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Input-to-Output Stability for Systems Described by Retarded Functional Differential Equations

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In this work characterizations of external notions of output stability for uncertain time-varying systems described by retarded functional differential equations are provided. Particularly, characterizations by means of Lyapunov and Razumikhin functions of (uniform) Weighted Input-to-Output Stability are given. The results of this work have been developed for systems with outputs in abstract normed linear spaces in order to allow outputs with no delay, with discrete or distributed delay or functional outputs with memory.

Keywords: Lyapunov functionals, Razumikhin functions, time-delay systems, input-to-output stability.

1. Introduction

The introduction of the notion of Input-to-State Stability (ISS) in [30] for finite-dimensional systems described by ordinary differential equations, led to an exceptionally rich period of progress in mathematical systems and control theory. The notion of ISS and its characterizations given in [31,32] were proved to be extremely useful for the expression of small-gain results (see [2,7,8,9,14,19,36]) and for the construction of robust feedback stabilizers (see for instance the textbook [22]). The notion of ISS was extended to the notion of Input-to-Output Stability (IOS) in [5,7,34,35] and to the non-uniform in time notions of ISS and IOS in [12–14] (which extended the applicability of ISS to time-varying systems). Recently, semi-uniform notions of ISS have been proposed in [25]. The notions of ISS and IOS were recently proposed and characterized for discrete-time systems (see [10,11,15]) as well as to a wide class of systems with outputs (see [14]). It is our belief that the notions of ISS and IOS have become one of the most important conceptual tools for the development of nonlinear robust stability and control theory for a wide class of dynamical systems.

The importance of Retarded Functional Differential Equations (RFDEs) for engineering applications is well known (see for instance [4,23,26]). In this work we develop characterizations of various robust external stability notions for uncertain systems described by RFDEs, including uniform and non-uniform in time ISS and IOS. We provide necessary and sufficient Lyapunov criteria based on functionals for the Weighted Input-to-Output Stability for systems described by very general RFDEs (time-varying, with time-varying non-commensurate discrete and distributed time delays, uncertain). The obtained results are general and can be applied to a wide range of stability problems. The robust external stability notions proposed in the present work are parallel to the robust external stability notions used for finite-dimensional systems. Thus, it is expected that the results of the present paper will play an important role

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in mathematical systems and control theory for the important case of systems described by RFDEs. Particularly, exactly as in the finite-dimensional case, the results of the present work can be used:

- for the proof of necessary and sufficient Lyapunov-like conditions for the existence of feedback stabilizers which guarantee the IOS property for the corresponding closed-loop system,
- for the development of robust solutions to tracking control problems (even in the case where the desired trajectory is unbounded),
- for the proof of "adding an integrator" results (see for instance [1,7,17,37,38] for the finite-dimensional case),
- in order to develop methodologies for feedback construction for (time-varying) triangular systems,
- in order to develop Lyapunov redesign methodologies for the construction of feedback stabilizers which are robust with respect to control actuator and measurement errors,
- for the proof of necessary and sufficient Lyapunov-like conditions for the existence of observers which are robust with respect to measurement errors.

Let $D \subseteq \Re^l$ be a non-empty set, $U \subseteq \Re^m$ a non-empty set with $0 \in U$ and Y a normed linear space. We denote by $x(t)$ with $t \geq t_0$ the unique solution of the initial-value problem:

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, u(t), d(t)) \ , \\ &Y(t) = H(t, T_r(t)x) \\ &x(t) \in \Re^n \ , \ Y(t) \in Y, \ d(t) \in D \, , \ u(t) \in U \end{aligned} \tag{1.1}$$

with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$, where $r > 0$ is a constant, $T_r(t)x := x(t+\theta)$; $\theta \in [-r,0]$ and the mappings $f: \Re^+ \times C^0([-r,0]\,;\,\Re^n) \times U \times D \to \Re^n, H: \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to Y$ satisfy $f(t,0,0,d) = 0, H(t,0) = 0$ for all $(t,d) \in \Re^+ \times D$. In Section 3, we provide characterizations of the (uniform or non-uniform) Weighted IOS property for systems of the form (1.1) under weak hypotheses, which are usually satisfied in applications. The study of the uniform ISS property for autonomous systems described by RFDEs, was recently initiated in [28,39]. The important technical issues that arise in the study of the case (1.1) are solved with a combination of the Lyapunov-like characterization given in [21] and an extension of the results in [27].

A major advantage of allowing the output to take values in abstract normed linear spaces is that using the framework of the case (1.1) we are in a position to consider:

- outputs with no delays, e.g. $Y(t)=h(t,x(t))$ with $Y=\Re^k$,
- outputs with discrete or distributed delay, e.g. $Y(t) = h(x(t), x(t-r))$ or $Y(t) = \int_{t-r}^{t} h(t,\theta,x(\theta))d\theta$ with $Y = \Re^k$,
- functional outputs with memory, e.g. $Y(t) = h(t,\theta,x(t+\theta))$; $\theta \in [-r,0]$ or the identity output $Y(t) = T_r(t)x = x(t+\theta)$; $\theta \in [-r,0]$ with $Y = C^0([-r,0];\Re^k)$.

Moreover, it should be emphasized that our assumptions for (1.1) are very weak, since we do not assume boundedness or continuity of the right-hand side of the differential equation with respect to time or a Lipschitz condition for f. Furthermore, we do not assume that the disturbance set $D \subseteq \Re^l$ is compact.

In Section 4, we develop Razumikhin conditions for the stability notions introduced in previous sections. The use of Razumikhin functions in the study of qualitative properties of the solutions of time-delay systems is emphasized in [4] (see also [6] for control applications). Recently, in [39] a major observation was established: Razumikhin theorems are "essentially" small-gain results. This idea is exploited in the present work, in order to produce novel results that are easily applicable.

Finally, in Section 5 we provide the concluding remarks of this work.

Notations: Throughout this paper we adopt the following notations:

* Let $I \subseteq \Re$ be an interval. By $C^0(I\,;\,\Omega)$, we denote the class of continuous functions on I, which take values in Ω. By $C^1(I\,;\,\Omega)$, we denote the class of functions on I with continuous derivative, which take values in Ω.
* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. For $x \in C^0([-r,0];\Re^n)$ we define $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$.
* We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. By K we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho : \Re^+ \to \Re^+$ is of class K_∞ if $\lim_{s\to+\infty} \rho(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s,t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot\,,t)$ is of class K ; (ii) for each $s \geq 0$, the mapping $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s,t) = 0$.
* Let $U \subseteq \Re^m$ be a non-empty set with $0 \in U$. By $B_U[0,r] := \{\, u \in U\,;\, |u| \leq r \,\}$ we denote the intersection of $U \subseteq \Re^m$ with the closed sphere of radius $r \geq 0$ in $\Re^m$, centered at $0 \in U$.

* Let $D \subseteq \Re^l$ be a non-empty set. By M_D we denote the class of all Lebesgue measurable and locally essentially bounded mappings $d : \Re^+ \to D$. By $\tilde{M}_D$ we denote the class of all right-continuous mappings $d : \Re^+ \to D$, with the property that there exists a countable set $J_d \subset \Re^+$ which is either finite or $J_d = \{t_k^d \; ; \; k = 1, ..., \infty\}$ with $t_{k+1}^d > t_k^d > 0$ for all $k = 1, 2, ...$ and $\lim t_k^d = +\infty$, such that the mapping $t \in \Re^+ \backslash J_d \to d(t) \in D$ is continuous.
* Let $x : [a - r, b) \to \Re^n$ with $b > a > -\infty$ and $r > 0$. By $T_r(t)x$ we denote the "r-history" of x at time $t \in [a, b)$, i.e., $T_r(t)x := x(t + \theta) \; ; \; \theta \in [-r, 0]$. Notice that $T_r(t)x \in C^0([-r, 0]; \Re^n)$ if x is continuous.
* By $\| \|_Y$, we denote the norm of the normed linear space Y.

2. Main Assumptions and Preliminaries for Systems Described by RFDEs

In this section we provide background material needed for the study of systems described by RFDEs. Although the results of this section are technical, they play a fundamental role in the proofs of the main results of the present work.

2.1. Main Assumptions for Systems Described by RFDEs

Concerning systems of the form (1.1) the following hypotheses will be valid throughout the text:

(S1) The mapping $(x, u, d) \to f(t, x, u, d)$ is continuous for each fixed $t \geq 0$ and such that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]\,;\, \Re^n) \times U$, there exists a constant $L \geq 0$ such that:

$$\begin{aligned}&(x(0) - y(0))'(f(t, x, u, d) - f(t, y, u, d))\\ &\quad \leq L \max_{\tau \in [-r,0]} |x(\tau) - y(\tau)|^2 = L\|x - y\|_r^2\\ &\forall t \in I\,,\ \forall (x, u, y, u) \in S \times S\,,\ \forall d \in D\end{aligned}$$

Hypothesis (S1) is equivalent to the existence of a continuous function $L : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$ the mappings $L(t, \cdot)$ and $L(\cdot, t)$ are non-decreasing, with the following property:

$$\begin{aligned}&(x(0) - y(0))'(f(t, x, u, d) - f(t, y, u, d))\\ &\quad \leq L(t, \|x\|_r + \|y\|_r + |u|)\ \|x - y\|_r^2\\ &\forall (t, x, y, d, u) \in \Re^+ \times C^0([-r, 0]\,;\, \Re^n)\\ &\quad \times C^0([-r, 0]\,;\, \Re^n) \times D \times U \end{aligned} \tag{2.1}$$

(S2) For every bounded $\Omega \subset \Re^+ \times C^0([-r, 0]\,;\, \Re^n) \times U$ the image set $f(\Omega \times D) \subset \Re^n$ is bounded.

(S3) There exists a countable set $A \subset \Re^+$, which is either finite or $A = \{t_k \; ; \; k = 1, ..., \infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1, 2, ...$ and $\lim t_k = +\infty$, such that mapping $(t, x, u, d) \in (\Re^+ \backslash A) \times C^0([-r, 0]; \Re^n) \times U \times D \to f(t, x, u, d)$ is continuous. Moreover, for each fixed $(t_0, x, u, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D$, we have $\lim_{t \to t_0^+} f(t, x, u, d) = f(t_0, x, u, d)$.

(S4) For every $\varepsilon > 0$, $t \in \Re^+$, there exists $\delta := \delta(\varepsilon, t) > 0$ such that $\sup\{\, |f(\tau, x, u, d)| \; ; \; \tau \in \Re^+\,,\ d \in D\,,\ u \in U,\ |\tau - t| + \|x\|_r + |u| < \delta \,\} < \varepsilon$.

