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Price stabilization using buffer stocks

George Athanasiou[a], Iasson Karafyllis[b], Stelios Kotsios[a,*]

[a]Department of Economics, University of Athens, 8 Pesmazoglou Street, 10559 Athens, Greece
[b]Department of Environmental Engineering, Technical University of Crete, 73100 Chania, Greece

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Abstract

The price stabilization problem is stated and solved for a nonlinear cobweb model with government stocks. It is shown that if the storage capacity for the commodity is sufficiently large then there exists a simple stabilization policy, called the 'keep supply at equilibrium (KSE)' policy, such that the equilibrium price is a global attractor for the corresponding closed-loop system. In addition, it is shown that if the government approximates the equilibrium supply with the average supply, stabilization is guaranteed. We refer to this policy as 'keep supply at average (KSA)'.

JEL classification: C60; C61; C62; E32; D84

Keywords: Buffer stocks; Nonlinear cobweb model; Nonlinear feedback control; Price stabilization

1. Introduction

The purpose of the present work can be summarized in the following points:

- We extend the nonlinear cobweb model literature by including government intervention in the form of buffer stocks. We assume that in every period the government buys or sells a certain quantity of the commodity in order to stabilize the price.

*Corresponding author. Tel.: +30 2103689818.
E-mail address: skotsios@econ.uoa.gr (S. Kotsios).

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- Under the assumption of naive expectations for the producers, we derive an analytical solution for the computation of the storage capacity of the stabilizing authority, which is necessary for the stabilization of the price. In addition, we compute the government's cost of the buffer stock program.
- The solution to our problem is a feedback law, i.e., a rule which updates in every period based on the information available in the previous period. We refer to this policy rule as 'keep supply at equilibrium (KSE)'. As expected, this law is proven to be robust in modeling and measurement errors.
- We consider the case that the government uses average supply as an observable proxy for the equilibrium level. We refer to this policy rule as 'keep supply at average (KSA)' and we test its effectiveness.
- We show that the necessary storage capacity for a successful application of the KSE and the KSA policies is usually a small percentage of the equilibrium supply. Thus, at least in theory, the government can stabilize the commodity price by releasing (or buying) small quantities of the commodity. However, as the degree of nonlinearity of the supply function increases, the necessary storage capacity, as a percentage of the equilibrium supply, increases too.

Nowadays, there is a large and increasing literature on the application of mathematical control theory in economics, which has provided us with a very good insight on our ability to control complex economic systems. One of the economic applications of control theory is to the study of the problem of price stabilization using various instruments as inputs. It is a well-known fact of economic reality that commodity prices are extremely volatile (Deaton and Laroque, 1992). One of the instruments for price stabilization, which is found frequently in the economic literature, is the so-called buffer stock scheme. The basic function of such a program is to store a certain amount of the commodity in boom periods, when the price is low, and to release a certain amount of the stored commodity in bust periods, when the price is high. Undoubtedly, there has been a lot of work over the past decades in the specific field (e.g., Arzac, 1979; Newbery and Stiglitz, 1981; Wright and Williams, 1982; Miranda and Helmberger, 1988; Van Groenendaal and Vingerhoets, 1995; Brennan, 2003) and important results have been derived. Given the existing literature, we consider that recent developments in the field of mathematical control theory for nonlinear systems (existence of Lyapunov functions, e.g., Jiang and Wang, 2002; utilization of Lyapunov functions and difference inequalities, e.g., Lakshmikantham and Trigiante, 2002; difference equations given through a system-theoretic framework, e.g., Sontag, 1998; attractor theory, e.g., Stuart and Humphries, 1998; nonlinear feedback control stabilization, e.g., Tsinias et al., 1989; Tsinias, 1989; Kotsios and Leventidis, 2004), can shed more light on the analysis of the buffer stock mechanism. Following this branch of research, we contribute to the economic literature by adopting a genuinely nonlinear approach, using Lyapunov functions, in order to ensure global results for our model. The motivation for this approach stems from the fact that even though in some cases the initial conditions of an economic system are close to the steady state, in the presence of small perturbations the system does not exhibit the desired behavior. In such cases a genuinely nonlinear approach can help the modeler to accomplish his scope by providing a robust solution to his problem.

In the thirties, the observation of regularly recurring cycles in the production and prices of particular commodities gave birth to the cobweb model (Ezekiel, 1938). The model describes price fluctuations in a single market for a commodity that takes one unit of time to produce. Thus, supply depends on the expectations of the producers about next period's price and market prices are driven by these expectations. When both the demand and the supply curves are linear, then only three types of price dynamics can occur: (i) convergence to a stable equilibrium (convergent fluctuation), (ii) convergence to a period-2 cycle, the so-called 'hog-cycle' (continuous fluctuation) or (iii) exploding oscillations (divergent fluctuation). However, these cases can hardly explain the continued existence of irregular cycles in real markets. During the last 20 years, due to the progress in nonlinear analysis, the cobweb literature flourished and models with chaotic dynamic behavior appeared. For example, nonlinear cobweb models were examined by Artstein (1983), Jensen and Urban (1984), Lichtenberg and Ujihara (1989), Hommes (1991, 1994, 1998), Finkenstädt (1995), Boussard (1996), Gallas and Nusse (1996), Onozaki et al. (2003), Chiarella and He (2003), Lasselle et al. (2005).

In this paper we consider a nonlinear cobweb model with a piecewise linear supply function and naive expectations. Under this setting, we examine the effect of government intervention, in the form of buffer stocks, on the behavior of the commodity price. Our analysis differs from the standard rational expectations competitive storage model in that it assumes that supply is endogenous. More specifically, in the deterministic cobweb model fluctuations arise because producers (due to the production lag) are assumed to form expectations based upon time series observations. Moreover, the complexity of the price fluctuations depends on the nonlinearity of the demand and supply functions. On the other hand, under the assumption of rational expectations, sustained oscillations arise only in the presence of exogenous random shocks. Regarding economic policy in a cobweb setting, Matsumoto (1998) examines the effect of government subsidies on output dynamics in agricultural markets. He concludes that, although subsidies prevent explosive output oscillations, they also cause bounded, highly irregular output fluctuations. He and Westerhoff (2005) develop a behavioral commodity market model and explore the effectiveness of price limiters. The main result of their paper is that simple price limits reduce the variability of prices quite strongly but under the possibility of an unsustainable buffer stock.

The remainder of this paper is organized as follows. In Section 2 a nonlinear cobweb model is developed for a single commodity with government stocks and naive expectations. The model is based on a piecewise linear S-shaped supply function and takes into account the constraints which must be satisfied by the stockpiling quantity. In Section 3 the reader is introduced to the price stabilization problem where it is shown that if the storage capacity for the particular commodity is sufficiently large then the KSE policy is successful (the equilibrium price is a global attractor for the corresponding closed-loop system). Numerical studies are also presented in Section 4, in which the efficiency of the KSE and the KSA policies is tested. Finally the conclusions of the present work are given in Section 5.

2. The cobweb model with Stocks

Let D_{t+1} be the demand and S_{t+1} be the supply of the commodity, where P_{t+1} is the actual price and P^{e}_{t+1} is the expected price, all at period $t+1$.

In order to keep the model as simple as possible, throughout the paper we assume a linear, strictly decreasing, demand function given by

$$D_{t+1} = a - bP_{t+1}, \quad a, b > 0. \tag{2.1}$$

In addition, we consider an S-shaped (monotonic) piecewise linear supply function of the private sector (for economic considerations see Hommes, 1994, p. 319) given by

$$S_{t+1} = g(P^{e}_{t+1}), \tag{2.2a}$$

$$g(P) = \max\{0; \min\{S^{M}; -c + dP\}\}, \quad c, d, S^{M} > 0, \tag{2.2b}$$

where S^{M} is the maximum supply of the commodity.

