

Global Output Stability for Systems Described by Retarded Functional Differential Equations: Lyapunov Characterizations

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In this work characterizations of internal notions of output stability for uncertain time-varying systems described by retarded functional differential equations are provided. Particularly, characterizations by means of Lyapunov functionals of uniform and non-uniform in time Robust Global Asymptotic Output Stability are given. The results of this work have been developed for systems with outputs in abstract normed linear spaces in order to allow outputs with no delay, with discrete or distributed delay or functional outputs with memory.

Keywords: Lyapunov functionals, time-delay systems, global asymptotic stability.

1. Introduction-Motivation

In this work we develop Lyapunov characterizations of various internal robust stability notions for uncertain systems described by Retarded Functional Differential Equations (RFDEs). The internal robust stability notions proposed in the present work are parallel to the internal robust stability notions used for finite-dimensional systems and the framework used in this work allows the study of systems with outputs with no delays, outputs with discrete or distributed delay or functional outputs with memory.

It should be emphasized that our assumptions for systems described by RFDEs are very weak, since we do not assume boundedness or continuity of the right-hand side of the differential equation with respect to time or a Lipschitz condition. Furthermore, we do not assume that the disturbance set is compact.

Notions of output stability have been studied for finite-dimensional systems described by ordinary differential equations (see [11,15,17,32,33]) or difference equations (see [12,16]). For systems described by RFDEs the notion of partial stability (which is a special case of the notion of global asymptotic output stability) has been studied in [2,3,10,34]. Particularly in [2], the authors provide Lyapunov characterizations of local partial stability for systems described by RFDEs without disturbances under the assumptions of the invariance of the attractive set and boundedness of the right-hand side of the differential equation with respect to time.

In this work we provide Lyapunov characterizations of Robust Global Asymptotic Output Stability (RGAOS) for systems described by RFDEs with disturbances, without the hypothesis that the attractive set is invariant and without the assumption that the right-hand side of the differential equation is bounded with respect to time. Particularly, we consider uniform and non-uniform notions of RGAOS, which directly extend the corresponding notions of Robust Global Asymptotic Stability of an equilibrium point (see

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[3,4,5,9,18,23,24,25,27,28]). The reader should notice that the notion of non-uniform in time (asymptotic) stability is a classical stability notion arising in time-varying differential equations (see for instance [8, 25]). The usefulness of the non-uniform in time stability notions in Mathematical Control Theory was recently shown in [13–20]: time-varying feedback will induce a time-varying closed-loop system even if the open-loop control system is autonomous. The use of time-varying feedback provides certain advantages which cannot be guaranteed by time-invariant feedback (see [13,14] and references therein). Finally, in [14,16,20] it was shown that non-uniform in time stability notions are useful even for autonomous systems (see for instance Proposition 3.7 in [14], Proposition 3.3 in [16] and Theorem 3.1 in [20]) and can be utilized in order to study robustness to perturbations for control systems.

The results of the present work are expected to have numerous applications for Mathematical Control Theory. For example, the characterizations presented in this work can be directly used (exactly as in the finite-dimensional case) in order to:

- obtain necessary and sufficient Lyapunov-like conditions for the existence of robust continuous feedback stabilizers for control systems described by RFDEs (use of Control Lyapunov Functionals),
- develop backstepping methods for the feedback design for triangular control systems described by RFDEs,
- develop Lyapunov redesign methodologies which guarantee robustness to disturbance inputs,
- study the solution of tracking control problems where the signal to be tracked is not necessarily bounded with respect to time,
- study the existence/design observer problem for systems described by RFDEs by means of Lyapunov-like conditions (e.g., Observer Lyapunov Function, Lyapunov characterizations of observability/detectability).

However, the most important application of the results presented in this work is the development of Lyapunov characterizations of the external stability notions of Input-to-Output Stability (IOS) and Input-to-State Stability (ISS) for systems described by RFDEs. Related findings are reported on in a companion paper [21].

The structure of the paper is as follows: Section 2 is devoted to the presentation of the class of systems studied in this work. The stability notions used in the present paper as well as other important notions concerning Lyapunov functionals are provided in Section 3. Section 4 contains the main results of this

work. Two important examples are presented in Section 5: Example 5.1 shows the applicability of the main results to feedback stabilization problems and Example 5.2 is an academic example which illustrates the use of the Lyapunov characterizations provided in the present paper. Finally, the main results are proved in the Appendix. For the proofs of the main results some important technical results are stated and proved in the Appendix. It should be noticed that the capability of dealing with measurable (and not piecewise continuous) disturbances is provided by the three technical results proved in the Appendix (Lemma A.1, Lemma A.2 and Lemma A.3).

Notations Throughout this paper we adopt the following notations:

- * Let $I \subseteq \mathfrak{R}^n$ be an interval. By $C^0(I; \Omega)$, we denote the class of continuous functions on I , which take values in $\Omega \subseteq \mathfrak{R}^n$. By $C^1(I; \Omega)$, we denote the class of functions on I with continuous derivative, which take values in Ω .
- * For a vector $x \in \mathfrak{R}^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. For $x \in C^0([-r, 0]; \mathfrak{R}^n)$ we define $\|x\|_r := \max_{\theta \in [-r, 0]} |x(\theta)|$.
- * N denotes the set of positive integers and \mathfrak{R}^+ denotes the set of non-negative real numbers.
- * We denote by $[R]$ the integer part of the real number R , i.e., the greatest integer, which is less than or equal to R .
- * \mathbf{E} denotes the class of non-negative C^0 functions $\mu: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, for which it holds: $\int_0^{+\infty} \mu(t) dt < +\infty$ and $\lim_{t \rightarrow +\infty} \mu(t) = 0$.
- * We denote by K^+ the class of positive C^0 functions defined on \mathfrak{R}^+ . We say that a function $\rho: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. By K we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is of class K_∞ if $\lim_{s \rightarrow +\infty} \rho(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s, t): \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class K ; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t \rightarrow +\infty} \sigma(s, t) = 0$.
- * Let $U \subseteq \mathfrak{R}^m$ be a non-empty set with $0 \in U$. By $B_U[0, r] := \{u \in U; |u| \leq r\}$ we denote the closed sphere in $U \subseteq \mathfrak{R}^m$ with radius $r \geq 0$, centered at $0 \in U$.
- * Let $D \subseteq \mathfrak{R}^l$ be a non-empty set. By M_D we denote the class of all Lebesgue measurable and locally essentially bounded mappings $d: \mathfrak{R}^+ \rightarrow D$. By \tilde{M}_D we denote the class of all right-continuous

mappings $d: \mathfrak{R}^+ \rightarrow D$, with the property that there exists a countable set $A_d \subset \mathfrak{R}^+$ which is either finite or $A_d = \{t_k^d; k = 1, \dots, \infty\}$ with $t_{k+1}^d > t_k^d > 0$ for all $k = 1, 2, \dots$ and $\lim t_k^d = +\infty$, such that the mapping $t \in \mathfrak{R}^+ \setminus A_d \rightarrow d(t) \in D$ is continuous.

* Let $x: [a-r, b) \rightarrow \mathfrak{R}^n$ with $b > a > -\infty$ and $r > 0$. By $T_r(t)x$ we denote the “ r -history” of x at time $t \in [a, b)$, i.e., $T_r(t)x := x(t + \theta); \theta \in [-r, 0]$. Notice that $T_r(t)x \in C^0([-r, 0]; \mathfrak{R}^n)$ if x is continuous.

* By $\|\cdot\|_Y$, we denote the norm of the normed linear space Y .

2. Main Assumptions and Preliminaries for Systems Described by RFDEs

Let $D \subseteq \mathfrak{R}^l$ be a non-empty set and Y a normed linear space. We denote by $x(t)$ the solution of the initial-value problem:

$$\begin{aligned} \dot{x}(t) &= f(t, T_r(t)x, d(t)), \\ Y(t) &= H(t, T_r(t)x) \\ x(t) &\in \mathfrak{R}^n, d(t) \in D, Y(t) \in Y \end{aligned} \quad (2.1)$$

with initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathfrak{R}^n)$, where $r > 0$ is a constant and the mappings $f: \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D \rightarrow \mathfrak{R}^n$, $H: \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ satisfy $f(t, 0, d) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \mathfrak{R}^+ \times D$. The vector $d(t) \in D$ represents a time-varying uncertainty of the model.