(S5) The mapping $u \to f(t, x, u, d)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]\,;\, \Re^n) \times U$, there exists a constant $L_U \geq 0$ such that:

$$\begin{aligned}&|f(t, x, u, d) - f(t, x, v, d)| \leq L_U |u - v|,\\ &\forall t \in I\,,\ \forall (x, u, x, v) \in S \times S\,,\ \forall d \in D\end{aligned}$$

Hypothesis (S5) is equivalent to the existence of a continuous function $L_U : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$ the mappings $L_U(t, \cdot)$ and $L_U(\cdot, t)$ are non-decreasing, with the following property:

$$\begin{aligned}&|f(t, x, u, d) - f(t, x, v, d)|\\ &\leq L_U(t, \|x\|_r + |u| + |v|\,)\ |u - v|\\ &\forall (t, x, d, u, v) \in \Re^+ \times C^0([-r, 0]\,;\, \Re^n)\\ &\quad \times D \times U \times U\end{aligned} \tag{2.2}$$

(S6): U is a positive cone, i.e., for all $u \in U$ and $\lambda \geq 0$ it follows that $(\lambda u) \in U$.

(S7) The mapping $H(t, x)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r, 0]\,;\, \Re^n)$, there exists a constant $L_H \geq 0$ such that:

$$\begin{aligned}&\|H(t, x) - H(\tau, y)\|_Y \leq L_H \left(|t - \tau| + \|x - y\|_r\right),\\ &\quad \forall (t, \tau) \in I \times I\,,\ \forall (x, y) \in S \times S\end{aligned} \tag{2.3}$$

Hypothesis (S7) is equivalent to the existence of a continuous function $L_H : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $t \geq 0$ the mappings $L_H(t, \cdot)$ and $L_H(\cdot, t)$ are non-decreasing, with the following property:

$$\begin{aligned}&\|H(t,x) - H(\tau,y)\|_Y \leq L_H\big(\max\{\,t, \tau\}, \|x\|_r + \|y\|_r\big)\\ &\big(|t - \tau| + \|x - y\|_r\big)\\ &\forall (t, \tau, x, y) \in \Re^+ \times \Re^+ \times C^0([-r, 0]; \Re^n)\\ &\times C^0([-r, 0]; \Re^n)\end{aligned} \tag{2.4}$$

Using hypotheses (S1–7) above, Theorem 2.1 in [4] (and its extension given in paragraph 2.6 of the same book) and Theorem 3.2 in [4], we may conclude that for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0]\,;\Re^n) \times M_D \times M_U$ there exists $t_{\max} \in (t_0, +\infty]$, such that the unique solution $x(t)$ of (1.1) with initial condition $T_r(t_0)x = x_0$ is defined on $[t_0 - r, t_{\max})$ and cannot be further continued. Moreover, if $t_{\max} < +\infty$ then we must necessarily have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$. We denote by $\phi(t, t_0, x_0; u, d)$ the "r-history" of the unique solution of (1.1), i.e., $\phi(t, t_0, x_0; u, d) := T_r(t)x$, with initial condition $T_r(t_0)x = x_0$ corresponding to $(d, u) \in M_D \times M_U$. Moreover, the following inequality holds for every pair $\phi(\cdot, t_0, x_0; u, d) : [t_0, t^x_{\max}) \to C^0([-r,0];\Re^n)$, $\phi(\cdot, t_0, y_0; u, d) : [t_0, t^y_{\max}) \to C^0([-r,0];\Re^n)$ of solutions of (1.1) with initial conditions $T_r(t_0)x = x_0$, $T_r(t_0)y = y_0$, corresponding to the same $(d, u) \in M_D \times M_U$ and for all $t \in [t_0, t_1)$ with $t_1 = \min\{t^x_{\max}; t^y_{\max}\}$:

$$\begin{aligned}
&\|\phi(t,t_0,x_0;u,d)-\phi(t,t_0,y_0;u,d)\|_r \\
&\quad \le \|x_0-y_0\|_r \exp(L(t,a(t))(t-t_0)) \\
&\|H(t,\phi(t,t_0,x_0;u,d))-H(t,\phi(t,t_0,y_0;u,d))\|_Y \\
&\quad \le L_H(t,a(t))\|x_0-y_0\|_r \exp(L(t,a(t))(t-t_0)) \\
&a(t)=\sup_{\tau\in[t_0,t]}\big(\|\phi(t,t_0,x_0;u,d)\|_r \\
&\qquad +\|\phi(t,t_0,y_0;u,d)\|_r\big)+\sup_{\tau\in[t_0,t]}|u(\tau)|
\end{aligned} \tag{2.5}$$

2.2. Important Notions for Systems Described by RFDEs

An important property for systems of the form (1.1) is Robust Forward Completeness (RFC) from an external input (see [14,18,19]). This property will be used extensively in the following sections of the present work. Notice that the notion of Robust Forward Completeness (RFC) from the input $u \in M_U$ coincides with the notion of Robust Forward Completeness (RFC) for systems without external inputs (see [21]).

Definition 2.1: *We say that (1.1) under hypotheses (S1–7) is* ***robustly forward complete (RFC) from the input*** $u \in M_U$ *if for every* $s \ge 0, T \ge 0$, *it holds that*

$$\begin{aligned}
&\sup\{\ \|\phi(t_0+\xi, t_0, x_0; u, d)\|_r; \\
&u \in M_{B_U[0,s]},\ \xi \in [0,T]\ ,\ \|x_0\|_r \le s, \\
&t_0 \in [0,T]\ ,\ d \in M_D\ \} < +\infty
\end{aligned}$$

In order to study the asymptotic properties of the solutions of systems of the form (1.1), we will use Lyapunov functionals and functions. Therefore, certain notions and properties concerning functionals are needed. Let $x \in C^0([-r,0]\,;\Re^n)$ and $V : \Re^+ \times C^0([-r,0]\,;\Re^n) \to \Re$. By $E_h(x; v)$, where $0 \le h < r$ and $v \in \Re^n$ we denote the following operator:

$$E_h(x;v) := \begin{cases} x(0)+(\theta+h)\,v & \text{for } \ -h<\theta\le 0 \\ x(\theta+h) & \text{for } \ -r\le\theta\le -h \end{cases} \tag{2.6}$$

and we define

$$V^0(t,x;v) := \limsup_{\substack{h\to 0^+ \\ y\to 0,\, y\in C^0([-r,0];\Re^n)}} \frac{V(t+h,E_h(x;v)+hy)-V(t,x)}{h} \tag{2.7}$$

The class of functionals which are "almost Lipschitz on bounded sets" is introduced in [21] and is used extensively in the present work. For reasons of completeness we repeat the definition here.

Definition 2.2: *We say that a continuous functional* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ *is "almost Lipschitz on bounded sets", if there exist non-decreasing functions* $M : \Re^+ \to \Re^+, P : \Re^+ \to \Re^+, G : \Re^+ \to [1,+\infty)$ *such that for all* $R \ge 0$, *the following properties hold:*

(P1) *For every* $x, y \in \{x \in C^0([-r,0];\Re^n)\ ;\ \|x\|_r \le R\}$, *it holds that:*

$$|V(t,y) - V(t,x)| \le M(R)\ \|y - x\|_r, \forall t \in [0,R]$$

(P2) *For every absolutely continuous function* $x : [-r,0] \to \Re^n$ *with* $\|x\|_r \le R$ *and essentially bounded derivative, it holds that:*

$$|V(t+h,x)-V(t,x)| \le hP(R)\left(1+\sup_{-r\le\tau\le 0}|\dot{x}(\tau)|\right),$$
$$\text{for all } t\in[0,R] \text{ and } 0\le h\le 1/G\left(R+\sup_{-r\le\tau\le 0}|\dot{x}(\tau)|\right)$$

If the continuous functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, is "almost Lipschitz on bounded sets" then the derivative $V^0(t,x;v)$ defined by (2.7) is simplified in the following way:

$$V^0(t,x;v) := \limsup_{h\to 0^+} \frac{V(t+h,E_h(x;v)) - V(t,x)}{h}$$

The following definition introduces an important relation between output mappings. The equivalence relation defined next, will be used extensively in the following sections of the present work (see also [21]).

Definition 2.3: *Suppose that there exists a continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$ *with* $h(t,0) = 0$ *for* *all $t \geq -r$ and functions $a_1, a_2 \in K_\infty$ such that* $a_1(|h(t, x(0))|) \leq \|H(t,x)\|_Y \leq a_2\left(\sup_{\theta\in[-r,0]} |h(t+\theta, x(\theta))|\right)$ for *all $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$. Then we say that $H : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to Y$ is equivalent to the finite-dimensional mapping h.*

For example the identity output mapping $H(t,x) = x \in C^0([-r,0]\,;\,\Re^n)$ is equivalent to finite-dimensional mapping $h(t,x) = x \in \Re^n$.

2.3. Useful Technical Results

The following lemma presents some elementary properties of the generalized derivative given above. Its proof is almost identical with Lemma A.1 in [21] and is omitted.

Lemma 2.4: *Let $V : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to \Re$ and let $x \in C^0([t_0 - r, t_{\max}); \Re^n)$ a solution of (1.1) under hypotheses (S1–7) corresponding to certain $(d,u) \in M_D \times M_U$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then it holds that*

$$\limsup_{h\to 0^+} h^{-1}(V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \leq V^0(t, T_r(t)x; D^+x(t)),\ a.e. \text{ on } [t_0, t_{\max}) \quad (2.8)$$

where $D^+x(t) = \lim_{h\to 0^+} h^{-1}(x(t+h) - x(t))$. Moreover, if $(d,u) \in \tilde{M}_D \times \tilde{M}_U$ then (2.8) holds for all $t \in [t_0, t_{\max})$.

The following results are direct extensions of the similar results in [27]. Their proofs are almost identical with the proof of Lemma A.2 and Lemma A.3 in [21] and are omitted. More specifically, the proof of Lemma 2.6 utilizes inequality (2.5), which guarantees continuity of the solution with respect to the initial conditions.

Lemma 2.5: *Let $V : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to \Re$ be a functional which is almost Lipschitz on bounded sets and let $x \in C^0([t_0 - r, t_{\max}); \Re^n)$ a solution of (1.1) under hypotheses (S1–7) corresponding to certain $(d,u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0]; \Re^n)$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T\,] \to V(t, T_r(t)x)$ is absolutely continuous.*

Lemma 2.6: *Suppose that there exist mappings $\beta_1 : \Re^+ \times C^0([-r,0]; \Re^n) \to \Re$, $\beta_2 : \Re^+ \times \Re^+ \times C^0([-r,0]; \Re^n) \times \Theta \to \Re$, where $\Theta \subseteq M_D \times M_U$, with the following properties:*

(i) *for each $(t, t_0, d, u) \in \Re^+ \times \Re^+ \times \Theta$, the mappings $x \to \beta_1(t,x), x \to \beta_2(t, t_0, x, d, u)$ are continuous,*

(ii) *there exists a continuous function $M : \Re^+ \times \Re^+ \to \Re^+$ such that for all $T,\ s \geq 0$*

$$\sup\Big\{\beta_2(t_0+\xi, t_0, x_0, d, u); \sup_{t\geq 0}|u(\tau)| \leq s\,, \xi \in [0,T]\,,\ x_0 \in C^0([-r,0]; \Re^n)\,, \|x_0\|_r \leq s\,,\ t_0 \in [0,T]\,,\ (d,u) \in \Theta\Big\} \leq M(T,s)$$

(iii) *for every $(t_0, x_0, d, u) \in \Re^+ \times C^1([-r,0]; \Re^n) \times \Theta$ the solution $x(t)$ of (1.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in \Theta$ satisfies:*

$$\beta_1(\,t\,,\ T_r(t)x) \leq \beta_2(\,t\,,\ t_0, x_0, d, u), \forall t \geq t_0 \quad (2.9)$$

Moreover, suppose that one of the following properties holds:

(iv) $\sup\Big\{\|T_r(t_0+\xi)x\|_r; \sup_{t\geq 0}|u(\tau)| \leq s, \xi \in [0,T],\ x_0 \in C^0([-r,0]; \Re^n), \|x_0\|_r \leq s,\ t_0 \in [0,T]\ (d,u) \in \Theta\Big\} < +\infty$ *for all $T, s \geq 0$*

(v) *there exist functions $a \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq \beta_1(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$*

Then for every $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0]; \Re^n) \times \Theta$ the solution $x(t)$ of (1.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d,u) \in \Theta$ exists for all $t \geq t_0$ and satisfies (2.9).

