases the closed-loop system presents fast convergence of the states to the equilibrium point. The sampling time τ_i $(i = 0, 1, 2, \ldots)$ that the state trajectory enters the set $\Theta = \{x \in \Re^2 \colon V(x) < (457/2) + \varepsilon\}$ is the time where the derivative $\dot{x}_2(t)$ presents an abrupt jump (from the value -1 to a negative value with large absolute value).

It should be emphasised that other feedback laws can be constructed (using different control Lyapunov functions from the quadratic one that we used in this work).

Example 4.2: This example illustrates the use of Theorem 3.1 for the construction of a globally stabilising sampled-data feedback. Consider the scalar system:

$$\begin{aligned} &\dot{x} = a(x) + u \\ &x \in \Re, u \in (-\infty, 0] \end{aligned} \tag{26}$$

where $a : \Re \to \Re$ is a locally Lipschitz function with $a(0) = 0$ and $a(x) > 0$ for all $x \neq 0$. We claim that system (26) satisfies hypotheses (P1), (P3) and consequently, (by virtue of Theorem 3.1) it is robustly globally stabilisable by means of sampled-data control with positive sampling rate.

In order to show the validity of hypothesis (P1), define $\Theta = (-\infty, 2)$, $\tilde{h} := (1/(L+1))$ and

$$\tilde{k}(x) := \begin{cases} 0 & \text{for } x \in (-\infty, 0] \\ -(L+1)x & \text{for } x > 0 \end{cases} \tag{27}$$

where $L > 0$ is the Lipschitz constant that satisfies

$$a(x) \leq Lx, \quad \forall x \in [0, 2] \tag{28}$$

The solution of $\dot{x} = a(x) + \tilde{k}(x_0)$ starting at $x(0) = x_0 \in (0, 2)$ satisfies $x(t) \leq x_0$ as long as the solution exists, since by virtue of (28) we have $a(x_0) - (L+1)x_0 \leq -x_0 < 0$. Moreover, as long as the solution satisfies $x(t) \geq 0$, it holds that $-(L+1)x_0 \leq \dot{x} \leq -x_0$, which directly implies $(1 - (L+1)t)x_0 \leq x(t) \leq (1-t)x_0$. A simple contradiction argument shows that $0 \leq x(t) \leq (1-t)x_0$ for all $t \in [0, \tilde{h}]$. Consequently, $x(t) \in \Theta$ and $|x(t)| \leq \exp(-t)|x_0|$, for all $t \in [0, \tilde{h}]$. Working by induction it can be shown that for all $\bar{d} \in L^{\infty}_{\text{loc}}(\Re^+; \Re^+)$ the solution of $\dot{x}(t) = a(x(t)) + \tilde{k}(x(\tau_i))$, $\tau_{i+1} = \tau_i + \tilde{h}\exp(-\bar{d}(\tau_i))$ starting at $x(0) = x_0 \in (0, 2)$ satisfies $x(t) \in \Theta$ and $|x(t)| \leq \exp(-t)|x_0|$, for all $t \geq 0$.

On the other hand, the solution of $\dot{x} = a(x) + \tilde{k}(x_0)$ starting at $x(0) = x_0 < 0$ satisfies $x_0 \leq x(t) \leq 0$ for all $t \geq 0$. Consequently, it holds that $(d/dt)|x(t)| = -a(-|x(t)|)$ for all $t \geq 0$ and Lemma 4.4 in Lin, Sontag and Wang (1996) implies the existence of $\sigma \in KL$ with $\sigma(\sigma(s,t),\tau) = \sigma(s, t+\tau)$ for all s, t, $\tau \geq 0$ such that $|x(t)| \leq \sigma(|x_0|, t)$ for all $t \geq 0$. Working inductively, it can be shown that for all $\bar{d} \in L^{\infty}_{\text{loc}}(\Re^+; \Re^+)$ the solution of $\dot{x}(t) = a(x(t)) + \tilde{k}(x(\tau_i))$, $\tau_{i+1} = \tau_i + \tilde{h}\exp(-\bar{d}(\tau_i))$ starting at $x(0) = x_0 < 0$ satisfies $x(t) \in \Theta$ and $|x(t)| \leq \sigma(|x_0|, t)$, for all $t \geq 0$.

Therefore, hypothesis (P1) holds for system (26).

We next show that hypothesis (P3) holds as well. Consider the sets $\Omega_1 = \Theta$, $\Omega_j = [j-1, j]$ for $j = 2, 3, \ldots$. We will show that for all $j = 2, 3, \ldots$ and $r > 0$, the set $\cup_{i=1}^{j-1}\Omega_i \subseteq \Re$ is r-robustly reachable from the set $\Omega_j \subseteq \Re$ for system (26) with constant control. Notice that $\cup_{i=1}^{j-1}\Omega_i = \Theta$ for $j = 2$ and $\cup_{i=1}^{j-1}\Omega_i = (-\infty, j-1]$ for $j \geq 3$. Let $v_j = -1 - \max_{j-1 \leq s \leq j} a(s)$ and consider the solution of $\dot{x} = a(x) + v_j$ with initial condition $x(0) = x_0 \in \Omega_j$. As long as the solution exists, the following inequalities hold: $\dot{x} \geq v_j$ and $x(t) \leq x_0$. Consequently, it holds that $x_0 + tv_j \leq x(t) \leq x_0$. A simple contradiction argument shows that the solution exists for all $t \geq 0$ and satisfies $|x(t)| \leq \exp(t)(|x_0| + g(2|x_0|))$, where $g(s) = s + \max_{0 \leq x \leq s} a(x)$ (a function of class K_∞). For $j \geq 3$, the fact that the set $\cup_{i=1}^{j-1}\Omega_i = (-\infty, j-1]$ is r-robustly reachable from the set $\Omega_j \subseteq \Re$ can be shown by following the procedure in

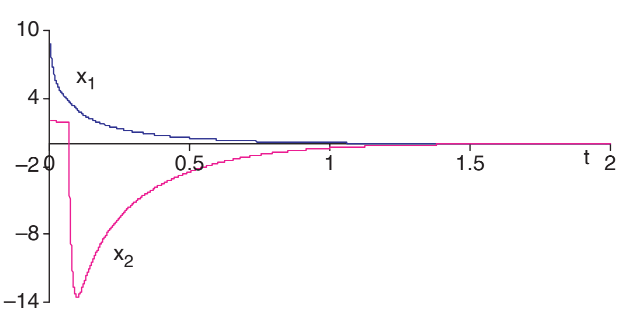

Figure 2. The evolution of the states of the closed-loop system (9) with (25) corresponding to inputs $d_1(t) \equiv 1$, $d_2(t) = \sin(t)$, $\tilde{d}(t) \equiv 0$.

