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Necessary and sufficient conditions for the existence of stabilizing feedback for control systems

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We prove that the existence of a non-smooth control Lyapunov function is a necessary and sufficient condition for the existence of an ordinary smooth time-varying feedback that stabilizes an affine time-varying control system. Results concerning the non-affine case are also provided.

Keywords: time-varying feedback; affine control systems; global stabilization; Lyapunov functions.

1. Introduction

In this paper we consider affine control systems of the form

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)u \\ x &\in \Re^n, t \geqslant 0, u \in \mathcal{U} \end{aligned} \tag{1.1}$$

where $f(t,x)$ and $g(t,x)$ are C^0 mappings on $\Re^+ \times \Re^n$, locally Lipschitz with respect to $x \in \Re^n$, with $f(t,0) = 0$ for all $t \geqslant 0$ and $\mathcal{U} \subseteq \Re^m$ is a convex set that contains $0 \in \Re^m$. Our objective is to give necessary and sufficient conditions for the existence of a C^0 function $k : \Re^+ \times \Re^n \to \mathcal{U}$, with $k(\cdot,0) = 0$, $k(t,x)$ being locally Lipschitz with respect to $x \in \Re^n$, such that $0 \in \Re^n$ is globally asymptotically stable (GAS) for the closed-loop system (1.1) with

$$u = k(t,x). \tag{1.2}$$

Most of the existing works concerning feedback stabilization deal with uniform-in-time global asymptotic stability (Artstein, 1983; Sontag, 1989; Tsinias, 1989) and the concept of the control Lyapunov function (CLF, a framework introduced by E. D. Sontag) has proved to be useful. Recently, it was proved that the existence of a continuous CLF is a necessary and sufficient condition for the existence of a discontinuous feedback that stabilizes an autonomous control system (Clarke *et al.*, 1997). Currently, many papers are concerned with the issue of robustness for such control laws (Clarke *et al.*, 2000; Prieur, 2001). Moreover, in Rifford (2001) it was proved that the existence of a locally Lipschitz CLF is equivalent to the existence of a stabilizing feedback of Krasovskii or Filippov type.

In this paper we are interested in non-uniform-in-time global asymptotic stability and the paper is a continuation of recent papers that present properties and application of this notion (see Karafyllis & Tsinias, 2003a,b,c; Karafyllis, 2002; Karafyllis & Tsinias, 2003). The notion of non-uniform-in-time global asymptotic stability was introduced in Karafyllis & Tsinias (2003a,b) and Lyapunov characterizations for this notion were given in Karafyllis & Tsinias (2003b). In Karafyllis & Tsinias (2003b) we gave a set of Lyapunov-like necessary and sufficient conditions for the existence of a time-varying stabilizer of the form (1.2), for the case $\mathcal{U} = \Re^n$. Particularly, it was proved that the existence of a time-varying stabilizer is equivalent to the existence of a C^1 CLF or is equivalent to the existence of a robust time-varying stabilizer. In this paper we relax the regularity requirements of Karafyllis & Tsinias (2003b) and we show that the existence of a time-varying stabilizer of the form (1.2) is equivalent to the existence of a lower semi-continuous CLF (Theorem 2.8). This result implies that the main issue for the existence of a time-varying feedback stabilizer is not the regularity of the CLF but the type of the derivative used to express the 'decrease condition', i.e. the Lyapunov differential inequality.

In Section 4, we consider the special case of non-affine single-input control systems of the form

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)a(t,x,u) \\ x &\in \Re^n, t \geqslant 0, u \in \Re \end{aligned} \tag{1.3}$$

where $f(t,x)$ and $g(t,x)$ are C^0 mappings on $\Re^+ \times \Re^n$, locally Lipschitz with respect to $x \in \Re^n$, with $f(t,0) = 0$ for all $t \geqslant 0$. We establish a necessary and sufficient condition (Proposition 4.1) for the existence of a time-varying stabilizer for (1.3), under some mild assumptions concerning the nature of the function $a(\cdot)$. The obtained result includes the so-called 'power-integrator' case, namely the case $a(t,x,u) = u^p$, where p is an odd positive integer. The stabilization of such systems was recently investigated in Lin & Qian (2000), Tsinias (1997).

In Section 5, we establish that all control systems that can be uniformly stabilized by means of continuous time-varying feedback, can also be (non-uniformly) stabilized by means of smooth time-varying feedback. Moreover, we discover the links between the asymptotic behaviour of system

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)\tilde{k}(t,x,w) \\ \dot{w} &= h(t,x,w) \\ x &\in \Re^n, w \in \Re^l, t \geqslant 0 \end{aligned}$$

where $\tilde{k} \in C^0(\Re^+ \times \Re^n \times \Re^l; \mathcal{U})$, $h \in C^0(\Re^+ \times \Re^n \times \Re^l; \Re^l)$, and the existence of a feedback state stabilizer for (1.1). By an immediate application of our main results we find necessary and sufficient conditions for the existence of such a stabilizer (Proposition 5.6).

We believe that the results of this paper will be used in future research in order to prove the connection of the existence of a time-varying stabilizer to the concept of asymptotic controllability (appropriately modified) for general time-varying affine systems. Moreover, since the value function of a solvable optimal control problem is usually proved to be lower semi-continuous, we believe that the results of this paper will provide a link between the existence of a time-varying stabilizer and the solvability of an optimal control problem.

Notation.

- We denote by $C^i(A; B)$ the class of functions $a : A \to B$, with continuous derivatives of order $i \geqslant 0$.

- We denote by B the unit sphere of $\mathfrak{R}^m$.
- We denote by $\mathcal{E}$ the class of functions $\mu \in C^0(\mathfrak{R}^+;\mathfrak{R}^+)$ that satisfy $\int_0^{+\infty}\mu(t)\mathrm{d}t < +\infty$ and $\lim_{t\to+\infty}\mu(t) = 0$.
- We denote by K^+ the class of positive C^∞ functions defined on $\mathfrak{R}^+$. We say that a function $\rho : \mathfrak{R}^+ \to \mathfrak{R}^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. We say that a positive definite, increasing and continuous function $\rho : \mathfrak{R}^+ \to \mathfrak{R}^+$ is of class K_∞ if $\lim_{s\to+\infty}\rho(s) = +\infty$.
- For a vector field $f(t, x)$, which is defined on $\mathfrak{R}^+ \times \mathfrak{R}^n$ and appears in the right-hand side of a system of differential equations, we say that $f(\cdot)$ is locally Lipschitz with respect to $x \in \mathfrak{R}^n$ if for every compact $S \subset \mathfrak{R}^+ \times \mathfrak{R}^n$ there exists a constant $L \geqslant 0$ such that $|f(t, x) - f(t, x)| \leqslant L|x - y|$ for all $(t, x) \in S$ and $(t, y) \in S$.
- For a scalar function $v(t)$ we define the lower right-hand side Dini derivative $Dv(t) := \liminf_{h\to 0^+} \frac{v(t+h)-v(t)}{h}$. For a lower semi-continuous function $V : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}$, we define $DV(t, x; v) := \liminf_{\substack{h\to 0^+\\ w\to v}} \frac{V(t+h,x+hw)-V(t,x)}{h}$.
- Let $V : A \to \mathfrak{R}$ be locally Lipschitz on an open set $A \subseteq \mathfrak{R}^n$. Then by Rademacher's theorem we know that $V(\cdot)$ is Frechet differentiable a.e. on A. We denote by $\Omega_V \subset A$ the set of all points where $V(\cdot)$ fails to be differentiable.
- Let $V : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}$ be lower semi-continuous and let $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$. We denote by $\partial_P V(t, x)$ the proximal subgradient of V at $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ (which may be empty): $(\theta, \zeta) \in \mathfrak{R} \times \mathfrak{R}^n$ belongs to $\partial_P V(t, x)$ iff there exists σ and $\eta > 0$ such that

  $$V(\tau, y) \geqslant V(t, x) + \theta(\tau - t) + \langle \zeta, y - x\rangle - \sigma|\tau - t|^2 - \sigma|y - x|^2$$

  for all $(\tau, y) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ with $|(\tau - t, y - x)| < \eta$. It is known from Theorem 3.1 in Clarke *et al.* (1998) that the domain of $\partial_P V$, denoted by A_V, is dense in $\mathfrak{R}^+ \times \mathfrak{R}^n$. Furthermore, if $\partial_P V(t, x) \neq \emptyset$, it follows that $\sup\{\theta + \langle\zeta, v\rangle; (\theta, \zeta) \in \partial_P V(t, x)\} \leqslant DV(t, x; v)$.

2. Definitions and main results for affine systems

DEFINITION 2.1 Let $V : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}$ be lower semi-continuous and bounded on a neighbourhood of $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$. We define

$$V^0(t, x; v) = \limsup_{\substack{(\tau,y)\to(t,x)\\ (\tau,y)\in A_V\\ w\to v}} \sup\{\theta + \langle\zeta, w\rangle; (\theta, \zeta) \in \partial_P V(\tau, y)\} \tag{2.1}$$

The following lemma presents some elementary properties of this generalized derivative. Notice that the function $(t, x, v) \to V^0(t, x; v)$ may take values in the extended real number system $\mathfrak{R}^* = [-\infty, +\infty]$.

LEMMA 2.2 Let $V : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}$ be lower semi-continuous and let $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$. Then

(i) The function $(t, x, v) \to V^0(t, x; v)$ is upper semi-continuous at $(t, x, v) \in \Re^+ \times \Re^n \times \Re^n$.

(ii) Let $v_i \in \Re^n$ with $V^0(t, x; v_i) < +\infty$ (or $V^0(t, x; v_i) > -\infty$) for $i = 1, 2$. Then it holds that

$$V^0(t, x; \lambda v_1 + (1-\lambda)v_2) \leqslant \lambda V^0(t, x; v_1) + (1-\lambda)V^0(t, x; v_2), \quad \forall \lambda \in (0, 1). \tag{2.2}$$

Moreover, let $x(\cdot) : [a, b) \to \Re^n$ be any C^1 function defined on the non-empty interval $[a, b) \subseteq \Re^+$. Then it holds that

$$\liminf_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \leqslant V^0(t, x(t); \dot{x}(t)), \quad \forall t \in [a, b). \tag{2.3}$$

Proof. (i) This is obvious since $V^0(t, x; v)$ is the upper limit of a function defined on a dense subset of $\Re^+ \times \Re^n \times \Re^n$.

(ii) Let $v_1, v_2 \in \Re^n$ and $\lambda \in (0, 1)$. Clearly, by Definition 2.1 we have

$$\begin{aligned}
&V^0(t, x; \lambda v_1 + (1-\lambda)v_2) = \\
&\limsup_{\substack{(\tau,y)\to(t,x)\\(\tau,y)\in A_V\\ w\to\lambda v_1+(1-\lambda)v_2}} \sup\left\{\lambda\theta+(1-\lambda)\theta+\lambda\left\langle\zeta, \frac{w-(1-\lambda)v_2}{\lambda}\right\rangle+(1-\lambda)\langle\zeta, v_2\rangle; (\theta,\zeta)\in\partial_P V(\tau,y)\right\} \\
&\leqslant \limsup_{\substack{(\tau,y)\to(t,x)\\(\tau,y)\in A_V\\ w\to\lambda v_1+(1-\lambda)v_2}} [\lambda a_1(\tau, y, w) + (1-\lambda)a_2(\tau, y, w)]
\end{aligned}$$

where

$$\begin{aligned}
a_1(\tau, y, w) &:= \sup\left\{\theta + \left\langle \zeta, \frac{w-(1-\lambda)v_2}{\lambda}\right\rangle; (\theta, \zeta) \in \partial_P V(\tau, y)\right\} \\
a_2(\tau, y, v_2) &:= \sup\{\theta + \langle\zeta, v_2\rangle; (\theta, \zeta) \in \partial_P V(\tau, y)\}.
\end{aligned}$$

The previous inequality in conjunction with subadditivity of the upper limit shows that (2.2) holds.