The following technical small-gain lemma will be used in the proofs of our main results. It is a direct corollary of Theorem 1 in [36] and is closely related to Lemma A.1 in [7].

Lemma 2.7: *For every $\sigma \in KL$ and $a \in K$ with $a(s) < s$ for all $s > 0$, there exists $\tilde{\sigma} \in KL$ with the following property: if $y : [t_0, t_1) \to \Re^+, u : \Re^+ \to \Re^+$ are locally bounded functions and $M \geq 0$ a constant such that the following inequality holds for all $t \in [t_0, t_1)$:*

$$y(t) \leq \inf_{t_0\leq\xi\leq t} \max\left\{\sigma(M, t-\xi);\ a\left(\sup_{\xi\leq\tau\leq t} y(\tau)\right);\ u(t)\right\} \quad (2.10)$$

then the following estimate holds for all $t \in [t_0, t_1)$:

$$y(t) \leq \max\left\{\tilde{\sigma}(\,M\,,\ t - t_0\,)\ ;\ \sup_{t_0\leq\tau\leq t} u(\tau)\right\} \quad (2.11)$$

Finally, we end this section by presenting a comparison lemma, which provides a "fading memory" estimate for an absolutely continuous mapping. The proof of Lemma 2.8 is given in [20].

Lemma 2.8: *For each positive definite locally Lipschitz function $\rho : \Re^+ \to \Re^+$ there exists a function σ of class KL, with $\sigma(s,0) = s$ for all $s \geq 0$ with the following property: if $y : [t_0, t_1] \to \Re^+$ is an absolutely continuous function, $u : \Re^+ \to \Re^+$ is a locally bounded mapping and $I \subset [t_0, t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0, t_1]\backslash I$ and such that the following implication holds for all $t \in [t_0, t_1]\backslash I$:*

$$y(t) \geq u(t) \quad \Rightarrow \quad \dot{y}(t) \leq -\rho(y(t)) \tag{2.12}$$

then the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \max\left\{ \sigma(\, y(t_0)\,,\, t - t_0\,),\; \sup_{t_0 \leq s \leq t} \sigma(\, u(s)\,,\, t - s\,) \right\} \tag{2.13}$$

3. Input-to-Output Stability (IOS) and its Equivalent Characterizations

In this section we introduce the reader to the notion of non-uniform and uniform Weighted Input-to-Output Stability (IOS) for systems described by RFDEs and we provide estimates for the solutions of such systems. Notice that the notion of IOS is an "External Stability" property since it is applied to systems which operate under the effect of external non-vanishing perturbations.

Definition 3.1: *We say that (1.1) under hypotheses (S1–7) satisfies the **Weighted Input-to-Output Stability property (WIOS) from the input** $u \in M_U$ **with gain** $\gamma \in K$ **and weight** $\delta \in K^+$, if (1.1) is robustly forward complete (RFC) from the input $u \in M_U$ and there exist functions $\sigma \in KL, \beta \in K^+$, such that for all $(d,u) \in M_D \times M_U, (t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$ satisfies the following estimate for all $t \geq t_0$:*

$$\|H(t, T_r(t)x)\|_Y \leq \max\left\{ \sigma(\beta(t_0)\|x_0\|_r \,, t - t_0)\,,\; \sup_{t_0 \leq \tau \leq t} \gamma(\delta(\tau)|u(\tau)|) \right\} \tag{3.1}$$

Moreover,

(i) *if $\beta(t) \equiv 1$, then we say that (1.1) satisfies the **Uniform Weighted Input-to-Output Stability property (UWIOS) from the input** $u \in M_U$ **with gain** $\gamma \in K$ **and weight** $\delta \in K^+$,*

(ii) *if $\delta(t) \equiv 1$, then we say that (1.1) satisfies the **Input-to-Output Stability property (IOS) from the input** $u \in M_U$ **with gain** $\gamma \in K$,*

(iii) *if $\beta(t) = \delta(t) \equiv 1$, then we say that (1.1) satisfies the **Uniform Input-to-Output Stability property (UIOS) from the input** $u \in M_U$ **with gain** $\gamma \in K$,*

(iv) *if $\|x\|_r \leq \|H(t,x)\|_Y$ for all $(t,x) \in \Re+ \times C^0([-r,0];\Re^n)$, then we say that (1.1) satisfies the **Weighted Input-to-State Stability property (WISS) from the input** $u \in M_U$ **with gain** $\gamma \in K$ **and weight** $\delta \in K^+$,*

(v) *if $\|x\|_r \leq \|H(t,x)\|_Y$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\beta(t) \equiv 1$, then we say that (1.1) satisfies the **Uniform Weighted Input-to-State Stability property (UWISS) from the input** $u \in M_U$ **with gain** $\gamma \in K$ **and weight** $\delta \in K^+$,*

(vi) *if $\|x\|_r \leq \|H(t,x)\|_Y$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\delta(t) \equiv 1$, then we say that (1.1) satisfies the **Input-to-State Stability property (ISS) from the input** $u \in M_U$ **with gain** $\gamma \in K$,*

(vii) *if $\|x\|_r \leq \|H(t,x)\|_Y$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $\beta(t) = \delta(t) \equiv 1$, then we say that (1.1) satisfies the **Uniform Input-to-State Stability property (UISS) from the input** $u \in M_U$ **with gain** $\gamma \in K$,*

It should be emphasized that for periodic systems estimate (3.1) leads to a simpler estimate. We say that (1.1) under hypotheses (S1−7) is T−periodic, if there exists $T > 0$ such that $f(t+T, x, u, d) = f(t, x, u, d)$ and $H(t+T, x) = H(t,x)$ for all $(t, x, u, d) \in \Re^+ \times C^0([-r, 0]\,;\, \Re^n) \times U \times D$. Lemma 2.19 and Lemma 2.20 in [19] show that if system (1.1) is T−periodic and satisfies the WIOS property with gain γ and weight δ from the input $u \in M_U$, then system (1.1) satisfies the UWIOS property from the input $u \in M_U$ with gain γ and weight $\tilde{\delta}$, where $\tilde{\delta}(t) := \max\{\delta(s)\,;\; s \in [0, t]\,\}$.

We are now in a position to state characterizations for the WIOS property for time-varying uncertain systems. The proof of the following theorem is provided in the Appendix. For the definition of non-uniform in time Robust Global Asymptotic Output Stability (RGAOS), see the companion paper [21].

Theorem 3.2: *The following statements are equivalent for system (1.1) under hypotheses (S1–7):*

(a) *System (1.1) is robustly forward complete (RFC) from the input $u \in M_U$ and there exist functions $\sigma \in KL, \beta, \phi \in K^+, \rho \in K$ such that that for all $(d,u) \in M_D \times M_U$, $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$*

corresponding to $(d,u) \in M_D \times M_U$ *satisfies the following estimate for all* $t \geq t_0$:

$$\|H(t, T_r(t)x)\|_Y \leq \max\left\{ \sigma\big(\beta(t_0)\,\|x_0\|_r\,,\ t - t_0\big)\,, \sup_{t_0 \leq \tau \leq t} \sigma(\,\rho(\phi(\tau)\,|u(\tau)|)\,,\ t - \tau\,)\right\} \tag{3.2}$$

(b) *System (1.1) satisfies the WIOS property from the input* $u \in M_U$.

(c) *There exist a locally Lipschitz function* $\theta \in K_\infty$, *functions* $\phi, \mu \in K^+$ *such that the following system is non-uniformly in time RGAOS with disturbances* $(d', d) \in \tilde{M}_\Delta$:

$$\dot{x}(t) = f\left(t, T_r(t)x, \frac{\theta\big(\|T_r(t)x\|_r\big)}{\phi(t)}d'(t), d(t)\right); \quad Y(t) = \tilde{H}(t, T_r(t)x) \tag{3.3}$$

where $\Delta := B_U[0,1] \times D, \tilde{H}(t,x) := (H(t,x), \mu(t)x) \in Y \times C^0([-r,0];\Re^n)$.

(d) *There exist a Lyapunov functional* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, *which is almost Lipschitz on bounded sets, functions* $a_1,\ a_2,\ a_3$ *of class* $K_\infty, \beta\ ,\ \delta, \mu$ *of class* K^+ *such that:*

$$a_1(\|H(t,x)\|_Y + \mu(t)\|x\|_r) \leq V(t,x) \leq a_2(\beta(t)\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.4}$$

$$V^0(t,x;f(t,x,u,d)) \leq -V(t,x) + a_3(\delta(t)|u|), \quad \forall (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \tag{3.5}$$

(e) *System (1.1) is RFC from the input* $u \in M_U$ *and there exist a Lyapunov functional* $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, *which is almost Lipschitz on bounded sets, functions* $a_1,\ a,\ \zeta$ *of class* $K_\infty, \beta,\ \delta$ *of class* K^+ *and a locally Lipschitz positive definite function* $\rho : \Re^+ \to \Re^+$ *such that:*

$$a_1(\|H(t,x)\|_Y) \leq V(t,x) \leq a(\beta(t)\|x\|_r), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.6}$$

$$V^0(t,x;f(t,x,u,d)) \leq -\rho(V(t,x)), \quad \text{for all} (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \text{ with } \zeta(\delta(t)|u|) \leq V(t,x) \tag{3.7}$$

(f) *System (1.1) is RFC from the input* $u \in M_U$ *and system (1.1) with* $u \equiv 0$ *is non-uniformly in time RGAOS with disturbances* $d \in M_D$.

Moreover,

i) *if* $H : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to Y$ *is equivalent to the finite-dimensional continuous mapping* $h : [-r,+\infty) \times \Re^n \to \Re^p$ *then inequality (3.6) in the above statement (e) can be replaced by the following inequality:*

$$a(\,|h(t,x(0))|) \leq V(t,x) \leq a\big(\beta(t)\|x\|_r\big), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.8}$$

ii) *if there exist functions* $p \in K_\infty, \mu \in K^+$ *and a constant* $R \geq 0$ *such that* $p(\mu(t)|x(0)|) \leq \|H(t,x)\|_Y + R$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$ *then the requirement that (1.1) is RFC from the input* $u \in M_U$ *is not needed in statement (a) above.*

iii) *if there exist functions* $p \in K_\infty, \mu \in K^+$ *and a constant* $R \geq 0$ *such that* $p(\mu(t)|x(0)|) \leq V(t,x) + R$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$ *then the requirement that (1.1) is RFC from the input* $u \in M_U$ *is not needed in statement (e) above.*

In order to obtain characterizations of the UIOS property, we need an extra hypothesis for system (1.1).