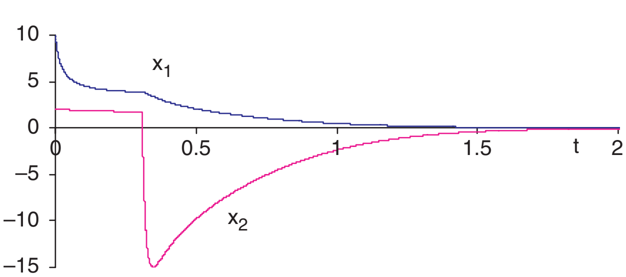

Figure 3. The evolution of the states of the closed-loop system (9) with (25) corresponding to inputs $d_1(t) \equiv 1$, $d_2(t) \equiv 1$, $\tilde{d}(t) = |\sin(t)|$.

the proof of Lemma 2.7 (with $h(x) = x$). For $j = 2$, the fact that the set $\cup_{i=1}^{j-1}\Omega_i = (-\infty, 2)$ is r-robustly reachable from the set $\Omega_j \subseteq \Re$ can be shown by the fact that the solution satisfies $x(t) < x_0$ for all $t > 0$.

Thus, system (26) is robustly globally stabilisable by means of sampled-data control with positive sampling rate. A possible selection of the feedback is:

$$k(x) = \tilde{k}(x), \quad \text{for } x \in (-\infty, 2)$$
$$k(2) = -1 - \max_{1 \le s \le 2} a(s)$$
$$k(x) = -1 - \max_{j-1 \le s \le j} a(s), \quad \text{for } x \in (j-1, j],\ j \ge 3$$

where $\tilde{k}(x)$ is defined by (27) and $h = 1/(L+1)$ where $L > 0$ is the Lipschitz constant that satisfies (28).

Example 4.3: Consider the bilinear control system

$$\begin{aligned} \dot{x} &= Ax + Bu + u_1 C_1 x + \cdots + u_m C_m x \\ x &\in \Re^n, \quad u = (u_1, \ldots, u_m)' \in \Re^m \end{aligned} \tag{29}$$

where $A, C_1, \ldots, C_m \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ are constant matrices. We assume the following:

(A1) There exists $u^* = (u_1^*, \ldots, u_m^*) \in \Re^m$ such that the matrix $(A + u_1^* C_1 + \cdots + u_m^* C_m)$ is Hurwitz.

(A2) The pair of matrices (A, B) is stabilisable. Particularly, there exist $K \in \Re^{m\times n}$, a symmetric, positive definite matrix $P \in \Re^{n\times n}$, constants $\varepsilon > 0$ and $R > (x^*)'Px^*$, where $x^* = (A + u_1^* C_1 + \cdots + u_m^* C_m)^{-1} Bu^*$, such that $\max\{x'P(A+BK)x + \sum_{i=1}^m (x'K'g_i) \times x'PC_i x + \varepsilon|x|^2 : x \in \Re^n, x'Px \le R\} \le 0$, where $g_1' := (1, 0, \ldots, 0) \in \Re^m$, $g_2' := (0, 1, \ldots, 0) \in \Re^m, \ldots, g_m' := (0, 0, \ldots, 1) \in \Re^m$.

We next utilise the result of Theorem 3.1 in order to show that system (29) is robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate. A possible selection of the bounded sampled-data feedback stabiliser is:

$$k(x) = Kx, \quad \text{for} \quad x \in \Theta := \{x \in \Re^n : x'Px < R\}$$
$$k(x) = u^*, \quad \text{for} \quad x \notin \Theta,$$

where $K \in \Re^{m\times n}$, $P \in \Re^{n\times n}$ are the matrices involved in hypothesis (A2) and $u^* = (u_1^*, \ldots, u_m^*) \in \Re^m$ is the vector involved in hypothesis (A1). First, notice that by using hypothesis (A2) and performing simple computations we are in a position to guarantee that the requirements of Lemma 3.4 hold with $\tilde{k}(x) := Kx$, $\gamma(s) := (|A| + |BK|)s + \mu s^2$, $V(z) := z'Pz$, $L := |A| + (3/2)\mu b$, $q := (1/2)\mu b$, $M = 2\varepsilon^{-1}|P|\mu b$, $\rho(s) := \varepsilon|P|^{-1}s$, where $\varepsilon > 0$ is the constant involved in hypothesis (A2) and $\mu := \sum_{i=1}^m |g_i'K||C_i|$, $b := \max\{|x| : x \in \Re^n, x'Px \le R\}$. Consequently, hypothesis (P1) of Theorem 3.1 holds with $\Theta := \{x \in \Re^n : x'Px < R\}$. Moreover, hypothesis (P2) of Theorem 3.1 holds with $N = 2$, $\Omega_1 = \Theta$, $\Omega_2 = \Re^n$ for every $r > 0$. Indeed, by virtue of hypothesis (A1) above, it follows that system (32) with $u \equiv u^*$ has a compact global attractor (namely the set $A := \{x^*\}$). Consequently, Lemma 2.9 implies that for every $\delta, r > 0$ the δ-neighbourhood of $A \subset \Re^n$, $\mathsf{N}(A, \delta) := \{x \in \Re^n : |x - x^*| < \delta\}$ is r-robustly reachable from $\Re^n$ for system (29) with constant control. The reader should notice that since $R > (x^*)'Px^*$, there exists $\delta > 0$ sufficiently small such that $\mathsf{N}(A, \delta) := \{x \in \Re^n : |x - x^*| < \delta\} \subseteq \Theta := \{x \in \Re^n : x'Px < R\}$. Therefore, for every $r > 0$ the set $\Omega_1 = \Theta$ is r-robustly reachable from $\Omega_2 = \Re^n$ for system (29) with constant control (namely, the control $u \equiv u^*$).