The proof of the last statement is made by contradiction. Suppose that there exists $l \in \Re$, $\varepsilon > 0$ and $t \in [a, b)$ such that

$$\liminf_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \geqslant l$$
$$V^0(t, x(t); \dot{x}(t)) \leqslant l - 4\varepsilon.$$

Without loss of generality we may assume that $\varepsilon < 1$. Then by definitions of the upper and lower limits and the fact that $x(\cdot) : [a, b) \to \Re^n$ is C^1, we obtain the existence of

$0 < \delta_1 \leqslant \delta_2 \leqslant 1$ such that

$$V(t+h, x(t+h)) \geqslant V(t, x(t)) + (l - 2\varepsilon)h \quad \forall h \in [0, 2\delta_1) \tag{2.4a}$$

$$\begin{gathered} \theta + \langle \zeta, w \rangle \leqslant l - 3\varepsilon \\ \forall(\theta, \zeta) \in \partial_P V(\tau, y), \quad \forall(\tau, y, w) \in A_V \times \mathfrak{R}^n \\ \text{with } |(\tau - t, y - x(t))| < 2\delta_2 \text{ and } |w - \dot{x}(t)| < 2\delta_2 \end{gathered} \tag{2.4b}$$

$$\left| \frac{x(t+h) - x(t)}{h} \dot{x}(t) \right| + |x(t+h) - x(t)| \leqslant \delta_2(1 - \varepsilon) \quad \forall h \in [0, 2\delta_1). \tag{2.4c}$$

Furthermore, by the mean value inequality (Clarke *et al.*, 1998, Theorem 2.6) we obtain that for all $(\tau, y, t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^+ \times \mathfrak{R}^n$ and for all $\rho > 0$, there exists $(T, z) \in A_V$, $\lambda \in [0,\ 1]$ and $(\theta, \zeta) \in \partial_P V(T, z)$ with

$$\begin{gathered} V(\tau, y) - V(t, x) < \theta(\tau - t) + \langle \zeta, y - x \rangle + \rho \\ |(T - \lambda t - (1-\lambda)\tau, z - \lambda x - (1-\lambda)y)| < \rho \end{gathered} \tag{2.4d}$$

Applying the mean value inequality for the selection $\tau = t + \delta_1$, $y = x(t + \delta_1)$, $x = x(t)$ and $\rho = \delta_1 \varepsilon$, we get from (2.4d) in conjunction with (2.4a) that there exists $(T, z) \in A_V$ and $(\theta, \zeta) \in \partial_P V(T, z)$ with

$$\begin{gathered} (l - 2\varepsilon)\delta_1 \leqslant V(t + \delta_1, x(t + \delta_1)) - V(t, x(t)) < \theta\delta_1 + \langle \zeta, x(t+\delta_1) - x(t) \rangle + \delta_1 \varepsilon \\ |(T - t, z - x(t))| < \delta_1(1 + \varepsilon) + |x(t + \delta_1) - x(t)|. \end{gathered} \tag{2.4e}$$

Clearly, by virtue of (2.4c), (2.4e) and the facts that $\varepsilon < 1$, $0 < \delta_1 \leqslant \delta_2 \leqslant 1$, we conclude that there exists $(T, z) \in A_V$ and $(\theta, \zeta) \in \partial_P V(T, z)$ with

$$\begin{gathered} l - 3\varepsilon < \theta + \left\langle \zeta, \frac{x(t+\delta_1) - x(t)}{\delta_1} \right\rangle \\ |(T - t, z - x(t))| < 2\delta_2 \text{ and } \left| \frac{x(t+\delta_1) - x(t)}{\delta_1} - \dot{x}(t) \right| < 2\delta_2 \end{gathered} \tag{2.4f}$$

which contradicts (2.4b). The proof is complete. □

The following corollary clarifies the relation between the generalized derivative of Definition 2.1 and Clarke's derivative $V^0(t, x; (1, v)) = \limsup_{\substack{h \to 0^+ \\ (\tau,y)\to(t,x)}} \frac{V(\tau+h, y+hv) - V(\tau, y)}{h}$, when $V(\cdot)$ is Lipschitz around $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ (following the notation in Clarke *et al.*, 1998). It is known (Clarke *et al.*, 1998) that Clarke's derivative can be characterized by the following equality:

$$\limsup_{\substack{h \to 0^+ \\ (\tau,y)\to(t,x)}} \frac{V(\tau + h, y + hv) - V(\tau, y)}{h} = \limsup_{(\tau,y)\to(t,x)} DV(\tau, y; v).$$

Using the results of Lemma 2.2, we can establish that for the case of locally Lipschitz functions the generalized derivative of Definition 2.1 is identically equal to Clarke's derivative at the direction $(1, v)$. Particularly, we have the following corollary.

COROLLARY 2.3 Let $V : \Re^+ \times \Re^n \to \Re$ be lower semi-continuous and let $(t, x) \in \Re^+ \times \Re^n$. Then it holds that

$$DV(t, x; v) \leqslant V^0(t, x; v), \quad \forall v \in \Re^n. \tag{2.5a}$$

Moreover, if $V : \Re^+ \times \Re^n \to \Re$ is Lipschitz around $(t, x) \in \Re^+ \times \Re^n$, then for all $v \in \Re^n$ it holds that

$$\begin{aligned} \limsup_{\substack{h \to 0^+ \\ (\tau, y) \to (t, x)}} \frac{V(\tau + h, y + hv) - V(\tau, y)}{h} &= V^0(t, x; v) \\ &= \limsup_{\substack{(\tau, y) \to (t, x) \\ (\tau, y) \in A_V}} \sup\{\theta + \langle \zeta, v \rangle; (\theta, \zeta) \in \partial_P V(\tau, y)\}. \end{aligned} \tag{2.5b}$$

We next give the notion of the CLF. Moreover, our regularity requirements are minimal, compared to the corresponding definitions given in Karafyllis & Tsinias (2003b), Rifford (2001), Sontag (1989), Tsinias (1989).

DEFINITION 2.4 We say that $V : \Re^+ \times \Re^n \to \Re^+$ is a CLF for system (1.1), if $V(\cdot)$ is lower semi-continuous on $\Re^+ \times \Re^n$ and there exists function $W : \Re^+ \times \Re^n \to \Re^+$ being upper semi-continuous, $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \beta(t)\mathrm{d}t = +\infty$, $\mu \in \boldsymbol{\mathcal{E}}$ and $\rho : \Re^+ \to \Re^+$ being positive definite and lower semi- continuous, such that the following inequalities hold:

$$a_1(|x|) \leqslant V(t, x) \leqslant a_2(\gamma(t)|x|), \quad \forall (t, x) \in \Re^+ \times \Re^n \tag{2.6}$$

$$\inf_{u \in \mathcal{U}} V^0(t, x; f(t, x) + g(t, x)u) \leqslant -W(t, x) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t, x) \in \Re^+ \times (\Re^n \backslash \{0\}) \tag{2.7}$$

$$W(t, x) \geqslant \beta(t)\rho(V(t, x)), \quad \forall (t, x) \in \Re^+ \times \Re^n. \tag{2.8}$$

Notice by virtue of Corollary 2.3 that, if $V(\cdot)$ is locally Lipschitz on $\Re^+ \times (\Re^n \backslash \{0\})$, then inequality (2.7) can be expressed as

$$\begin{aligned} &\inf_{u \in \mathcal{U}} \max_{(\theta, \zeta) \in \partial_C V(t, x)} \theta + \langle \zeta, f(t, x) + g(t, x)u \rangle \\ &\leqslant -W(t, x) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t, x) \in \Re^+ \times (\Re^n \backslash \{0\}) \end{aligned} \tag{2.7$'$}$$

where $\partial_C V(t, x)$ denotes Clarke's generalized gradient (Clarke *et al.*, 1998). When $\mathcal{U}(x) \subseteq \mathcal{U}$ is a compact convex subset of $\mathcal{U} \subseteq \Re^m$ that satisfies

$$\inf_{u \in \mathcal{U}(x)} \max_{(\theta, \zeta) \in \partial_C V(t, x)} \theta + \langle \zeta, f(t, x) + g(t, x)u \rangle \leqslant -W(t, x) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t, x) \in \Re^+ \times (\Re^n \backslash \{0\})$$

then using the Minimax Theorem (Aubin & Cellina, 1991), we can express this relation as

$$\inf_{u \in \mathcal{U}(x)} \theta + \langle \zeta, f(t, x) + g(t, x)u \rangle \leqslant -W(t, x) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t, x) \in \Re^+ \times (\Re^n \backslash \{0\}), (\theta, \zeta) \in \partial_C V(t, x)$$

which is obviously weaker than the corresponding condition used in Rifford (2001). Moreover, if $V(\cdot)$ is C^1 on $\Re^+ \times (\Re^n \backslash \{0\})$, then inequality (2.7) can be expressed as

$$\inf_{u\in\mathcal{U}} \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)(f(t,x)+g(t,x)u)$$
$$\leqslant -W(t,x) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \forall (t,x) \in \Re^+ \times (\Re^n\backslash\{0\}). \tag{2.7''}$$

We next recall the notions of global asymptotic stability. Consider the system

$$\dot{x} = f(t,x), \quad x \in \Re^n, t \geqslant 0 \tag{2.9}$$

where $f : \Re^+ \times \Re^n \to \Re^n$ is measurable in $t \geqslant 0$ and locally Lipschitz in $x \in \Re^n$, satisfying $f(t,\ 0) = 0$, for all $t \geqslant 0$. Let us denote its solution by $x(t)$ initiated from x_0 at time t_0. We say that $0 \in \Re^n$ is *(non-uniformly in time) GAS* with respect to (2.9), if for any initial (t_0, x_0), $x(\cdot)$, is defined for all $t \geqslant t_0$ and the following conditions hold.

(P1) For any $\varepsilon > 0$ and $T \geqslant 0$, it holds that $\sup\{|x(t)|; t \geqslant t_0, |x_0| \leqslant \varepsilon, t_0 \in [0,T]\} < +\infty$ and there exists a $\delta = \delta(\varepsilon, T) > 0$, such that

$$|x_0| \leqslant \delta, t_0 \in [0,T] \Rightarrow \sup_{t\geqslant t_0} |x(t)| \leqslant \varepsilon \text{ (stability).}$$

(P2) For any $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists a $\tau = \tau(\varepsilon, T, R) \geqslant 0$, such that

$$|x_0| \leqslant R, t_0 \in [0,T] \Rightarrow \sup_{t\geqslant t_0+\tau} |x(t)| \leqslant \varepsilon \text{ (attractivity).}$$

We say that $0 \in \Re^n$ is *uniformly GAS* (UGAS) with respect to (2.9), if for any initial (t_0, x_0), $x(\cdot)$ is defined for all $t \geqslant t_0$ and the following conditions hold.

(P1′) For every $\varepsilon > 0$, it holds that $\sup\{|x(t)|; t \geqslant t_0, |x_0| \leqslant \varepsilon, t_0 \geqslant 0\} < +\infty$ and there exists a $\delta = \delta(\varepsilon) > 0$, such that for all $t_0 \geqslant 0$ it holds that

$$|x_0| \leqslant \delta \Rightarrow \sup_{t\geqslant t_0} |x(t)| \leqslant \varepsilon \text{ (uniform stability).}$$

(P2′) For any $\varepsilon > 0$ and $R \geqslant 0$, there exists a $\tau = \tau(\varepsilon, R) \geqslant 0$, such that for all $t_0 \geqslant 0$ it holds that

$$|x_0| \leqslant R \Rightarrow \sup_{t\geqslant t_0+r} |x(t)| \leqslant \varepsilon \text{ (uniform attractivity).}$$

The following lemma provides Lyapunov-like criteria for global asymptotic stability. Its proof can be found in the Appendix.