(S8) There exists a constant $R \geq 0$ and a function $a \in K_\infty$ such that the inequality $\|x\|_r \leq a\big(\|H(t,x)\|_Y\big) + R$ holds for all $(t,x) \in \Re^+ \times C^0([-r,o];\Re^n)$.

Hypothesis (S8) holds for the important case of the output map $H(t,x) := d(x(\theta),\Gamma)\ ;\ \theta \in [-r,0]$, where $\Gamma \subset \Re^n$ is a compact set which contains $0 \in \Re^n$ and $d(x,\Gamma)$ denotes the distance of the point $x \in \Re^n$ from the set $\Gamma \subset \Re^n$. Notice that it is not required that $\Gamma \subset \Re^n$ is positively invariant for (1.1) with $u \equiv 0$.

Hypothesis (S8) allows us to provide characterizations for the UIOS property for periodic uncertain systems. The proof of the following theorem is given in the Appendix. For the definition of Uniform Robust Global Asymptotic Output Stability (URGAOS), see the companion paper [21].

Theorem 3.3: *Suppose that system (1.1) under hypotheses (S1–8) is T− periodic. The following statements are equivalent:*

(a) *There exist functions* $\sigma \in KL, \rho \in K_\infty$ *such that for all* $(d,u) \in M_D \times M_U$, $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ *the solution* $x(t)$ *of (1.1) with* $T_r(t_0)x = x_0$ *corresponding to* $(d,u) \in M_D \times M_U$, *satisfies the following estimate for all* $t \geq t_0$:

$$\|H(t, T_r(t)x)\|_Y \leq \max\left\{ \sigma(\,\|x_0\|_r\,,\ t - t_0\,)\,, \sup_{t_0 \leq \tau \leq t} \sigma(\,\rho(\,|u(\tau)|)\,,\ t - \tau\,)\right\} \tag{3.9}$$

(b) *System (1.1) satisfies the UIOS property.*

(c) *There exists a locally Lipschitz function $\theta \in K_\infty$ such that $0 \in C^0([-r,0];\Re^n)$ is URGAOS with disturbances $(d',d) \in \tilde{M}_\Delta$ for the system:*

$$\dot{x}(t) = f\big(t, T_r(t)x, \theta\big(\|H(t, T_r(t)x)\|_Y\big)d'(t), d(t)\big);\quad Y(t) = H(t, T_r(t)x) \tag{3.10}$$

where $\Delta := B_U[0,1] \times D$.

(d) *There exist a $T-$ periodic Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1, a_2, a_3 of class K_∞ such that:*

$$a_1(\|H(t,x)\|_Y) \le V(t,x) \le a_2(\|x\|_r),\quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.11}$$

$$V^0(t,x;f(t,x,u,d)) \le -V(t,x) + a_3(|u|),\quad \forall (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \tag{3.12}$$

(e) *There exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a_1, a_2, ζ of class K_∞ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that:*

$$a_1(\|H(t,x)\|_Y) \le V(t,x) \le a_2(\|x\|_r),\quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.13}$$

$$V^0(t,x;f(t,x,u,d)) \le -\rho(V(t,x)),\quad \text{for all } (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \text{ with } \zeta(|u|) \le V(t,x) \tag{3.14}$$

Finally, if $H : \Re^+ \times C^0([-r,0];\Re^n) \to Y$ is equivalent to the finite-dimensional continuous $T-$ periodic mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ then inequalities (3.11), (3.13) in the above statements (d) and (e), respectively, can be replaced by the following inequality:

$$a_1(|h(t,x(0))|) \le V(t,x) \le a_2\big(\|x\|_r\big),\quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.15}$$

Remark 3.4: *A statement like (e) of Theorem 3.3 was extensively used as a tool of proving the UISS property for autonomous time-delay systems in [28]. Moreover, Sontag and Wang formulated IOS in [34,35] for continuous-time finite-dimensional systems using an estimate of the form (3.1) with $\beta(t) \equiv \delta(t) \equiv 1$. On the other hand, estimates of the form (3.9) ("fading memory estimates") were first used by Praly and Wang in [29] for the formulation of exp-ISS and by L. Grune in [2,3] for the formulation of Input-to-State Dynamical Stability (ISDS) with $H(t,x) = x$, $\beta(t) \equiv \gamma(t) \equiv 1$, which was proved to be qualitatively equivalent to (3.1) for finite-dimensional continuous-time systems.*

The following theorem provides sufficient Lyapunov-like conditions for the (U)WIOS property. The proof of implications (e)⇒(a) of Theorem 3.2 and (e)⇒(a) of Theorem 3.3 are based on the result of Theorem 3.5, which gives quantitative estimates of the solutions of (1.1) under hypotheses (S1−7). The gain functions and the weights of the WIOS property can be determined **explicitly** in terms of the functions involved in the assumptions of Theorem 3.5.

Theorem 3.5: *Consider system (1.1) under hypotheses (S1−7) and suppose that there exist a Lyapunov functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which is almost Lipschitz on bounded sets, functions a, ζ of class K_∞, β, δ of class K^+ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ such that:*

$$V(t,x) \le a(\beta(t)\|x\|_r),\ \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.16}$$

$$V^0(t,x;f(t,x,u,d)) \le -\rho(V(t,x)),\quad \text{for all } (t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \text{ with } \zeta(\delta(t)|u|) \le V(t,x) \tag{3.17}$$

Moreover, suppose that one of the following holds:

a) *system (1.1) is RFC from the input $u \in M_U$*
b) *there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \ge 0$ such that $p(\mu(t)|x(0)|) \le V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$*

Then system (1.1) is RFC from the input $u \in M_U$ and there exist a function $\sigma \in KL$ with $\sigma(s,0) = s$ for all $s \ge 0$, such that that for all $(d,u) \in M_D \times M_U$, $(t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ the solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ corresponding to $(d,u) \in M_D \times M_U$, satisfies the following estimate for all $t \ge t_0$:

$$V(t, T_r(t)x) \le \max\left\{ \sigma\big(a\big(\beta(t_0)\|x_0\|_r\big), t - t_0\big),\ \sup_{t_0 \le \tau \le t} \sigma\Big(\zeta(\delta(\tau)\,|u(\tau)|),\ t-\tau\Big) \right\} \tag{3.18}$$

Finally,
(i) *if there exist a function a_1 of class K_∞ such that $a_1(\|H(t,x)\|_Y) \le V(t,x)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$, then system (1.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ. Moreover, if in addition it*

holds that $\beta(t) \equiv 1$, then system (1.1) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s))$ and weight δ.

(ii) *if $H : \Re^+ \times C^0([-r,0]\,;\Re^n) \to Y$ is equivalent to the finite-dimensional continuous mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$ and there exist functions a_1, a_2 of class K_∞ such that $a_1(|h(t,x(0))|) \le V(t,x)$, $\|H(t,x)\|_Y \le a_2\left(\sup_{\theta\in[-r,0]} |h(t+\theta,\ x(\theta))|\right)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$ then system (1.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_2(a_1^{-1}(\zeta(s)))$ and weight δ.*

The following example presents an autonomous time-delay system, which satisfies the UWIOS property and does not satisfy the UIOS property. The analysis is performed with the help of Theorem 3.2 and Theorem 3.5.

Example 3.6: *Consider the following autonomous time-delay system:*

$$
\begin{aligned}
&\dot{x}_1(t) = d(t)x_1(t)\\
&\dot{x}_2(t) = -x_2(t) + x_1(t-r)u(t)\\
&Y(t) = x_2(t)\\
&(x_1(t),x_2(t))' \in \Re^2\,,\ d(t) \in D := [-1,1]\,,\\
&u(t) \in U := \Re\,,\ Y(t) \in \Re
\end{aligned}
\tag{3.19}
$$

Consider the functional:

$$
V(t,x_1,x_2) := \exp(-8\,t)x_1^4(0) + \exp(-4\,t)x_1^2(0) + \frac{1}{2}x_2^2(0) + \frac{1}{4}\exp(-8\,t)\int_{-r}^{0} x_1^4(s)ds \tag{3.20}
$$

First notice that the functional $V : \Re^+ \times C^0([-r,0];\Re^2) \to \Re^+$ defined by (3.20) is almost Lipschitz on bounded sets. Moreover, inequalities (3.6), (3.16) are satisfied for this functional with $a(s) = a_2(s) := \left(1+\frac{r}{4}\right)s^4 + s^2$, $a_1(s) := \frac{1}{2}s^2$ and $\beta(t) \equiv 1$. We next estimate an upper bound for the Dini derivative of the functional $V : \Re^+ \times C^0([-r,0];\Re^2) \to \Re^+$ along the solutions of system (3.19). We have for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^2) \times \Re \times [-1,1]$:

$$
\begin{aligned}
&V^0(t,x_1,x_2;(dx_1(0),-x_2(0)+x_1(-r)u))\\
&= -8\exp(-8\,t)x_1^4(0) + 4d\exp(-8\,t)x_1^4(0)\\
&-4\exp(-4t)x_1^2(0) + 2d\exp(-4t)x_1^2(0)\\
&-x_2^2(0) + x_2(0)x_1(-r)u - 2\exp(-8t)\int_{-r}^{0} x_1^4(s)ds\\
&+\frac{1}{4}\exp(-8t)x_1^4(0) - \frac{1}{4}\exp(-8t)x_1^4(-r)
\end{aligned}
$$

Using the inequalities $|d| \le 1$, $x_2(0)x_1(-r)u \le \frac{1}{2}x_2^2(0) + \frac{1}{2}x_1^2(-r)u^2$, $\frac{1}{2}x_1^2(-r)u^2 \le \frac{1}{4}\exp(-8t)x_1^4(-r) + \frac{1}{4}\exp(8t)u^4$, we are in a position to estimate for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^2) \times \Re \times [-1,1]$:

$$
\begin{aligned}
&V^0(t,x_1,x_2;(dx_1(0),-x_2(0)+x_1(-r)u))\\
&\le -3\exp(-8\,t)x_1^4(0) - 2\exp(-4t)x_1^2(0) - \frac{1}{2}x_2^2(0)\\
&+\frac{1}{4}\exp(8t)u^4 - 2\exp(-8t)\int_{-r}^{0} x_1^4(s)ds
\end{aligned}
$$

Finally, using the above inequality and definition (3.20) we obtain for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^2) \times \Re \times [-1,1]$:

$$
\begin{aligned}
&V^0(t,x_1,x_2;(dx_1(0),-x_2(0)+x_1(-r)u))\\
&\le -V(t,x_1,x_2) + \frac{1}{4}\exp(8t)u^4
\end{aligned}
\tag{3.21}
$$

Inequality (3.21) guarantees that (3.7) and (3.17) hold with $\rho(s) := \frac{1}{2}s$, $\zeta(s) := \frac{1}{2}s^4$ and $\delta(t) := \exp(2t)$. Definition (3.20) guarantees that there exist functions $p \in K_\infty$, $\mu \in K^+$ and a constant $R \ge 0$ such that $p(\mu(t)|x(0)|) \le V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0]\,;\Re^2)$ (e.g., $p(s) := \frac{1}{2}s^2$, $\mu(t) := \exp(-2t)$ and $R := 0$). It follows from Theorem 3.2 (statement (e)) that system (3.19) satisfies the UWIOS property from the input $u \in M_U$. In order to be able to determine the gain and weight functions we utilize the result of Theorem 3.5. Indeed, it follows from Theorem 3.5 that system (3.19) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_1^{-1}(\zeta(s)) = s^2$ and weight $\delta(t) := \exp(2t)$.