As a more specific example we consider the bilinear system:

$$\begin{aligned} &\dot{x}_1 = x_2 + u_1 \quad \dot{x}_2 = x_2 + u_2(1 + x_1 + x_2) \\ &x = (x_1, x_2)' \in \Re^2, \quad u = (u_1, u_2)' \in \Re^2 \end{aligned} \tag{30}$$

which corresponds to the form (29) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_1 = 0 \in \Re^{2\times 2}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Hypothesis (A1) holds with $u^* = (0, -2)'$. The reader can verify that hypothesis (A2) is satisfied as well with

$$P = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}, \quad K = -\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad R > \frac{1}{2}$$

and appropriate $\varepsilon > 0$ for sufficiently large $M > 0$, $k_1, k_2 > 0$. Notice that $x^* = (-1, 0)' \in \Re^2$.

Example 4.4: Continuous stirred microbial bioreactors, often called chemostats, cover a wide range of applications; specialised 'pure culture' biotechnological processes for the production of specialty chemicals (proteins, antibiotics etc.) as well as large-scale environmental technology processes of mixed cultures such as wastewater treatment. The dynamics of the chemostat is often adequately represented by a simple dynamic model involving two state variables, the microbial biomass x and the limiting organic substrate s (Smith and Waltman 1995). For control purposes, the dilution rate $D \ge 0$ is considered as the manipulated input. A general model for microbial growth on a limiting substrate in a chemostat is of the form:

$$\begin{aligned} \dot{x} &= (\mu(s) - D)x \\ \dot{s} &= D(S_0 - s) - \frac{1}{Y_{x/s}}\mu(s)x \end{aligned} \tag{31}$$

where S_0 is the feed substrate concentration, $\mu(s)$ is the specific growth rate and $Y_{x/s} > 0$ is a biomass yield factor. The specific growth rate function $\mu : \Re^+ \to \Re^+$ is a non-negative, globally Lipschitz, smooth, bounded function which satisfies $\mu(0) = 0$ and $\mu(s) \in (0, \mu_{\max}]$ for all $s > 0$. The state space of system (31) is the positively invariant set $(0, +\infty)^2 \subset \Re^2$. For a constant value of the dilution rate $D_s > 0$, we assume the existence of an equilibrium point $(s_s, x_s) \in (0, S_0) \times (0, +\infty)$ with $\mu(s_s) = D_s$ and $x_s = Y_{x/s}(S_0 - s_s)$. The need for the stabilisation of the equilibrium point is explained in Karafyllis et al. (2008) for the case of non-monotone specific growth rate functions (but see also references therein). Here we consider the global stabilisation problem for the equilibrium point $(s_s, x_s) \in (0, S_0) \times (0, +\infty)$ by means of *bounded sampled-data control with positive sampling rate*. First, we perform the following transformation:

$$x_1 = \ln\left(\frac{x}{x_s}\right); \quad x_2 = \ln\left(\frac{s}{s_s}\right); \quad D = D_s e^u \tag{32}$$

System (31) under transformation (32) is expressed by the following system:

$$\begin{aligned}
&\dot{x}_1 = \mu(s_s e^{x_2}) - \mu(s_s)e^u \\
&\dot{x}_2 = \mu(s_s)e^u(Qe^{-x_2} - 1) - \mu(s_s e^{x_2})(Q-1)e^{x_1 - x_2} \\
&x = (x_1, x_2)' \in \Re^2, u \in \Re
\end{aligned} \tag{33}$$

where $Q := (S_0/s_s) > 1$. We will show that hypotheses (P1) and (P2) of Theorem 3.1 hold for system (33) by establishing the following facts:

Fact 1: There exists $v \in \Re$ sufficiently small, such that the dynamical system (33) with $u(t) \equiv v$ has a compact global attractor $A \subset \Re^2$.

Fact 2: For every $R > 0$, there exist a constant $\tilde{h} > 0$ and a function $\sigma \in KL$ such that the following estimate holds for all $(x_0, \bar{d}) \in \{x \in \Re^2: |x|^2 < 2R\} \times \mathsf{L}^{\infty}_{\mathrm{loc}}(\Re^+; \Re^+)$ and $t \geq 0$:

$$|x(t)| \leq \sigma(|x_0|, t), \quad x(t) \in \{x \in \Re^2: |x|^2 < 2R\}, \tag{34}$$

where $x(t)$ denotes the solution of the system (33) with:

$$\begin{aligned}
&u(t) = \ln\left(\frac{\mu(s_s \exp(x_2(\tau_i)))}{\mu(s_s)} \exp(x_1(\tau_i))\right), \quad t \in [\tau_i, \tau_{i+1}) \\
&\tau_0 = 0, \quad \tau_{i+1} = \tau_i + \tilde{h}\exp(-\bar{d}(\tau_i)), \\
&\qquad i = 0, 1, \ldots
\end{aligned} \tag{35}$$

with initial condition $x(0) = x_0 \in \{x \in \Re^2: |x|^2 < 2R\}$.

Indeed, by selecting $R > 0$ sufficiently large, we can guarantee that $\mathsf{N}(A, \varepsilon) \subset \Omega_1$ for certain $\varepsilon > 0$, where $\Omega_1 = \{x \in \Re^2: |x|^2 < 2R\}$ and $A \subset \Re^2$ is the compact global attractor, whose existence is guaranteed by Fact 1. Consequently, Lemma 2.9 implies that for every $r > 0$ the set $\Omega_1 = \{x \in \Re^2: |x| < R\}$ is r-robustly reachable from $\Omega_2 = \Re^2$ for system (33) with constant control. Therefore, hypothesis (P2) of Theorem 3.1 holds for system (33). Fact 2 implies that hypothesis (P1) of Theorem 3.1 holds as well for system (33) with $\Theta = \{x \in \Re^2: |x|^2 < 2R\}$ and

$$\tilde{k}(x) := \ln\left(\frac{\mu(s_s \exp(x_2))}{\mu(s_s)} \exp(x_1)\right).$$

Hence, Theorem 3.1 guarantees that system (33) is robustly globally stabilisable by means of bounded sampled-data control with positive sampling rate. A possible selection of the bounded sampled-data feedback stabiliser is:

$$\begin{aligned}
&k(x) = \ln\left(\frac{\mu(s_s \exp(x_2))}{\mu(s_s)} \exp(x_1)\right), \\
&\qquad \text{for } x \in \Theta := \{x \in \Re^2: |x|^2 < 2R\} \\
&\qquad k(x) = v, \quad \text{for } x \notin \Theta
\end{aligned}$$

where $v \in \Re$ is the input value involved in Fact 1.