LEMMA 2.5 Let $V : \Re^+ \times \Re^n \to \Re^+$ be lower semi-continuous on $\Re^+ \times \Re^n$ and suppose there exist functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \beta(t)\mathrm{d}t = +\infty$, $\mu \in \boldsymbol{\mathcal{E}}$ and $\rho \in C^1(\Re^+; \Re^+)$ being positive definite, such that the following inequalities hold

$$a_1(|x|) \leqslant V(t,x) \leqslant a_2(\gamma(t)|x|), \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{2.10}$$

$$V^0(t,x;\ f(t,x)) \leqslant -\beta(t)\rho(V(t,x)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t,x) \in S \tag{2.11}$$

where the set S is defined by

$$S := \left\{ (t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n;\, a_2(\gamma(t)|x|) \geqslant \eta \left(\int_0^t \beta(s)\mathrm{d}s, 0, c \right) \right\} \tag{2.12}$$

for certain constant $c > 0$ and $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial value problem

$$\begin{aligned} \dot{\eta} &= -\rho(\eta) + \mu(t) \\ \eta(t_0) &= \eta_0 \geqslant 0. \end{aligned} \tag{2.13}$$

Suppose, furthermore, that $f \in C^0(\mathfrak{R}^+ \times \mathfrak{R}^n; \mathfrak{R}^n)$. Then $0 \in \mathfrak{R}^n$ is GAS for system (2.9).

The proof of Lemma 2.5 is based on the following comparison principle (Lemma 2.6) as well as Corollary 2.7. Lemma 2.6 is a direct extension of the corresponding comparison principle given in Khalil (1996) and its proof is given in the Appendix. The proof of Corollary 2.7 is an immediate consequence of Lemma 5.2 in Karafyllis & Tsinias (2003b) and is left to the reader.

LEMMA 2.6 (Comparison principle) Consider the scalar differential equation

$$\begin{aligned} \dot{w} &= f(t, w) \\ w(t_0) &= w_0 \end{aligned} \tag{2.14}$$

where $f(t, w)$ is continuous in $t \geqslant 0$ and locally Lipschitz in $w \in J \subseteq \mathfrak{R}$. Let $[t_0, T)$ be the maximal interval of existence of the solution $w(t)$ and suppose that $w(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a lower semi-continuous and right-continuous function that satisfies the differential inequality

$$Dv(t) \leqslant f(t, v(t)), \quad \forall t \in [t_0, T) \tag{2.15}$$

Suppose, furthermore,

$$v(t_0) \leqslant w_0 \tag{2.16a}$$

$$v(t) \in J, \quad \forall t \in [t_0, T). \tag{2.16b}$$

Then $v(t) \leqslant w(t)$, for all $t \in [t_0, T)$.

COROLLARY 2.7 The solution $\eta(t, t_0, \eta_0)$ of the initial-value problem (2.13), with $\mu \in \mathcal{E}$ and $\rho \in C^1(\mathfrak{R}^+; \mathfrak{R}^+)$ being positive definite, exists for all $t \geqslant t_0$ and there exist a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that the following properties are satisfied for all $t_0 \geqslant 0$:

$$0 \leqslant \eta_0 < \eta_1 \Rightarrow \eta(t, t_0, \eta_0) < \eta(t, t_0, \eta_1), \quad \forall t \geqslant t_0 \tag{2.17a}$$

$$0 \leqslant \eta(t, t_0, \eta_0) \leqslant \sigma(\eta_0 + M, t - t_0), \quad \forall t \geqslant t_0, \quad \forall \eta_0 \geqslant 0. \tag{2.17b}$$

We are now in a position to state our main result.

THEOREM 2.8 The following statements are equivalent:

(i) There exists a CLF for (1.1) and an upper semi-continuous function $W : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}^+$ such that (2.6), (2.7) and (2.8) are satisfied for some functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \beta(t)\mathrm{d}t = +\infty$, $\mu \in \mathcal{E}$ and $\rho : \mathfrak{R}^+ \to \mathfrak{R}^+$ being lower semi-continuous and positive definite.

(ii) There exists a function $k \in C^\infty(\mathfrak{R}^+ \times \mathfrak{R}^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, such that $0 \in \mathfrak{R}^n$ is GAS for the closed-loop system (1.1) with (1.2).

(iii) There exists a function $k \in C^0(\mathfrak{R}^+ \times \mathfrak{R}^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, $k(t, x)$ being locally Lipschitz with respect to $x \in \mathfrak{R}^n$, such that $0 \in \mathfrak{R}^n$ is GAS for the closed-loop system (1.1) with (1.2).

(iv) There exists a CLF for (1.1) of class $C^\infty(\mathfrak{R}^+ \times \mathfrak{R}^n)$ and a function $W : \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}^+$ with $W(\cdot, 0) = 0$ of class $C^\infty(\mathfrak{R}^+ \times \mathfrak{R}^n)$, such that (2.6)–(2.8) are satisfied for some functions $a_1, a_2 \in K_\infty$, $\gamma \in K^+$, $\beta(t) \equiv 1 \in K^+$, $\mu \equiv 0 \in \mathcal{E}$ and $\rho(s) := s$.

REMARK 2.9 We emphasize that Theorem 2.8 gives necessary and sufficient conditions for the existence of an ordinary feedback stabilizer. This explains the difference in the definition of the CLF with the definitions given in Clarke *et al.* (1997, 2000), because in these papers stabilization is achieved in a different way (see Clarke *et al.*, 1997, where the difference is explained). Finally, notice that Corollary 5.4 in Karafyllis & Tsinias (2003b) in conjunction with Theorem 2.8 implies that the existence of a CLF as defined in this paper is a necessary and sufficient condition for the robust stabilization of (1.1), for the case $\mathcal{U} = \mathfrak{R}^m$.

REMARK 2.10 Theorem 2.8 is also valid if in the definition of the CLF the following Dini derivative is used:

$$\overline{V'}(t, x; v) := \limsup_{\substack{(\tau,y)\to(t,x)\\ h\to 0^+\\ w\to v}} \frac{V(\tau + h, y + hw) - V(\tau, y)}{h} \tag{2.18}$$

instead of $V^0(t, x; v)$. It can be proved that Lemma 2.2 holds for this construct. However, we did not use it for two reasons:

(1) It is clear by definitions (2.1) and (2.18) that the following inequality can be established: $V^0(t, x; v) \leqslant \overline{V'}(t, x; v)$, for all $(t, x, v) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^n$.

(2) Using $V^0(t, x; v)$ we have shown clearly the difference between our definition of a CLF and the one used in Clarke *et al.* (1997). Particularly, the difference lies in the operator $\limsup_{\substack{(\tau,y)\to(t,x)\\ (\tau,y)\in A_V\\ w\to v}}$ used in the definition of $V^0(t, x; v)$.

3. Proof of Theorem 2.8

(i) *implies* (ii) Notice first that without loss of generality we may assume that the function ρ involved in (2.8) is of class $C^1(\mathfrak{R}^+; \mathfrak{R}^+)$. If this is not the case then we can replace ρ by any C^1 positive definite function $\tilde{\rho}$ that satisfies $\tilde{\rho}(s) \leqslant \rho(s)$ for all $s \geqslant 0$. By Lemma 2.2, we know that $V^0(t, x; v)$ is upper semi-continuous in (t, x, v) for all $(t, x, v) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^n$. Furthermore, without loss of generality we may assume that (2.7) holds for certain $\mu \in \mathcal{E}$ that satisfies $\mu(t) > 0$ for all $t \geqslant 0$. For convenience we define

$$\phi(t) := \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right)$$

which is clearly a continuous function. We proceed by noticing some facts.

Fact I. For all $(t_0, x_0) \in \Re^+ \times (\Re^n \backslash \{0\})$, there exists $u_0 \in \mathcal{U}$ and a neighbourhood $\mathcal{N}(t_0, x_0) \subset \Re^+ \times (\Re^n \backslash \{0\})$, such that

$$(t, x) \in \mathcal{N}(t_0, x_0) \Rightarrow V^0(t, x; v)|_{v=f(t,x)+g(t,x)u_0} \leqslant -W(t, x) + 8\phi(t). \tag{3.1}$$

Proof of Fact I. By virtue of (2.7) it follows that for all $(t_0, x_0) \in \Re^+ \times (\Re^n \backslash \{0\})$, there exists $u_0 \in \mathcal{U}$ such that

$$V^0(t_0, x_0; \dot{x})|_{\dot{x}=f(t_0,x_0)+g(t_0,x_0)u_0} \leqslant -W(t_0, x_0) + 2\phi(t_0). \tag{3.2}$$

Since $V^0(t, x; v)$ and $W(t, x)$ are upper semi-continuous and since $f \in C^0(\Re^+ \times \Re^n; \Re^n)$, $g \in C^0(\Re^+ \times \Re^n; \Re^{n\times m})$, $\phi \in C^0(\Re^+; (0, +\infty))$, there exists a neighborhood $\mathcal{N}(t_0, x_0) \subset \Re^+ \times (\Re^n \backslash \{0\})$ around (t_0, x_0) such that for all $(t, x) \in \mathcal{N}(t_0, x_0)$

$$\begin{aligned} &V^0(t, x; v)|_{v=f(t,x)+g(t,x)u_0} \leqslant -W(t_0, x_0) + \phi(t_0) \\ &W(t, x) \leqslant W(t_0, x_0) + \phi(t_0) \\ &\phi(t_0) \leqslant 2\phi(t). \end{aligned} \tag{3.3}$$

Therefore, (3.2) and (3.3) imply (3.1) for all $(t, x) \in \mathcal{N}(t_0, x_0)$.

Fact II. There exists a family of open sets $(\Omega_j)_{j\in J}$ with $\Omega_j \subset \Re^+ \times (\Re^n \backslash \{0\})$ for all $j \in J$, which consists a locally finite open covering of $\Re^+ \times (\Re^n \backslash \{0\})$ and a family of points $(u_j)_{j\in J}$ with $u_j \in \mathcal{U}$ for all $j \in J$, such that

$$(t, x) \in \Omega_j \Rightarrow V^0(t, x; v)|_{v=f(t,x)+g(t,x)u_j} \leqslant -W(t, x) + 8\phi(t). \tag{3.4}$$

The proof of this fact is an immediate consequence of Fact I and the obvious inclusion $\Re^+ \times \Re^n \subset \Re^{n+1}$.

Fact III. There exists a $C^\infty(\Re^+ \times \Re^n; \mathcal{U})$ function $k(t, x)$ with $k(\cdot, 0) = 0$ such that

$$V^0(t, x; v)|_{v=f(t,x)+g(t,x)k(t,x)} \leqslant -W(t, x) + 8\phi(t), \quad \forall (t, x) \in S \tag{3.5}$$

where the set S is defined in (2.12) for certain constant $c > 0$ and $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial value problem

$$\begin{aligned} &\dot{\eta} = -\rho(\eta) + 8\mu(t) \\ &\eta(t_0) = \eta_0 \geqslant 0. \end{aligned} \tag{3.6}$$

Proof. By virtue of Fact II and standard partition of unity arguments, there exists a family of functions $\theta_0 : \Re^+ \times \Re^n \to [0, 1]$, $\theta_j : \Re^+ \times \Re^n \to [0, 1]$, with $\theta_j(t, x) = 0$ if $(t, x) \notin \Omega_j \subset \Re^+ \times (\Re^n \backslash \{0\})$ and $\theta_0(t, x) = 0$ if $(t, x) \in S$, $\theta_0(t, x) + \sum_j \theta_j(t, x)$ being locally finite and $\theta_0(t, x) + \sum_j \theta_j(t, x) = 1$ for all $(t, x) \in \Re^+ \times \Re^n$. We set

$$k(t, x) := \sum_j \theta_j(t, x)u_j. \tag{3.7}$$

Notice that $(t, 0) \notin \Omega_j$ for all $j \in J$ and consequently by definition (3.7) we have $k(t, 0) = 0$ for all $t \geqslant 0$. Since each u_j is a member of the convex set $\mathcal{U}$ and $0 \in \mathcal{U}$, it follows from (3.7) that $k(t, x) \in \mathcal{U}$ for all $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$. It also follows from (2.2) and (3.7) that for all $(t, x) \in S$ and $J'(t, x) = \{j \in J; \theta_j(t, x) \neq 0\}$

$$\begin{aligned} V^0(t, x; v)|_{v=f(t,x)+g(t,x)k(t,x)} &= V^0(t, x; v)|_{v=\sum\limits_{i \in J'(t,x)} \theta_i(t,x)(f(t,x)+g(t,x)u_i)} \\ &\leqslant \sum_{i \in J'(t,x)} \theta_i(t, x) V^0(t, x; v_i)|_{v_i=f(t,x)+g(t,x)u_i} \end{aligned} \tag{3.8}$$

where the last inequality follows from the fact that $\theta_0(t, x) + \sum_j \theta_j(t, x)$ is locally finite and $V^0(t, x; v_i)|_{v_i=f(t,x)+g(t,x)u_i} < +\infty$ for all $i \in J'(t, x)$. Combining (3.4) with (3.8), we have the desired (3.5).