We assume that the government is responsible for the operation of a buffer stock program for the commodity and that private stockpiling is negligible. To make the analysis tractable we assume that the interest rate is zero and that there are no inventory losses. In order to introduce stockpiling into our model we define two variables: Q_t is a state variable which denotes the government inventory at period t and G_t is a control variable which denotes the quantity of the commodity released to the market by the government at period $t+1$. When $G_t < 0$ then the government actually buys and stores $|G_t|$ commodity units at period $t+1$. Furthermore, the inventory must satisfy the following difference equation for all periods:

$$Q_{t+1} = Q_t - G_t. \tag{2.3}$$

In a completely competitive market, closed to external trade, the market clearing condition is described by

$$D_{t+1} = S_{t+1} + G_t. \tag{2.4}$$

In order to close the model, we consider, as a first approximation, the so-called naive (or myopic) price expectations. According to this expectations scheme, producers are assumed to expect the last observed price, i.e.,

$$P^{e}_{t+1} = P_t. \tag{2.5}$$

At first sight, this hypothesis seems implausible because it implies that producers do not learn from experience. However, recent work on bounded rationality (e.g., Conlisk, 1996; Chavas, 2000; Hommes et al., 2007) recognizes the cost involved in the process of gathering and processing information and argues that individuals may form expectations based upon simple habitual rules of thumb instead of perfectly optimal decision rules. Conlisk (1996) reports that psychology and economics provide wide-ranging evidence that bounded rationality is important. In addition, Chavas (2000) develops and estimates an econometric model of market prices in the U.S. beef market. He finds that about 47% of beef producers behave in a way consistent with naive expectations. Furthermore, Hommes et al. (2007) conduct cobweb experiments and show that within an unstable environment subjects are not able to learn the rational expectations equilibrium price, but instead use simple forecasting rules.

Taking into account (2.1)–(2.5), the unique solution of (2.4) gives the following discrete-time control system:

$$P_{t+1} = b^{-1}(a - G_t - g(P_t)),$$

$$Q_{t+1} = Q_t - G_t,$$

$$\max\{Q_t - Q^{\mathrm{M}}; -g(P_t)\} \leqslant G_t \leqslant Q_t, \tag{2.6}$$

where P_{t+1} is the output of the system, $g(P_t)$ is defined by (2.2b) and $Q^{\mathrm{M}} > 0$ denotes the government's storage capacity for the particular commodity. Examination of (2.6) shows that if $a > Q^{\mathrm{M}} + S^{\mathrm{M}}$ then the above evolution equation defines a control system with $(P_t, Q_t) \in (0, +\infty) \times [0, Q^{\mathrm{M}}]$, for all t. Observe that the cobweb model (2.6) is nonlinear. The piecewise linear S-shaped supply function given in (2.2b) as well as the constraints satisfied by the control variable G_t, explain the nonlinear character of the model.

Moreover, it should be emphasized that the control system (2.6) is a deterministic control system. However, in order to be able to capture uncertainties in the nominal model (2.6) as well as random perturbations (e.g., demand shocks) we also considered the system under the presence of additive noise, that is, the 'perturbed' system:

$$P_{t+1} = b^{-1}(a - G_t - g(P_t) + w_t),$$

$$Q_{t+1} = Q_t - G_t,$$

$$\max\{Q_t - Q^{\mathrm{M}}; -g(P_t)\} \leqslant G_t \leqslant Q_t, \tag{2.7}$$

where the noise term w_t is normally distributed with mean zero and constant standard deviation, for all t.

3. The price stabilization problem

In the absence of government intervention ($G_t \equiv 0$) the corresponding dynamical system (2.6) reduces to the standard cobweb model with a piecewise linear supply function (the state variable Q_t does not affect the system). In this case the price behavior is described by the difference equation:

$$P_{t+1} = h(P_t); \quad P_t \in (0, +\infty), \tag{3.1}$$

where

$$h(P) := \begin{cases} \dfrac{a}{b} & \text{if } P \leqslant \dfrac{c}{d}, \\ \dfrac{a+c-dP}{b} & \text{if } \dfrac{c}{d} < P < \dfrac{c+S^{\mathrm{M}}}{d}, \\ \dfrac{a-S^{\mathrm{M}}}{b} & \text{if } P \geqslant \dfrac{c+S^{\mathrm{M}}}{d}. \end{cases} \tag{3.2}$$

The dynamics of the system without intervention were examined and the results are summarized in Table 1 (for proofs of the statements made in Table 1, see Appendix A). System (3.1) admits a unique equilibrium price, which is given by the following equation:

$$P^{\mathrm{o}} = \begin{cases} \dfrac{a}{b} & \text{if } ad \leqslant bc, \\ \dfrac{a+c}{b+d} & \text{if } S^{\mathrm{M}} > \dfrac{ad-bc}{b+d} \quad \text{and} \quad ad > bc, \\ \dfrac{a-S^{\mathrm{M}}}{b} & \text{if } S^{\mathrm{M}} \leqslant \dfrac{ad-bc}{b+d}. \end{cases} \tag{3.3}$$

Notice that if $ad > bc$ in (3.2) then a/b is not an equilibrium price, demand is positive and the producers are active in the market. Thus, in order to rule out the degenerate case of no production we will assume that $ad > bc$. Furthermore, it is verified that with naive expectations and a monotonic nonlinear supply curve, prices either converge to a stable steady state or to a stable period-2 'hog-cycle' (Hommes, 1998). The relation between the introduction of piecewise linear functions and the

Table 1
The asymptotic behaviour of (3.1)

Cases	Asymptotic behaviour
1. $S^{\mathrm{M}} > \dfrac{ad-bc}{b+d}$ and $\dfrac{d}{b} < 1$	$P^{\mathrm{o}} = \dfrac{a+c}{b+d}$ is globally asymptotically stable
2. $S^{\mathrm{M}} \leqslant \dfrac{ad-bc}{b+d}$	$P^{\mathrm{o}} = \dfrac{a-S^{\mathrm{M}}}{b}$ is globally asymptotically stable
3. $S^{\mathrm{M}} > \dfrac{ad-bc}{b+d}$ and $\dfrac{d}{b} \geqslant 1$	Period-2 solution:
(a) If $S^{\mathrm{M}} > \dfrac{ad-bc}{b}$	$P_1 = \dfrac{a}{b}, \quad P_2 = \dfrac{a+c}{b} - \dfrac{ad}{b^2}$
(b) If $\dfrac{ad-bc}{d} \leqslant S^{\mathrm{M}} \leqslant \dfrac{ad-bc}{b}$	$P_1 = \dfrac{a}{b}, \quad P_2 = \dfrac{a-S^{\mathrm{M}}}{b}$
(c) If $S^{\mathrm{M}} < \dfrac{ad-bc}{d}$	$P_1 = \dfrac{a+c}{b} + \dfrac{d(S^{\mathrm{M}}-a)}{b^2}, \quad P_2 = \dfrac{a-S^{\mathrm{M}}}{b}$
(d) If $b = d$	Infinite number of period-2 solutions: $P_1 \in \left(\dfrac{1}{b}\max(c; a-S^{\mathrm{M}}), \dfrac{1}{b}\min(a; c+S^{\mathrm{M}})\right)$ arbitrary, $P_2 = \dfrac{a+c}{b} - P_1$

appearance of cycles was investigated by Hicks (1950), Simonovits (1991), Hommes et al. (1995).

Our first concern is to examine whether there is a stockpiling policy rule which attempts to lead the commodity price to a predefined target price. We refer to this problem as the tracking control problem and we define it as follows:

The tracking control problem for (2.6): Let $P^* > 0$ be the desired price. Is there a static feedback law (or price stabilization policy) $\phi : (0, +\infty) \times [0, Q^{\mathrm{M}}] \to [-Q^{\mathrm{M}}, Q^{\mathrm{M}}]$:

$$G_t = \phi(P_t, Q_t) \tag{3.4}$$

such that the solution of the closed-loop system (2.6) with (3.4) satisfies $\lim_{t\to+\infty} P_t = P^*$ for all initial conditions $(P_0, Q_0) \in (0, +\infty) \times [0, Q^{\mathrm{M}}]$?

The following lemma presents the necessary condition for solvability of the tracking control problem.

Lemma 3.1. *If the tracking control problem for* (2.6) *is solvable then the following condition must hold*:

$$P^* = P^{\mathrm{o}},$$

where P^{o} *is the equilibrium price of the open-loop system* (3.1) *as given by* (3.3).