It should be emphasized that system (3.19) does not satisfy the UIOS property from the input $u \in M_U$. This can be shown by considering the solution of (3.19) corresponding to inputs $d(t) \equiv 1$ and $u(t) \equiv 1$. It can be shown that for $x_1(0) \ne 0$ the output of (3.19) is not bounded and satisfies $\lim_{t\to+\infty} |Y(t)| = +\infty$. Consequently, bounded inputs can produce unbounded output responses, which contradicts the requirements of the UIOS property from the input $u \in M_U$.

In order to understand how important the above conclusions are, consider the output regulation problem for the system:

$$
\begin{aligned}
&\dot{x}_1(t) = d_1(t)x_1(t)\\
&\dot{x}_2(t) = -x_2(t) + d_2(t)x_1(t-r)p(\|T_r(t)x_3\|_r)\\
&\dot{x}_3(t) = u(t)\\
&Y(t) = x_2(t)\\
&(x_1(t),x_2(t),x_3(t))' \in \Re^3\,,\ d(t) = (d_1(t),d_2(t))\\
&\in D := [-1,1]^2\,,\ u(t) \in \Re\,,\ Y(t) \in \Re
\end{aligned}
\tag{3.22}
$$

where $p : \Re \to \Re$ is a polynomial with $p(0) = 0$. Using the main result in [19], and the fact that system (3.19) satisfies the UWIOS property from the input $u \in M_U$ with gain $\gamma(s) := s^2$ and weight $\delta(t) := \exp(2t)$, we are in a position to show that the linear feedback $u(t) = -k\, x_3(t) + v(t)$, where $v \in M_{\Re}$ represents the control actuator error, achieves the regulation of the output $Y(t) = x_2(t)$ for $k > 2$. More specifically, the closed-loop system (3.22) with $u(t) = -k\, x_3(t) + v(t)$ and $k > 2$ satisfies the UWIOS property from the input $v \in M_{\Re}$. ◁

4. Razumikhin Method for WIOS

Let $V : [-r, +\infty) \times \Re^n \to \Re$ be a locally Lipschitz mapping and let $(t, x, v) \in \Re^+ \times \Re^n \times \Re^n$. We define

$$D^+ V(t, x; v) := \limsup_{h \to 0^+} \frac{V(t + h, x + hv) - V(t, x)}{h} \tag{4.1}$$

The following proposition provides conditions in terms of Razumikhin functions for the (U)WIOS property.

Proposition 4.1: *Consider system (1.1) under hypotheses (S1–7) and suppose that $H : \Re^+ \times C^0([-r, 0]; \Re^n) \to Y$ is equivalent to the finite-dimensional mapping $h : [-r, +\infty) \times \Re^n \to \Re^p$. Moreover, suppose that there exist a locally Lipschitz function $V : [-r, +\infty) \times \Re^n \to \Re^+$, functions a_1, a_2, a, ζ of class K_∞ with $a(s) < s$ for all $s > 0$, β, δ of class K^+ and a locally Lipschitz positive definite function ρ such that:*

$$a_1(|h(t-r, x)|) \leq V(t-r, x) \leq a_2(\beta(t)|x|), \qquad \forall (t, x) \in \Re^+ \times \Re^n \tag{4.2}$$

$$\begin{aligned} &D^+ V(t, x(0); f(t, x, u, d)) \\ &\quad \leq -\rho(V(t, x(0))), \text{ for all } (t, x, u, d) \in \Re^+ \\ &\qquad \times C^0([-r, 0]; \Re^n) \times U \times D \text{ with} \\ &\quad \max\left\{ \zeta(\delta(t)|u|) \,,\, a\left(\sup_{\theta \in [-r, 0]} V(t + \theta, x(\theta)) \right) \right\} \\ &\quad \leq V(t, x(0)) \end{aligned} \tag{4.3}$$

Finally, suppose that one of the following holds:

(i) system (1.1) is RFC from the input $u \in M_U$

(ii) there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x|) \leq V(t - r, x) + R$ for all $(t, x) \in \Re^+ \times \Re^n$

Let $a_3 \in K_\infty$ be the function with the property $\|H(t, x)\|_Y \leq a_3\left(\sup_{\theta \in [-r, 0]} |h(t + \theta, x(\theta))| \right)$ for all $(t, x) \in \Re^+ \times C^0([-r, 0]; \Re^n)$. Then system (1.1) satisfies the WIOS property with gain $\gamma(s) := a_3\left(a_1^{-1}(\zeta(s))\right)$ and weight δ. Moreover, if β is bounded then system (1.1) satisfies the UWIOS property from the input $u \in M_U$. Finally, if β, δ are bounded then system (1.1) satisfies the UIOS property from the input $u \in M_U$.

Proof: Consider a solution $x(t)$ of (1.1) under hypotheses (S1–7) corresponding to arbitrary $(u, d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \Re^n)$. It follows from (4.3) and Lemma 2.8 that there exists a continuous function σ of class KL, with $\sigma(s, 0) = s$ for all $s \geq 0$ such that for all $t \in [t_0, t_{\max})$ we have:

$$\begin{aligned} &V(t, x(t)) \\ &\leq \max \left\{ \begin{aligned} &\sigma(\, V(t_0, x(t_0)) \,,\, t - t_0 \,); . \\ &\sup_{t_0 \leq s \leq t} \sigma\left(a\left(\sup_{\theta \in [-r, 0]} V(s + \theta, x(s + \theta)) \right) ,\, t - s \right); \\ &\sup_{t_0 \leq s \leq t} \sigma(\, \zeta(\delta(s)|u(s)|) \,,\, t - s \,) \end{aligned} \right\} \end{aligned} \tag{4.4}$$

An immediate consequence of estimate (4.4) and the fact that $\sigma(s, 0) = s$ for all $s \geq 0$ is the following estimate for all $t \in [t_0, t_{\max})$:

$$\begin{aligned} &\sup_{\theta \in [-r, 0]} V(t + \theta, x(t + \theta)) \\ &\leq \max \left\{ \begin{aligned} &\bar{\sigma}\left(\sup_{\theta \in [-r, 0]} V(t_0 + \theta, x(t_0 + \theta)) \,,\, t - t_0 \right); \\ &a\left(\sup_{t_0 \leq s \leq t} \sup_{\theta \in [-r, 0]} V(s + \theta, x(s + \theta)) \right); \\ &\sup_{t_0 \leq s \leq t} \zeta(\delta(s)|u(s)|) \end{aligned} \right\} \end{aligned} \tag{4.5}$$

where $\bar{\sigma}(s, t) := s$ for $t \in [0, r]$ and $\bar{\sigma}(s, t) := \sigma(s, t - r)$ for $t > r$. Using the fact that $a(s) < s$ for all $s > 0$ and estimate (4.5) it may be shown that:

$$\begin{aligned} &\sup_{\theta \in [-r, 0]} V(t + \theta, x(t + \theta)) \\ &\quad \leq \max \left\{ \begin{aligned} &\sup_{\theta \in [-r, 0]} V(t_0 + \theta, x(t_0 + \theta)); \\ &\sup_{t_0 \leq s \leq t} \zeta(\delta(s)|u(s)|) \end{aligned} \right\}, \qquad \forall t \in [t_0, t_{\max}) \end{aligned} \tag{4.6}$$

In case that system (1.1) is RFC from the input $u \in M_U$, we have $t_{\max} = +\infty$. In case that there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x|) \leq V(t - r, x) + R$ for all

$(t,x) \in \Re^+ \times \Re^n$, inequality (4.6) in conjunction with (4.2) implies:

$$\|T_r(t)x\|_r \le \frac{1}{\min_{0\le\tau\le t+r} \mu(t)} p^{-1}\left(R + a_2\left(\max_{0\le\tau\le t_0+r} \beta(\tau)\|x_0\|_r\right) + \sup_{t_0\le s\le t} \zeta\left(\delta(s)|u(s)|\right)\right), \ \forall t\in[t_0, t_{\max}) \quad (4.7)$$

Estimate (4.7) implies that system (1.1) is RFC from the input $u \in M_U$. Therefore we conclude that in any case system (1.1) is RFC from the input $u \in M_U$ and that estimates (4.5), (4.6) hold for all $t \ge t_0$. Combining (4.5) with (4.6) we obtain for all $t \ge t_0$:

$$\sup_{\theta\in[-r,0]} V(t+\theta, x(t+\theta)) \le \inf_{t_0\le\xi\le t} \max\left\{ \bar{\sigma}\left(\sup_{\theta\in[-r,0]} V(t_0+\theta, x(t_0+\theta)), t-\xi \right); a\left(\sup_{\xi\le s\le t} \sup_{\theta\in[-r,0]} V(s+\theta, x(s+\theta)) \right); \sup_{t_0\le s\le t} \zeta(\delta(s)|u(s)|) \right\} \quad (4.8)$$

Lemma 2.7 in conjunction with inequality (4.8) implies the existence of $\tilde{\sigma} \in KL$ such that:

$$\sup_{\theta\in[-r,0]} V(t+\theta, x(t+\theta)) \le \max\left\{ \tilde{\sigma}\left(\sup_{\theta\in[-r,0]} V(t_0+\theta, x(t_0+\theta)),\ t-t_0 \right); \sup_{t_0\le s\le t} \zeta(\delta(s)|u(s)|) \right\}, \forall t \ge t_0 \quad (4.9)$$

Since $H : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to Y$ is equivalent to the finite-dimensional mapping h, there exists a function $a_3 \in K_\infty$ such that $\|H(t,x)\|_Y \le a_3\left(\sup_{\theta\in[-r,0]} |h(t+\theta, x(\theta))| \right)$ for all $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$. Using the previous inequality in conjunction with (4.9) and (4.2) we obtain:

$$\|H(t, T_r(t)x)\|_Y \le \max\left\{ a_3\left(a_1^{-1}\left(\tilde{\sigma}\left(a_2\left(\max_{0\le\tau\le t_0+r} \beta(\tau)\|x_0\|_r \right),\ t-t_0 \right)\right)\right);\ \sup_{t_0\le s\le t} a_3\left(a_1^{-1}(\zeta(\delta(s)|u(s)|))\right) \right\}, \quad \forall t \ge t_0 \quad (4.10)$$

Estimate (4.10) implies that system (1.1) satisfies the WIOS property from the input $u \in M_U$ with gain $\gamma(s) := a_3\left(a_1^{-1}(\zeta(s))\right)$ and weight δ. Moreover, if β is bounded then estimate (4.10) implies that system (1.1) satisfies the UWIOS property from the input $u \in M_U$. Finally, if β, δ are bounded then estimate (4.10) implies that system (1.1) satisfies the UIOS property from the input $u \in M_U$. The proof is complete. ◁

The following corollary extends the classical Razumikhin theorem to systems with disturbances as well as to the case of output asymptotic stability. The notion of Robust Global Asymptotic Output Stability (RGAOS) is presented in the companion paper [21].