Now consider the trajectory $x(t)$ of the solution of the closed-loop system (1.1) with (1.2), namely

$$\dot{x} = f(t, x) + g(t, x)k(t, x). \tag{3.9}$$

It follows from (2.8) and (3.9) that the following inequality holds:

$$\begin{aligned} V^0(t, x;\ f(t, x) + g(t, x)k(t, x)) &\leqslant -\beta(t)\rho(V(t, x)) \\ &\quad +8\beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad \forall (t, x) \in S. \end{aligned} \tag{3.10}$$

Since $8\mu(\cdot) \in \mathcal{E}$, Lemma 2.5 and (3.10) guarantee that $0 \in \mathfrak{R}^n$ is GAS for (3.9).

(ii) *implies* (iii). This implication is obvious.

(iii) *implies* (iv). By Theorem 3.1 in Karafyllis & Tsinias (2003b) we have that there exists a C^∞ function $V(\cdot)$, functions $a_1, a_2 \in K_\infty$, $\gamma \in K^+$, such that (2.6) is satisfied, as well as the following inequality for all $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$:

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) + \frac{\partial V}{\partial x}(t, x)g(t, x)k(t, x) \leqslant -V(t, x). \tag{3.11}$$

It is clear that (2.7) and (2.8) are satisfied for

$$\begin{aligned} W(t, x) &\equiv V(t, x), \quad \forall (t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n \\ \beta(t) &\equiv 1, \quad \forall t \geqslant 0 \\ \mu(t) &\equiv 0, \quad \forall t \geqslant 0 \\ \rho(s) &= s, \quad \forall s \geqslant 0. \end{aligned} \tag{3.12}$$

(iv) *implies* (i). This implication is obvious.

The proof is complete. □

4. Some results on non-affine systems

The following proposition gives necessary and sufficient conditions for the existence of a time-varying stabilizer for systems (1.3).

PROPOSITION 4.1 Consider the system (1.3), where $a : \Re^+ \times \Re^n \times \Re \to \Re$ is a C^0 function with $a(t,\ x,\ 0) = 0$ for all $(t, x) \in \Re^+ \times \Re^n$, which is locally Lipschitz with respect to (x, u) and in such a way that there exist functions $a^{-1} : \Re^+ \times (\Re^n \backslash\{0\}) \times (\Re\backslash\{0\}) \to \Re$, which is of class $C^\infty(\Re^+ \times (\Re^n\backslash\{0\}) \times (\Re\backslash\{0\}))$ and $\rho \in C^\infty(\Re^+ \times \Re^n; \Re^+)$ such that

$$a\left(t, x, a^{-1}(t, x, u)\right) = u, \quad \forall (t, x, u) \in \Re^+ \times (\Re^n\backslash\{0\}) \times (\Re\backslash\{0\}) \tag{4.1}$$

$$|a(t, x, \lambda u)| \leqslant \rho(t, x)|a(t, x, u)|, \quad \forall (t, x, u) \in \Re^+ \times \Re^n \times \Re, \quad \forall \lambda \in [0,\ 1]. \tag{4.2}$$

Then the following statements are equivalent:

(i) There exists a C^0 function $k(t, x)$ with $k(t,\ 0) = 0$ for all $t \geqslant 0$, which is locally Lipschitz with respect to x, such that $0 \in \Re^n$ is GAS for the closed-loop system (1.1) with (1.2).

(ii) There exists a C^∞ function $\tilde{k} : \Re^+ \times \Re^n \to \Re$ with $\tilde{k}(t,\ 0) = 0$ for all $t \geqslant 0$, such that $0 \in \Re^n$ is GAS for (1.3) with

$$u = \tilde{k}(t, x). \tag{4.3}$$

Proof (i) $\Rightarrow$ (ii). Since system (1.1) is stabilizable by a locally Lipschitz time-varying feedback law, then by Theorem 2.8, there exists a C^∞ function $\bar{k} : \Re^+ \times \Re^n \to \Re$ with $\bar{k}(t,\ 0) = 0$ for all $t \geqslant 0$, such that $0 \in \Re^n$ is GAS with respect to (1.1) with

$$u = \tilde{k}(t, x). \tag{4.4}$$

Furthermore, by Theorem 3.1 in Karafyllis & Tsinias (2003b), there exists a C^∞ Lyapunov function $V : \Re^+ \times \Re^n \to \Re^+$, a pair of K_∞ functions a_1, a_2 and a function β of class K^+, such that for all $(t, x) \in \Re^+ \times \Re^n$ we have

$$a_1(|x|) \leqslant V(t, x) \leqslant a_2(\beta(t)|x|) \tag{4.5a}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) + \frac{\partial V}{\partial x}(t, x) g(t, x)\bar{k}(t, x) \leqslant -V(t, x). \tag{4.5b}$$

Let $\mu : \Re^+ \times \Re^n \to (0, +\infty)$ be a positive C^∞ function that satisfies for all $(t, x) \in \Re^+ \times \Re^n$:

$$\mu(t, x) \leqslant \frac{V(t, x) + 2\exp(-t)}{4\left(1 + \left|\frac{\partial V}{\partial x}(t, x) g(t, x)\right|\right)}. \tag{4.5c}$$

Let $\theta : \Re \to [0,\ 1]$ be a C^∞ function that satisfies $\theta(s) = 1$ for $|s| \geqslant 1$ and $\theta(s) = 0$ for $|s| \leqslant \frac{1}{2}$. Notice that by (4.5b), (4.5c) we have

$$|\bar{k}(t, x)| \leqslant \mu(t, x) \Rightarrow \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) \leqslant -\frac{3}{4} V(t, x) + \frac{1}{2}\exp(-t). \tag{4.5d}$$

Define

$$\tilde{k}(t, x) := \begin{cases} \theta\left(\dfrac{1 + \rho(t, x)}{\mu(t, x)} \bar{k}(t, x)\right) a^{-1}(t, x, \bar{k}(t, x)) & \text{for } \bar{k}(t, x) \neq 0 \\ 0 & \text{for } \bar{k}(t, x) = 0 \end{cases} \tag{4.6}$$

where $\rho(\cdot)$ is the function involved in (4.2). Notice that $\tilde{k}$ is a C^{∞} function that satisfies $\tilde{k}(t,\ 0) = 0$ for $|\overline{k}(t,x)| \leqslant \frac{\mu(t,x)}{2(1+\rho(t,x))}$.

Claim: For all $(t,x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ it holds that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) + \frac{\partial V}{\partial x}(t,x) g(t,x) a(t,x,\tilde{k}(t,x)) \leqslant -\frac{1}{2} V(t,x) + \exp(-t). \tag{4.7}$$

To prove the claim, consider the following cases.

(I) $|\overline{k}(t,x)| \geqslant \frac{\mu(t,x)}{1+\rho(t,x)}$.
Then by definition (4.6) we have: $a(t,x,\tilde{k}(t,x)) = \overline{k}(t,x)$. Consequently for this case (4.5b) implies (4.7).

(II) $0 < |\overline{k}(t,x)| \leqslant \frac{\mu(t,x)}{1+\rho(t,x)}$.
Since $\theta\left(\frac{1+\rho(t,x)}{\mu(t,x)}\overline{k}(t,x)\right) \leqslant 1$, by virtue of (4.1), (4.2) and definition (4.6) it follows that $|a(t,x,\tilde{k}(t,x))| \leqslant \rho(t,x)|\overline{k}(t,x)| \leqslant \mu(t,x)$. Clearly, by virtue of (4.5c) we have

$$\begin{aligned} \frac{\partial V}{\partial x}(t,x) g(t,x) a(t,x,\tilde{k}(t,x)) &\leqslant \left| \frac{\partial V}{\partial x}(t,x) g(t,x) \right| |a(t,x,\tilde{k}(t,x))| \\ &\leqslant \left(1 + \left| \frac{\partial V}{\partial x}(t,x) g(t,x) \right| \right) \mu(t,x) \\ &\leqslant \frac{1}{4} V(t,x) + \frac{1}{2}\exp(-t). \end{aligned} \tag{4.8}$$

Furthermore, in this case we have $0 < |\overline{k}(t,x)| \leqslant \frac{\mu(t,x)}{1+\rho(t,x)} \leqslant \mu(t,x)$ and consequently (4.5d) holds. Clearly, (4.5d) in conjunction with (4.8) implies (4.7).

(III) $\overline{k}(t,x) = 0$.
In this case by definition (4.6) we have $\tilde{k}(t,x) = 0$ and consequently $a(t,x,\tilde{k}(t,x)) = 0$. Clearly in this case (4.5b) implies (4.7).

Now consider the solution $x(t)$ of (1.3) with $u = \tilde{k}(t,x)$, initiated at time $t_0 \geqslant 0$ from $x_0 \in \mathfrak{R}^n$. Inequalities (4.5a) and (4.7) imply the following estimate

$$|x(t)| \leqslant a_1^{-1}\left(\exp\left(-\frac{1}{2}(t-t_0)\right)(2 + a_2(\beta(t_0)|x_0|))\right), \quad \forall t \geqslant t_0. \tag{4.9}$$

We define for all $(t, t_0, s) \in (\mathfrak{R}^+)^3$ the continuous function

$$\Delta(s,t_0,t) := \begin{cases} a_1^{-1}\left(\exp\left(-\frac{1}{2}(t-t_0)\right)(2 + a_2(\overline{\beta}(t_0)s))\right) & \text{if } t \geqslant t_0 \\ a_1^{-1}(2 + a_2(\overline{\beta}(t_0)s)) & \text{if } t < t_0 \end{cases}$$

where $\overline{\beta}(t) := \max_{0 \leqslant \tau \leqslant t} \beta(\tau)$. Notice that by virtue of (4.9) and the definition above we obtain

$$|x(t)| \leqslant \Delta(|x_0|, t_0, t), \quad \forall t \geqslant t_0.$$

Using Lemma 2.5 in Karafyllis & Tsinias (2003a), we conclude that $0 \in \mathfrak{R}^n$ is GAS for (1.3) with $u = \tilde{k}(t,x)$.