Standard hypotheses (see (H1), (H3), (H4) below) are employed in order to guarantee uniqueness of solutions for (2.1), Lipschitz continuity of the solution with respect to the initial conditions and continuity of the output map. An additional hypothesis will be used in order to guarantee the “Boundedness-Implies-Continuation” property (see (H2) below). Particularly, in this work we consider systems of the form (2.1) under the following hypotheses:

(H1) The mapping $(x, d) \rightarrow f(t, x, d)$ is continuous for each fixed $t \geq 0$ and there exists a symmetric, positive definite matrix $P \in \mathfrak{R}^{n \times n}$ such that for every bounded $I \subseteq \mathfrak{R}^+$ and for every bounded $S \subset C^0([-r, 0]; \mathfrak{R}^n)$, there exists a constant $L \geq 0$ satisfying the following inequality:

$$\begin{aligned} &(x(0) - y(0))' P (f(t, x, d) - f(t, y, d)) \\ &\leq L \max_{\tau \in [-r, 0]} |x(\tau) - y(\tau)|^2 = L \|x - y\|_r^2 \\ &\forall t \in I, \forall (x, y) \in S \times S, \forall d \in D \end{aligned}$$

Hypothesis (H1) is equivalent to the existence of a continuous function $L: \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ such that for each fixed $t \geq 0$ the mappings $L(t, \cdot)$ and $L(\cdot, t)$ are non-decreasing, with the following property:

$$\begin{aligned} &(x(0) - y(0))' P (f(t, x, d) - f(t, y, d)) \\ &\leq L(t, \|x\|_r + \|y\|_r) \|x - y\|_r^2 \forall (t, x, y, d) \in \mathfrak{R}^+ \\ &\times C^0([-r, 0]; \mathfrak{R}^n) \times C^0([-r, 0]; \mathfrak{R}^n) \times D \end{aligned} \quad (2.2)$$

(H2) For every bounded $\Omega \subset \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ the image set $f(\Omega \times D) \subset \mathfrak{R}^n$ is bounded.

(H3) There exists a countable set $A \subset \mathfrak{R}^+$, which is either finite or $A = \{t_k; k = 1, \dots, \infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1, 2, \dots$ and $\lim t_k = +\infty$, such that the mapping $(t, x, d) \in (\mathfrak{R}^+ \setminus A) \times C^0([-r, 0]; \mathfrak{R}^n) \times D \rightarrow f(t, x, d)$ is continuous. Moreover, for each fixed $(t_0, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D$, we have $\lim_{t \rightarrow t_0^+} f(t, x, d) = f(t_0, x, d)$.

(H4) The mapping $H(t, x)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \mathfrak{R}^+$ and for every bounded $S \subset C^0([-r, 0]; \mathfrak{R}^n)$, there exists a constant $L_H \geq 0$ such that:

$$\begin{aligned} &\|H(t, x) - H(\tau, y)\|_Y \leq L_H (|t - \tau| + \|x - y\|_r) \\ &\forall (t, \tau) \in I \times I, \forall (x, y) \in S \times S \end{aligned}$$

Hypothesis (H4) is equivalent to the existence of a continuous function $L_H: \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ such that for each fixed $t \geq 0$ the mappings $L_H(t, \cdot)$ and $L_H(\cdot, t)$ are non-decreasing, with the following property:

$$\begin{aligned} &\|H(t, x) - H(\tau, y)\|_Y \\ &\leq L_H (\max\{t, \tau\}, \|x\|_r + \|y\|_r) (|t - \tau| + \|x - y\|_r) \\ &\forall (t, \tau, x, y) \in \mathfrak{R}^+ \times \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \\ &\times C^0([-r, 0]; \mathfrak{R}^n) \end{aligned} \quad (2.3)$$

It should be emphasized at this point that a major advantage of allowing the output to take values in abstract normed linear spaces in (2.1), is that we are in a position to consider:

- outputs with no delays, e.g. $Y(t) = h(t, x(t))$ with $Y = \mathfrak{R}^k$,
- outputs with discrete or distributed delay, e.g. $Y(t) = h(x(t), x(t-r))$ or $Y(t) = \int_{t-r}^t h(t, \theta, x(\theta)) d\theta$ with $Y = \mathfrak{R}^k$,
- functional outputs with memory, e.g. $Y(t) = h(t, \theta, x(t + \theta)); \theta \in [-r, 0]$ or the identity output $Y(t) = T_r(t)x = x(t + \theta); \theta \in [-r, 0]$ with $Y = C^0([-r, 0]; \mathfrak{R}^k)$.

It is clear that (by virtue of hypotheses (H1–3) above and Lemma 1 in [7], page 4) for every $d \in M_D$ the composite map $f(t, x, d(t))$ satisfies the Carathéodory condition on $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ and consequently, by virtue of Theorem 2.1 in [9] (and its extension given in paragraph 2.6 of the same book), for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$ there exists $h > 0$ and at least one continuous function $x : [t_0 - r, t_0 + h] \rightarrow \mathfrak{R}^n$, which is absolutely continuous on $[t_0, t_0 + h]$ with $T_r(t_0)x = x_0$ and $\dot{x}(t) = f(t, T_r(t)x, d(t))$ almost everywhere on $[t_0, t_0 + h]$. Let $x : [t_0 - r, t_0 + h] \rightarrow \mathfrak{R}^n$ and $y : [t_0 - r, t_0 + h] \rightarrow \mathfrak{R}^n$ be two solutions of (2.1) with initial conditions $T_r(t_0)x = x_0$ and $T_r(t_0)y = y_0$ and corresponding to the same $d \in M_D$. Evaluating the derivative of the absolutely continuous map $z(t) = (x(t) - y(t))'P(x(t) - y(t))$ on $[t_0, t_0 + h]$ in conjunction with hypothesis (H1) above, we obtain the integral inequality:

$$\begin{aligned} |x(t) - y(t)|^2 &\leq \frac{K_2}{K_1} |x(t_0) - y(t_0)|^2 \\ &+ 2 \int_{t_0}^t \tilde{L} \|T_r(\tau)x - T_r(\tau)y\|_r^2 d\tau, \quad \forall t \in [t_0, t_0 + h] \end{aligned}$$

where $\tilde{L} := K_1^{-1}L(t_0 + h, a(t_0 + h))$, $L(\cdot)$ is the function involved in (2.2), $a(t) := \sup_{\tau \in [t_0 - r, t]} |x(\tau)| + \sup_{\tau \in [t_0 - r, t]} |y(\tau)|$ and $K_2 \geq K_1 > 0$ are the constants that satisfy $K_1|x|^2 \leq x'Px \leq K_2|x|^2$ for all $x \in \mathfrak{R}^n$. Consequently, we obtain:

$$\begin{aligned} \|T_r(t)(x - y)\|_r^2 &\leq \frac{K_2}{K_1} \|x_0 - y_0\|_r^2 \\ &+ 2 \int_{t_0}^t \tilde{L} \|T_r(\tau)(x - y)\|_r^2 d\tau, \quad \forall t \in [t_0, t_0 + h] \end{aligned}$$

and a direct application of the Gronwall-Bellman inequality gives:

$$\begin{aligned} \|T_r(t)(x - y)\|_r &\leq \sqrt{\frac{K_2}{K_1}} \|x_0 - y_0\|_r \exp(\tilde{L}(t - t_0)), \\ \forall t \in [t_0, t_0 + h] \end{aligned} \quad (2.4)$$

Thus, we conclude that under hypotheses (H1–4), for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$ there exists $h > 0$ and exactly one continuous function $x : [t_0 - r, t_0 + h] \rightarrow \mathfrak{R}^n$, which is absolutely continuous on $[t_0, t_0 + h]$ with $T_r(t_0)x = x_0$ and $\dot{x}(t) = f(t, T_r(t)x, d(t))$ almost everywhere on $[t_0, t_0 + h]$. We denote by $\phi(t, t_0, x_0; d)$ the “ r -history” of the unique solution of (2.1), i.e., $\phi(t, t_0, x_0; d) := T_r(t)x$, with initial condition $T_r(t_0)x = x_0$ corresponding to $d \in M_D$. Using hypothesis (H2) above and Theorem