The reader may verify that the requirements of Lemma 3.5 hold with

$$\tilde{k}(x) := \ln\left(\frac{\mu(s_s \exp(x_2))}{\mu(s_s)} \exp(x_1)\right)$$

and $V(x) = (1/2)x_1^2 + (1/2)x_2^2$. Therefore, Fact 2 is a direct consequence of Lemma 3.5.

Thus we are left with the proof of Fact 1. Let $\varepsilon > 0$ sufficiently small $(3\varepsilon < Q - 1)$ and $v \in \Re$ such that

$$2\mu(s_s)\exp(v) \leq \min_{S_0 - 3\varepsilon s_s \leq S \leq S_0} \mu(S) \tag{36}$$

For every $x_0 = (x_{10}, x_{20}) \in \Re^2$ there exists $t_{\max} > 0$ such that the solution of (33) with $u(t) \equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 = (x_{10}, x_{20}) \in \Re^2$ exists for all $t \in [0, t_{\max})$. Using the fact that $\mu(s) \geq 0$ for all $s \geq 0$ and differential equations (33), we obtain for all $t \in [0, t_{\max})$:

$$\begin{aligned}
(Q-1)e^{x_1(t)} + e^{x_2(t)} &= Q(1 - e^{-Dt}) \\
&\quad + e^{-Dt}((Q-1)e^{x_{10}} + e^{x_{20}})
\end{aligned} \tag{37a}$$

$$x_{10} - Dt \leq x_1(t) \leq \ln\left(\frac{Q + (Q-1)e^{x_{10}} + e^{x_{20}}}{Q-1}\right) \tag{37b}$$

$$x_2(t) \leq \ln(Q + e^{-Dt}(e^{x_{20}} - Q)) \leq \max\{x_{20}, \ln(Q)\} \tag{37c}$$

where $D = \mu(S_s)e^v$. Notice that the upper bound in (37b) is a direct consequence of (37a), the lower bound in (37b) is obtained from the differential inequality $\dot{x}_1 \geq -D$ and inequality (37c) is a consequence of inequalities (37a, b). Exploiting (37a) and differential equations (33), we get the following differential equation for all $t \in [0, t_{\max})$:

$$\frac{d}{dt}e^{x_2(t)} = \big(D - \mu(s_s e^{x_2(t)})\big)\big(Q - e^{x_2(t)}\big) + \mu\big(s_s e^{x_2(t)}\big)e^{-Dt}(Q - (Q-1)e^{x_{10}} - e^{x_{20}}) \quad (38)$$

Since $\mu(0) = 0$, $D > 0$, it follows from continuity of $\mu : \Re^+ \to \Re^+$ that for every $x_0 = (x_{10}, x_{20}) \in \Re^2$ there exists $S_{\min}(x_0) \in (0, (S_0/2)]$ such that:

$$\mu(S) \le \frac{D}{8\exp(|x_0|)}, \quad \text{for all } S \in [0, S_{\min}(x_0)], \quad (39)$$

Notice that inequality (39) in conjunction with differential equation (38) implies that $(d/dt)e^{x_2(t)} \ge (5/16)DQ$ for all $x_2 \le \ln((S_{\min}(x_0))/s_s)$. Therefore, the globally Lipschitz function $Y(t) = \min(x_2(t), (S_{\min}(x_0)/s_s))$ satisfies $\limsup_{h\to 0^+}[(Y(t+h) - Y(t))/h] \ge 0$ for all $t \in [0, t_{\max})$. Thus the mapping $t \to Y(t)$ is non-decreasing; hence $Y(t) \ge Y(0)$ for all $t \in [0, t_{\max})$. By distinguishing the cases $x_2(t) \ge \ln[(S_{\min}(x_0))/s_s]$ and $x_2(t) < \ln[(S_{\min}(x_0))/s_s]$ in conjunction with $Y(t) \ge Y(0)$ and definition $Y(t) = \min(x_2(t), \ln(S_{\min}(x_0)/s_s))$ we obtain for all $t \in [0, t_{\max})$:

$$x_2(t) \ge \min\left(x_{20}, \ln\left(\frac{S_{\min}(x_0)}{s_s}\right)\right) \quad (40)$$

Estimates (37b, c) and (40) guarantee that for every $x_0 = (x_{10}, x_{20}) \in \Re^2$ the solution of (33) with $u(t) \equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 = (x_{10}, x_{20}) \in \Re^2$ exists for all $t \ge 0$ (i.e. $t_{\max} = +\infty$).

Since $\mu(0) = 0$, $D > 0$, it follows from continuity of $\mu : \Re^+ \to \Re^+$ that there exists $S^- \in (0, s_s)$ such that:

$$\mu(S) \le \frac{D}{2}, \quad \text{for all } S \in [0, S^-] \quad (41)$$

We next show that for every $x_0 = (x_{10}, x_{20}) \in \Re^2$ the solution of (33) with $u(t) \equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 \in \Re^2$ satisfies:

$$x_2(t) \ge \ln\left(\frac{S^-}{s_s}\right), \quad \text{for all } t \ge D^{-1}\ln\left(\frac{4\mu_{\max}e^{|x_0|}}{D}\right) + \frac{4S^-}{s_s D(Q-1)} \quad (42)$$

Let $t_1 := D^{-1}\ln(4\mu_{\max}e^{|x_0|}/D)$ and notice that by virtue of (38) we get $(d/dt)e^{x_2(t)} \ge (D - \mu(S_s e^{x_2(t)})) \times (Q - e^{x_2(t)}) - [(D(Q-1))/4]$ for all $t \ge t_1$. If in addition $s_s e^{x_2} \le S^-$ holds then $(d/dt)e^{x_2(t)} \ge [(D(Q-1))/4]$. Using the previous differential inequality and following the same arguments as in the proof of Lemma 2.7, we may conclude that (42) holds. A direct consequence of (37a) and (42) is the following inequality:

$$x_1(t) \le \ln\left(\frac{Q}{Q-1}\left(1 + \frac{D}{4\mu_{\max}}\right) - \frac{S^-}{(Q-1)s_s}\right), \quad \text{for all } t \ge D^{-1}\ln\left(\frac{4\mu_{\max}e^{|x_0|}}{D}\right) + \frac{4S^-}{s_s D(Q-1)} \quad (43)$$