(ii) ⇒ (i). Simply define $k(t,x) := a(t,x,\tilde{k}(t,x))$. The rest of proof is obvious. □

EXAMPLE 4.2 Consider system (1.3) for the so-called 'power-integrator' case, i.e. the case $a(t, x, u) = u^p$, where p is an odd positive integer, namely

$$\begin{aligned} &\dot{x} = f(t, x) + g(t, x)u^p \\ &x \in \Re^n, t \geqslant 0, u \in \Re. \end{aligned} \tag{4.10}$$

Clearly we have

$$|a(t, x, \lambda u)| = \lambda^p |u|^p \leqslant |u|^p = |a(t, x, u)|, \quad \forall (t, x, u) \in \Re^+ \times \Re^n \times \Re, \quad \forall \lambda \in [0,\ 1] \tag{4.11}$$

and consequently (4.2) is satisfied for $\rho(t, x) \equiv 1$. Moreover, we define

$$a^{-1}(t, x, u) := \mathrm{sgn}(u)|u|^{\frac{1}{p}} \tag{4.12}$$

which is of class $C^\infty(\Re\backslash\{0\})$ and notice that (4.1) also holds. Therefore, by virtue of Proposition 4.1 we conclude that (4.10) is stabilizable at zero if and only if (1.1) is stabilizable at zero.

5. Comments on the issue of stabilization by means of time-varying feedback

Using the main result in Bacciotti & Rosier (2001) we may establish that all control systems that can be uniformly stabilized by means of continuous time-varying feedback can also be (non-uniformly) stabilized by means of smooth time-varying feedback. Specifically, consider the system

$$\begin{aligned} &\dot{x} = f(t, x, u) \\ &x \in \Re^n, t \geqslant 0, u \in \mathcal{U} \end{aligned} \tag{5.1}$$

where $\mathcal{U} \subseteq \Re^m$ is a convex set with $0 \in \mathcal{U}$, $f(\cdot) \in C^0(\Re^+ \times \Re^n \times \Re^m; \Re^n)$ is locally Lipschitz with respect to (x, u) with $f(t,\ 0,\ 0) = 0$ for all $t \geqslant 0$. Then we have the following proposition.

PROPOSITION 5.1 Suppose that there exists a function $\tilde{k} \in C^0(\Re^+ \times \Re^n; \mathcal{U})$, such that $0 \in \Re^n$ is UGAS for the closed-loop system (5.1) with $u = \tilde{k}(t, x)$. Then there exists a function $k \in C^\infty(\Re^+ \times \Re^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, such that $0 \in \Re^n$ is GAS for the closed-loop system (5.1) with $u = k(t, x)$.

Proof. Using Theorem 4.5 in Bacciotti & Rosier (2001), there exists a C^∞ Lyapunov function $V : \Re^+ \times \Re^n \to \Re^+$, and a pair of K_∞ functions a_1, a_2 such that for all $(t, x) \in \Re^+ \times \Re^n$ we have

$$a_1(|x|) \leqslant V(t, x) \leqslant a_2(|x|) \tag{5.1a}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, \tilde{k}(t, x)) \leqslant -V(t, x). \tag{5.1b}$$

Following the proof of Lemma 2.7 in Karafyllis & Tsinias (2003c), we may conclude that there exists a pair of K_∞ functions a_3, a_4 and a function $\kappa(\cdot) \in K^+$ such that for all $(t, x) \in \Re^+ \times \Re^n$ we have

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, \tilde{k}(t, x) + v) \leqslant -V(t, x) + a_3(|x|)a_4(\kappa(t)|v|). \tag{5.2}$$

We define the continuous, positive functions

$$\theta(t,s) := \begin{cases} \dfrac{\exp(-t)}{a_3(1)} & \text{if } 0 \leqslant s \leqslant 1 \\ \dfrac{\exp(-t)}{a_3(s)} & \text{if } s > 1 \end{cases}$$

$$\gamma(t,s) := \frac{1}{\kappa(t)} a_4^{-1}(\theta(t,s)) \tag{5.3}$$

and notice that by virtue of inequality (5.2) and definitions (5.3) we obtain

$$|u - \tilde{k}(t,x)| \leqslant \gamma(t,|x|) \Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,u) \leqslant -V(t,x) + \exp(-t). \tag{5.4}$$

By standard partition of unity arguments, we obtain the existence of a function $k \in C^\infty(\mathfrak{R}^+ \times \mathfrak{R}^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, such that

$$\left| k(t,x) - \tilde{k}(t,x) \right| \leqslant \gamma(t,|x|),$$
$$\forall (t,x) \in S := \left\{ (t,x) \in \mathfrak{R}^+ \times \mathfrak{R}^n;\, a_2(|x|) \geqslant (t+1)\exp(-t) \right\}. \tag{5.5}$$

The rest is a consequence of Lemma 2.5. □

The question that arises is, does the converse of Proposition 5.1 hold? The answer to this question is negative, as the following example shows. There exist systems that can be (non-uniformly) stabilized by means of a smooth time- varying feedback and cannot be uniformly stabilized by means of a continuous time-varying feedback.

EXAMPLE 5.2 Consider the system

$$\begin{aligned} \dot{x} &= \exp(t)x + y \\ \dot{y} &= u \\ (x,y) &\in \mathfrak{R}^2, u \in \mathfrak{R}, t \geqslant 0. \end{aligned} \tag{5.6}$$

Suppose that there exists a function $\tilde{k} \in C^0(\mathfrak{R}^+ \times \mathfrak{R}^2; \mathfrak{R})$, with $\tilde{k}(\cdot, 0, 0) = 0$, such that $0 \in \mathfrak{R}^2$ is UGAS for the closed-loop system (5.6) with $u = \tilde{k}(t,x,y)$. Then there exists a function $a(\cdot) \in K_\infty$ such that for every trajectory $(x(t), y(t))$ of the closed-loop system (5.6) with $u = \tilde{k}(t,x,y)$, initiated at time $t_0 \geqslant 0$ from $(x_0, y_0) \in \mathfrak{R}^2$, the following estimate holds

$$|(x(t), y(t))| \leqslant a(|(x_0, y_0)|), \quad \forall t \geqslant t_0. \tag{5.7}$$

Let $s > 0$ be a positive constant and consider any trajectory $(x(t), y(t))$ of the closed-loop system (5.6) with $u = \tilde{k}(t,x,y)$, initiated at time $t_0 \geqslant 0$ from $(x_0, y_0) = (s, 0)$. Clearly the set

$$N := \{t \geqslant t_0;\, \exp(t)x(t) + y(t) < 0\} \tag{5.8}$$

cannot be empty (because otherwise we would have $\dot{x}(t) \geqslant 0$ for all $t \geqslant t_0$ and consequently $\liminf_{t\to+\infty} |x(t)| \geqslant s > 0$, which contradicts our assumptions). Let

$$T := \inf\{t \in N\}. \tag{5.9}$$

In other words T is the first time that we have $\dot{x}(t) \leqslant 0$. Notice that due to our assumptions we have $T > t_0$ and furthermore by continuity of the solution we obtain: $\exp(T)x(T) = -y(T)$. Moreover, since $\dot{x}(t) \geqslant 0$, for all $t \in [t_0, T]$, we have $|x(T)| = x(T) \geqslant s$. By virtue of these observations and inequality (5.7) we obtain

$$\sqrt{1+\exp(2t_0)}s \leqslant \sqrt{1+\exp(2T)}|x(T)| = |(x(T), y(T))| \leqslant a(s).$$

The latter inequality implies that for all $t_0 \geqslant 0$ we must have $\sqrt{1+\exp(2t_0)} \leqslant \frac{a(s)}{s}$, which obviously cannot hold. Thus there is no function $\tilde{k} \in C^0(\Re^+ \times \Re^2; \Re)$, with $\tilde{k}(\cdot\, 0, 0) = 0$, such that $0 \in \Re^2$ is UGAS for the closed-loop system (5.6) with $u = \tilde{k}(t, x, y)$. On the other hand, there exists a function $k \in C^\infty(\Re^+ \times \Re^2; \Re)$, with $k(\cdot, 0, 0) = 0$, such that $0 \in \Re^2$ is (non-uniformly in time) GAS for the closed-loop system (5.6) with $u = k(t, x, y)$. To see this, consider the function

$$V(t, x, y) := 3\exp(2t)x^2 + \frac{1}{2}(y + 2\exp(t)x)^2. \tag{5.10}$$

Clearly, $V(\cdot)$ is of class $C^\infty(\Re^+ \times \Re^2)$ and satisfies the estimate

$$\frac{1}{4}(x^2+y^2) \leqslant V(t, x, y) \leqslant 6\exp(2t)(x^2+y^2). \tag{5.11}$$

It is obvious that the following estimates hold:

$$y \neq -2\exp(t)x \Rightarrow$$
$$\inf_{u\in\Re}\left(\frac{\partial V}{\partial t}(t, x, y) + \frac{\partial V}{\partial x}(t, x, y)(\exp(t)x + y) + \frac{\partial V}{\partial y}(t, x, y)u\right) = -\infty \tag{5.12a}$$
$$y = -2\exp(t)x \Rightarrow$$
$$\inf_{u\in\Re}\left(\frac{\partial V}{\partial t}(t, x, y) + \frac{\partial V}{\partial x}(t, x, y)(\exp(t)x + y) + \frac{\partial V}{\partial y}(t, x, y)u\right)$$
$$= -2(\exp(t)-1)V(t, x, y) \leqslant -\frac{V(t, x, y)}{1+V^2(t, x, y)} + \exp(-t). \tag{5.12b}$$

Thus, by virtue of (5.11), (5.12a) and (5.12b), $V(\cdot)$ is a CLF for (5.6) and satisfies (2.6)–(2.8) with $a_1(s) = \frac{1}{4}s^2$, $a_2(s) = 6s^2$, $\gamma(t) = \exp(t)$, $\beta(t) \equiv 1$, $\rho(s) = \frac{s}{1+s^2}$, $\mu(t) = \exp(-t)$ and $W(t, x, y) = \rho(V(t, x, y))$. Consequently, Theorem 2.8 implies the existence of a function $k \in C^\infty(\Re^+ \times \Re^2; \Re)$, with $k(\cdot, 0, 0) = 0$, such that $0 \in \Re^2$ is (non-uniformly in time) GAS for the closed-loop system (5.6) with $u = k(t, x, y)$.

However, notice that for the above example the dynamics of the system are time-varying and not bounded with respect to $t \geqslant 0$. We do not know if the converse of Proposition 5.1 holds for autonomous control systems. This is an open problem in mathematical control theory.

At this point, we emphasize the fact that for Proposition 5.1, it is not required that the uniform stabilizer $\tilde{k} \in C^0(\Re^+ \times \Re^n; \mathcal{U})$ vanishes at zero. However, for uniform global asymptotic stability it is necessary that $0 \in \Re^n$ is an equilibrium point for (5.1): namely, we must have $f(t, 0, \tilde{k}(t, 0)) = 0$ for all $t \geqslant 0$. This requirement may be guaranteed by the structure of the vector field $f(\cdot) \in C^0(\Re^+ \times \Re^n \times \Re^m; \Re^n)$, as the following example shows.

EXAMPLE 5.3 Consider the system

$$\begin{aligned} &\dot{x} = x + xu \\ &x \in \Re, u \in [-2, 0]. \end{aligned} \tag{5.13}$$

Notice that the feedback law $u = \tilde{k}(t, x) := -2$, globally uniformly asymptotically stabilizes the equilibrium point $x = 0$ of system (5.13). Moreover, system (5.13) cannot be stabilized by a continuous time invariant feedback $\tilde{k}(t, x) = \tilde{k}(x)$ that vanishes at zero. By virtue of Proposition 5.1 there exists a function $k \in C^\infty(\Re^+ \times \Re; [-2, 0])$, with $k(\cdot, 0) = 0$, such that $x = 0$ is GAS for the closed-loop system (5.13) with $u = k(t, x)$. For example, the following feedback law:

$$k(t, x) := \begin{cases} -\exp(t)x^2 & \text{if } \exp(t)x^2 \leqslant 2 \\ \quad -2 & \text{if } \exp(t)x^2 > 2 \end{cases} \tag{5.14}$$

is locally Lipschitz on $\Re^+ \times \Re$, taking values in $[-2, 0]$ with $k(\cdot, 0) = 0$. Moreover, if we define $V(x) = \frac{1}{2}x^2$ then immediate calculations show that

$$\dot{V}|_{(5,13)} \leqslant -\rho(V(x)) + \frac{1}{2}\exp(-t), \quad \forall (t, x) \in \Re^+ \times \Re \tag{5.15}$$

where $\rho(s) := 2\min\{s, s^2\}$. Thus by virtue of Lemma 2.5, the feedback law given by (5.14) globally asymptotically stabilizes $x = 0$ for (5.13).