3.2 in [9], we conclude that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$ there exists $t_{\max} \in (t_0, +\infty]$, such that the unique solution $x(t)$ of (2.1) is defined on $[t_0 - r, t_{\max}]$ and cannot be further continued. Moreover, if $t_{\max} < +\infty$ then we must necessarily have $\limsup_{t \rightarrow t_{\max}} |x(t)| = +\infty$. A direct consequence of inequalities (2.4) and (2.3) is the following inequality which holds for every pair $\phi(\cdot, t_0, x_0; d) : [t_0, t_{\max}^x] \rightarrow C^0([-r, 0]; \mathfrak{R}^n)$, $\phi(\cdot, t_0, y_0; d) : [t_0, t_{\max}^y] \rightarrow C^0([-r, 0]; \mathfrak{R}^n)$ of solutions of (2.1) with initial conditions $T_r(t_0)x = x_0$, $T_r(t_0)y = y_0$, corresponding to the same $d \in M_D$ and for all $t \in [t_0, t_1)$ with $t_1 = \min\{t_{\max}^x; t_{\max}^y\}$:

$$\begin{aligned} &\|\phi(t, t_0, x_0; d) - \phi(t, t_0, y_0; d)\|_r \\ &\leq G \|x_0 - y_0\|_r \exp(\tilde{L}(t, a(t))(t - t_0)) \\ &\|H(t, \phi(t, t_0, x_0; d)) - H(t, \phi(t, t_0, y_0; d))\|_Y \\ &\leq GL_H(t, a(t)) \|x_0 - y_0\|_r \\ &\exp(\tilde{L}(t, a(t))(t - t_0)) \\ &a(t) = \sup_{\tau \in [t_0, t]} (\|\phi(\tau, t_0, x_0; d)\|_r + \|\phi(\tau, t_0, y_0; d)\|_r) \end{aligned} \quad (2.5)$$

where $G := \sqrt{\frac{K_2}{K_1}}$ and $\tilde{L} := K_1^{-1}L(t, a(t))$. Since $f(t, 0, d) = 0$ for all $(t, d) \in \mathfrak{R}^+ \times D$, it follows that $\phi(t, t_0, 0; d) = 0 \in C^0([-r, 0]; \mathfrak{R}^n)$ for all $(t_0, d) \in \mathfrak{R}^+ \times M_D$ and $t \geq t_0$. Furthermore, (2.5) implies that for every $\varepsilon > 0$, $T, h \geq 0$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that:

$$\begin{aligned} \|x\|_r < \delta &\Rightarrow \sup\{\|\phi(\tau, t_0, x; d)\|_r; \\ &\in M_D, \tau \in [t_0, t_0 + h], t_0 \in [0, T]\} < \varepsilon \end{aligned}$$

Thus $0 \in C^0([-r, 0]; \mathfrak{R}^n)$ is a robust equilibrium point for (2.1) in the sense described in [15].

It should be emphasized that if $d \in \tilde{M}_D$ then the map $t \rightarrow f(t, x, d(t))$ is right-continuous on \mathfrak{R}^+ and continuous on $\mathfrak{R}^+ \setminus (A \cup A_d)$. Applying repeatedly Theorem 2.1 in [9] on each one of the intervals contained in $[t_0, t_{\max}) \setminus (A \cup A_d)$, we conclude that the solution satisfies $\dot{x}(t) = f(t, T_r(t)x, d(t))$ for all $t \in [t_0, t_{\max}) \setminus (A \cup A_d)$. By virtue of the mean value theorem, it follows that $\lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h} = f(t, T_r(t)x, d(t))$ for all $t \in [t_0, t_{\max})$.

3. Definitions of Important Notions

An important property for systems of the form (2.1) is Robust Forward Completeness (RFC) (see [15]). This property will be used extensively in the following sections of the present work.

Definition 3.1: We say that (2.1) under hypotheses (H1–4) is **Robustly Forward Complete (RFC)** if for every $s \geq 0$, $T \geq 0$, it holds that

$$\sup \left\{ \|\phi(t_0 + \xi, t_0, x_0; d)\|_r; \right. \\ \left. \xi \in [0, T], \|x_0\|_r \leq s, t_0 \in [0, T], d \in M_D \right\} < +\infty$$

In what follows the reader is introduced to the notions of non-uniform in time and uniform Robust Global Asymptotic Output Stability (RGAOS) for systems described by RFDEs. Notice that the notion of RGAOS is applied to uncertain systems with a robust equilibrium point (vanishing perturbations) and is an ‘‘Internal Stability’’ property.

Definition 3.2: Consider system (2.1) under hypotheses (H1–4). We say that (2.1) is **non-uniformly in time Robustly Globally Asymptotically Output Stable (RGAOS)** with disturbances $d \in M_D$ if (2.1) is RFC and the following properties hold:

P1 (2.1) is **Robustly Lagrange Output Stable**, i.e., for every $\varepsilon > 0$, $T \geq 0$, it holds that

$$\sup \left\{ \|H(t, \phi(t, t_0, x_0; d))\|_Y; \right. \\ \left. t \in [t_0, +\infty), \|x_0\|_r \leq \varepsilon, t_0 \in [0, T], d \in M_D \right\} < +\infty$$

(Robust Lagrange Output Stability)

P2 (2.1) is **Robustly Lyapunov Output Stable**, i.e., for every $\varepsilon > 0$ and $T \geq 0$ there exists a $\delta := \delta(\varepsilon, T) > 0$ such that:

$$\|x_0\|_r \leq \delta, t_0 \in [0, T] \Rightarrow \|H(t, \phi(t, t_0, x_0; d))\|_Y \\ \leq \varepsilon, \forall t \geq t_0, \forall d \in M_D$$

(Robust Lyapunov Output Stability)

P3(2.1) satisfies the **Robust Output Attractivity Property**, i.e. for every $\varepsilon > 0$, $T \geq 0$ and $R \geq 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geq 0$, such that:

$$\|x_0\|_r \leq R, t_0 \in [0, T] \Rightarrow \|H(t, \phi(t, t_0, x_0; d))\|_Y \\ \leq \varepsilon, \forall t \geq t_0 + \tau, \forall d \in M_D$$

Moreover, if there exists a function $a \in K_\infty$ such that $a(\|x\|_r) \leq \|H(t, x)\|_Y$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, then we say that (2.1) is **non-uniformly in time Robustly Globally Asymptotically Stable (RGAS)** with disturbances $d \in M_D$.

We say that (2.1) is **non-uniformly in time Robustly Globally Asymptotically Output Stable (RGAOS)** with disturbances $d \in \tilde{M}_D$ if (1.1) is RFC and properties P1–3 above hold with $d \in \tilde{M}_D$ instead of $d \in M_D$.

The next lemma provides an estimate of the output behavior for non-uniformly in time RGAOS systems. It is a direct corollary of Lemma 3.4 in [15].

Lemma 3.3: System (2.1) under hypotheses (H1–4) is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$) if and only if system (2.1) is RFC and there exist functions $\sigma \in KL$, $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geq t_0$:

$$\|H(t, \phi(t, t_0, x_0; d))\|_Y \leq \sigma(\beta(t_0)\|x_0\|_r, t - t_0) \quad (3.1)$$

We next provide the definition of Uniform Robust Global Asymptotic Output Stability, in terms of KL functions, which is completely analogous to the finite-dimensional case (see [22,26,32,33]). It is clear that such a definition is equivalent to a $\delta - \varepsilon$ definition (analogous to Definition 3.2).

Definition 3.4: Suppose that (2.1) under hypotheses (H1–4) is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$) and there exist $\sigma \in KL$ such that estimate (3.1) holds for all $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geq t_0$ with $\beta(t) \equiv 1$. Then we say that (2.1) is **Uniformly Robustly Globally Asymptotically Output Stable (URGAOS)** with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$).

The following lemma must be compared to Lemma 1.1, page 131 in [9]. It shows that for periodic systems RGAOS is equivalent to URGAOS. Its proof can be found at the Appendix. We say that (2.1) under hypotheses (H1–4) is T -periodic, if there exists $T > 0$ such that $f(t + T, x, d) = f(t, x, d)$ and $H(t + T, x) = H(t, x)$ for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D$. We say that (2.1) under hypotheses (H1–4) is autonomous if $f(t, x, d) = f(0, x, d)$ and $H(t, x) = H(0, x)$ for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D$.

Lemma 3.5: Suppose that (2.1) under hypotheses (H1–4) is T -periodic. If (2.1) is non-uniformly in time RGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$), then (2.1) is URGAOS with disturbances $d \in M_D$ (or $d \in \tilde{M}_D$).

In order to study the asymptotic properties of the solutions of systems of the form (2.1), we will use Lyapunov functionals and functions. Therefore, certain notions and properties concerning functionals are needed.