It should be emphasised that inequality (37c) implies that if $x_{20} \le \ln(Q)$ then $x_2(t) \le \ln(Q)$ for all $t \ge 0$. On the other hand, if $x_{20} > \ln(Q)$ then $T := \sup\{t \ge 0 : \min_{0\le\tau\le t} x_2(\tau) > \ln(Q)\} > 0$. Differential equation (33) in conjunction with the left-hand side inequality (37b), inequality (37c) implies the differential inequality $(d/dt)e^{x_2(t)} \le D(Q - e^{x_2(t)}) - \mu_{\min}(x_0)\times (Q-1)e^{x_{10} - Dt}$ for all $t \in [0, T)$, where $\mu_{\min}(x_0) := \min\{\mu(S),\ S_0 \le S \le s_s e^{x_{20}} + S_0\}$ The previous differential inequality implies $Q < e^{x_2(t)} \le Q + \exp(-Dt) \times [e^{x_{20}} - Q - \mu_{\min}(Q-1)e^{x_{10}}t]$ for all $t \in [0, T)$. Clearly, we must have $T \le [(\max\{0, e^{x_{20}} - Q\})/(\mu_{\min} \times (Q-1)e^{x_{10}})]$. Continuity of the solution implies $x_2(T) = \text{In}(Q)$. Exploiting inequality (37c) (with initial time $T \le \max\{0, e^{x_{20}} - Q\}/(\mu_{\min}(x_0)(Q-1)e^{x_{10}})$ we obtain:

$$x_2(t) \le \ln(Q), \quad \text{for all } t \ge \frac{\max\{0, e^{x_{20}} - Q\}}{\mu_{\min}(x_0)(Q-1)e^{x_{10}}} \quad (44)$$

Moreover, it follows from (36) and (38) that the following implication holds for the solution of (33) with $u(t) \equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 = (x_{10}, x_{20}) \in \Re^2$:

$$\begin{aligned} &\text{If } x_2(\xi) \le \ln(Q - 2\varepsilon) \quad \text{for certain} \\ &\xi \ge \max\left\{0, D^{-1}\ln\left(\frac{2\mu_{\max}Q}{\varepsilon D}\right)\right\} \\ &\text{then } x_2(t) \le \ln(Q - 2\varepsilon) \text{ for all } t \ge \xi \end{aligned} \quad (45)$$

Implication (45) follows from the differential inequality $(d/dt)e^{x_2(t)} \le -\varepsilon D + \mu_{\max}e^{-Dt}Q$, which holds for all $x_2 \in [\ln(Q - 3\varepsilon), \ln(Q - \varepsilon)]$ and is a direct consequence of (36) and (38). Notice that for $t \ge \xi \ge \max\{0, D^{-1}\ln[(2\mu_{\max}R)/\varepsilon D]\}$ and $x_2 \in [\ln(Q - 3\varepsilon), \ln(Q - \varepsilon)]$ we obtain $(d/dt)e^{x_2(t)} \le -(1/2)\varepsilon D$.

Finally, we claim that for every $x_0 = (x_{10}, x_{20}) \in \Re^2$ the solution of (33) with $u(t) \equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 \in \Re^2$ satisfies:

$$x_2(t) \le \ln(Q - 2\varepsilon), \quad \text{for all } t \ge D^{-1}\left[2 + \frac{6\varepsilon}{Q-1}e^{|x_0| + D\tilde{T}(x_0)}\right] + \tilde{T}(x_0) \quad (46a)$$

where

$$\tilde{T}(x_0) := \max\left\{0, D^{-1}\ln\left(\frac{2\mu_{\max}Q}{\varepsilon D}\right), \frac{e^{|x_0|}(e^{|x_0|} + Q)}{(Q-1)\mu_{\min}(x_0)}\right\} \quad (46b)$$

The proof of this claim is made by contradiction. Suppose that there exists $x_0 \in \Re^2$ and

$$t \geq D^{-1}\left[2+\frac{6\varepsilon}{Q-1}e^{|x_0|+D\tilde{T}(x_0)}\right]+\tilde{T}(x_0)$$

such that the solution of (33) with $u(t)\equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0$ satisfies $x_2(t) > \ln(Q-2\varepsilon)$. By virtue of (44) and (45) it is clear that $x_2(\tau) \in (\ln(Q-2\varepsilon), \ln(Q)]$ for all $\tau \in [\tilde{T}(x_0), t]$. By virtue of (33) and (36) we obtain $\dot{x}_1(\tau) \geq D$ for all $\tau \in [\tilde{T}(x_0), t]$, which in conjunction with the left-hand side inequality (37b) implies:

$$\exp(x_1(\tau)) \geq \exp(x_{10} - 2D\tilde{T}(x_0) + D\tau), \quad \text{for all } \tau \in [\tilde{T}(x_0), t] \tag{47}$$

Notice that (33) in conjunction with (36), (47) and the fact $x_2(\tau) \in (\ln(Q-2\varepsilon), \ln(Q)]$ for all $\tau \in [\tilde{T}(x_0), t]$, implies that $(d/d\tau)e^{x_2(\tau)} \leq 2\varepsilon D - (Q-1)\times De^{x_{10}-2D\tilde{T}(x_0)+D\tau}$ for all $\tau \in [\tilde{T}(x_0), t]$. The previous differential inequality implies $Q-2\varepsilon < e^{x_2(t)} \leq Q + 2\varepsilon D(t-\tilde{T}(x_0)) - (Q-1)e^{x_{10}-2D\tilde{T}(x_0)}(e^{Dt}-e^{D\tilde{T}(x_0)})$. Since $t-\tilde{T}(x_0) \geq D^{-1}[2+(6\varepsilon/Q-1)e^{|x_0|+D\tilde{T}(x_0)}]$, the previous inequality gives a contradiction (since we get $2\varepsilon D(t-\tilde{T}(x_0)) - (Q-1)e^{x_{10}-2D\tilde{T}(x_0)}(e^{Dt}-e^{D\tilde{T}(x_0)}) \leq -2\varepsilon$).