Next we consider the following problem: Suppose that system (1.1) is dynamically stabilizable. Is system (1.1) stabilizable by a state feedback of the form (1.2)? In order to answer this question, we first have to give the precise definition of dynamic stabilization.

DEFINITION 5.4 We say that (1.1) is dynamically stabilizable if there exist functions $\tilde{k} \in C^0(\Re^+ \times \Re^n \times \Re^l; \mathcal{U})$, $h \in C^0(\Re^+ \times \Re^n \times \Re^l; \Re^l)$, with $\tilde{k}(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$ for all $t \geqslant 0$, $\tilde{k}(t, x, w)$ and $h(t, x, w)$ being locally Lipschitz with respect to (x, w) such that the origin $(x, w) = (0, 0) \in \Re^n \times \Re^l$ is GAS for the system

$$\begin{aligned} &\dot{x} = f(t, x) + g(t, x)\tilde{k}(t, x, w) \\ &\dot{w} = h(t, x, w) \\ &x \in \Re^n, w \in \Re^l, t \geqslant 0. \end{aligned} \tag{5.16}$$

If system (1.1) is dynamically stabilizable, then by virtue of Theorem 2.8, there exists a function $\Psi(\cdot) \in C^1(\Re^+ \times \Omega; \Re^+)$, where $\Omega := (\Re^n \backslash \{0\}) \times \Re^l$, functions $a_1, a_2 \in K_\infty$,

$\beta, \gamma \in K^+$ with $\int_0^{+\infty} \beta(s)\mathrm{d}s = +\infty$, $\mu \in \mathcal{E}$ and $\rho(\cdot) \in C^0(\Re^+; \Re^+)$ being positive definite such that

$$a_1(|(x, w)|) \leqslant \Psi(t, x, w) \leqslant a_2(\gamma(t)|(x, w)|), \quad \forall (t, x, w) \in \Re^+ \times \Re^n \times \Re^l \tag{5.17a}$$

$$\frac{\partial \Psi}{\partial t}(t, x, w) + \frac{\partial \Psi}{\partial x}(t, x, w)(f(t, x) + g(t, x)\tilde{k}(t, x, w)) + \frac{\partial \Psi}{\partial w}(t, x, w)h(t, x, w)$$
$$\leqslant -\beta(t)\rho(\Psi(t, x, w)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right)$$
$$\text{for all } (t, x, w) \in \Re^+ \times \Omega. \tag{5.17b}$$

The following Lemma shows the existence of a locally Lipschitz function $V(\cdot) : \Re^+ \times \Re^n \to \Re^+$, that can be regarded 'almost' as a CLF for (1.1). Its proof can be found in the Appendix.

LEMMA 5.5 Consider the function defined as

$$V(t, x) := \inf_{w \in \Re^l} \Psi(t, x, w). \tag{5.18}$$

Then $V(\cdot) : \Re^+ \times \Re^n \to \Re^+$ is continuous everywhere, locally Lipschitz on $\Re^+ \times (\Re^n \backslash \{0\})$ and satisfies

$$a_1(|x|) \leqslant V(t, x) \leqslant a_2(\gamma(t)|x|), \quad \forall (t, x) \in \Re^+ \times \Re^n \tag{5.19a}$$

$$DV(t, x; v) \leqslant \min_{w \in \mathcal{M}(t,x)} \left(\frac{\partial \Psi}{\partial t}(t, x, w) + \frac{\partial \Psi}{\partial x}(t, x, w)v\right),$$
$$\forall (t, x, v) \in \Re^+ \times (\Re^n \backslash \{0\}) \times \Re^n \tag{5.19b}$$

$$V^0(t, x, v) \leqslant -\beta(t)\rho(V(t, x)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right)$$
$$+ \max_{w \in \mathcal{M}(t,x)} \left(\frac{\partial \Psi}{\partial x}(t, x, w)(v - f(t, x) - g(t, x)\tilde{k}(t, x, w))\right)$$
$$\text{for all } (t, x, v) \in \Re^+ \times (\Re^n \backslash \{0\}) \times \Re^n \tag{5.19c}$$

where the set-valued map

$$\mathcal{M}(t, x) := \{w \in \Re^l : V(t, x) = \Psi(t, x, w)\} \tag{5.20}$$

is non-empty, with compact images and upper semi-continuous.

The existence of $V(\cdot)$ with the above properties cannot guarantee the existence of a function $k \in C^0(\Re^+ \times \Re^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, $k(t, x)$ being locally Lipschitz with respect to $x \in \Re^n$, such that $0 \in \Re^n$ is GAS for the closed-loop system (1.1) with (1.2). However, a necessary and sufficient condition for the existence of a state stabilizer is that the set-valued map $\mathcal{M}(t, x)$, as defined by (5.20), is a singleton. Specifically, we have the following proposition.

PROPOSITION 5.6 The following statements are equivalent:

(i) There exists a function $k \in C^0(\Re^+ \times \Re^n; \mathcal{U})$, with $k(\cdot, 0) = 0$, $k(t, x)$ being locally Lipschitz with respect to $x \in \Re^n$ such that $0 \in \Re^n$ is GAS for the closed-loop system (1.1) with (1.2).

(ii) System (1.1) is dynamically stabilizable and there exist functions $\Psi(\cdot) \in C^\infty(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$, $\gamma \in K^+$, $\tilde{k} \in C^\infty(\Re^+ \times \Re^n \times \Re^l; \mathcal{U})$, $h \in C^\infty(\Re^+ \times \Re^n \times \Re^l; \Re^l)$ with $\tilde{k}(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$ for all $t \geqslant 0$, such that inequalities (5.17a), (5.17b) hold for all $(t, x, w) \in \Re^+ \times \Re^n \times \Re^l$ with $\rho(s) := s$, $\mu(t) \equiv 0$, $\beta(t) \equiv 1$. Moreover, the set-valued map $\mathcal{M}(t, x)$, as defined by (5.20), is a singleton.

(iii) There exist functions $\Psi(\cdot) \in C^1(\Re^+ \times \Omega; \Re^+)$, $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \beta(t)\mathrm{d}t = +\infty$, $\tilde{k} \in C^0(\Re^+ \times \Omega; \mathcal{U})$, $h : \Re^+ \times \Omega \to \Re^l$ being locally bounded, $\mu \in \mathcal{E}$ and $\rho(\cdot) \in C^0(\Re^+; \Re^+)$ being positive definite, such that inequalities (5.17a), (5.17b) hold. Moreover, the set-valued map $\mathcal{M}(t, x)$, as defined by (5.20), is a singleton.

Proof (i) $\Rightarrow$ (ii). Notice that by virtue of the equivalence of statements (ii) and (iii) of Theorem 2.8, we may suppose that $k \in C^\infty(\Re^+ \times \Re^n; \mathcal{U})$. By virtue of Theorem 3.1 in Karafyllis & Tsinias (2003b) there exists a function $V(\cdot) \in C^\infty(\Re^+ \times \Re^n; \Re^+)$, functions $\tilde{a}_1, \tilde{a}_2 \in K_\infty$, $\tilde{\gamma} \in K^+$ such that for all $(t, x) \in \Re^+ \times \Re^n$ we have

$$\tilde{a}_1(|x|) \leqslant V(t, x) \leqslant \tilde{a}_2(\tilde{\gamma}(t)|x|) \tag{5.21a}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)(f(t, x) + g(t, x)k(t, x)) \leqslant -V(t, x). \tag{5.21b}$$

Define for all $w \in \Re^l$

$$\begin{aligned} &\Psi(t, x, w) := V(t, x) + \frac{1}{2}|w|^2 \\ &\tilde{k}(t, x, w) := k(t, x) \\ &h(t, x, w) := -w \end{aligned} \tag{5.22}$$

and notice that inequalities (5.17a), (5.17b) are satisfied with $\rho(s) := s$, $\beta(t) \equiv 1 \in K^+$, $\mu(t) \equiv 0 \in \mathcal{E}$ and $a_1(s) := \min\left\{\tilde{a}_1\left(\frac{s}{2}\right), \frac{1}{8}s^2\right\}$, $a_2(s) := \tilde{a}_2(s) + \frac{1}{2}s^2$, $\gamma(t) := \tilde{\gamma}(t) + 1$. Moreover, the set-valued map $\mathcal{M}(t, x)$, as defined by (5.20), is a singleton, since we have $\mathcal{M}(t, x) = \{0 \in \Re^l\}$.

(ii) $\Rightarrow$ (iii). This implication is obvious.

(iii) $\Rightarrow$ (i). Let $V(\cdot) : \Re^+ \times \Re^n \to \Re^+$ be the locally Lipschitz function on $\Re^+ \times (\Re^n \backslash \{0\})$ defined by (5.18). Clearly, upper semi-continuity of $\mathcal{M}(t, x)$ implies the existence of a continuous function $\varphi : \Re^+ \times \Re^n \to \Re^l$ such that $\mathcal{M}(t, x) = \{\varphi(t, x)\}$. Clearly, by virtue of (5.19c), for all $(t, x) \in \Re^+ \times (\Re^n \backslash \{0\})$ we have

$$\begin{aligned} \inf_{u \in \mathcal{U}} V^0(t, x;\ f(t, x) + g(t, x)u) &\leqslant V^0(t, x;\ f(t, x) + g(t, x)\tilde{k}(t, x, \varphi(t, x))) \\ &\leqslant -\beta(t)\rho(V(t, x)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right). \end{aligned}$$

Thus $V(\cdot) : \Re^+ \times \Re^n \to \Re^+$ is a CLF for (1.1). The rest is a consequence of Theorem 2.8. $\square$

REMARK 5.7 Notice that statement (iii) can be replaced by the weaker hypothesis that instead of (5.17b) the following inequality holds for all $(t, x, w) \in \Re^+ \times \Omega$:

$$\frac{\partial \Psi}{\partial t}(t,x,w) + \frac{\partial \Psi}{\partial x}(t,x,w)(f(t,x) + g(t,x)\tilde{k}(t,x,w)) + \frac{\partial \Psi}{\partial w}(t,x,w)h(t,x,w)$$

$$\leqslant -\beta(t)\rho(V(t,x)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right) \qquad (5{\cdot}17\mathrm{b})$$

where $V(\cdot)$ is defined by (5.18).