Proposition 4.2: *Consider system (1.1) under hypotheses (S1–7) with $u \equiv 0$ and suppose that $H : \Re^+ \times C^0([-r,0]\,;\,\Re^n) \to Y$ is equivalent to the finite-dimensional mapping $h : [-r,+\infty) \times \Re^n \to \Re^p$. Moreover, suppose that there exist a locally Lipschitz function $V : [-r,+\infty) \times \Re^n \to \Re^+$, functions a_1 , a_2 , a of class K_∞, with $a(s) < s$ for all $s > 0$, β of class K^+ and a locally Lipschitz positive definite function ρ such that:*

$$a_1(|h(t-r,x)|) \le V(t-r,x) \le a_2(\beta(t)|x|), \quad \forall (t,x) \in \Re^+ \times \Re^n \quad (4.11)$$

$$D^+V(t,x(0); f(t,x,0,d)) \le -\rho(V(t,x(0))), \text{ for all } (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D \text{ with } a\left(\sup_{\theta\in[-r,0]} V(t+\theta, x(\theta))\right) \le V(t,x(0)) \quad (4.12)$$

Finally, suppose that one of the following holds:

(i) system (1.1) with $u \equiv 0$ is RFC

(ii) there exist functions $\zeta \in K_\infty$, $\mu \in K^+$ and a constant $R \ge 0$ such that $\zeta(\mu(t)|x|) \le V(t-r,x) + R$ for all $(t,x) \in \Re^+ \times \Re^n$

Then system (1.1) with $u \equiv 0$ is non-uniformly in time RGAOS. Moreover, if β is bounded then system (1.1) with $u \equiv 0$ is URGAOS.

The following example illustrates the application of Proposition 4.1 to an autonomous time-delay system.

Example 4.3: *Consider the following autonomous time-delay system:*

$$\dot{x}(t) = d(t)x(t-r) - x^3(t) + u(t)$$
$$Y(t) = H(T_r(t)x)$$
$$x(t) \in \Re,\ d(t) \in D := [-R,R],\ u(t) \in U := \Re,\ Y(t) \in C^0([-r,0];\Re) \quad (4.13)$$

where $R > 0, H(x) := h(x(\theta))\,;\, \theta \in [-r,0]$, $h(x) := x\ (1 - 2\sqrt{R}|x|^{-1})$ *for* $|x| > 2\sqrt{R}$ *and* $h(x) := 0$ *for*

$|x| \le 2\sqrt{R}$. Notice that $H : C^0([-r,0]\,;\,\Re) \to Y := C^0([-r,0]\,;\,\Re)$ is equivalent to the finite-dimensional mapping $h : \Re \to \Re$. Consider the locally Lipschitz function:

$$V(x) := \max\left\{\,0\,;\, x^2 - 4R\,\right\} \tag{4.14}$$

which satisfies (4.2) with $a_1(s) = a_2(s) := s^2$ and $\beta(t) \equiv 1$. Notice that for all $|d| \le R$ and $x \in C^0([-r,0]\,;\,\Re)$ with $|x(0)| > 2\sqrt{R}$, we have:

$$\begin{aligned} D^+V(x(0); dx(-r) - x^3(0) + u) &= 2dx(0)x(-r) \\ &\quad - 2x^4(0) + 2x(0)u \le 2R|x(0)|\,|x(-r)| \\ &\quad - 2x^4(0) + 2|x(0)|\,|u| \end{aligned}$$

Using the Young inequality $2|x(0)|\,|u| \le x^4(0) + 3\,|u|^{\frac{4}{3}}$ and completing the squares, we obtain for all $|d| \le R$ and $x \in C^0([-r,0]\,;\,\Re)$ with $|x(0)| > 2\sqrt{R}$:

$$\begin{aligned} D^+V(x(0); dx(-r) - x^3(0) + u) &\le 2R\,x^2(0) \\ &\quad + \frac{R}{2}\,x^2(-r) - x^4(0) + 3\,|u|^{\frac{4}{3}} \end{aligned} \tag{4.15}$$

Let $a(s) := \frac{1}{4}s, \rho(s) := 2R\,s$ and $\zeta(s) := \frac{3}{2R}s^{\frac{4}{3}}$. Notice that if $a\left(\sup_{\theta\in[-r,0]} V(x(\theta))\right) \le V(x(0))$ and $|x(0)| > 2\sqrt{R}$ then $x^2(-r) \le 4x^2(0)$. Consequently, previous definitions, definition (4.14) and inequality (4.15) implies that the following inequality holds for all $|d| \le R$ and $x \in C^0([-r,0]\,;\,\Re)$ with $\max\left\{\zeta(|u|)\,,\, a\left(\sup_{\theta\in[-r,0]} V(x(\theta))\right)\right\} \le V(x(0)), |x(0)| > 2\sqrt{R}$:

$$D^+V(x(0); dx(-r) - x^3(0) + u) \le -\rho(V(\,x(0))) \tag{4.16}$$

Notice that (4.16) holds also for all $|d| \le R$ and $x \in C^0([-r,0]\,;\,\Re)$ with $\max\left\{\zeta(|u|)\,,\, a\left(\sup_{\theta\in[-r,0]} V(x(\theta))\right)\right\} \le V(x(0))$, $|x(0)| \le 2\sqrt{R}$. Thus V satisfies (4.3) with $a(s) := \frac{1}{4}s, \rho(s) := 2Rs, \zeta(s) := \frac{3}{2R}s^{\frac{4}{3}}$ and $\delta(t) \equiv 1$. Finally, notice that there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $K \ge 0$ such that $p(\mu(t)|x|) \le V(t-r,x) + K$ for all $(t,x) \in \Re^+ \times \Re^n$ (for instance, $p(s) := s^2, \mu(t) \equiv 1$ and $K = 4R$). It follows from Proposition 4.1 that system (4.13) satisfies the UIOS property from the input $u \in M_U$ with gain $\gamma(s) := \sqrt{\frac{3}{2R}}\,s^{\frac{2}{3}}$. ◁

5. Conclusions

In this work Lyapunov-like characterizations of the external stability notions of (uniform) weighted Input-to-State Stability (ISS) and Input-to-Output Stability (IOS) for uncertain time-varying systems described by Retarded Functional Differential Equations (RFDEs) are developed. Necessary and sufficient conditions in terms of Lyapunov functionals and sufficient conditions in terms of Razumikhin functions are provided for these notions. The framework of the present work allows outputs with no delays, outputs with discrete or distributed delays and functional outputs with memory.

The robust stability notions and properties proposed in the present work are parallel to those recently developed for dynamical systems described by finite-dimensional ordinary differential equations. Just as the popularity gained by the notions of uniform and non-uniform in time ISS and IOS in the context of deterministic systems, it is our firm belief that the stability results of this paper will play an important role in mathematical systems and control theory for important classes of systems described by RFDEs.

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Appendix—Proofs

Proof of Theorem 3.2: We prove implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (e), (e) $\Rightarrow$ (a). The equivalence between (f) and (b) is a direct consequence of Theorem 3.1 in [18].

(a) $\Rightarrow$ (b) : Suppose that there exist functions $\sigma \in KL, \beta, \phi \in K^+, \rho \in K_\infty$ such that the estimate (3.2) holds for all $(d,u) \in M_D \times M_U$, $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$ and $t \geq t_0$. If we set $\gamma(s) := \sigma(\rho(s), 0)$ (that obviously is of class *K*), the desired (3.1) is a consequence of (3.2) and the previous definition. Thus statement (b) holds if (1.1) is RFC from the input $u \in M_U$. If the hypothesis that (1.1) is RFC from the input $u \in M_U$ is not included in statement (a) then there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x(0)|) \leq \|H(t,x)\|_Y + R$ for all $(t,x) \in \Re^+ \times\ C^0([-r,0]\,;\,\Re^n)$. It follows from (3.2) and previous definitions that for every $(t_0, x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, $(d,u) \in M_D \times M_U$ the corresponding solution $x(t)$ of (1.1) with $T_r(t_0)x = x_0$ satisfies the following estimate for all $t \geq t_0$:

$$p(\mu(t)|x(t)|) \leq R + \max\left\{ \sigma(\beta(t_0)\|x_0\|_r\,, t - t_0), \sup_{t_0 \leq \tau \leq t} \gamma(\delta(\tau)|u(\tau)|) \right\}$$

The above estimate in conjunction with Definition 2.1 implies that (1.1) is RFC from the input $u \in M_U$.

(b) $\Rightarrow$ (c) : The methodology of the proof of this implication follows closely the methodology used in [16] for finite-dimensional systems. Since (1.1) is RFC from the input $u \in M_U$, by virtue of Lemma 3.5 in [14], there exist functions $q \in K^+, a \in K_\infty$ and a constant $R \geq 0$ such that the following estimate holds for all $u \in M_U$ and $(t_0, x_0, d) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times M_D$:

$$\|T_r(t)x\|_r \quad \leq q(t)\, a\left(R + \|x_0\|_r + \sup_{\tau\in[t_0,t]} |u(\tau)| \right), \quad \forall t \geq t_0 \tag{A1}$$

Using Corollary 10 and Remark 11 in [33], we obtain $\kappa \in K_\infty$ such that $a(rs) \leq \kappa(r)\kappa(s)$ for all $(r,s) \in (\Re^+)^2$. Let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \leq \min\{\kappa^{-1}(s)\ ;\ s\}$ for all $s \geq 0$. Moreover, let $\phi(t) := \frac{4}{\kappa^{-1}\left(\frac{\exp(-t)}{2q(t)}\right)} + \exp(t)\; q(t)\; \delta(t)$, $\mu(t) := \frac{\exp(-t)}{q(t)}$, where $\delta \in K^+$ is the function involved in (3.1). The previous definitions guarantee that:

$$\text{if } \phi(t)|u| \leq \theta(\|x\|_r) \text{ then } a(4|u|) \leq \frac{1}{2}\mu(t)\|x\|_r \text{ and } \gamma(\delta(t)|u|) \leq \gamma(\mu(t)\|x\|_r) \tag{A2}$$

By virtue of (A1), (3.1) and (A2) it follows that the solution $x(\cdot)$ of (1.1) satisfies the following implication:

$$\begin{aligned}\phi(\tau)|u(\tau)| \leq \theta(\|T_r(\tau)x\|_r),\ \text{a.e. in } [t_0,t] \\ \Rightarrow \mu(t)\|T_r(t)x\|_r \leq \exp(-t)a(2R) \\ + \exp(-t)a(4\|x_0\|_r) \\ + \frac{1}{2}\exp(-t) \sup_{t_0\leq\tau\leq t} \mu(\tau)\|T_r(\tau)x\|_r\end{aligned} \tag{A3}$$

$$\begin{aligned}\phi(\tau)|u(\tau)| \leq \theta(\|T_r(\tau)x\|_r),\ \text{a.e. in } [t_0,t] \\ \Rightarrow \|H(t, T_r(t)x)\|_Y \leq \sigma(\beta(t_0)\|x_0\|_r, \\ t - t_0) + \sup_{t_0\leq\tau\leq t} \gamma(\mu(\tau)\|T_r(\tau)x\|_r)\end{aligned} \tag{A4}$$