Proof. See Appendix A. □

Lemma 3.1 indicates a major limitation imposed by the application of a price stabilization policy: the price dynamics can only have a unique accumulation point, which is no other than the equilibrium price. This important limitation may be used to explain the failure of buffer stock policies: if the government tries to lead the commodity price to values different from the equilibrium values, the buffer stock policy will fail (Van Groenendaal and Vingerhoets, 1995; p. 259). Since the desired price must coincide with the equilibrium price, we may state next the price stabilization problem.

The price stabilization problem for (2.6): Is there a static feedback law (or price stabilization policy) $\phi : (0, +\infty) \times [0, Q^{\mathrm{M}}] \to [-Q^{\mathrm{M}}, Q^{\mathrm{M}}]$ given by (3.4) with $\phi(P^{\mathrm{o}}, Q) = 0$ for all $Q \in [0, Q^{\mathrm{M}}]$, such that the equilibrium set $\{(P^{\mathrm{o}}, Q); Q \in [0, Q^{\mathrm{M}}]\}$ is a global attractor for the closed-loop system (2.6) with (3.4), i.e., $\lim_{t\to+\infty} P_t = P^{\mathrm{o}}$ for all initial conditions $(P_0, Q_0) \in (0, +\infty) \times [0, Q^{\mathrm{M}}]$?

This problem is particularly interesting for Case 3 in Table 1, i.e., when $S^{\mathrm{M}} > (ad - bc)/(b + d)$ and $d/b \geqslant 1$, since in this case the equilibrium set $\{(P^{\mathrm{o}}, Q); Q \in [0, Q^{\mathrm{M}}]\}$ is not a global attractor without government intervention, i.e., when $G_t \equiv 0$.

In order to solve the problem we set:

$$G_t = \min\{Q_t; \max\{S^{\mathrm{o}} - g(P_t); Q_t - Q^{\mathrm{M}}\}\}, \tag{3.5}$$

where $S^{\mathrm{o}} = (ad - bc)/(b + d)$ is the equilibrium supply that corresponds to the equilibrium price $P^{\mathrm{o}} = (a + c)/(b + d)$ and $g(P_t)$ is the estimated supply at period $t + 1$.

The feedback law described by (3.5) is particularly simple and attempts to bring the total quantity of the commodity available in the market at period $t+1$ (given by $S_{t+1}+G_t$) as close as possible to the equilibrium supply S^{o}. In the rest of the paper we will refer to this policy rule as KSE. According to this policy rule, when the difference between the equilibrium supply and available supply is positive then the market runs short of the commodity and the government intervenes by selling a certain quantity of the commodity. Of course the quantity released at period $t+1$ cannot exceed the quantity stored at period t. When the difference between the equilibrium supply and the available supply is negative, then there is abundance of the commodity and the government buys a certain quantity of the commodity. In this case the purchased quantity at period $t+1$ cannot exceed the quantity $Q^{\mathrm{M}}-Q_t$ (i.e., the storage capacity at period t).

Furthermore, formula (3.5) shows an important feature. For the implementation of the KSE policy the government need to know four things: the inventory Q_t, the storage capacity Q^{M}, the equilibrium supply S^{o} and the supply of the private sector $g(P_t)$. Consequently, if the government can estimate the supply of the private sector at any period, it is not necessary to know the exact functional form of $g(P_t)$. The following proposition provides criteria for the success of the KSE buffer stock policy.

Proposition 3.2. *Consider the price stabilization problem under the hypotheses* $S^{\mathrm{M}}>S^{\mathrm{o}}$ *and* $d/b\geqslant 1$. *The equilibrium set* $\{(P^{\mathrm{o}},Q); Q\in[0,Q^{\mathrm{M}}]\}$ *is a global attractor for the closed-loop system* (2.6) *with* (3.5) *if and only if* $Q^{\mathrm{M}}>R$, *where*

$$R:=\max\{R_1;R_2;R_3;R_4\}, \tag{3.6}$$

$$R_1:=\min\left\{\left(\frac{d-b}{d}\right)S^{\mathrm{o}};S^{\mathrm{M}}-\left(\frac{b+d}{d}\right)S^{\mathrm{o}}\right\},$$

$$R_2:=\left(\frac{d-b}{d}\right)\min\{S^{\mathrm{o}};S^{\mathrm{M}}-S^{\mathrm{o}}\}\geqslant 0,$$

$$R_3:=\min\left\{S^{\mathrm{M}}-\left(\frac{b+d}{d}\right)S^{\mathrm{o}};\left(\frac{b+d}{d}\right)S^{\mathrm{o}}-\frac{b}{d}S^{\mathrm{M}}\right\},\quad R_4:=\left(\frac{d-b}{d}\right)(S^{\mathrm{M}}-S^{\mathrm{o}}).$$

Particularly, for every initial condition $(P_0,Q_0)\in(0,+\infty)\times[0,Q^{\mathrm{M}}]$ *there exists* $T>0$ *such that the solution of the price stabilization problem satisfies*:

$$P_t=P^{\mathrm{o}},\quad \forall t\geqslant T \tag{3.7}$$

Proof. See Appendix A. □

Remark. We call the quantity R the 'critical storage capacity'. Moreover, R can be computed analytically using (3.6). For the reasonable case of $S^{\mathrm{M}}<2S^{\mathrm{o}}$, $\min\{R_1,R_2,R_3,R_4\}=S^{\mathrm{M}}-(\frac{b+d}{d})S^{\mathrm{o}}$ and $\max\{R_1,R_2,R_3,R_4\}=(\frac{d-b}{d})(S^{\mathrm{M}}-S^{\mathrm{o}})$. Proposition 3.2 provides the sharpest characterization of the property of global attractivity of the equilibrium point under (3.5). Clearly, if the storage capacity is larger than the critical storage capacity, i.e., if $Q^{\mathrm{M}}>R$, then the KSE policy is successful. Indeed, when $Q^{\mathrm{M}} \leqslant R$ the control action given by (3.5) is not able to cancel the periodic orbits of the open-loop system. Table 2 provides non-trivial period-2 solutions for the closed-loop system (2.6) with (3.5) and $Q^{\mathrm{M}} \leqslant R$. Consequently, if inequality $Q^{\mathrm{M}} > R$ is violated the equilibrium set is not a global attractor.

The conclusions of Proposition 3.2 may be used in order to calculate the storage cost needed for a successful stabilization policy (since the storage cost is directly related to Q^{M}). Notice that the result of Proposition 3.2 may be interpreted as a trade off between the efficiency of the price stabilization policy and the storage cost. It should be noted here that the total cost of the buffer stock policy includes also the purchase cost, i.e., the cost of purchasing quantities of the commodity defined by

$$\text{Total Purchase Cost up to period } t+1 = -\sum_{i=0}^{t} G_i P_{i+1}.$$

More specifically, if $-\sum_{i=0}^{t} G_i P_{i+1} > 0$ then the government has a cost in order to achieve price stabilization and if $-\sum_{i=0}^{t} G_i P_{i+1} < 0$ then the government increase its revenues from the function of the buffer stock program (storage cost is not included). The total purchase cost up to period $t+1$ in general depends on the initial conditions P_0 and Q_0. The evolution of the total purchase cost is crucial for the success of the buffer stock policy. Thus, if the total purchase cost increases (decreases) with time then the government has to increase (decrease) its spending in order to achieve price stabilization. The dynamic behavior of the total purchase cost will be studied in the next section.