Let $x \in C^0([-r, 0]; \mathfrak{R}^n)$. By $E_h(x; v)$, where $0 \leq h < r$ and $v \in \mathfrak{R}^n$ we denote the following operator:

$$E_h(x; v) := \begin{cases} x(0) + (\theta + h)v & \text{for } -h < \theta \leq 0 \\ x(\theta + h) & \text{for } -r \leq \theta \leq -h \end{cases} \quad (3.2)$$

Let $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$. We define

$$V^0(t, x; v) := \limsup_{\substack{h \rightarrow 0^+ \\ y \rightarrow 0, y \in C^0([-r, 0]; \mathfrak{R}^n)}} \frac{V(t+h, E_h(x; v) + hy) - V(t, x)}{h} \quad (3.3)$$

Notice that the function $(t, x, v) \rightarrow V^0(t, x; v)$ may take values in the extended real number set $\mathfrak{R}^* = [-\infty, +\infty]$.

An important class of functionals is presented next.

Definition 3.6: We say that a continuous functional $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, is ‘‘almost Lipschitz on bounded sets’’, if there exist non-decreasing functions $M : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, $P : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, $G : \mathfrak{R}^+ \rightarrow [1, +\infty)$ such that for all $R \geq 0$, the following properties hold:

(P1) For every $x, y \in \{x \in C^0([-r, 0]; \mathfrak{R}^n); \|x\|_r \leq R\}$, it holds that:

$$|V(t, y) - V(t, x)| \leq M(R) \|y - x\|_r, \quad \forall t \in [0, R]$$

(P2) For every absolutely continuous function $x : [-r, 0] \rightarrow \mathfrak{R}^n$ with $\|x\|_r \leq R$ and essentially bounded derivative, it holds that:

$$|V(t+h, x) - V(t, x)| \leq hP(R) \left(1 + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)| \right),$$

$$\text{for all } t \in [0, R] \text{ and } 0 \leq h \leq \frac{1}{G \left(R + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)| \right)}$$

Remark 3.7: For mappings $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$, which are Lipschitz on bounded sets of $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, the derivative defined in (3.3) coincides with the derivative introduced in [6] and was used later in [4]. Particularly, we have:

$$V^0(t, x; v) := \limsup_{h \rightarrow 0^+} \frac{V(t+h, E_h(x; v)) - V(t, x)}{h}$$

Finally, the following definition introduces an important relation between output mappings. The equivalence relation defined next, will be used extensively in the following sections of the present work.

Definition 3.8: Suppose that there exists a continuous mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ with $h(t, 0) = 0$ for all $t \geq -r$ and functions $a_1, a_2 \in K_\infty$ such that

$$a_1(|h(t, x(0))|) \leq \|H(t, x)\|_\gamma \leq a_2 \left(\sup_{\theta \in [-r, 0]} |h(t+\theta, x(\theta))| \right)$$

for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$. Then we say that $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{Y}$ is equivalent to the finite-dimensional mapping h .

For example, the identity output mapping $H(t, x) = x \in C^0([-r, 0]; \mathfrak{R}^n)$ is equivalent to finite-dimensional mapping $h(t, x) = x \in \mathfrak{R}^n$.

4. Main Results

We are now in a position to present Lyapunov-like characterizations for non-uniform in time RGAOS and URGAOS. The proofs are provided in the Appendix.

Theorem 4.1: Consider system (1.1) under hypotheses (H1–4). The following statements are equivalent:

- (a) (2.1) is non-uniformly in time RGAOS with disturbances $d \in M_D$.
- (b) (2.1) is non-uniformly in time RGAOS with disturbances $d \in \tilde{M}_D$.
- (c) (2.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_{-\infty}^0 \gamma(t) dt = +\infty$, a positive definite locally Lipschitz function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ and a mapping $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, which is almost Lipschitz on bounded sets, such that:

$$a_1(\|H(t, x)\|_\gamma) \leq V(t, x) \leq a_2(\beta(t)\|x\|_r),$$

$$\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \quad (4.1)$$

$$V^0(t, x; f(t, x, d)) \leq -\gamma(t)\rho(V(t, x)),$$

$$\forall (t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D \quad (4.2)$$

- (d) (2.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$ and a mapping $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, which is almost Lipschitz on bounded sets, such that inequalities (4.1), (4.2) hold with $\gamma(t) \equiv 1$ and $\rho(s) := s$.
- (e) (2.1) is RFC and there exist a lower semi-continuous mapping $V : \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, a constant $\tau \geq 0$, functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_{-\infty}^0 \gamma(t) dt = +\infty$, $\mu \in \mathbf{E}$ (see Notations) and a positive definite locally Lipschitz function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, such that the following inequalities hold:

$$a_1(\|H(t, x)\|_\gamma) \leq V(t, x) \leq a_2(\beta(t)\|x\|_{r+\tau}),$$

$$\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) \quad (4.3)$$

$$\begin{aligned}
V^0(t, x; f(t, T_r(0)x, d)) &\leq -\gamma(t)\rho(V(t, x)) \\
&+ \gamma(t)\mu \left(\int_0^t \gamma(s) ds \right), \quad \forall (t, d) \in [\tau, +\infty) \times D, \\
\forall x \in S(t) & \quad (4.4)
\end{aligned}$$

where the set-valued map $S(t)$ is defined for $t \geq \tau$ by $S(t) := \bigcup_{d \in \tilde{M}_D} S(t, d)$ and the set-valued map $S(t, d)$ is defined for $t \geq \tau$ and $d \in \tilde{M}_D$ by:

$$\begin{aligned}
S(t, d) := & \left\{ x \in C^0([-r - \tau, 0]; \mathfrak{R}^n); x(\theta) = x(-\tau) \right. \\
& \left. + \int_{-\tau}^{\theta} f(t + s, T_r(s)x, d(\tau + s)) ds, \forall \theta \in [-\tau, 0] \right\} \quad (4.5)
\end{aligned}$$

Moreover,

- (i) if $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is equivalent to the finite-dimensional continuous mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ then inequalities (4.1) in statements (c) and (d) can be replaced by the following inequalities:

$$\begin{aligned}
a_1(|h(t, x(0))|) &\leq V(t, x) \leq a_2(\beta(t)\|x\|_r), \\
\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) & \quad (4.6)
\end{aligned}$$

- (ii) if $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is equivalent to the finite-dimensional continuous mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ then inequalities (4.3) in statement (e) can be replaced by the following inequalities:

$$\begin{aligned}
a_1(|h(t, x(0))|) &\leq V(t, x) \leq a_2(\beta(t)\|x\|_{r+\tau}), \\
\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) & \quad (4.7)
\end{aligned}$$

- (iii) if there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq V(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ then the requirement that (2.1) is RFC is not needed in statements (c) and (d) above.

Theorem 4.2: Consider system (2.1) under hypotheses (H1–4). The following statements are equivalent:

- (a) (2.1) is URGAOS with disturbances $d \in M_D$.
(b) (2.1) is URGAOS with disturbances $d \in \tilde{M}_D$.
(c) (2.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$, a positive definite locally Lipschitz function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ and a mapping $V : \mathfrak{R}^+ \times C^0$

$([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, which is almost Lipschitz on bounded sets, such that:

$$\begin{aligned}
a_1(\|H(t, x)\|_Y) &\leq V(t, x) \leq a_2(\|x\|_r), \\
\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) & \quad (4.8)
\end{aligned}$$

$$\begin{aligned}
V^0(t, x; f(t, x, d)) &\leq -\rho(V(t, x)), \\
\forall (t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times D & \quad (4.9)
\end{aligned}$$

- (d) (2.1) is RFC and there exist functions $a_1, a_2 \in K_\infty$ and a mapping $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, which is almost Lipschitz on bounded sets, such that inequalities (4.8), (4.9) hold with $\rho(s) := s$. Moreover, if system (2.1) is T -periodic, then V is T -periodic (i.e. $V(t + T, x) = V(t, x)$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$) and if (2.1) is autonomous then V is independent of t .