Using (42), (43), (46a) and (37a), we conclude that for every $x_0 = (x_{10}, x_{20}) \in \Re^2$ the solution of (33) with $u(t)\equiv v$ and initial condition $(x_1(0), x_2(0)) = x_0 \in \Re^2$ satisfies:

$$\ln\left(\frac{S^-}{s_s}\right) \leq x_2(t) \leq \ln(Q-2\varepsilon) \tag{48a}$$

$$\ln\left(\frac{\varepsilon}{Q-1}\right) \leq x_1(t) \leq \ln\left(\frac{Q}{Q-1}\left(1+\frac{D}{4\mu_{\max}}\right)-\frac{S^-}{(Q-1)s_s}\right) \tag{48b}$$

for all

$$t \geq \max\left\{D^{-1}\ln\left(\frac{4\mu_{\max}e^{|x_0|}}{D}\right)+\frac{4S^-}{s_sD(Q-1)},\; D^{-1}\ln\left(\frac{Q}{\varepsilon}\right),\; D^{-1}\left[2+\frac{6\varepsilon}{Q-1}e^{|x_0|+D\tilde{T}(x_0)}\right]+\tilde{T}(x_0)\right\}.$$

By virtue of Theorem 1.1 in Temam (1998; p. 23), the ω-limit set of the absorbing set:

$$\mathsf{B} = \left[\ln\left(\frac{\varepsilon}{Q-1}\right), \ln\left(\frac{Q}{Q-1}\left(1+\frac{D}{4\mu_{\max}}\right)-\frac{S^-}{(Q-1)s_s}\right)\right] \times \left[\ln\left(\frac{S^-}{s_s}\right), \ln(Q-2\varepsilon)\right]$$

is a compact global attractor $A \subset \Re^2$ for the dynamical system (33) with $u(t)\equiv v$. Moreover, by virtue of (37a) and Lemma 3.1 in Khalil (1996, p. 114), it follows that the inclusion

$$A \subseteq \left\{(x_1, x_2) \in \left[\ln\left(\frac{\varepsilon}{Q-1}\right), \ln\left(\frac{Q}{Q-1}\left(1+\frac{D}{4\mu_{\max}}\right)-\frac{S^-}{(Q-1)s_s}\right)\right] \times \left[\ln\left(\frac{S^-}{s_s}\right), \ln(Q-2\varepsilon)\right] : (Q-1)e^{x_1}+e^{x_2} = Q\right\}$$

holds for the compact global attractor $A \subset \Re^2$ of the dynamical system (33) with $u(t)\equiv v$.

5. Concluding remarks

A novel notion of robust reachability of one set from another set under constant control is proposed in the present work. This notion is used to construct a control strategy, involving sequential set-to-set reachability, which guarantees robust global stabilisation of non-linear sampled data systems with positive sampling rate. Sufficient conditions for robust reachability of one set from another under constant control are also provided.

Whenever applicable, the proposed sampled-data feedback design methodology based on the main result of the present work (Theorem 3.1), has very desirable features, including that it:

- provides a *simple formula* for a stabilising sampled-data feedback,
- guarantees *global* asymptotic stability for the closed-loop system,
- guarantees *robustness* to perturbations of the sampling schedule,
- provides means to *determine the maximum allowable sampling period*,
- is *not limited* to special cases where the solution map is available,
- is *not limited* to special cases where the non-linear term is homogeneous or globally Lipschitz.

No other sampled-data feedback design methodology available in the literature can provide all the above features simultaneously. On the other hand, the proposed methodology requires further work, based on knowledge of the dynamics of the system, in order to come up with a concrete control strategy in a specific application. This was accomplished in §4 in two important engineering applications, a jet engine system and the chemostat. Future research can provide guidelines to expand the range of practical applicability of the proposed theory for the development of concrete control strategies.

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Appendix

Proof of Lemma 3.4: Lemma 4.4 in Lin et al. (1996), guarantees the existence of a continuous function σ of class KL, with $\sigma(s,0)=s$ for all $s\geq 0$ which satisfies $(\partial/\partial t)\sigma(s,t)=-\rho(\sigma(s,t))$ for all s, $t\geq 0$ with the following property: if $y:[t_0,t_1]\to\Re^+$ is an absolutely continuous function and $I\subset[t_0,t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0,t_1]\backslash I$ and such that the following differential inequality holds for all $t\in[t_0,t_1]\backslash I$:

$$\dot{y}(t)\leq -\rho(y(t)) \tag{A1}$$

then the following estimate holds for all $t\in[t_0,t_1]$:

$$y(t)\leq \sigma(y(t_0),t-t_0) \tag{A2}$$

Actually, the statement of Lemma 4.4 in Lin et al. (1996) does not guarantee that σ is continuous or that $\sigma(s,0)=s$ for all $s\geq 0$, but a close look at the proof of Lemma 4.4 in Lin et al. (1996) shows that this is the case when $\rho:\Re^+\to\Re^+$ is a positive definite continuous function.

Let $x_0\in\Theta$, $x_0\neq 0\in\Re^n$, $(d,\tilde{d})\in L^\infty_{\mathrm{loc}}(\Re^+;D)\times L^\infty_{\mathrm{loc}}(\Re^+;\Re^+)$. The solution $x(t)$ of (7) with $k\equiv\tilde{k}$, $h=\tilde{h}$ exists locally and satisfies $x(t)\in\Theta$ and $M|x(t)-x_0|\leq|x(t)|$ for $t>0$ sufficiently small. Let $\tau_1=h\exp(-\tilde{d}(0))$ and $T:=\sup\{t\in[0,\tau_1):\max_{0\leq\tau\leq t}\times V(x(\tau))<R,\ \max_{0\leq\tau\leq t}(M|x(\tau)-x_0|-|x(\tau)|)<0\}>0$. Notice that the previous definition of T combined with (19) and (20) gives:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))\leq -\rho(V(x(t)))\quad\text{and}\\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|x(t)-x_0|^2\right)\leq L|x(t)-x_0|^2+q|x_0|^2,\\ &\quad\text{a.e. for } t\in[0,T)\end{aligned} \tag{A3}$$