Table 2
Non-trivial period-2 solutions for the closed-loop system (2.6) with (3.5) and $Q^{\mathrm{M}} \leqslant R$

Cases	Non-trivial period-2 solutions
$\max\left\{0; S^{\mathrm{o}} + \frac{b}{d}(S^{\mathrm{o}} - S^{\mathrm{M}})\right\} \leqslant Q^{\mathrm{M}} \leqslant R_1$	$(P_1, Q_1) = \left(\frac{a - Q^{\mathrm{M}}}{b}, 0\right)$
	$(P_2, Q_2) = \left(\frac{a + c + Q^{\mathrm{M}}}{b} + \frac{d(Q^{\mathrm{M}} - a)}{b^2}, Q^{\mathrm{M}}\right)$
$0 < Q^{\mathrm{M}} \leqslant R_2$	$(P_1, Q_1) = \left(P^{\mathrm{o}} - \frac{Q^{\mathrm{M}}}{d - b}, Q^{\mathrm{M}}\right)$
	$(P_2, Q_2) = \left(P^{\mathrm{o}} + \frac{Q^{\mathrm{M}}}{d - b}, 0\right)$
$0 \leqslant Q^{\mathrm{M}} \leqslant R_3$	$(P_1, Q_1) = \left(\frac{a - S^{\mathrm{M}} + Q^{\mathrm{M}}}{b}, Q^{\mathrm{M}}\right)$
	$(P_2, Q_2) = \left(\frac{a - Q^{\mathrm{M}}}{b}, 0\right)$
$\max\left\{0; S^{\mathrm{M}} - S^{\mathrm{o}}\left(\frac{b + d}{d}\right)\right\} \leqslant Q^{\mathrm{M}} \leqslant R_4$	$(P_1, Q_1) = \left(\frac{a - S^{\mathrm{M}} + Q^{\mathrm{M}}}{b}, Q^{\mathrm{M}}\right)$
	$(P_2, Q_2) = \left(\frac{a + c - Q^{\mathrm{M}}}{b} - \frac{d(a - S^{\mathrm{M}} + Q^{\mathrm{M}})}{b^2}, 0\right)$

4. Numerical simulations

In this section we present numerical investigations for the dynamic behavior of the price under the feedback law (3.5). We study the cobweb model developed in Section 2 for the following parameter values:

$$a = 50, \quad b = 3,$$

$$c = 10, \quad d = 4,$$

$$S^{\mathrm{M}} = 25. \tag{4.1}$$

Under this setting, the equilibrium price (P^{o}) is 8.5714 and the equilibrium supply (S^{o}) is 24.2857. Notice that these parameter values ensure that the conditions for positive price and demand, $a > S^{\mathrm{M}}$ and $ad > bc$, respectively, are satisfied. Moreover, it can be verified that the conditions $S^{\mathrm{M}} > S^{\mathrm{o}}$ and $d/b \geqslant 1$ are satisfied and thus the equilibrium price is not a global attractor for the nominal system (2.6) without intervention (see Table 1).

Let us examine the effect of the KSE policy on the behavior of the price, for the case of naive expectations. Under this expectation rule, the well-known phenomenon of the 'hog-cycles' (Hommes, 1998) appears and it has been numerically verified that this period-2 solution attracts every solution, starting from initial conditions that do not coincide with the equilibrium price. Fig. 1 shows the period-2 cycle, where $P_1 = 8.33$ and $P_2 = 8.88$.

Proposition 3.2 guarantees that price stabilization by means of buffer stocks is feasible if Q^{M} exceeds R, where the constant R is determined by (3.6). In this case we find that the critical storage capacity (R) is 0.17857. Notice that the critical storage capacity amounts to approximately only 0.73% of the equilibrium supply ($S^{\mathrm{o}} = 24.285$). Fig. 2 shows the price dynamics of (2.6) under the KSE rule (3.5), $Q^{\mathrm{M}} = 0.18$, $P_0 = 6.5$, and $Q_0 = 0$. As expected by Proposition 3.2, the equilibrium price is guaranteed to be the global attractor of this system. Moreover, we applied (3.5) to the perturbed case (2.7), with additive noise term $w_t \sim \mathrm{N}(0, 0.1)$, and the result

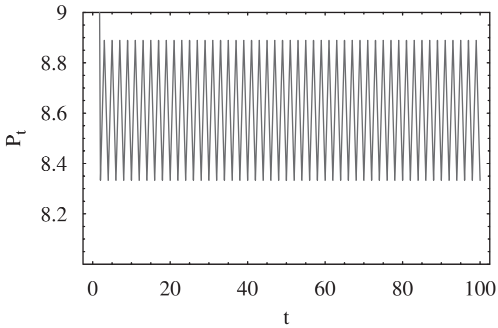

Fig. 1. Price dynamics for cobweb model (2.6) without government intervention.

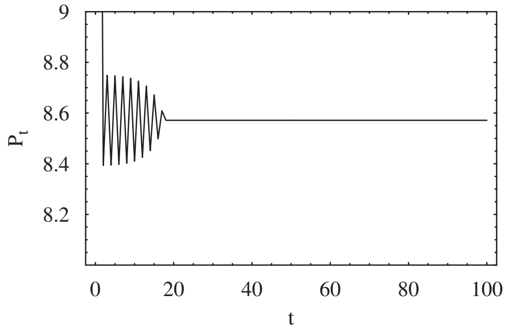

Fig. 2. Price behavior for case (2.6) under the KSE policy rule (3.5).

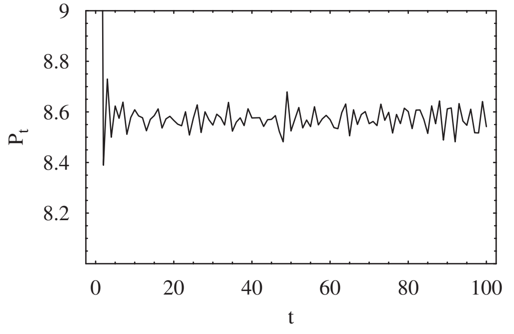

Fig. 3. Price behavior for case (2.7) under the KSE policy rule (3.5).

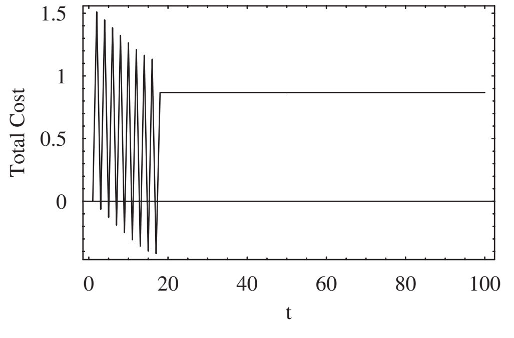

Fig. 4. Total purchase cost for case (2.6) under the KSE policy rule (3.5).

is depicted in Fig. 3. It is evident that the fluctuation of the price is more intense when the government does not apply the KSE policy.

Fig. 4 shows the evolution of the total purchase cost for the nominal case (2.6) under (3.5) with $Q^{\mathrm{M}} = 0.18$ and initial conditions $P_0 = 6.5$, $Q_0 = 0$. It is clear that the total purchase cost is bounded from above and that the government must subsidize the function of the stabilization program. The program is costly because on

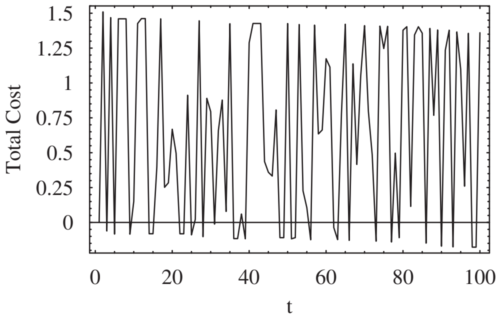

Fig. 5. Total purchase cost for case (2.7) under the KSE policy rule (3.5).

the one hand the government buys a certain quantity of the commodity at a low price, but on the other hand sells only a part of its inventory at a higher price. In order to have some reference point for the total purchase cost, we calculated the revenue at equilibrium, i.e., $RV^{\mathrm{o}} = P^{\mathrm{o}}S^{\mathrm{o}} = 208.163$. Clearly, the purchase cost corresponds to a small percentage of RV^{o}. Fig. 5 depicts the evolution of the total purchase cost for the perturbed case (2.7) under (3.5) and the same values for Q^{M}, P_0, Q_0, as in Fig. 3. Again, the government in general has a cost from the operation of the stabilization program which is a small percentage of RV^{o}.