Clearly, (A3) implies:

$$\begin{aligned}\mu(\tau)\|T_r(\tau)x\|_r \leq \sup_{t_0\leq s\leq t} \mu(s)\|T_r(s)x\|_r \leq 2a(2R) \\ + 2a(4\|x_0\|_r),\ \forall \tau \in [t_0, t], \\ \text{provided that } \phi(\tau)|u(\tau)| \leq \theta(\|T_r(\tau)x\|_r), \\ \text{a.e. in } [t_0, t]\end{aligned} \tag{A5}$$

Notice that every solution $x(\cdot)$ of (3.3) corresponding to some $(d', d) \in \tilde{M}_\Delta$ coincides (as long as it exists) with the solution of (1.1) corresponding to $u(\cdot) = \frac{\theta(\|T_r(\cdot)x\|_r)}{\phi(\cdot)} d'(\cdot)$ initiated from same initial $x_0 \in C^0([-r, 0]; \Re^n)$ at time $t_0 \geq 0$ and corresponding to same $d \in \tilde{M}_D$. Thus, by taking into account (A3), (A4) and (A5), it follows that the solution $x(\cdot)$ of (3.3) exists for all $t \geq t_0$ and satisfies:

$$\begin{aligned}\mu(t)\|T_r(t)x\|_r \leq \exp(-t)a(2R) \\ + \exp(-t)a(4\|x_0\|_r) + \frac{1}{2}\exp(-t) \\ \sup_{t_0\leq\tau\leq t} \mu(\tau)\|T_r(\tau)x\|_r, \forall t \geq t_0, (d', d) \in \tilde{M}_\Delta, \\ (t_0, x_0) \in \Re^+ \times C^0([-r, 0]; \Re^n)\end{aligned} \tag{A6}$$

$$\begin{aligned}\|H(t, T_r(t)x)\|_Y \leq \sigma(\beta(t_0)\|x_0\|_r, t - t_0) \\ + \sup_{t_0\leq\tau\leq t} \gamma(\mu(\tau)\|T_r(\tau)x\|_r), \forall t \geq t_0, (d', d) \\ \in \tilde{M}_\Delta, (t_0, x_0) \in \Re^+ \times C^0([-r, 0]; \Re^n)\end{aligned} \tag{A7}$$

$$\begin{aligned}\mu(t)\|T_r(t)x\|_r \leq 2a(2R) + 2a(4\|x_0\|_r), \\ \forall t \geq t_0, (d', d) \in \tilde{M}_\Delta, (t_0, x_0) \in \Re^+ \\ \times C^0([-r, 0]; \Re^n)\end{aligned} \tag{A8}$$

Consider the functions $c(h, T, s) := \sup\{\ \mu(t_0 + h) \|T_r(t_0 + h)x\|_r\ ;\ \ (d', d) \in \tilde{M}_\Delta,\ \|x_0\|_r \leq s,\ t_0 \in [0, T]\}$ and $b(h, T, s) := \sup\{\|H(t_0 + h, T_r(t_0 + h)x)\|_Y\ ;\ (d', d) \in \tilde{M}_\Delta,\ \|x_0\|_r \leq s,\ t_0 \in [0, T]\}$, where $x(\cdot)$ denotes the solution of (3.3) corresponding to some $(d', d) \in \tilde{M}_\Delta$. Next we show that $\lim_{h\to+\infty} b(h, T, s) = \lim_{h\to+\infty} c(h, T, s) = 0$, for all $(T, s) \in (\Re^+)^2$. Clearly, by (A7), (A8) and definitions of c, b we have

$$\begin{aligned}&c(t,T,s) \leq 2a(2R) + 2a(4s);\\ &b(t,T,s) \leq \sigma(\max_{0\leq\tau\leq T} \beta(\tau)s, 0) + \gamma(2a(2R) + 2a(4s)),\\ &\forall t \geq 0\end{aligned}$$

By virtue of the above estimates the mappings c, b are bounded for each fixed $(T, s) \in (\Re^+)^2$ and thus the limits $\limsup_{h\to+\infty} c(h, T, s) = \rho$ and $\limsup_{h\to+\infty} b(h, T, s) = l$ are well defined and finite. We show that $\rho = l = 0$. Indeed, for every $\varepsilon > 0$ there exists $\tau = \tau(\varepsilon, T, s) \geq 0$ such that

$$c(h, T, s) \leq \rho + \varepsilon, \forall h \geq \tau \tag{A9}$$

Again recall definitions of c, b above and (A6), (A7), (A8) in conjunction with (A9), which imply

$$\begin{aligned}b(h, T, s) \leq \sigma\Big(\exp(T + \tau) \max_{0\leq\xi\leq T+\tau} \beta(\xi) \max_{0\leq\xi\leq T+\tau} \\ q(\xi)\left[2a(2R) + 2a(4s)\right], h - \tau\Big) \\ + \gamma(\rho + \varepsilon),\ \text{for all} \quad h \geq \tau\end{aligned}$$

$$c(h,T,s) \le \exp(-h)a(2R) + \exp(-h)a\Big(4\exp(T+\tau)\max_{0\le\xi\le T+\tau} q(\xi)[2a(2R)+2a(4s)]\Big) + \frac{1}{2}\exp(-h)(\rho+\varepsilon),$$
for all $h \ge \tau$

Clearly, the above inequalities imply that $\rho = 0$ as well as $l \le \gamma(\varepsilon)$ for all $\varepsilon > 0$. Consequently, we must have $\limsup_{h\to+\infty} b(h,T,s) = l = 0$. The fact that $\lim_{h\to+\infty} b(h,T,s) = \lim_{h\to+\infty} c(h,T,s) = 0$ in conjunction with Lemma 3.3 in [14] shows that (3.3) is non-uniformly in time RGAOS with disturbances $(d',d) \in \tilde{M}_\Delta$.

(c) $\Rightarrow$ (d) : Suppose that (3.3) is non-uniformly in time RGAOS with disturbances $(d',d) \in \tilde{M}_\Delta$. Theorem 4.1 (statement (d)) in [21] implies that there exists a continuous mapping $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, which is almost Lipschitz on bounded sets, with the following properties:

-- there exist functions $a_1, a_2 \in K_\infty, \beta \in K^+$ such that:

$$a_1(\|H(t,x)\|_Y + \mu(t)\|x\|_r) \le V(t,x) \le a_2\big(\beta(t)\|x\|_r\big), \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{A10}$$

-- it holds that:

$$V^0\left(t,x;f\left(t,x,\frac{\theta(\|x\|_r)}{\phi(t)}d',d\right)\right) \le -V(t,x), \forall (t,x,d',d) \in \Re^+ \times C^0([-r,0];\Re^n) \times \Delta \tag{A11}$$

Notice that inequality (A11) implies the following inequality:

$$V^0(t,x;f(t,x,u,d)) \le -V(t,x), \text{for all}(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \text{ with } \phi(t)|u| \le \theta(\|x\|_r) \tag{A12}$$

Using property (P1) of Definition 2.2 for the continuous mapping $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, we obtain for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$:

$$\big|V^0(t,x;f(t,x,u,d)) - V^0(t,x;f(t,x,0,d))\big| \le M\big(t+\|x\|_r+1\big)\,|f(t,x,u,d) - f(t,x,0,d)|$$

The above inequality in conjunction with (2.2) implies that the following inequality holds for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$:

$$\big|V^0(t,x;f(t,x,u,d)) - V^0(t,x;f(t,x,0,d))\big| \le M\big(t+\|x\|_r+1\big)\,L_U(t,\|x\|_r+|u|\,)\,|u| \tag{A13}$$

Define

$$\psi(t,s) := \sup\big\{\, M\big(t+\|x\|_r+1\big) L_U(t,\|x\|_r+|u|\,)\,|u|;\, \|x\|_r \le \theta^{-1}(\phi(t)s)\,,\ |u| \le s \,\big\} \tag{A14}$$

Without loss of generality we may assume that the function $\phi \in K^+$ is non-decreasing. Clearly, $\psi : \Re^+ \times \Re^+ \to \Re^+$ is a mapping with $\psi(t,0) = 0$ for all $t \ge 0$, such that (i) for each fixed $t \ge 0$, the mapping $\psi(t,\cdot)$ is non-decreasing; (ii) for each fixed $s \ge 0$, the mapping $\psi(\cdot,s)$ is non-decreasing and (iii) $\lim_{s\to 0^+} a(t,s) = 0$, for all $t \ge 0$. Hence, by employing Lemma 2.3 in [12], we obtain functions $a_3 \in K_\infty$ and $\delta \in K^+$ such that $\psi(t,s) \le a_3(\delta(t)s)$.

We next establish inequality (3.5), with a_3 as previously, by considering the following two cases:

* $\theta^{-1}(\phi(t)|u|) \le \|x\|_r$. In this case inequality (3.5) is a direct consequence of (A12).
* $\theta^{-1}(\phi(t)|u|) \ge \|x\|_r$. In this case, by virtue of inequalities (A12), (A13), definition (A14) and definition of a_3, we have: $V^0(t,x;f(t,x,u,d)) \le V^0(t,x;f(t,x,0,d)) + \psi(t,|u|) \le -V(t,x) + a_3(\delta(t)|u|)$, which implies (3.5).

(d) $\Rightarrow$ (e) : Notice that (3.5) implies (3.7) with $\zeta(s) := 2a_3(s)$ and $\rho(s) := \frac{1}{2}s$. The fact that system (1.1) is RFC follows directly from Theorem 3.5.

(e) $\Rightarrow$ (a) : Theorem 3.5 implies that system (1.1) is RFC from the input $u \in M_U$ and that (3.18) holds. Next, we distinguish the following cases:

1) If (3.6) holds, then (3.2) is a direct consequence of (3.18) and (3.6).
2) If (3.8) holds, then (3.18) implies the following estimate:

$$|h(t,x(t))| \le \max\Big\{ a_1^{-1}\big(\sigma\big(a\big(\beta(t_0)\|x_0\|_r\big),\, t-t_0\,\big)\big),\ \sup_{t_0\le s\le t} a_1^{-1}(\sigma(\,\zeta(\delta(s)|u(s)|),\, t-s))\Big\},$$
for all $t \ge t_0$

Since $h : [-r,+\infty) \times \Re^n \to \Re^p$ is continuous with $h(t,0) = 0$ for all $t \ge -r$, it follows from Lemma 3.2 in [14] implies that there exist functions $p \in K_\infty$ and $\phi \in K^+$ such that:

$$|h(t-r,x)| \le p(\phi(t)|x|), \forall (t,x) \in \Re^+ \times \Re^n$$

Combining the two previous inequalities we obtain:

$$\sup_{\theta\in[-r,0]} |h(t+\theta,x(t+\theta))| \le \max\Big\{ \omega\big(\,q(t_0)\|x_0\|_r\,,\ t-t_0\big),\ \sup_{t_0\le s\le t} \omega(\,\zeta(\delta(s)|u(s)|)\,,\ t-s\,)\Big\},\ \forall t \ge t_0$$

where $q(t) := \beta(t) + \max_{t \le \tau \le t+r} \phi(\tau)$ $\omega(s,t) := \max\{ p(s)\,,\ a_1^{-1}(\sigma(s + a(s)\ ,\ 0\ ))\}$ for $t \in [0,r)$ and $\omega(s,t) := \max\left\{ \exp(r-t)p(s)\,,\ a_1^{-1}(\sigma(s + a(s)\ ,\ t-r\ ))\right\}$ for $t \ge r$. The above estimate, in conjunction with the fact that $H : \Re^+ \times C^0([-r,0]\,;\Re^n) \to Y$ is equivalent to the finite-dimensional mapping h shows that (1.1) satisfies inequality (3.2).