EXAMPLE 5.8 Suppose that $0 \in \Re^n \times \Re^l$ is GAS for the linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)(k_1(t)x + k_2(t)w) \\ \dot{w} &= C_1(t)x + C_2(t)w \\ x &\in \Re^n,\, w \in \Re^l,\, t \geqslant 0 \end{aligned} \qquad (5.23)$$

where $A(\cdot)$, $B(\cdot)$, $k_1(\cdot)$, $k_2(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$ are matrices of dimensions $n \times n$, $n \times m$, $m \times n$, $m \times l$, $l \times n$, $l \times l$, respectively, whose elements are continuous functions. Then Proposition 2.3 in Karafyllis & Tsinias (2003) guarantees the existence of a C^1 positive definite matrix $P(t) := \begin{bmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & P_3(t) \end{bmatrix}$ (^{T} denotes transposition) and a function $\beta(\cdot) \in K^+$ with $\int_0^{+\infty} \beta(t)\mathrm{d}t = +\infty$ such that for the quadratic Lyapunov function $\Psi(t,x,w) := x^T P_1(t)x + 2x^T P_2(t)w + w^T P_3(t)w$ and for all $(t,x,w) \in \Re^+ \times \Re^n \times \Re^l$ it holds that

$$\Psi(t,x,w) \geqslant |x|^2 + |w|^2 \qquad (5.24\mathrm{a})$$

$$\frac{\partial \Psi}{\partial t}(t,x,w) + \frac{\partial \Psi}{\partial x}(t,x,w)(A(t)x + B(t)k_1(t)x + B(t)k_2(t)w)$$
$$+\frac{\partial \Psi}{\partial w}(t,x,w)(C_1(t)x + C_2(t)w) \leqslant -2\beta(t)\Psi(t,x,w). \qquad (5.24\mathrm{b})$$

Moreover, the set-valued map $\mathcal{M}(t,x)$, as defined by (5.20), is a singleton, since we have $\mathcal{M}(t,x) = \{-P_3^{-1}(t)P_2^T(t)x\}$. Thus, by virtue of Proposition 5.6 there exists a function $k \in C^0(\Re^+ \times \Re^n; \Re^m)$, with $k(\cdot, 0) = 0$, $k(t,x)$ being locally Lipschitz with respect to $x \in \Re^n$ such that $0 \in \Re^n$ is GAS for the following system:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)k(t,x) \\ x &\in \Re^n, \quad t \geqslant 0. \end{aligned} \qquad (5.25)$$

Moreover, notice that the function $V(\cdot)$, as defined by (5.18), satisfies

$$V(t,x) := x^T(P_1(t) - P_2(t)P_3^{-1}(t)P_2^T(t))x \geqslant |x|^2 \qquad (5.26)$$

and that we can actually stabilize the system using the linear feedback law,

$$k(t,x) := (k_1(t) - k_2(t)P_3^{-1}(t)P_2^T(t))x. \qquad (5.27)$$

We conclude that if a linear time-varying system can be dynamically stabilized by linear integrators and feedback then it can also be statically stabilized by a linear state time-varying feedback.

6. Conclusions

In this paper we show that the existence of a time-varying stabilizer for an affine control system is equivalent to the existence of a lower semi-continuous CLF. This result shows that the main issue for the existence of a time-varying feedback stabilizer is not the regularity of the CLF but the type of the derivative used to express the 'decrease condition', i.e. the Lyapunov differential inequality. Some results about non-affine control systems are also given, which include the so-called 'power-integrator' case.

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Appendix

Proof of Lemma 2.5 First, notice that as long as $x(t)$ exists, $t \to x(t)$ is a C^1 function. Since $V(t, x)$ is lower semi-continuous, it follows that $t \to V(t, x(t))$ is a lower semi-continuous function as long as $x(t)$ exists. We proceed by observing the following facts

Fact I. Suppose that $(t, x(t)) \in S$ for all $t \in [a, b)$. Then it holds that

$$DV(t, x(t)) \leqslant V^0(t, x(t); \dot{x}(t)) \leqslant -\beta(t)\rho(V(t, x(t))) + \beta(t)\mu\left(\int_0^t (s)\mathrm{d}s\right), \quad \forall t \in [a, b). \tag{A.1}$$

This fact can be shown easily using Lemma 2.2, inequality (2.11), Definition 2.1 and the fact that $\dot{x}(t) = f(t, x(t))$.

Fact II. Suppose that $(t, x(t)) \in S$ for all $t \in [a, b)$. Then the function $t \to V(t, x(t))$ is right-continuous for all $t \in [a, b)$.

To prove this fact notice that, as long as $x(t)$ exists, the function $\eta(t) := V(t, x(t)) - \int_0^t \beta(\tau)\mu\left(\int_0^\tau \beta(s)\mathrm{d}s\right)\mathrm{d}\tau$ is lower semi-continuous and by virtue of (A.1) it satisfies the following differential inequality for all $t \in [a, b)$:

$$D\eta(t) = DV(t, x(t)) - \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right) \leqslant 0. \tag{A.2}$$

Thus by Lemma 6.3 in Bacciotti & Rosier (2001), it follows that $\eta(t)$ is non-increasing. This implies for all $t \in [a, b)$ and $h \geqslant 0$ sufficiently small, such that $t + h \in [a, b)$:

$$V(t+h, x(t+h)) \leqslant V(t, x(t)) + \int_t^{t+h} \beta(\tau)\mu\left(\int_0^\tau \beta(s)\mathrm{d}s\right)\mathrm{d}\tau. \tag{A.3}$$

Inequality (A.3) in conjunction with the lower semi-continuity of $V(t, x(t))$ implies right-continuity.

Fact III. Suppose that $(t, x(t)) \in S$ for all $t \in [a, b)$. Then the following estimate holds:

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, \int_0^a \beta(s)\mathrm{d}s, V(a, x(a))\right), \quad \forall t \in [a, b). \tag{A.4}$$

This fact is an immediate consequence of Facts I–II and Lemma 2.6 (comparison principle).

Let $[t_0, T)$ denote the maximal interval of existence of the solution of (2.9). We define the following disjoint sets:

$$A^+ := \left\{t \in [t_0, T);\, a_2(\gamma(t)|x(t)|) > \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right)\right\} \tag{A.5}$$

$$A^- := \left\{t \in [t_0, T);\, a_2(\gamma(t)|x(t)|) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right)\right\} \tag{A.6}$$

where $c > 0$ and $\eta(t, t_0, \eta_0)$ are defined in (2.12) and (2.13), respectively. Obviously $[t_0, T) = A^+ \cup A^-$. Notice that by virtue of definitions (2.12) and (A.5) if $t \in A^+$ then $(t, x(t)) \in S$. Moreover, notice that the set $A^+\backslash(A^+ \cap \{t_0\})$ is open. Thus $A^+\backslash(A^+ \cap \{t_0\})$ is either empty or it decomposes into a finite number or a denumerable infinity of open and disjoint intervals (a_k, b_k) with $a_k < b_k$. When $t_0 \in A^+$ we obviously have the latter case. We distinguish the following cases.

Case A. $t_0 \notin A^+$ and $A^+\backslash(A^+ \cap \{t_0\})$ is not empty. In this case the set $A^+\backslash(A^+ \cap \{t_0\})$ decomposes into a finite number or a denumerable infinity of open and disjoint intervals (a_k, b_k) with $a_k < b_k$ for $k = 1, \ldots$. Furthermore, by continuity of the solution $x(t)$ it follows that $(a_k, x(a_k)) \in S$ and thus $(t, x(t)) \in S$ for all $t \in [a_k, b_k)$. Clearly, by Fact III, the following estimate will hold:

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, \int_0^{a_k} \beta(s)\mathrm{d}s, V(a_k, x(a_k))\right), \quad \forall t \in [a_k, b_k). \tag{A.7}$$

The fact that $a_k \notin A^+$ implies that $a_k \in A^-$, and consequently by virtue of (2.10) and definition (A.6) we have

$$V(a_k, x(a_k)) \leqslant \eta\left(\int_0^{a_k} \beta(s)\mathrm{d}s, 0, c\right). \tag{A.8}$$

Using Property (2.17a) in conjunction with (A.7) and (A.8) gives the following estimate:

$$\begin{aligned} V(t, x(t)) &\leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, \int_0^{a_k} \beta(s)\mathrm{d}s, \eta\left(\int_0^{a_k} \beta(s)\mathrm{d}s, 0, c\right)\right) \\ &= \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right), \quad \forall t \in [a_k, b_k). \end{aligned} \tag{A.9}$$

When $t \notin [a_k, b_k)$, it follows that $t \in A^-$ and consequently by virtue of (2.10) and definition (A.6) we have

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right), \quad \forall t \notin [a_k, b_k). \tag{A.10}$$

Estimates (A.9) and (A.10) provide the following estimate:

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right), \quad \forall t \in [t_0, T). \tag{A.11}$$

Case B. The set $A^+\backslash(A^+ \cap \{t_0\})$ is empty. In this case we have $t_0 \notin A^+$ and consequently it follows that $A^- = [t_0, T)$. Therefore by virtue of (2.10) and definition (A.6) we have that estimate (A.11) holds.

Case C. $t_0 \in A^+$ and $A^+\backslash(A^+ \cap \{t_0\})$ is not empty. In this case there exists a time $b > t_0$ and an open set $\tilde{A}$ such that $A^+ = [t_0, b) \cup \tilde{A}$. For $t \in [t_0, b)$ it follows that $(t, x(t)) \in S$ and thus by Fact III we obtain the estimate:

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, \int_0^{t_0} \beta(s)\mathrm{d}s, V(t_0, x_0)\right), \quad \forall t \in [t_0, b). \tag{A.12}$$

For the case $b = T$, the estimate above holds for all $t \in [t_0, T)$. For the case $b < T$, we have $b \notin A^+$ and thus we may repeat the analysis in cases A and B for the rest of the interval.

The analysis above shows that in any case the following estimate holds:

$$V(t, x(t)) \leqslant \eta\left(\int_0^t \beta(s)\mathrm{d}s, \int_0^{t_0} \beta(s)\mathrm{d}s, V(t_0, x_0)\right) + \eta\left(\int_0^t \beta(s)\mathrm{d}s, 0, c\right), \quad \forall t \in [t_0, T). \tag{A.13}$$

Furthermore, by virtue of Corollary 2.7, there exist a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that

$$V(t, x(t)) \leqslant 2\sigma\left(V(t_0, x_0) + c + M, \int_{t_0}^t \beta(s)\mathrm{d}s\right), \quad \forall t \in [t_0, T). \tag{A.14}$$

A standard contradiction argument in conjunction with (A.14) shows that $T = +\infty$. Thus estimate (A.14) holds for all $t \geqslant t_0$. We define for all $(t, t_0, s) \in (\mathfrak{R}^+)^3$ the continuous function

$$\Delta(s, t_0, t) := \begin{cases} a_1^{-1}\left(2\sigma\left(a_2(\overline{\gamma}(t_0)s) + c + M, \int_{t_0}^t \beta(s)\mathrm{d}s\right)\right) & \text{if } t \geqslant t_0 \\ a_1^{-1}(2\sigma(a_2(\overline{\gamma}(t_0)s) + c + M, 0)) & \text{if } t < t_0 \end{cases}$$

where $\overline{\gamma}(t) := \max_{0 \leqslant \tau \leqslant t} \gamma(\tau)$. Notice that by virtue of (2.10), (A.14) and the definition above we obtain

$$|x(t)| \leqslant \Delta(|x_0|, t_0, t), \quad \forall t \geqslant t_0.$$

Using Lemma 2.5 in Karafyllis & Tsinias (2003a), we conclude that $0 \in \mathfrak{R}^n$ is GAS for (2.9). The proof is complete.

Proof of Lemma 2.6: Consider the scalar differential equation

$$\begin{aligned} \dot{z} &= f(t, z) + \lambda \\ z(t_0) &= w_0 \end{aligned} \tag{A.15}$$

where λ is a positive constant. On any compact interval $[t_0, t_1] \subset [t_0, T)$, we conclude from Theorem 2.6 in Khalil (1996) that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \lambda < \delta$ then (A.7) has a unique solution $z(t, \lambda)$ defined on $[t_0, t_1]$ and satisfies

$$|z(t, \lambda) - w(t)| < \varepsilon, \quad \forall t \in [t_0, t_1]. \tag{A.16}$$

Fact I. $v(t) \leqslant z(t, \lambda)$, for all $t \in [t_0, t_1)$.
This fact is shown by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $v(t) - z(t, \lambda) > 0$. Clearly, the function $m(t) := v(t) - z(t, \lambda)$ is lower semi-continuous and therefore the set

$$A^+ := \{\tau \in (t_0, t_1) : m(\tau) > 0\} \tag{A.17}$$

is open and non-empty. Thus A^+ decomposes into a finite number or a denumerable infinity of open and disjoint intervals (a_k, b_k) with $a_k < b_k$. Since $a_k \notin A^+$ and consequently we have $v(a_k) \leqslant z(a_k, \lambda)$. On the other hand, for every non-increasing sequence $\{\tau_{i,k} \in (a_k, b_k)\}_{i=1}^{\infty}$ with $\tau_{i,k} \to a_k$, we obtain

$$v(\tau_{i,k}) - v(a_k) > z(\tau_{i,k}, \lambda) - z(a_k, \lambda). \tag{A.18}$$

This implies

$$Dv(a_k) = \liminf_{i\to\infty} \frac{v(\tau_{i,k}) - v(a_k)}{\tau_{i,k} - a_k} \geqslant \dot{z}(a_k, \lambda) = f(a_k, z(a_k, \lambda)) + \lambda. \tag{A.19}$$

Moreover, the function $m(t) := v(t) - z(t, \lambda)$ is right-continuous and by definition (A.17) we obtain $v(a_k) \geqslant z(a_k, \lambda)$. Thus we have $v(a_k) = z(a_k, \lambda)$. Then using (A.19) we get

$$Dv(a_k) \geqslant f(a_k, v(a_k)) + \lambda > f(a_k, v(a_k))$$

which contradicts (2.15).