Consequently, by virtue of (A1), (A2) and (A3) we obtain the following inequalities which hold for all $t\in[0,T]$:

$$V(x(t))\leq\sigma(V(x_0),t)\ \text{ and }\ |x(t)-x_0|\leq|x_0|\sqrt{q\frac{\exp(2Lt)-1}{L}}\quad\text{for the case } L>0 \tag{A4}$$

or

$$\begin{aligned}&V(x(t))\leq\sigma(V(x_0),t)\quad\text{and}\\ &|x(t)-x_0|\leq|x_0|\sqrt{2qt}\ \text{for the case}\ L=0\end{aligned} \tag{A5}$$

By using (A4), (A5) in conjunction with the inequality $|x_0|\leq|x(t)-x_0|+|x(t)|$, we obtain the following inequalities which hold for all $t\in[0,T]$:

$$\begin{aligned}&V(x(t))\leq\sigma(V(x_0),t)\quad\text{and}\\ &|x(t)-x_0|\leq\frac{\sqrt{q(\exp(2Lt)-1)}}{\sqrt{L}-\sqrt{q(\exp(2Lt)-1)}}|x(t)|\\ &\quad\text{for the case } L>0\end{aligned} \tag{A6}$$

or

$$\begin{aligned}&V(x(t))\leq\sigma(V(x_0),t)\quad\text{and}\\ &|x(t)-x_0|\leq\frac{\sqrt{2qt}}{1-\sqrt{2qt}}|x(t)|\quad\text{for the case } L=0\end{aligned} \tag{A7}$$

Notice that since $t\leq T\leq\tau_1\leq h=\tilde{h}<(1/2L)\ln(1+(L/q)\times[1/(1+M)]^2)$, for the case $L>0$ or $t\leq T\leq\tau_1\leq h=\tilde{h}<(1/2q)(1/(1+M))^2$ for the case $L=0$, we conclude that

$$M\frac{\sqrt{q(\exp(2Lt)-1)}}{\sqrt{L}-\sqrt{q(\exp(2Lt)-1)}}<1,\quad\sqrt{q\frac{\exp(2Lt)-1}{L}}<1$$

for the case $L>0$ or $M(\sqrt{2qt}/1-\sqrt{2qt})<1$, $\sqrt{2q\,t}<1$ for the case $L=0$. Consequently, combining the previous inequalities with (A4)–(A7), we obtain the following estimate for the solution $x(t)$ of (7) with $k\equiv\tilde{k}$, $h=\tilde{h}$, which holds for all $t\in[0,T]$:

$$V(x(t))\leq\sigma(V(x_0),t)\quad\text{and}\quad M|x(t)-x_0|<|x(t)|$$

A simple contradiction argument shows that the above estimate holds for all $t\in[0,\tau_1]$ (i.e. $T=\tau_1$). Notice that estimate $V(x(t))\leq\sigma(V(x_0),t)$ holds for the case $x_0=0$ as well.

Using induction and the semi-group property for σ, we obtain for all non-negative integers i:

$$V(x(t)) \le \sigma(V(x_0), t), \quad \text{for all } t \in [\tau_i, \tau_{i+1}] \tag{A8}$$

Since $V:\Re^n \to \Re^+$ is a positive definite, continuously differentiable and radially unbounded function, there exist functions $a_1, a_2 \in K_\infty$ such that $a_1(|x|) \le V(x) \le a_2(|x|)$, for all $x \in \Re^n$ (Lemma 3.5, Khalil (1996, p. 138)). The conclusion of Lemma 3.2 is an immediate consequence of (A8), the previous inequality and the properties of KL functions. The proof is complete. □

Proof of Lemma 3.5: Let $R > 0$ (arbitrary) and notice that since $V \in C^2(\Re^n;\Re^+)$ is radially unbounded, it follows that the set $\Theta := \{x \in \Re^n\colon V(x) < R\}$ is bounded. Let $S \subseteq \Re^n$ be a convex compact set with $\{x \in \Re^n\colon |x| \le 2|z|,\ z \in \Theta\} \subseteq S$ and let the compact set $W := \{\tilde{k}(x)\colon x \in S\} \subseteq U$. Let $C, K > 0$ be the constants that satisfy inequalities (21), (22) for z, $x \in S$, u, $v \in W$ and let $G \ge 0$ be the Lipschitz constant for $\tilde{k}: \Re^n \to U \subseteq \Re^m$ on the compact $S \subseteq \Re^n$, i.e. $G \ge 0$ satisfies $|\tilde{k}(z) - \tilde{k}(x)| \le G|z - x|$ for all $z, x \in S$. Notice that by virtue of (21) we get for all $z \in \Theta$, $x \in \Theta$ and $d \in D$:

$$\begin{aligned}
&(z-x)'f(d,z,\tilde{k}(x)) \\
&\quad = (z-x)'(f(d,z,\tilde{k}(x)) - f(d,x,\tilde{k}(x))) + (z-x)'f(d,x,\tilde{k}(x)) \\
&\quad \le C|z-x|^2 + (z-x)'f(d,x,\tilde{k}(x)) \\
&\quad \le \left(C + \frac{1}{2}\right)|z-x|^2 + \frac{1}{2}|f(d,x,\tilde{k}(x))|^2
\end{aligned} \tag{A9}$$

Inequality (21) and the fact that $\tilde{k}(0) = 0$ implies:

$$\sup_{d \in D} \left|f\left(d, x, \tilde{k}(x)\right)\right| \le C(1+G)|x|, \quad \text{for all } x \in S \tag{A10}$$

By virtue of (A9), (A10), we conclude that inequality (19) holds with $L := C + (1/2)$ and $q := (1/2)C^2(1+G)^2$. Since $V \in C^2(\Re^n;\Re^+)$, there exists a constant $B \ge 0$ such that $|\nabla V(z)| \le B|z|$ for all $z \in S$. Moreover, notice that if $z \in \Theta$ and $M \ge 1$ then every $x \in \Re^n$ with $M|z - x| \le |z|$ belongs to the compact set $S \subseteq \Re^n$. Consequently, we get from (21) and (22) for all $z \in \Theta$ and $M \ge 1 + 2(BCG/K)$:

$$\begin{aligned}
&\sup\left\{\nabla V(z) f\left(d, z, \tilde{k}(x)\right) : d \in D, M|z-x| \le |z|\right\} \\
&\quad = \sup\Big\{\nabla V(z) f\left(d, z, \tilde{k}(z)\right) + \nabla V(z)\Big(f\left(d, z, \tilde{k}(x)\right) \\
&\qquad - f\left(d, z, \tilde{k}(z)\right)\Big)\colon d \in D, M|z-x| \le |z|\Big\} \\
&\quad \le \sup\left\{-K|z|^2 + |\nabla V(z)|C|\tilde{k}(x) - \tilde{k}(z)|\colon M|z-x| \le |z|\right\} \\
&\quad \le \sup\left\{-K|z|^2 + BCG|z||z-x|\colon M|z-x| \le |z|\right\} \\
&\quad \le -\left(K - \frac{BCG}{M}\right)|z|^2 \le -\frac{K}{2}|z|^2
\end{aligned}$$

The above inequality shows that inequality (20) holds for the continuous, positive definite function $\rho(s) := \frac{Ks}{2(1+B)}$ (since $V(z) \le B|z|^2$ for all $z \in S$).