- (e) (2.1) is RFC and there exist constants $\tau, \beta \geq 0$, a lower semi-continuous mapping $V : \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$, functions $a_1, a_2 \in K_\infty$ and a positive definite locally Lipschitz function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, such that the following inequalities hold:

$$\begin{aligned}
a_1(\|H(t, x)\|_Y) &\leq V(t, x) \leq a_2(\|x\|_{r+\tau}), \\
\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) & \quad (4.10)
\end{aligned}$$

$$\begin{aligned}
V^0(t, x; f(t, T_r(0)x, d)) &\leq \beta V(t, x), \\
\forall (t, x, d) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n) \times D & \quad (4.11a)
\end{aligned}$$

$$\begin{aligned}
V^0(t, x; f(t, T_r(0)x, d)) &\leq -\rho(V(t, x)), \\
\forall (t, d) \in [\tau, +\infty) \times D, \forall x \in S(t) & \quad (4.11b)
\end{aligned}$$

where the set-valued map $S(t)$ is defined for $t \geq \tau$ by $S(t) := \bigcup_{d \in \tilde{M}_D} S(t, d)$ and the set-valued map $S(t, d)$ is defined for $t \geq \tau$ and $d \in \tilde{M}_D$ by (4.5).

Moreover,

- (i) if $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is equivalent to the finite-dimensional continuous T -periodic mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ then inequalities (4.8) in statements (c) and (d) can be replaced by the following inequalities:

$$\begin{aligned}
a_1(|h(t, x(0))|) &\leq V(t, x) \leq a_2(\|x\|_r), \\
\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) & \quad (4.12)
\end{aligned}$$

(ii) if $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is equivalent to the finite-dimensional continuous T -periodic mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ then inequalities (4.10) in statement (e) can be replaced by the following inequalities:

$$a_1(|h(t, x(0))|) \leq V(t, x) \leq a_2(\|x\|_{r+\tau}), \quad (4.13)$$

$$\forall (t, x) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^n)$$

(iii) if there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq V(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ then the requirement that (2.1) is RFC is not needed in statements (c) and (d) above.

Remark 4.3: The set-valued map $S(t, d)$ defined by (4.5) can be equivalently described for given $t \geq \tau$ and $d \in \tilde{M}_D$ as

“the set of all $x \in C^0([-r - \tau, 0]; \mathfrak{R}^n)$, which are arbitrary on $[-r - \tau, -\tau]$ (i.e., $T_r(-\tau)x$ is arbitrary) and coincide on $[-\tau, 0]$ with the unique solution $y(t)$ of $\dot{y}(t) = f(t, T_r(t)y, d(t))$ with initial condition $T_r(t - \tau)y = T_r(-\tau)x$, i.e., $T_\tau(0)x = T_\tau(t)y$ and $x = T_{r+\tau}(t)y$ ”

Statements (e) of Theorem 4.1 and Theorem 4.2 are important, since they can be used efficiently when some information about the solution of (1.1) is available (e.g., we have analytical expressions for some components of the solution vector). In this case, the Lyapunov differential inequality is required to hold only for all $(t, d) \in [\tau, +\infty) \times D$ and $x \in S(t)$ since the solution of (1.1) initiated from $t_0 \geq 0$ and corresponding to input $d \in \tilde{M}_D$ satisfies $T_{r+\tau}(t)x \in S(t, d)$ for all $t \geq t_0 + \tau$. In the following section an important example is presented, where statement (e) of Theorem 4.1 is used in conjunction with additional information for the solution (see Example 5.1 below). Moreover, statements (e) of Theorem 4.1 and Theorem 4.2 have an additional advantage: the Lyapunov functional is not required to be almost Lipschitz on bounded sets (lower semi-continuity is sufficient). Consequently, value functionals of optimal control problems can be used for verification of RGAOS (usually value functionals are not continuous).

5. Illustrative Examples

In this section we present examples which illustrate the use of our main results. Our first example shows an important application of Theorem 4.1 to feedback design problems.

Example 5.1: Consider the state stabilization problem for the system:

$$\dot{z} = F(t, z, x, u) \quad (5.1a)$$

$$z \in \mathfrak{R}^l, x \in \mathfrak{R}^n, u \in \mathfrak{R}, t \geq 0$$

$$\dot{x}_i = f_i(t, x_1, \dots, x_i) + x_{i+1}, \quad i = 1, \dots, n-1$$

$$\dot{x}_n = f_n(t, x) + u$$

$$x := (x_1, \dots, x_n) \in \mathfrak{R}^n, u \in \mathfrak{R}, t \geq 0 \quad (5.1b)$$

where $F : \mathfrak{R} \times \mathfrak{R}^l \times \mathfrak{R}^n \times \mathfrak{R} \rightarrow \mathfrak{R}^l$, $f_i : \mathfrak{R}^+ \times \mathfrak{R}^i \rightarrow \mathfrak{R}$ ($i = 1, \dots, n$) are smooth mappings with $F(t, 0, 0, 0) = 0$, $f_i(t, 0) = 0$ ($i = 1, \dots, n$) for all $t \in \mathfrak{R}$. We assume the following:

(S1) System (5.1a) is RFC from the input $(x, u) \in \mathfrak{R}^n \times \mathfrak{R}$, i.e., there exist functions $\beta \in K^+$, $a \in K_\infty$ such that for every $(t_0, z_0) \in \mathfrak{R}^+ \times \mathfrak{R}^l$ and for every pair of Lebesgue measurable and locally essentially bounded mappings $x : \mathfrak{R}^+ \rightarrow \mathfrak{R}^n$, $u : \mathfrak{R}^+ \rightarrow \mathfrak{R}$ the unique solution of (5.1a) with initial condition $z(t_0) = z_0$ corresponding to $x : \mathfrak{R}^+ \rightarrow \mathfrak{R}^n$, $u : \mathfrak{R}^+ \rightarrow \mathfrak{R}$ satisfies the following estimate for all $t \geq t_0$:

$$|z(t)| \leq \beta(t)a \left(|z_0| + \sup_{t_0 \leq s \leq t} |x(s)| + \sup_{t_0 \leq s \leq t} |u(s)| \right) \quad (5.2)$$

(S2) $0 \in \mathfrak{R}^l$ is non-uniformly in time GAS for the system $\dot{z} = F(t, z, 0, 0)$.

We will next show that under hypotheses (S1), (S2), we are in a position to design a stabilizing feedback law for (5.1) depending only on $x \in \mathfrak{R}^n$ (partial state feedback) which involves delays (retarded feedback). To this purpose we will employ statement (e) of Theorem 4.1.

Theorem 3.1 in [19] guarantees the existence of functions $\varphi \in K^+$, $q \in K_\infty$, constants $T, r > 0$ and the existence of a locally Lipschitz mapping $k : \mathfrak{R} \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$, which maps bounded subsets of $\mathfrak{R} \times C^0([-r, 0]; \mathfrak{R}^n)$ into bounded subsets of \mathfrak{R} such that for every $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ the solution of the closed-loop system (5.1b) with $u(t) = k(t, T_r(t)x)$ and initial condition $T_r(t_0)x = x_0$ satisfies

$$|x(t)| \leq \varphi(t)q(\|x_0\|_r), \quad \forall t \geq t_0 \quad (5.3)$$

$$x(t) = 0, \quad \forall t \geq t_0 + T \quad (5.4)$$

Moreover, if the mappings $f_i : \mathfrak{R}^+ \times \mathfrak{R}^i \rightarrow \mathfrak{R}$ ($i = 1, \dots, n$) are independent of time then $k : \mathfrak{R} \times$

$C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ is time-periodic, i.e., there exists $\omega > 0$ such that $k(t + \omega, x) = k(t, x)$ for all $(t, x) \in \mathfrak{R} \times C^0([-r, 0]; \mathfrak{R}^n)$. The reader should notice that the closed-loop system (5.1) with $u(t) = k(t, T_r(t)x)$ is a system described by RFDEs. We will show that $0 \in C^0([-r, 0]; \mathfrak{R}^l \times \mathfrak{R}^n)$ is non-uniformly in time GAS for the closed-loop system (5.1) with $u(t) = k(t, T_r(t)x)$.