Let us now examine the robustness of the KSE policy rule. To this purpose, we consider the following cases:

4.1. Case 1: the keep supply at average policy rule

First, we consider that the government approximates the equilibrium supply (S^{o}) by the average supply, i.e., the average of the last K observed supplied quantities, expressed by

$$S_t^{\mathrm{A}} = \frac{\sum_{j=1}^{K} S_{t-j}}{K}. \tag{4.2}$$

Thus, we substitute in the feedback law (3.5) the equilibrium supply S^{o} with the average supply S_t^{A}:

$$G_t = \min\{Q_t; \max\{S_t^{\mathrm{A}} - g(P_{t+1}^{\mathrm{e}}); Q_t - Q^{\mathrm{M}}\}\}. \tag{4.3}$$

We refer to this policy as KSA.

Fig. 6 shows the evolution of the price for the nominal case (2.6) under the feedback law (4.3), for the parameter set (4.1), $K = 5$, $Q^{\mathrm{M}} = 0.18$, and initial conditions $P_0 = 6.5, Q_0 = 0, \{S_0, S_{-1}, \ldots, S_{-4}\} = \{16, 14, 10, 12, 14\}$. Fig. 8 depicts the evolution of the price for $K = 10$, $Q^{\mathrm{M}} = 0.18$ and initial conditions $P_0 = 6.5, Q_0 = 0, \{S_0, S_{-1}, \ldots, S_{-9}\} = \{16, 14, 10, 12, 14, 10, 11, 8, 12, 14\}$. In both cases, the price oscillations are mitigated by the application of the KSA policy rule (compared with Fig. 1). Finally, Figs. 7 and 9 show the total purchase cost for $K = 5$ and 10, respectively. In both cases, it is evident that the government has a cost

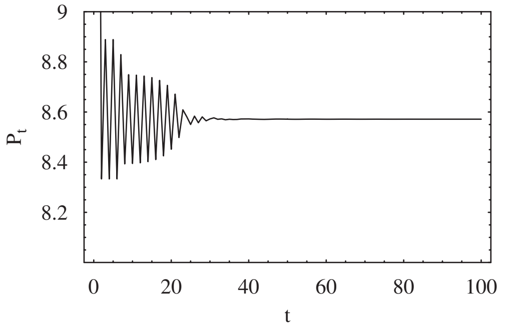

Fig. 6. Price behavior for case (2.6) under the KSA ($K = 5$) policy rule (4.3).

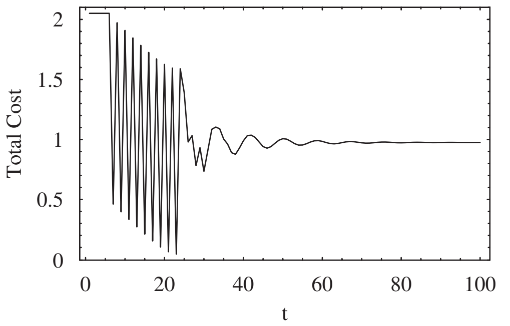

Fig. 7. Total purchase cost for case (2.6) under the KSA ($K = 5$) policy rule (4.3).

from the function of the stabilization program and that the total purchase cost is a small percentage of the revenue at equilibrium ($RV^{\text{o}} = 208.163$) (Figs. 7–9).

4.2. Case 2: hyperbolic tangent supply function

In this case the KSE and the KSA policies are tested for a strongly nonlinear supply function. Following Hommes (1994) we consider a hyperbolic tangent supply function given by

$$S(P^{\mathrm{e}}) = 1 + \tanh(\gamma(P^{\mathrm{e}} - \bar{p})) = 1 + \frac{\mathrm{e}^{\gamma(P^{\mathrm{e}}-\bar{p})} - \mathrm{e}^{-\gamma(P^{\mathrm{e}}-\bar{p})}}{\mathrm{e}^{\gamma(P^{\mathrm{e}}-\bar{p})} + \mathrm{e}^{-\gamma(P^{\mathrm{e}}-\bar{p})}}, \tag{4.4}$$

where the parameter γ tunes the steepness of the S-shape and $\bar{p}$ is the price corresponding to the inflection point. In addition, we consider linear backward-looking expectations with two lags (Collery, 1955; Hommes, 1998):

$$P^{\mathrm{e}}_{t+1} = \lambda P_t + (1 - \lambda)P_{t-1}, \tag{4.5}$$

where $\lambda \in [0, 1]$ is a parameter determining the weights of the past two commodity prices.

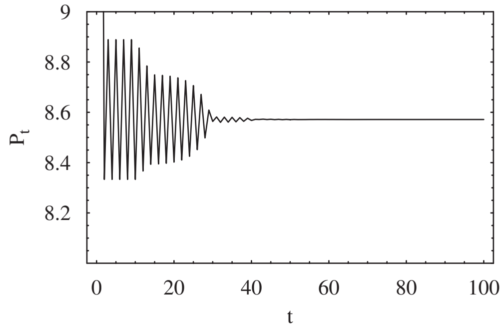

Fig. 8. Price behavior for case (2.6) under the KSA ($K = 10$) policy rule (4.3).

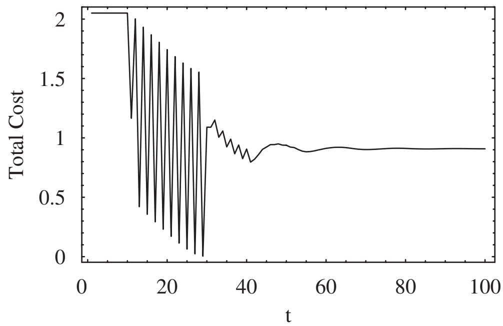

Fig. 9. Total purchase cost for case (2.6) under the KSA ($K = 10$) policy rule (4.3).

The discrete-time control system related to the hyperbolic tangent supply function is described by

$$
\begin{aligned}
&P_{t+1} = b^{-1}(a - G_t - g(\lambda P_t + (1-\lambda)P_{t-1})),\\
&Q_{t+1} = Q_t - G_t, \qquad (4.6)\\
&\max\{Q_t - Q^{\mathrm{M}}; -g(P_t)\} \leqslant G_t \leqslant Q_t,
\end{aligned}
$$

where $g(\lambda P_t + (1-\lambda)P_{t-1})$ is defined by (4.4).

In order to investigate the dynamic behavior of the price we assume a new parameter set, i.e., $\gamma = 5$, $\bar{p} = 1$, $a = 2$, $b = 0.8$, $\lambda = 0.59$ and new initial conditions $\{p_0, p_{-1}\} = \{0.6, 0.5\}$. Under this setting, the equilibrium price (P^{o}) is 1.0348, the equilibrium supply (S^{o}) is 1.17218, the maximum supply (S^{M}) is 1.99 and the equilibrium revenue (RV^{o}) is 1.2129. In the absence of intervention the model has a strange attractor (Hommes, 1998) which is depicted in Fig. 10. Fig. 11 shows the price dynamics for the system under consideration when $G_t \equiv 0$. Fig. 12 shows that the application of the KSE policy with $Q^{\mathrm{M}} = 1.4$ manages to suppress the fluctuation of the price. Notice that in this case the government's storage capacity (Q^{M}) exceeds the equilibrium supply (S^{o}). Fig. 13 shows that under the KSE policy the

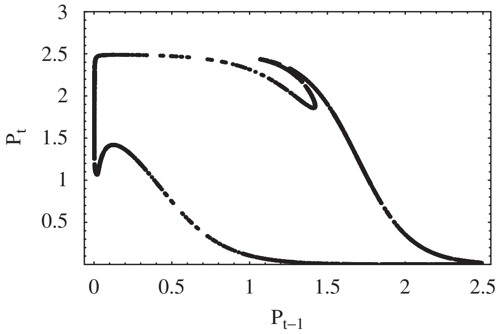

Fig. 10. Phase plane for cobweb model (4.6) with (4.4), without government intervention.

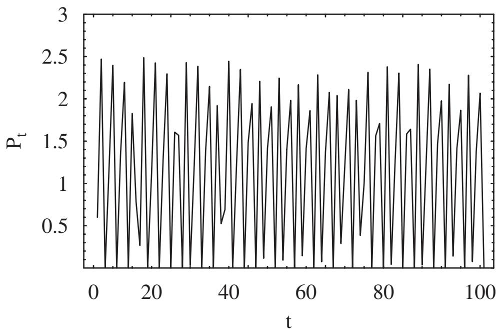

Fig. 11. Price dynamics for cobweb model (4.6) with (4.4), without government intervention.