The proof is complete. ◁

Proof of Theorem 3.3: The proof of implications (a) ⇒ (b), (d) ⇒ (e) and (e) ⇒ (a) follow the same methodology as in the proof of Theorem 3.2. Particularly, in the proof of implication (e) ⇒ (a), we use in addition the fact that since $h : [-r,+\infty) \times \Re^n \to \Re^p$ is continuous and $T-periodic$ with $h(t,0) = 0$ for all $t \ge -r$, it follows from Lemma 3.2 in [14] implies that there exist a function $p \in K_\infty$ such that:

$$|h(t-r,x)| \le p(|x|), \forall (t,x) \in \Re^+ \times \Re^n$$

The proof of implication (c) ⇒ (d) differs from the corresponding proof in Theorem 3.2 in the definition of ψ. Specifically, we first notice that the fact that V is $T-periodic$, implies that $V^0(t,x;f(t,x,u,d))$ is $T-periodic$. Using property (P1) of Definition 2.2 for the continuous mapping $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, we obtain for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$:

$$\begin{aligned}&\left|V^0(t,x;f(t,x,u,d)) - V^0(t,x;f(t,x,0,d))\right| \\ &\le M\left(T + \|x\|_r + 1\right) |f(t,x,u,d) - f(t,x,0,d)|\end{aligned}$$

for certain non-decreasing function $M : \Re^+ \to \Re^+$. The above inequality in conjunction with (2.2) and the fact that f is $T-periodic$ implies that the following inequality holds for all $(t,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$:

$$\begin{aligned}&\left|V^0(t,x;f(t,x,u,d)) - V^0(t,x;f(t,x,0,d))\right| \\ &\le M\left(T + \|x\|_r + 1\right) L_U(T, \|x\|_r + |u|\,)\ |u|\end{aligned}$$

We next define:

$$\begin{aligned}\psi(s) := \sup\{\, &M\left(T + \|x\|_r + 1\right) L_U(T, \|x\|_r + |u|\,)\,|u|; \\ &\|H(t,x)\|_Y \le \theta^{-1}(s)\,,\ |u| \le s\,\}\end{aligned}$$

Notice that hypothesis (S8) implies that $\psi(s) \le a_3(s)$ for all $s \ge 0$, where $a_3(s) := \tilde{M}\left(T + R + 1 + a\left(\theta^{-1}(s)\right)\right) L_U(T, R + a\left(\theta^{-1}(s)\right) + s)\,s$, $R \ge 0$ is the constant involved in hypothesis (S8), $a \in K_\infty$ is the function involved in hypothesis (S8) and $\tilde{M}(s)$ is a continuous positive function which satisfies $\tilde{M}(s) \ge M(s)$ for all $s \ge 0$. From this point the proof of implication (c) ⇒ (d) is exactly the same as in Theorem 3.2 (i.e., by distinguishing the cases $\theta^{-1}(|u|) \le \|H(t,x)\|_Y$ and $\theta^{-1}\ (|u|) \ge \|H(t,x)\|_Y$).

Finally, we continue with the proof of implication (b) ⇒ (c). Without loss of generality we may assume that $\gamma \in K_\infty$. Let $\theta \in K_\infty$ be a locally Lipschitz function that satisfies $\theta(s) \le \gamma^{-1}\left(\frac{1}{2}s\right)$ for all $s \ge 0$. By virtue of (3.1) and hypothesis (S8) it follows that the solution $x(\cdot)$ of (1.1) satisfies the following implications:

$$\begin{aligned}&|u(\tau)| \le \theta\left(\|H(\tau, T_r(\tau)x)\|_Y\right), \text{a.e. in } [t_0, t] \\ &\quad \Rightarrow \|H(t, T_r(t)x)\|_Y \le \max\left\{\sigma(\|x_0\|_r, t-t_0);\right. \\ &\quad \left.\frac{1}{2}\sup_{t_0 \le \tau \le t} \|H(\tau, T_r(\tau)x)\|_Y\right\}\end{aligned} \tag{A15}$$

$$\begin{aligned}&|u(\tau)| \le \theta\left(\|H(\tau, T_r(\tau)x)\|_Y\right), \text{a.e. in} [t_0, t] \\ &\quad \Rightarrow \|T_r(t)x\|_r \le R + a\Big(\sigma(\|x_0\|_r, t-t_0) \\ &\quad + \frac{1}{2}\sup_{t_0 \le \tau \le t} \|H(\tau, T_r(\tau)x)\|_Y\Big)\end{aligned} \tag{A16}$$

Proceeding in exactly the same way as in the proof of Theorem 3.2, it can be shown that for all $(d',d) \in M_\Delta, (t_0,x_0) \in \Re^+ \times C^0([-r,0];\Re^n)$, the corresponding solution $x(\cdot)$ of (3.10) satisfies the following estimates for all $t \ge t_0$:

$$\begin{aligned}&\|H(t, T_r(t)x)\|_Y \le \max\left\{\sigma(\|x_0\|_r, t-t_0)\,;\right. \\ &\left.\frac{1}{2}\sup_{t_0 \le \tau \le t} \|H(\tau, T_r(\tau)x)\|_Y\right\}\end{aligned} \tag{A17}$$

$$\begin{aligned}&\|T_r(t)x\|_r \le R + a\Big(\sigma(\|x_0\|_r, t-t_0) \\ &\quad + \frac{1}{2}\sup_{t_0 \le \tau \le t} \|H(\tau, T_r(\tau)x)\|_Y\Big)\end{aligned} \tag{A18}$$

$$\begin{aligned}&\|H(t, T_r(t)x)\|_Y \le \sigma\left(\|x_0\|_r\,,\, 0\right) \text{ and } \|T_r(t)x\|_r \\ &\quad \le R + a\left(2\sigma(\|x_0\|_r, 0)\right)\end{aligned} \tag{A19}$$

Consequently, (A19) and Definition 2.1 in [21] imply that system (3.10) is RFC. Moreover, using estimates (A17) and (A19), it follows that for all $t \ge t_0$ we have:

$$\begin{aligned}&\|H(t, T_r(t)x)\|_Y \\ &\quad \le \inf_{t_0 \le \xi \le t} \max\left\{\,\sigma\left(R + a\left(2\sigma\left(\|x_0\|_r, 0\right)\right), t-\xi\right);\right. \\ &\left.\frac{1}{2}\sup_{\xi \le \tau \le t} \|H(\tau, T_r(\tau)x)\|_Y\right\}\end{aligned} \tag{A20}$$

Lemma 2.7 in conjunction with inequality (A20) guarantees the existence of $\omega \in KL$ such that for all $t \geq t_0$ we have:

$$\|H(t, T_r(t)x)\|_Y \leq \omega\big(R + a\big(2\sigma\big(\|x_0\|_r, 0\big)\big), t - t_0\big) \tag{A21}$$

Combining estimate (A19) with (A21) we obtain for all $t \geq t_0$:

$$\|H(t, T_r(t)x)\|_Y \leq \kappa\big(\|x_0\|_r\, ,\ t - t_0\big) \tag{A22}$$

where $\kappa(s,t) := (\sigma(s,0))^{\frac{1}{2}}(\omega(R + a(2\sigma(s,0)), t))^{\frac{1}{2}}$ (notice that κ is of class KL). Estimate (A22) in conjunction with Definition 3.3 in [21] and the fact that system (3.10) is RFC shows that (3.10) is URGAOS with disturbances $(d', d) \in \tilde{M}_\Delta$. The proof is complete. ◁

Proof of Theorem 3.5: Consider a solution of (1.1) under hypotheses (S1–7) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r,0];\Re^n)$. By virtue of Lemma 2.5, for every $T \in (t_0, t_{\max})$, the mapping $[t_0,\ T] \ni t \to V(t, T_r(t)x)$ is absolutely continuous. It follows from (3.17) and Lemma 2.4 that there exists a set $I \subset [t_0, T]$ of zero Lebesgue measure such that the following implication holds for all $t \in [t_0, T]\backslash I$:

$$\begin{aligned} &V(t, T_r(t)x) \geq \zeta(\delta(t)|u(t)|) \\ &\quad \Rightarrow \frac{d}{dt}(V(t, T_r(t)x)) \leq -\rho(V(t, T_r(t)x)) \end{aligned}$$

Lemma 2.8 implies the existence of a continuous function σ of class KL, with $\sigma(s,0) = s$ for all $s \geq 0$ such that:

$$\begin{aligned} V(t, T_r(t)x) \leq \max\Bigg\{ &\sigma(\ V(t_0, T_r(t_0)x)\ ,\ t - t_0\ ), \\ &\sup_{t_0 \leq s \leq t} \sigma(\ \zeta(\delta(s)|u(s)|)\ ,\ t - s\ )\Bigg\}, \forall t \in [t_0, T] \end{aligned} \tag{A23}$$

with $T \in (t_0, t_{\max})$. Notice that for the case that (1.1) is RFC from the input $u \in M_U$ then $t_{\max} = +\infty$. For the case that there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x(0)|) \leq V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$, combining the previous inequality and (A23) we obtain for every $T \in (t_0, t_{\max})$:

$$\begin{aligned} &p(\mu(t)|x(t)|) \\ &\quad \leq R + \max\Bigg\{ \sigma(\ V(t_0, T_r(t_0)x)\ ,\ t - t_0\ ), \\ &\qquad \sup_{t_0 \leq s \leq t} \sigma(\ \zeta(\delta(s)|u(s)|)\ ,\ t - s\ )\Bigg\}, \forall t \in [t_0, T] \end{aligned} \tag{A24}$$

It follows from estimate (A24) that $t_{\max} = +\infty$. A direct consequence of Lemma 2.6 is that estimate (A23) holds for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$ and $t \geq t_0$. Moreover, if there exist functions $p \in K_\infty, \mu \in K^+$ and a constant $R \geq 0$ such that $p(\mu(t)|x(0)|) \leq V(t,x) + R$ for all $(t,x) \in \Re^+ \times C^0([-r,0]\,;\,\Re^n)$, then Lemma 2.6 implies that estimate (A24) also holds for all $(t_0, x_0, d, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D \times M_U$ and $t \geq t_0$. In this case the fact that system (1.1) is RFC from the input $u \in M_U$ is an immediate consequence of (A24) and Definition 2.1. Notice that (3.18) is an immediate consequence of (A23) and (3.16). Finally, (i) and (ii) are immediate consequences of (3.18). The proof is complete. ◁