Indeed, Theorem 3.1 in [13] in conjunction with hypothesis (S2) implies the existence of a smooth function $W: \mathfrak{R}^+ \times \mathfrak{R}^l \rightarrow \mathfrak{R}^+$, functions $a_1, a_2 \in K_\infty$, $\delta \in K^+$, such that the following inequalities hold:

$$\begin{aligned} a_1(|z|) &\leq W(t, z) \leq a_2(\delta(t)|z|), \\ \forall(t, z) &\in \mathfrak{R}^+ \times \mathfrak{R}^l \end{aligned} \quad (5.5)$$

$$\begin{aligned} \frac{\partial W}{\partial t}(t, z) + \frac{\partial W}{\partial z}(t, z)F(t, z, 0, 0) &\leq -W(t, z), \\ \forall(t, z) &\in \mathfrak{R}^+ \times \mathfrak{R}^l \end{aligned} \quad (5.6)$$

Let $\tilde{a}_1, \tilde{a}_2 \in K_\infty$ and let $Q: \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$ be any continuous functional that satisfies $\tilde{a}_1(|x(0)|) \leq Q(t, x) \leq \tilde{a}_2(\delta(t)\|x\|_r)$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ (e.g., $Q(t, x) := \tilde{a}_1(|x(0)|)$ or $Q(t, x) = \tilde{a}_2(\delta(t)\|x\|_r)$). Define the continuous functional for all $(t, z, x) \in \mathfrak{R}^+ \times C^0([-r - \tau, 0]; \mathfrak{R}^l \times \mathfrak{R}^n)$ with $\tau := r + T$:

$$V(t, z, x) := W(t, z(0)) + Q(t, T_r(0)x) \quad (5.7)$$

Using definition (4.5) and (5.4) it follows that for all $t \geq \tau = r + T$ each $(z, x) \in C^0([-r - \tau, 0]; \mathfrak{R}^l \times \mathfrak{R}^n)$ which belongs to $S(t)$ satisfies $T_r(0)x = 0$. Consequently, $V(t, z, x) := W(t, z(0))$ for all $t \geq \tau$ and for all $(z, x) \in S(t)$. Inequality (5.6) (combined with the fact that $F(t, z(0), x(0), k(t, T_r(0)x)) = F(t, z(0), 0, 0)$ for all $t \geq \tau$ and for all $(z, x) \in S(t)$) implies that inequality (4.4) holds with $\gamma(t) \equiv 1$, $\mu(t) \equiv 0$ and $\rho(s) := s$. Moreover, inequality (5.5) in conjunction with inequality $\tilde{a}_1(|x(0)|) \leq Q(t, x) \leq \tilde{a}_2(\delta(t)\|x\|_r)$, which holds for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, implies that inequality (4.7) holds with $h(t, z, x) := |(z, x)|$ for appropriate $a_1, a_2 \in K_\infty, \beta \in K^+$. Finally, we notice that hypothesis (S1) in conjunction with (5.3) implies that the closed-loop system (5.1) with $u(t) = k(t, T_r(t)x)$ is RFC. Notice that the finite-dimensional mapping $h(t, z, x) := |(z, x)|$ is equivalent to the output map $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^l \times \mathfrak{R}^n) \ni (t, z, x) \rightarrow H(t, z, x) := (z, x) \in C^0([-r, 0]; \mathfrak{R}^l \times \mathfrak{R}^n) = \mathcal{Y}$.

Therefore, statement (e) of Theorem 4.1 holds for the closed-loop system (5.1) with $u(t) = k(t, T_r(t)x)$

and thus we conclude that $0 \in C^0([-r, 0]; \mathfrak{R}^l \times \mathfrak{R}^n)$ is non-uniformly in time GAS. It should be noted that if the mappings $F: \mathfrak{R} \times \mathfrak{R}^l \times \mathfrak{R}^n \times \mathfrak{R} \rightarrow \mathfrak{R}^l, f_i: \mathfrak{R}^+ \times \mathfrak{R}^l \rightarrow \mathfrak{R} (i = 1, \dots, n)$ are independent of time, then the closed-loop system (5.1) with $u(t) = k(t, T_r(t)x)$ is time periodic (since $k: \mathfrak{R} \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ is time-periodic, i.e., there exists $\omega > 0$ such that $k(t + \omega, x) = k(t, x)$ for all $(t, x) \in \mathfrak{R} \times C^0([-r, 0]; \mathfrak{R}^n)$). In this case Lemma 3.5 guarantees that $0 \in C^0([-r, 0]; \mathfrak{R}^l \times \mathfrak{R}^n)$ is uniformly GAS. \triangleleft

Our following example is an academic example, which illustrates the use of Theorem 4.1 and Theorem 4.2.

Example 5.2: Consider the system:

$$\begin{aligned} \dot{x}_1(t) &= -2g_1(t)x_1(t) + d_1(t)g_2(t)x_1^2(t) \\ &\quad - g_3(t)x_1^3(t) + d_2(t)b(t)|x_2(t - \tau(t))|^p \\ \dot{x}_2(t) &= c(t)x_2(t) \\ Y(t) &= x_1(t) \in \mathfrak{R} \\ x(t) &= (x_1(t), x_2(t))' \in \mathfrak{R}^2, \\ d &= (d_1, d_2) \in D := [-1, 1] \times [-1, 1] \end{aligned} \quad (5.8)$$

where $g_1 \in K^+, g_2: \mathfrak{R}^+ \rightarrow \mathfrak{R}, g_3: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+, b: \mathfrak{R}^+ \rightarrow \mathfrak{R}, c: \mathfrak{R} \rightarrow \mathfrak{R}$ are continuous functions and $p > \frac{1}{2}$ is a constant. We consider system (5.8) under the following hypotheses:

(A1) There exist a continuous function $g \in K^+$ with $\int_0^{+\infty} g(s)ds = +\infty$ and a constant $M > 0$ such that $Mg(t) \leq g_1(t)$ for all $t \geq 0$. Moreover, $g_2^2(t) \leq 4g_1(t)g_3(t)$, for all $t \geq 0$.

(A2) The function $\tau: \mathfrak{R}^+ \rightarrow \mathfrak{R}$ is continuously differentiable, non-negative and bounded from above by a constant $r > 0$.

(A3) There exist constants $K, m > 0$ such that:

$$\begin{aligned} &\frac{|b(t)|^2}{g_1(t)} \exp\left(2p \int_0^{t-\tau(t)} c(s)ds\right) \\ &\leq K(1 - \dot{\tau}(t)) \exp\left(-mt - \int_0^t g(s)ds\right), \forall t \geq 0 \end{aligned} \quad (5.9)$$

For example hypotheses (A1–3) are satisfied for $p = 2$, $\tau(t) := 2 + \frac{1}{2}\sin(t)$, $c(t) \equiv 1$, $b(t) \equiv 1$, $g_1(t) = \exp(6t)$, $g_2(t) = \exp(t)$, $g_3(t) \equiv 1$, with $g(t) \equiv 1$, $M := 1$, $K := 2$, $m := 1$. We next show that system (5.8) under hypotheses (A1–3) is non-uniformly in time

RGAOS with disturbances $d \in M_D$. We consider the functional:

$$\begin{aligned} V(t, x) &:= \frac{1}{2}x_1^2(0) \\ &+ \exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s))ds\right)x_2^2(0) \\ &+ \frac{K}{2m}\exp\left(-mt - \int_0^t (g(s) + 2pc(s))ds\right)|x_2(0)|^{2p} \\ &+ \frac{K}{2}\exp\left(-mt - \int_0^t g(s)ds\right) \\ &\int_{-\tau(t)}^0 \exp\left(-ms - 2p \int_0^{t+s} c(\xi)d\xi\right)|x_2(s)|^{2p} ds \end{aligned} \quad (5.10)$$

Since $p > \frac{1}{2}$, it follows that the functional defined by (5.10) is almost Lipschitz on bounded sets. Moreover, inequalities (4.1) hold with $H(t, x) = x_1(0) \in Y := \mathfrak{R}$, $a_1(s) := 2^{-1}s^2$, $a_2(s) := s^2 + 2^{-1}K(m^{-1} + \text{rexp}(mr))s^{2p}$,

$$\begin{aligned} \beta(t) &:= 1 + \max\left\{\exp\left(-\frac{m}{2p}\xi - \frac{1}{2p}\int_0^\xi g(s)ds\right)\right. \\ &\times \left.\max\left\{\exp\left(-\int_0^{\xi+s} c(w)dw\right) : -\tau(\xi) \leq s \leq 0\right\} : 0 \leq \xi \leq t\right\} \end{aligned}$$