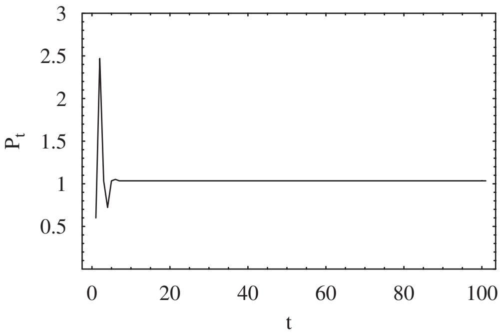

Fig. 12. Price dynamics for case (4.6) with (4.4) under the KSE policy rule (3.5).

government profits from the function of the stabilization program. This remarkable feature must not encourage the reader since a high storage capacity is related to a high storage cost. A more detailed analysis is needed in order to include storage costs.

Finally, we explored the effectiveness of the KSA policy rule (4.3) with the hyperbolic tangent supply function (4.4) and $Q^{\mathrm{M}} = 1.4$. The initial conditions for the

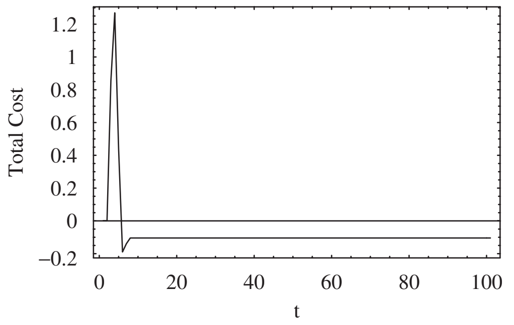

Fig. 13. Total purchase cost for case (4.6) with (4.4) under the KSE policy rule (3.5).

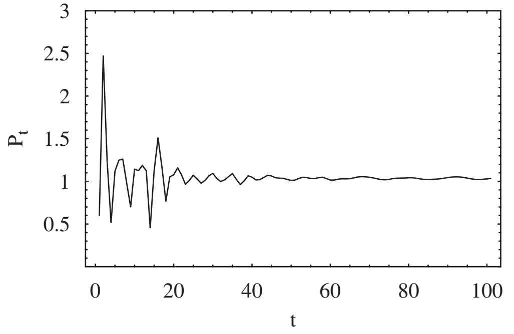

Fig. 14. Price dynamics for case (4.6) with (4.4) under the KSA ($K = 20$) policy rule (4.3).

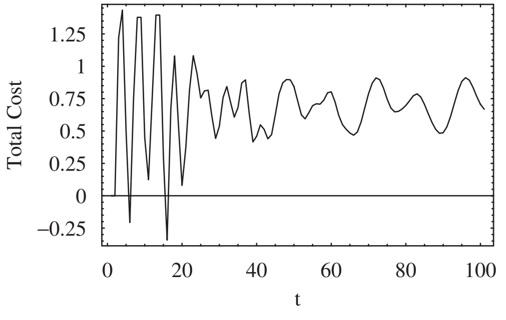

Fig. 15. Total purchase cost for case (4.6) with (4.4) under the KSA ($K = 20$) policy rule (4.3).

price and the supply are $\{p_0, p_{-1}\} = \{0.6, 0.5\}$ and $\{S_0, \ldots, S_{-19}\} = \{0.02, 1.99, 1.06, 0.08, 1.97, 0.91, 0.24, \ 1.98, 0.53, 1.37, 1.78, 0.01, 1.99, 1.11, 0.06, \ 1.98, 0.97, 0.16, 1.99, 0.71\}$, respectively. Fig. 14 shows the evolution of the price for $K = 20$, where it is evident the success of the KSA rule. Moreover, Fig. 15 shows that the operation of the stabilization program for $K = 20$, is costly for the government.

Notice that the total cost is in general lower than the equilibrium revenue ($RV^{\mathrm{o}} = 1.2129$), but their difference is relatively small.

5. Conclusions

In the present work a nonlinear cobweb model is developed for a single commodity with stocks and naive expectations. The model is based on a piecewise linear S-shaped supply function and takes into account the constraints which must be satisfied by the government stockpiling. It is shown that if the storage capacity for the particular commodity is sufficiently large (greater than the 'critical storage capacity') then the KSE policy is successful (the equilibrium price is a global attractor for the corresponding closed-loop system). Moreover, formulae for the computation of the 'critical storage capacity' are provided. The results indicate a trade off between the efficiency of the price stabilization policy and the storage cost.

The main result of the present work (Proposition 3.2) may be used in order to calculate the storage cost needed for a successful stabilization policy by means of buffer stocks. Numerical studies are also presented, which test the effectiveness of the KSE policy rule. Furthermore, we considered a policy rule where the government uses average supply as an observable proxy for the equilibrium level. We refer to this policy rule as KSA and we test its effectiveness.

Our work has certain limitations that must be overcome in order to provide a more complete analysis on the function of buffer stocks. Thus, future research should include private storage, other expectation rules, heterogeneous producers, discounting and international trade. In addition, the analysis could be extended to include more complicated price stabilization policies (possibly time-varying).

Acknowledgements

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Appendix A

We first perform a state transformation for model (2.6) in order to ease the derivation of the price dynamics. To this purpose, it is convenient to introduce the dimensionless variables: $x_t = a^{-1}bP_{t-1}$, $y_t = a^{-1}Q_t$ and the dimensionless constants: $r = b^{-1}d > 0$, $c_1 = (ad)^{-1}bc < 1$, $c_2 = (ad)^{-1}bS^{\mathrm{M}} < d^{-1}b$, $c_3 = a^{-1}Q^{\mathrm{M}}$. Thus the control system (2.6) expressed in state space form and in dimensionless coordinates is given by the following discrete-time control system:

$$\begin{aligned}
&x_{t+1} = 1 - f_1(x_t) - f_2(x_t, y_t, u_t),\\
&y_{t+1} = y_t - f_2(x_t, y_t, u_t),\\
&(x_t, y_t) \in (0, +\infty) \times [0, c_3], \quad u_t \in \Re,
\end{aligned} \tag{A1}$$

where

$$\begin{aligned}
f_1(x) &:= r \max\{0; \min\{c_2; x - c_1\}\},\\
f_2(x, y, u) &:= \min\{y; \max\{u; y - c_3; -f_1(x)\}\}.
\end{aligned}$$

The variable u_t is directly related to the quantity of the commodity released to the market by the government at period $t+1$ and is the control input of the system (i.e., u_t can be manipulated in order to achieve a certain control objective).

Notice that $f_2(x, y, 0) = 0$ for all $(x, y) \in (0, +\infty) \times [0, c_3]$. There exists an equilibrium set for the case $u_t \equiv 0$ (the so-called open-loop system) given by

$$(x, y) = (x^{\mathrm{o}}, y),$$

where $y \in [0, c_3]$ is arbitrary and

$$x^{\mathrm{o}} = \begin{cases} \dfrac{1 + rc_1}{1 + r} & \text{if } c_1 + c_2 > 1 - c_2 r,\\ 1 - c_2 r & \text{if } c_1 + c_2 \leqslant 1 - c_2 r. \end{cases} \tag{A2}$$

For the case of no government intervention (i.e., $u_t \equiv 0$) the corresponding dynamical system (A1) is one-dimensional and is given by the difference equation:

$$x_{t+1} = h(x_t); \quad x_t \in (0, +\infty), \tag{A3}$$

where

$$h(x) := 1 - f_1(x) = \begin{cases} 1 & \text{if } x \leqslant c_1,\\ 1 + c_1 r - r x & \text{if } c_1 < x < c_1 + c_2,\\ 1 - c_2 r & \text{if } x \geqslant c_1 + c_2. \end{cases} \tag{A4}$$

A.1. Proof of statements made in Table 1

We consider the following cases:

Case 1: If $c_1 + c_2 > 1 - c_2 r$ then $c_1 < x^{\mathrm{o}} < c_1 + c_2$ (notice that $c_1 < 1$) and consequently there exists an open neighborhood around x^{o} where h is continuously differentiable (in fact linear). Thus local asymptotic stability of the unique equilibrium point may be checked using linear theory, which demands $|h'(x^{\mathrm{o}})| < 1$ for local asymptotic stability of the equilibrium point or equivalently:

$$r < 1. \tag{A5}$$

However, it may be shown that the equilibrium point is globally asymptotically stable. We prove this fact by considering the Lyapunov function $V(x) := (x - x^{\mathrm{o}})^2$. It

Table A1
The asymptotic behavior of (A3) for Case 3

Cases	Period-2 solution
(a) $c_1 + c_2 > 1$	$x_1 = 1, \quad x_2 = 1 - r(1 - c_1) \leqslant c_1$
(b) $1 \geqslant c_1 + c_2$ and $c_1 + rc_2 \geqslant 1$	$x_1 = 1, \quad x_2 = 1 - rc_2 \leqslant c_1$
(c) $c_1 + rc_2 < 1$	$x_1 = 1 - r(1 - c_1) + r^2 c_2 \geqslant c_1 + c_2, \quad x_2 = 1 - rc_2 < c_1 + c_2$
(d) $r = 1$	There is an infinite number of non-trivial period-2 solutions: $x_1 \in (\max\{c_1; 1 - c_2\}, \min\{1; c_1 + c_2\})$ arbitrary, $x_2 = 1 + c_1 - x_1$

is a matter of simple but tedious calculations to show that, under hypothesis (A5), the following inequality holds:

$$V(h(x)) < V(x) \quad \text{for all } x \in (0, +\infty), \quad x \neq x^{\mathrm{o}},$$

which according to Corollary 3.3 in Jiang and Wang (2002), implies global asymptotic stability of the equilibrium point.

Case 2: If $c_1 + c_2 \leqslant 1 - c_2 r$ then we notice that $h(x) \geqslant c_1 + c_2$ for all $x \in (0, +\infty)$. Thus for every initial condition $x_0 \in (0, +\infty)$ we obtain

$$x_t = x^{\mathrm{o}} \quad \text{for all } t \geqslant 2. \tag{A6}$$

Thus the equilibrium point $x^{\mathrm{o}} = 1 - c_2 r$ is a global attractor for system (A3). In fact, in this case the phenomenon of finite-time stability appears as (A6) shows (i.e., the equilibrium point is approached in finite time).

Case 3: If $c_1 + c_2 > 1 - c_2 r$ and $r \geqslant 1$, the dynamics of the nominal system (A3) present the well-known phenomenon of the 'hog-cycles'. For the nonlinear case (A3) the 'hog-cycles' can be given explicitly (Table A1).

Clearly, the existence of a non-trivial period-2 solution excludes the possibility of the existence of an equilibrium point, which is a global attractor, i.e., the corresponding price equilibrium point $x^{\mathrm{o}} = (1 + rc_1)/(1 + r)$ is not a global attractor.

A.2. Proof of Lemma 3.1

By virtue of (A1) and continuity of the function f_1, it follows that the limit $\lim_{t \to +\infty} f_2(x_t, y_t, \phi(x_t, y_t))$ exists and is given by

$$\lim_{t \to +\infty} f_2(x_t, y_t, \phi(x_t, y_t)) = 1 - x^* - f_1(x^*).$$

The above equality implies that $\lim_{t \to +\infty} (y_{t+1} - y_t) = x^* - 1 + f_1(x^*)$. If $x^* - 1 + f_1(x^*) > 0$ then we obtain $\lim_{t \to +\infty} y_t = +\infty$, which contradicts the constraint $y_t \in [0, c_3]$. On the other hand, if $x^* - 1 + f_1(x^*) < 0$ then we obtain $\lim_{t \to +\infty} y_t = -\infty$, which again contradicts the constraint $y_t \in [0, c_3]$. Thus we must necessarily have $x^* - 1 + f_1(x^*) = 0$ and consequently the desired price $x^* > 0$ coincides with x^{o}, the unique equilibrium price of the open-loop system (A3) defined by (A2). The proof is complete. □

A.3. Proof of Proposition 3.2

Consider the price stabilization problem for (A1) under the hypotheses $c_1 + c_2 > 1 - c_2 r$ and $r \geqslant 1$. If $c_3 > R$, where

$$R := \max\{R_1; R_2; R_3; R_4\}. \tag{A7}$$

$$\begin{aligned}
&R_1 := \min\{(1 - r^{-1})(1 - x^{\mathrm{o}}); c_1 + c_2 r - 1\};\\
&R_2 := (r - 1)\min\{x^{\mathrm{o}} - c_1; c_1 + c_2 - x^{\mathrm{o}}\} \geqslant 0;\\
&R_3 := \min\{c_1 + c_2 r - 1; 1 - c_1 - c_2\};\\
&R_4 := (r + 1)^{-1} \min\{r(x^{\mathrm{o}} + c_2 r - 1); (r - 1)(c_1 + c_2 + c_2 r - 1)\},
\end{aligned}$$

then the equilibrium set $\{(x^{\mathrm{o}}, y); y \in [0, c_3]\}$ is a global attractor for the closed-loop system (A1) with $u_t = 1 - x^{\mathrm{o}} - f_1(x(t))$. Particularly, for every initial condition $(x_0, y_0) \in (0, +\infty) \times [0, c_3]$ there exists $T > 0$ such that the solution of (A1) with $u_t = 1 - x^{\mathrm{o}} - f_1(x_t)$ satisfies

$$x_t = x^{\mathrm{o}}, \quad \forall t \geqslant T. \tag{A8}$$

Notice that

$$f_2(x, y, 1 - x^{\mathrm{o}} - f_1(x)) = \min\{y; \max\{y - c_3; 1 - x^{\mathrm{o}} - f_1(x)\}\},$$

$$\begin{aligned}
&1 - f_1(x) - f_2(x, y, 1 - x^{\mathrm{o}} - f_1(x))\\
&\quad = \begin{cases} 1 - f_1(x) - y + c_3 & \text{if } y + f_1(x) > 1 - x^{\mathrm{o}} + c_3,\\ x^{\mathrm{o}} & \text{if } 1 - x^{\mathrm{o}} \leqslant y + f_1(x) \leqslant 1 - x^{\mathrm{o}} + c_3,\\ 1 - f_1(x) - y & \text{if } y + f_1(x) < 1 - x^{\mathrm{o}}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
&y - f_2(x, y, 1 - x^{\mathrm{o}} - f_1(x))\\
&\quad = \begin{cases} c_3 & \text{if } y + f_1(x) > 1 - x^{\mathrm{o}} + c_3,\\ y - 1 + x^{\mathrm{o}} + f_1(x) & \text{if } 1 - x^{\mathrm{o}} \leqslant y + f_1(x) \leqslant 1 - x^{\mathrm{o}} + c_3,\\ 0 & \text{if } y + f_1(x) < 1 - x^{\mathrm{o}}, \end{cases}
\end{aligned}$$

$$f_1(x) = \begin{cases} 0 & \text{if } x \leqslant c_1,\\ -c_1 r + r x & \text{if } c_1 < x < c_1 + c_2,\\ c_2 r & \text{if } x \geqslant c_1 + c_2. \end{cases}$$

Making use of the above equalities in conjunction with hypotheses $y_t \in [0, c_3]$, $c_1 + c_2 > 1 - c_2 r$ and $r \geqslant 1$, it can be shown that

$$\begin{aligned}
&x_{t+1} = 1 + c_1 r - r x_t + c_3 - y_t \quad \text{and} \quad y_{t+1} = c_3\\
&\quad \text{if } (x_t, y_t) \in B_1 := \{x^{\mathrm{o}} + r^{-1}(c_3 - y) < x < c_1 + c_2\},
\end{aligned} \tag{A9}$$

$$x_{t+1} = 1 - c_2 r + c_3 - y_t \quad \text{and} \quad y_{t+1} = c_3$$
$$\text{if } (x_t, y_t) \in B_2 \coloneqq \{x \geqslant c_1 + c_2 \text{ and } y > 1 - x^{\mathrm{o}} + c_3 - c_2 r\}, \tag{A10}$$

$$x_{t+1} = 1 - y_t \quad \text{and} \quad y_{t+1} = 0$$
$$\text{if } (x_t, y_t) \in B_3 \coloneqq \{0 < x \leqslant c_1 \text{ and } y < 1 - x^{\mathrm{o}}\}, \tag{A11}$$

$$x_{t+1} = 1 + c_1 r - r x_t - y_t \quad \text{and} \quad y_{t+1} = 0$$
$$\text{if } (x_t, y_t) \in B_4 \coloneqq \{c_1 < x < x^{\mathrm{o}} - r^{-1} y\}, \tag{A12}$$

$$x_{t+1} = x^{\mathrm{o}} \quad \text{if } (x_t, y_t) \in B_5 = (0, +\infty) \times [0, c_3] \Bigg\backslash \left(\bigcup_{i=1}^{4} B_i \right). \tag{A13}$$