Finally, we notice that the existence of a function $\gamma \in K_\infty$ with $|f(d,z,\tilde{k}(x))| \le \gamma(|z| + |x|)$ for all $(d,z,x) \in D \times \Re^n \times \Re^n$, follows directly from (21) and the fact that $\tilde{k}:\Re^n \to U \subseteq \Re^m$ with $\tilde{k}(0) = 0$ is locally Lipschitz. The proof is complete. □

Proof of fact in the proof of Theorem 3.1: Let $(d, \tilde{d}) \in L^\infty_{loc}(\Re^+; D) \times L^\infty_{loc}(\Re^+; \Re^+)$ and $x(\tau_i) \in C_k$ with $k > 1$ (arbitrary). By virtue of definition (21) it follows that $x(\tau_i) \in \Omega_k$. Let $v_k \in U$ be the constant control that guarantees property (Q) of Definition 2.4 with Ω replaced by Ω_k and $A = \cup_{i=1}^{k-1} \Omega_i = B_{k-1}$. The solution of (7) on $[\tau_i, \tau_{i+1}]$ coincides with the solution of (1) with $u(t) \equiv v_k$, same initial conditions and corresponding to the same input $d \in L^\infty_{loc}(\Re^+; D)$.

Let $T(d, x(\tau_i)) \in [0, c + b(|x(\tau_i)|)]$ the time involved in property (Q) of Definition 2.4. Since $\lim_{p\to\infty} \tau_{i+p} = +\infty$ (Karafyllis 2007a), there exists integer $p \ge 0$ such that $\tau_{i+p+1} - \tau_i > T(d, x(\tau_i)) \ge \tau_{i+p} - \tau_i$. We claim that there exists non-negative integer $q \le p+1$ such that $x(\tau_{i+q}) \in C_s$ for some $s < k$. Notice that an immediate consequence of the claim is that $\tau_{i+q} \le \tau_{i+p+1} \le \tau_{i+p} + r \le \tau_i + T(d, x(\tau_i)) + r \le \tau_i + c + b(|x(\tau_i)|) + r$.

By virtue of property (Q) of Definition 2.4 and since $\tau_{i+1} \le \tau_i + r$, the solution of (7) exists for all $t \in [\tau_i, \tau_{i+1}]$ and satisfies $|x(t)| \le a(|x(\tau_i)|)$, for all $t \in [\tau_i, \tau_{i+1}]$. In order to prove the above claim we distinguish the following cases:

(a) $T(d, x(\tau_i)) > \tau_{i+1} - \tau_i$. In this case, we have $x(\tau_{i+1}) \in \Omega_k \subseteq B_k$. Since $B_k = \cup_{s=1,\ldots,k} C_s$, it follows that there exists $s \in \{1, \ldots, k\}$ such that $x(\tau_{i+1}) \in C_s$.

(b) $T(d, x(\tau_i)) \le \tau_{i+1} - \tau_i$. Since $\tau_{i+1} - \tau_i \le r$, in this case there exists $m \in \{1, \ldots, k-1\}$ such that $x(\tau_{i+1}) \in \Omega_m \subseteq B_m$. Since $B_m = \cup_{s=1,\ldots,m} C_s$, there exists $s < k$ such that $x(\tau_{i+1}) \in C_s$.

In every case we obtain the existence of $s \in \{1, \ldots, k\}$ such that $x(\tau_{i+1}) \in C_s$. However, if $x(\tau_{i+1}) \in C_k \subseteq \Omega_k$, then $T(d, x(\tau_i)) > \tau_{i+1} - \tau_i$ and thus we can guarantee that property (Q) of Definition 2.4 holds with $T(d, x(\tau_{i+1})) = T(d, x(\tau_i)) - (\tau_{i+1} - \tau_i)$. Furthermore, since $\tau_{i+2} - \tau_{i+1} \le r$, the solution of (7) exists for all $t \in [\tau_i, \tau_{i+2}]$ and satisfies $|x(t)| \le a(|x(\tau_i)|)$, for all $t \in [\tau_i, \tau_{i+2}]$. By distinguishing cases (similarly as above), we conclude that there exists $s \in \{1, \ldots, k\}$ such that $x(\tau_{i+2}) \in C_s$.

However, if $x(\tau_{i+2}) \in C_k$, then $T(d, x(\tau_{i+1})) > \tau_{i+2} - \tau_{i+1}$ and thus we can guarantee that property (Q) of Definition 2.4 holds with $T(d, x(\tau_{i+2})) = T(d, x(\tau_{i+1})) - (\tau_{i+2} - \tau_{i+1}) = T(d, x(\tau_i)) - (\tau_{i+2} - \tau_i)$. Furthermore, since $\tau_{i+3} - \tau_{i+2} \le r$, the solution of (7) exists for all $t \in [\tau_i, \tau_{i+3}]$ and satisfies $|x(t)| \le a(|x(\tau_i)|)$, for all $t \in [\tau_i, \tau_{i+3}]$.

Continuing in the same way, we conclude that there exists non-negative integer $q \le p$ such that $x(\tau_{i+q}) \in C_s$ for some $s \in \{1, \ldots, k-1\}$, because otherwise we would have $T(d, x(\tau_{i+p+1})) = T(d, x(\tau_i)) - (\tau_{i+p+1} - \tau_i) < 0$ (a contradiction). Moreover, the solution of (7) satisfies $|x(t)| \le a(|x(\tau_i)|)$, for all $t \in [\tau_i, \tau_{i+q}]$. The proof is complete. □