Furthermore, we obtain for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$:

$$\begin{aligned} &V^0(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) \\ &+ d_2b(t)|x_2(-\tau(t))|^p, c(t)x_2(0)) \leq -2g_1(t)x_1^2(0) \\ &+ |g_2(t)||x_1(0)|^3 - g_3(t)x_1^4(0) \\ &+ |b(t)||x_1(0)||x_2(-\tau(t))|^p - p^{-1}(m+g(t)) \\ &\times \exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s))ds\right)x_2^2(0) \\ &- g(t)\frac{K}{2m}\exp\left(-mt - \int_0^t (g(s) + 2pc(s))ds\right)|x_2(0)|^{2p} \\ &- g(t)\frac{K}{2}\exp\left(-mt - \int_0^t g(s)ds\right) \\ &\int_{-\tau(t)}^0 \exp\left(-ms - 2p \int_0^{t+s} c(\xi)d\xi\right)|x_2(s)|^{2p} ds \\ &- \frac{K}{2}(1 - \dot{\tau}(t))\exp\left(-m(t - \tau(t)) - \int_0^t g(s)ds - 2p \int_0^{t-\tau(t)} c(s)ds\right)|x_2(-\tau(t))|^{2p} \end{aligned}$$

Hypothesis (A1) implies that $-g_1(t)x_1^2 + |g_2(t)||x_1|^3 - g_3(t)x_1^4 \leq 0$, for all $(t, x_1) \in \mathfrak{R}^+ \times \mathfrak{R}$. Using the inequality $|b(t)||x_1(0)||x_2(-\tau(t))|^p \leq 2^{-1}g_1(t)x_1^2(0) + \frac{|b(t)|^2}{2g_1(t)}|x_2(-\tau(t))|^{2p}$ in conjunction with (5.9) we obtain for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$:

$$\begin{aligned} &V^0\left(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) \right. \\ &\left. + d_2b(t)|x_2(-\tau(t))|^p, c(t)x_2(0)\right) \leq -\frac{1}{2}g_1(t)x_1^2(0) \\ &- p^{-1}g(t)\exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s))ds\right)x_2^2(0) \\ &- g(t)\frac{K}{2m}\exp\left(-mt - \int_0^t (g(s) + 2pc(s))ds\right)|x_2(0)|^{2p} \\ &- g(t)\frac{K}{2}\exp\left(-mt - \int_0^t g(s)ds\right) \\ &\int_{-\tau(t)}^0 \exp\left(-ms - 2p \int_0^{t+s} c(\xi)d\xi\right)|x_2(s)|^{2p} ds \end{aligned}$$

Since $Mg(t) \leq g_1(t)$ for all $t \geq 0$, the above inequality in conjunction with definition (5.10) gives for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$:

$$\begin{aligned} &V^0\left(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) \right. \\ &\left. + d_2b(t)|x_2(-\tau(t))|^p, c(t)x_2(0)\right) \\ &\leq -\min\{M, p^{-1}, 1\}g(t)V(t, x) \end{aligned}$$

Consequently, inequality (4.2) holds with $\rho(s) := s$ and $\gamma(t) := \min\{M, p^{-1}, 1\}g(t)$. If system (5.8) was RFC then we would have showed that statement (c) of Theorem 4.1 would hold. However, since inequality $a(\mu(t)|x(0)|) \leq V(t, x) + R$ holds for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2)$ with $\mu(t) := \left(\min\left(2^{-1}; \exp\left(-\frac{m}{p}t - \int_0^t (p^{-1}g(s) + 2c(s))ds\right)\right)\right)^{\frac{1}{2}}$, $R := 0$ and $a(s) := s^2$, the requirement that system (5.8) is RFC is not needed. Thus we can conclude that system (5.8) is non-uniformly in time RGAOS with disturbances $d \in M_D$.

Moreover, if in addition to hypotheses (A1–3) the following hypothesis holds as well:

(A4) There exist constants $\Gamma, A > 0$ such that $g(t) \geq \Gamma$ and $\frac{m}{2p}t + \frac{1}{2p} \int_0^t g(s)ds + \min_{-\tau(t) \leq w \leq 0} \int_0^{t+w} c(s)ds \geq -A$ for all $t \geq 0$.

then we can conclude that system (5.8) is URGAOS with disturbances $d \in M_D$. Notice that if (A4) holds inequalities (4.8), (4.9) hold with $\rho(s) := \min\{M, p^{-1}, 1\}\Gamma s$, $H(t, x) = x_1(0) \in Y := \mathfrak{R} a_1(s) := 2^{-1}s^2$, $a_2(s) := \exp(2A)s^2 + 2^{-1}K(m^{-1} + r \exp(mr)) \exp(2pA)s^{2p}$ and consequently statement (c) of Theorem 4.2 holds (without the requirement that system (5.8) is RFC). \triangleleft

6. Conclusions

In this work Lyapunov-like characterizations of non-uniform in time and uniform Robust Global Asymptotic Output Stability (RGAOS) for uncertain time-varying systems described by Retarded Functional Differential Equations (RFDEs) are developed. Necessary and sufficient conditions in terms of Lyapunov functionals are provided for these notions. The framework of the present work allows outputs with no delays, outputs with discrete or distributed delays and functional outputs with memory. The robust stability notions and properties proposed in the present work are parallel to those recently developed for dynamical systems described by finite-dimensional ordinary differential equations. The Lyapunov characterizations presented in this work can be directly used (exactly as in the finite-dimensional case) in order to:

- obtain necessary and sufficient Lyapunov-like conditions for the existence of robust continuous feedback stabilizers for control systems described by RFDEs (use of Control Lyapunov Functionals),
- develop backstepping methods for the feedback design for triangular control systems described by RFDEs,
- develop Lyapunov redesign methodologies which guarantee robustness to disturbance inputs,
- study the solution of tracking control problems where the signal to be tracked is not necessarily bounded with respect to time,
- study the existence/design observer problem for systems described by RFDEs by means of Lyapunov-like conditions (e.g., Observer Lyapunov Function, Lyapunov characterizations of observability/detectability).

However, the most important application of the results presented in this work is the development of Lyapunov characterizations of the external stability

notions of Input-to-Output Stability (IOS) and Input-to-State Stability (ISS) for systems described by RFDEs. Related findings are reported on in a companion paper [21].

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References

1. Bacciotti A, Rosier L. Lyapunov functions and stability in control theory, *Lecture Notes in Control and Information Sciences*, 267, Springer-Verlag, London, 2001.
2. Bernfeld SA, Corduneanu C, Ignatyev AO. On the stability of invariant sets of functional differential equations. *Nonlinear Anal Theory Methods Appl* 2003; 55: 641–656.
3. Burton TA. Uniform asymptotic stability in functional differential equations. *Proc Am Math Soc* 1978; 68(2): 195–199.
4. Burton TA. Stability and periodic solutions of ordinary and functional differential equations. *Mathematics in Science and Engineering*, vol. 178, Academic Press, 1985.
5. Burton TA, Hatvani L. Stability theorems for non-autonomous functional differential equations by Liapunov functionals, *Tohoku Math J* 1989; 41: 65–104.
6. Driver RD. Existence and stability of solutions of a delay-differential system. *Arch Rational Mech Anal* 1962; 10: 401–426.
7. Fillipov AV. *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer Academic Publishers, 1988.
8. Hahn W. *Stability of Motion*. Springer-Verlag, Berlin, 1967.
9. Hale JK, Lunel SMV. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.
10. Ignatyev AO. On the partial equiasymptotic stability in functional differential equations. *J Math Anal Appl* 2002; 268(2): 615–628.
11. Jiang ZP, Teel A, Praly L. Small-gain theorem for ISS systems and applications. *Math Control Signals Syst* 1994; 7: 95–120.
12. Jiang Z-P, Lin Y, Wang Y. Remarks on input-to-output stability for discrete-time systems, *Proceedings of 2005 IFAC World Congress*.
13. Karafyllis I, Tsinias J. A converse Lyapunov theorem for non-uniform in time global asymptotic stability and its application to feedback stabilization. *SIAM J Control Optim* 2003; 42(3): 936–965.
14. Karafyllis I, Tsinias J. Non-uniform in time input-to-state stability and the small-gain theorem. *IEEE Trans Autom Control* 2004; 49(2): 196–216.