Moreover, since $c_1 + c_2 > 1 - c_2 r$ which directly implies $c_1 < x^{\mathrm{o}} < c_1 + c_2$ (notice that $c_1 < 1$), it follows that the following implications hold:

$$(x_t, y_t) \in B_1 \cup B_2 \Rightarrow (x_{t+1}, y_{t+1}) \in B_3 \cup B_4 \cup B_5, \tag{A14}$$

$$(x_t, y_t) \in B_3 \cup B_4 \Rightarrow (x_{t+1}, y_{t+1}) \in B_1 \cup B_2 \cup B_5, \tag{A15}$$

$$(x_{t_0}, y_{t_0}) \in B_5 \Rightarrow x_t = x^{\mathrm{o}}, \quad \forall t \geqslant t_0 + 1. \tag{A16}$$

In order to show (A8) it suffices to show that for every initial condition $(x_0, y_0) \in (0, +\infty) \times [0, c_3]$ there exists $T \geqslant 0$ such that $(x_T, y_T) \in B_5$. The proof will be made by contradiction. Suppose on the contrary that there exists initial condition $(x_0, y_0) \notin B_5$ such that $(x_t, y_t) \notin B_5$ for all $t \geqslant 0$. Without loss of generality we may assume that $(x_0, y_0) \in B_1 \cup B_2$ (since if $(x_0, y_0) \in B_3 \cup B_4$ then by virtue of (A15) we would have $(x_1, y_1) \in B_1 \cup B_2$ and then consider the solution with initial condition $(x_1, y_1) \in B_1 \cup B_2$ which also satisfies $(x_t, y_t) \notin B_5$ for all $t \geqslant 0$). By virtue of implications (A14) and (A15) we must have:

$$(x_t, y_t) \in B_1 \cup B_2 \text{ if } t \text{ is even} \quad \text{and} \quad (x_t, y_t) \in B_3 \cup B_4 \text{ if } t \text{ is odd}. \tag{A17}$$

Moreover, by virtue of (A9–13) we obtain

$$y_t = 0 \text{ if } t \geqslant 1 \text{ is even} \quad \text{and} \quad y_t = c_3 \text{ if } t \geqslant 1 \text{ is odd}. \tag{A18}$$

By virtue of (A9–13), (A17–18) we must also have

$$x^{\mathrm{o}} < x^{\mathrm{o}} + \min\{r^{-1} c_3; c_1 + c_2 - x^{\mathrm{o}}\} \leqslant x_t \quad \text{if } t \geqslant 2 \text{ is even}. \tag{A19}$$

Next we show that there exists $\delta > 0$ such that

$$x_{t+2} \leqslant x_t - \delta \quad \text{for all even integers } t \geqslant 2. \tag{A20}$$

Thus we obtain a contradiction, since (A19) in conjunction with (A20) and the fact that $x_t \leqslant 1$ for all $t \geqslant 1$ will give

$$x^{\mathrm{o}} + \min\{r^{-1} c_3; c_1 + c_2 - x^{\mathrm{o}}\} \leqslant x_{2+2k} \leqslant 1 - k\delta \quad \text{for all integers } k \geqslant 1,$$

which cannot hold for $k \geqslant 1 + \delta^{-1} - \delta^{-1} x^{\mathrm{o}} - \delta^{-1} \min\{r^{-1} c_3; c_1 + c_2 - x^{\mathrm{o}}\}$.

Consequently, the rest part of proof is devoted to the determination of the constant $\delta > 0$ that satisfies (A20). For all $(x_t, 0) \in B_1 \cup B_2$, it can be shown that:

Case I: If $x^o + r^{-1}(x^o + c_3 - c_1) \leqslant x_t < c_1 + c_2$ and $c_3 < 1 - x^o$ then $x_{t+2} = 1 - c_3$.

Case II: If $x^o + r^{-2}(r+1)c_3 < x_t < c_1 + \min\{c_2; r^{-1}(1 + c_3 - c_1)\}$ then $x_{t+2} = x^o + r^2(x_t - x^o) - (r+1)c_3$.

Case III: If $x_t \geqslant c_1 + c_2$ and $c_3 < 1 - x^o$ and $c_3 \leqslant c_1 + c_2 r - 1$ then $x_{t+2} = 1 - c_3$.

Case IV: If $x_t \geqslant c_1 + c_2$ and $c_1 + c_2 r - 1 < c_3 < (r+1)^{-1} r(x^o + c_2 r - 1)$ then $x_{t+2} = 1 + c_1 r - r + c_2 r^2 - (r+1)c_3$.

Case V: If none of the above holds then $x_{t+2} = x^o$.

If Case I is possible (i.e., if $c_3 < 1 - x^o$ and $c_3 < c_1 + rc_2 - 1$) then by virtue of the inequality $c_3 > R_1$ we must necessarily have $c_3 > (1 - r^{-1})(1 - x^o)$ and consequently (A20) holds with $\delta := (1 + r^{-1})(x^o + c_3) - (1 + r^{-1}c_1) > 0$.

If Case II is possible (i.e., if $c_3 < r(x^o - c_1)$ and $c_3 < (r+1)^{-1} r^2 (c_1 + c_2 - x^o)$) then by virtue of the inequality $c_3 > R_2$ we must necessarily have $c_3 > (r-1)\min\{x^o - c_1; c_1 + c_2 - x^o\}$ and consequently (A20) holds with $\delta := (1+r)c_3 - (1+r)(r-1)\min\{c_1 + c_2 - x^o; c_1 + r^{-1}(1 + c_3 - c_1) - x^o\} > 0$.

If Case III is possible (i.e., if $c_3 < 1 - x^o$ and $c_3 \leqslant c_1 + rc_2 - 1$) then by virtue of the inequality $c_3 > R_3$ we must necessarily have $c_3 > 1 - c_1 - c_2$ and consequently (A20) holds with $\delta := c_1 + c_2 + c_3 - 1 > 0$.

If Case IV is possible (i.e., if $c_1 + c_2 r - 1 < c_3 < (r+1)^{-1} r(x^o + c_2 r - 1)$) then by virtue of the inequality $c_3 > R_4$ we must necessarily have $c_3 > (r+1)^{-1}(r-1)(c_1 + c_2 + c_2 r - 1)$ and consequently (A20) holds with $\delta := (1+r)c_3 + (r-1)(1 - c_1 - c_2 - c_2 r) > 0$.

If two or more cases are possible then the constant $\delta > 0$ that satisfies (A20) may be selected as the minimum of the corresponding constants given for each case. Thus property (A8) is proved.

Property (A8) implies that the ω-limit set of the bounded set $[1 - rc_2, 1] \times [0, c_3]$ is the equilibrium set $\{x^o\} \times [0, c_3]$. Moreover, notice that for every initial condition $(x_0, y_0) \in (0, +\infty) \times [0, c_3]$ we have $(x_1, y_1) \in [1 - rc_2, 1] \times [0, c_3]$. By virtue of Definitions 1.7.1, 1.8.4 and Theorem 1.7.2 in Stuart and Humphries (1998), we conclude that the equilibrium set $\{x^o\} \times [0, c_3]$ is a global attractor for the closed-loop system (A1) with $u_t = 1 - x^o - f_1(x_t)$. The proof is complete. $\square$

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