Fact II. $v(t) \leqslant w(t)$, for all $t \in [t_0, t_1)$.
Again, this claim may be shown by contradiction. Suppose that there exists $a \in (t_0, t_1)$ with $v(a) > w(a)$ and set $\varepsilon = \frac{1}{2}(v(a) - w(a)) > 0$. Furthermore, let $\lambda > 0$ be selected in such a way that (A.16) is satisfied with this particular selection of $\varepsilon > 0$. Then we obtain

$$v(a) = v(a) - w(a) + w(a) = 2\varepsilon + w(a) - z(a, \lambda) + z(a, \lambda) > \varepsilon + z(a, \lambda)$$

which contradicts Fact I.

Fact III. $v(t) \leqslant w(t)$, for all $t \in [t_0, T)$.
Suppose the contrary: there exists $a \in (t_0, T)$ with $v(a) > w(a)$. Let $t_1 := a + \frac{T-a}{2}$ for the case of finite T or $t_1 := a + 1$ for the case $T = +\infty$. Clearly, by Fact II we have a contradiction. The proof is complete. □

Proof of Lemma 5.5: Define

$$\delta(t, x) := a_1^{-1}(2a_2(\gamma(t)|x|) + 1) \tag{A.20}$$

where $a_1, a_2 \in K_\infty$, $\gamma \in K^+$ are the functions involved in (5.17a) and notice that $\delta(\cdot)$ is a continuous, positive function. By virtue of inequality (5.17a) and definition (5.18) we obtain:

$$a_1(|x|) \leqslant V(t, x) \leqslant \Psi(t, x, 0) \leqslant a_2(\gamma(t)|x|), \quad \forall (t, x) \in \Re^+ \times \Re^n. \tag{A.21}$$

Clearly, inequality (A.21) establishes (5.19a). Definitions (5.18), (A.20) and inequalities (5.17a), (A.21) imply that

$$\begin{aligned} V(t, x) &= \min(\inf\{\Psi(t, x, w);\ |w| \leqslant \delta(t, x)\}, \inf\{\Psi(t, x, w);\ |w| > \delta(t, x)\}) \\ &\geqslant \min(\inf\{\Psi(t, x, w);\ |w| \leqslant \delta(t, x)\}, \inf\{a_1(|w|);\ |w| > \delta(t, x)\}) \\ &\geqslant \min(\inf\{\Psi(t, x, w);\ |w| \leqslant \delta(t, x)\}, 2V(t, x) + 1). \end{aligned}$$

The latter inequality and continuity of $\Psi(\cdot)$ gives

$$V(t,x) = \min_{|w|\leqslant\delta(t,x)} \Psi(t,x,w). \tag{A.22}$$

Moreover, it follows that the set-valued map $\mathcal{M}(t,x) \subset \Re^l$, as defined by (5.20), is strict and bounded. By virtue of (5.19a) and definition (5.20) it follows that for all $t \geqslant 0$, we have

$$\mathcal{M}(t,0) = \{0\}. \tag{A.23}$$

We next establish that $V(\cdot)$ is locally Lipschitz on $\Re^+ \times (\Re^n\backslash\{0\})$. Let $A \subset \Re^+ \times (\Re^n\backslash\{0\})$ be a non-empty convex compact set and define

$$r := \max_{(t,x)\in A} \delta(t,x) \tag{A.24}$$

$$L := \max\left\{\left|\frac{\partial\Psi}{\partial t}(t,x,w)\right| + \left|\frac{\partial\Psi}{\partial x}(t,x,w)\right|;\, (t,x)\in A,\, |w|\leqslant r\right\}. \tag{A.25}$$

Let $(t,x), (\tau,y) \in A$ and $w_{(t,x)} \in \mathcal{M}(t,x)$, $w_{(\tau,y)} \in \mathcal{M}(\tau,y)$. We have, by virtue of (A.22), (A.24) and (A.25) that

$$\begin{aligned}
V(\tau,y) - V(t,x) &= \Psi(\tau,y,w_{(\tau,y)}) - \Psi(t,x,w_{(t,x)}) \leqslant \Psi(\tau,y,w_{(t,x)}) - \Psi(t,x,w_{(t,x)})\\
&= \int_0^1 \frac{\partial\Psi}{\partial t}(t+\lambda(\tau-t), x+\lambda(y-x), w_{(t,x)})\mathrm{d}\lambda(\tau-t)\\
&\quad + \int_0^1 \frac{\partial\Psi}{\partial x}(t+\lambda(\tau-t), x+\lambda(y-x), w_{(t,x)})\mathrm{d}\lambda(y-x)\\
&\leqslant L(|\tau-t| + |y-x|).
\end{aligned}$$

Reversing the roles of (t,x) and (τ,y) we get

$$|V(\tau,y) - V(t,x)| \leqslant L(|\tau-t| + |y-x|), \quad \forall (t,x), (\tau,y) \in A. \tag{A.26}$$

This establishes that $V(\cdot)$ is locally Lipschitz on $\Re^+ \times (\Re^n\backslash\{0\})$. It is also continuous on $\Re^+ \times \Re^n$, since continuity at $x = 0$ is guaranteed by (5.19a) with $V(t, 0) = 0$. Moreover, by continuity of $V(\cdot)$ on $\Re^+ \times \Re^n$, (A.23) and definition (5.20) it follows that for all $(t,x) \in \Re^+ \times \Re^n$ the set $\mathcal{M}(t,x) \subset \Re^l$ is compact.

Next we establish (5.19b). We have for all $(t,x,v) \in \Re^+ \times (\Re^n\backslash\{0\}) \times \Re^n$ and $w \in \mathcal{M}(t,x)$

$$\begin{aligned}
DV(t,x;v) &= \liminf_{h\to 0^+} \frac{V(t+h, x+hv) - V(t,x)}{h}\\
&= \liminf_{h\to 0^+} \left\{\frac{\Psi(t+h, x+hv, w') - \Psi(t,x,w)}{h};\, w' \in \mathcal{M}(t+h, x+hv)\right\}\\
&\leqslant \liminf_{h\to 0^+} \frac{\Psi(t+h, x+hv, w) - \Psi(t,x,w)}{h} = \frac{\partial\Psi}{\partial t}(t,x,w) + \frac{\partial\Psi}{\partial x}(t,x,w)v
\end{aligned}$$

which establishes inequality (5.19b). We continue the proof by establishing the following claim.

Claim. The set-valued map $\mathcal{M}(t,x)$, as defined by (5.20), is upper semi-continuous.

Proof of Claim. It suffices to prove that for every $(t,x) \in \Re^+ \times \Re^n$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\tau - t| + |y - x| < \delta \Rightarrow \mathcal{M}(\tau, y) \subset \mathcal{M}(t,x) + \varepsilon B. \tag{A.27}$$

The proof will be made by contradiction. Suppose the contrary: there exists $(t,x) \in \Re^+ \times \Re^n$ and $\varepsilon > 0$, such that for all $\delta > 0$, there exists $(\tau, y) \in (t,x) + \delta B$ and $w \in \mathcal{M}(\tau, y)$ with $|w - w'| \geqslant \varepsilon$, for all $w' \in \mathcal{M}(t,x)$. Clearly, this implies the existence of a sequence $\{(\tau_n, y_n, w_n)\}_{n=1}^{\infty}$ with $(\tau_n, y_n) \to (t,x)$, $w_n \in \mathcal{M}(\tau_n, y_n)$ and $|w_n - w'| \geqslant \varepsilon$, for all $w' \in \mathcal{M}(t,x)$ and $n = 1, 2 \ldots$. On the other hand, since w_n is bounded, it contains a convergent subsequence $w_k \to \overline{w} \notin \mathcal{M}(t,x)$. By continuity of $V(\cdot)$ and $\Psi(\cdot)$, we have

$$\begin{aligned} &V(\tau_k, y_k) \to V(t,x) \\ &V(\tau_k, y_k) = \Psi(\tau_k, y_k, w_k) \to \Psi(t, x, \overline{w}). \end{aligned}$$

Consequently, we must have $V(t,x) = \Psi(t,x,\overline{w})$, which, by virtue of definition (5.20) implies that $\overline{w} \in \mathcal{M}(t,x)$, a contradiction.

We finish the proof by proving inequality (5.19c). Let $(t,x,v) \in \Re^+ \times (\Re^n \backslash \{0\}) \times \Re^n$. Making use of the continuity properties of the maps $f(\cdot)$, $g(\cdot)$, $\rho(\cdot)$, $V(\cdot)$, $\mu(\cdot)$, $\beta(\cdot)$, inequalities (5.17b) and (5.19b), as well as the fact that for all $w \in \mathcal{M}(\tau, y)$ it holds that $\frac{\partial \Psi}{\partial w}(\tau, y, w) = 0$, we obtain

$$\begin{aligned} &V^0(t,x;v) = \limsup_{(\tau,y)\to(t,x)} DV(\tau,y;v) \\ &\leqslant \limsup_{(\tau,y)\to(t,x)} \min\left\{ \frac{\partial \Psi}{\partial t}(\tau,y,w') + \frac{\partial \Psi}{\partial x}(\tau,x,w')v;\ w' \in \mathcal{M}(\tau,y) \right\} \\ &\leqslant \limsup_{(\tau,y)\to(t,x)} \min\left\{ -\beta(\tau)\rho(\Psi(\tau,y,w')) + \beta(\tau)\mu\left(\int_0^\tau \beta(s)\mathrm{d}s\right) \right. \\ &\quad \left. + a(t,x,\tau,y,w') : w' \in \mathcal{M}(\tau,y) \right\} \\ &\leqslant -\beta(t)\rho(V(t,x)) + \beta(t)\mu\left(\int_0^t \beta(s)\mathrm{d}s\right) + \limsup_{(\tau,y)\to(t,x)} \min\{a(t,x,\tau,y,w');\ w' \in \mathcal{M}(\tau,y)\} \end{aligned} \tag{A.28}$$

where

$$a(t,x,\tau,y,w') := \frac{\partial \Psi}{\partial x}(\tau,y,w')(v - f(t,x) - g(t,x)\tilde{k}(\tau,y,w'))$$

Notice that by continuity of $\frac{\partial \Psi}{\partial x}(\cdot)$, $\tilde{k}(\cdot)$ and upper semi-continuity of the set-valued map $\mathcal{M}(t, x)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{gathered}|\tau - t| + |y - x| < \delta,\, w' \in \mathcal{M}(\tau, y) \Rightarrow \\ \frac{\partial \Psi}{\partial x}(\tau, y, w')(v - f(t, x) - g(t, x)\tilde{k}(\tau, y, w')) \\ \leqslant \max\left\{\frac{\partial \Psi}{\partial x}(t, x, w)(v - f(t, x) - g(t, x)\tilde{k}(t, x, w));\, w \in \mathcal{M}(t, x)\right\} + \varepsilon.\end{gathered} \tag{A.29}$$

Combining (A.28) and (A.29) we obtain (5.19c). The proof is complete. □