15. Karafyllis I. The non-uniform in time small-gain theorem for a wide class of control systems with outputs. *Eur J Control* 2004; 10(4): 307–323.
16. Karafyllis I. Non-uniform in time robust global asymptotic output stability for discrete-time systems. *Int J Robust Nonlinear Control* 2006; 16(4): 191–214.
17. Karafyllis I. Non-uniform in time robust global asymptotic output stability. *Syst Control Lett* 2005; 54(3): 181–193.
18. Karafyllis I. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Anal Theory Methods Appl* 2006; 64(3): 590–617.
19. Karafyllis I. Finite-time global stabilization by means of time-varying distributed delay feedback. *SIAM J Control Optim* 2006; 45(1): 320–342.
20. Karafyllis I. A system-theoretic framework for a wide class of systems II: input-to-output stability. *J Math Anal Appl* 2007; 328(1): 466–486.
21. Karafyllis I, Pepe P, Jiang Z-P. Input-to-output stability for systems described by retarded functional differential equations. *Eur J Control*; 2008; 14(6): 539–555.
22. Khalil HK, *Nonlinear Systems*, 2nd edn. Prentice-Hall, 1996.
23. Kharitonov VL, Melchor-Aguilar D. On delay dependent stability conditions for time-varying systems. *Syst Control Lett* 2002; 46: 173–180.
24. Kharitonov VL. Lyapunov-Krasovskii functionals for scalar time delay equations. *Syst Control Lett* 2004; 51: 133–149.
25. Krasovskii NN. *Stability of Motion*. Stanford University Press, Stanford, 1963.
26. Lin Y, Sontag ED, Wang Y. A smooth converse Lyapunov theorem for robust stability. *SIAM J Control and Optim* 1996; 34: 124–160.
27. Mazenc F, Niculescu S-I. Lyapunov stability analysis for nonlinear delay systems. *Syst Control Lett* 2001; 42: 245–251.
28. Niculescu SI. *Delay Effects on Stability: A Robust Control Approach*. Springer-Verlag, Heidelberg, Germany, 2001.
29. Pepe P. The problem of the absolute continuity for Liapunov-Krasovskii functionals. *IEEE Trans Autom Control* 2007; 52(5): 953–957.
30. Pepe P. On Liapunov-Krasovskii functionals under Carathéodory conditions. *Automatica* 2007; 43(4): 701–706.
31. Sontag ED. Comments on integral variants of ISS. *Syst Control Lett* 1998; 34: 93–100.
32. Sontag ED, Wang Y. Notions of input to output stability. *Syst Control Lett* 1999; 38: 235–248.
33. Sontag ED, Wang Y. Lyapunov characterizations of input-to-output stability. *SIAM J Control Optim* 2001; 39: 226–249.
34. Vorotnikov VI. *Partial stability and control*. Birkhauser, Boston, 1998.

Appendix—Proofs

For the proofs of Theorem 4.1 and Theorem 4.2 we need first to establish three auxiliary technical results, which allow us to derive useful estimates from the Lyapunov differential inequalities.

1st Auxiliary Result: Estimating the derivative of a Lyapunov functional

The following lemma presents some elementary properties of the generalized derivative defined in (3.3). Its proof is almost identical with Lemma 2.7 in [18]. Notice that we are not assuming that the mapping $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ is locally Lipschitz. The reader can see also the discussion in [30] for other cases of time-delay systems.

Lemma A.1: *Let $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ and let $x \in C^0([t_0 - r, t_{\max}); \mathfrak{R}^n)$ a solution of (2.1) under hypotheses (H1–4) corresponding to certain $d \in M_D$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then it holds that*

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} h^{-1}(V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \\ & \leq V^0(t, T_r(t)x; D^+x(t)), \text{ a.e. on } [t_0, t_{\max}) \end{aligned} \quad (\text{A.1})$$

where $D^+x(t) = \lim_{h \rightarrow 0^+} h^{-1}(x(t+h) - x(t))$. Moreover, if $d \in \tilde{M}_D$ then (A.1) holds for all $t \in [t_0, t_{\max})$.

Proof: It suffices to show that (A.1) holds for all $t \in [t_0, t_{\max}) \setminus I$ where $I \subset [t_0, t_{\max})$ is the set of zero Lebesgue measure such that $D^+x(t) = \lim_{h \rightarrow 0^+} h^{-1}(x(t+h) - x(t))$ is not defined on I . Let $h > 0$ and $t \in [t_0, t_{\max}) \setminus I$. We define:

$$T_r(t+h)x - E_h(T_r(t)x; D^+x(t)) = hy_h \quad (\text{A.2})$$

where

$$y_h = h^{-1} \begin{cases} x(t+h+\theta) - x(t) \\ -(\theta+h)D^+x(t) & \text{for } -h < \theta \leq 0 \\ 0 & \text{for } -r \leq \theta \leq -h \end{cases}$$

and notice that $y_h \in C^0([-r, 0]; \mathfrak{R}^n)$ (as difference of continuous functions, see (A.2) above). Equivalently y_h satisfies:

$$y_h := \begin{cases} \frac{\theta+h}{h} \left(\frac{x(t+\theta+h) - x(t)}{\theta+h} - D^+x(t) \right) & \text{for } -h < \theta \leq 0 \\ 0 & \text{for } -r \leq \theta \leq -h \end{cases}$$

with $\|y_h\|_{r,\leq} \leq \sup \left\{ \left| \frac{x(t+s) - x(t)}{s} - D^+x(t) \right|; 0 < s \leq h \right\}$.

Since $\lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h} = D^+x(t)$ we obtain that $y_h \rightarrow 0$ as $h \rightarrow 0^+$. Finally, by virtue of definitions (3.3), (A.2) and since $y_h \rightarrow 0$ as $h \rightarrow 0^+$, we have:

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} h^{-1}(V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \\ & = \limsup_{h \rightarrow 0^+} h^{-1}(V(t+h, E_h(T_r(t)x; D^+x(t)) + hy_h) \\ & \quad - V(t, T_r(t)x)) \leq V^0(t, T_r(t)x; D^+x(t)) \end{aligned}$$

The proof is complete. \triangleleft

2nd Auxiliary Result: Absolute continuity of the Lyapunov functional for differentiable initial conditions

For functionals which are almost Lipschitz on bounded sets, we are in a position to prove a novel result, which extends the result of Theorem 4 in [29].

Lemma A.2: *Let $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ be a functional which is almost Lipschitz on bounded sets and let $x \in C^0([t_0 - r, t_{\max}); \mathfrak{R}^n)$ a solution of (2.1) under hypotheses (H1–4) corresponding to certain $d \in M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \rightarrow V(t, T_r(t)x)$ is absolutely continuous.*

Proof: It suffices to show that for every $T \in (t_0, t_{\max})$ and $\varepsilon > 0$ there exists $\delta < 0$ such that $\sum_{k=1}^N |V(b_k, T_r(b_k)x) - V(a_k, T_r(a_k)x)| < \varepsilon$ for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ ($k = 1, \dots, N$) with $\sum_{k=1}^N (b_k - a_k) < \delta$. Let $T \in (t_0, t_{\max})$ and $\varepsilon > 0$ (arbitrary). Since the solution $x \in C^0([t_0 - r, T]; \mathfrak{R}^n)$ of (2.1) under hypotheses (H1–4) corresponding to certain $d \in M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ is bounded on $[t_0 - r, T]$, there exists $R_1 > 0$ such that $\sup_{t_0 \leq \tau \leq T} \|T_r(\tau)x\|_r \leq R_1$. Moreover, by virtue of hypothesis (H2) and since $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$, there exists $R_2 > 0$ such that $\sup_{t_0 - r \leq \tau \leq T} |\dot{x}(\tau)| \leq R_2$. The previous observations in conjunction with properties (P1), (P2) of Definition 3.6 imply for every interval $[a, b] \subset [t_0, T]$ with $b - a \leq \frac{1}{G(R_1 + R_2)}$:

$$\begin{aligned} & |V(b, T_r(b)x) - V(a, T_r(a)x)| \\ & \leq (b - a)P(R_1)(1 + R_2) \\ & + M(R_1)\|T_r(b)x - T_r(a)x\|_r \end{aligned}$$

In addition, the estimate $\sup_{t_0 - r \leq \tau \leq T} |\dot{x}(\tau)| \leq R_2$ implies $\|T_r(b)x - T_r(a)x\|_r \leq (b - a)R_2$ for every interval $[a, b] \subset [t_0, T]$. Consequently, we obtain for every interval $[a, b] \subset [t_0, T]$ with $b - a \leq \frac{1}{G(R_1 + R_2)}$:

$$\begin{aligned} & |V(b, T_r(b)x) - V(a, T_r(a)x)| \\ & \leq (b - a)[P(R_1)(1 + R_2) + M(R_1)R_2] \end{aligned}$$

The previous inequality implies that for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ ($k = 1, \dots, N$) with $\sum_{k=1}^N (b_k - a_k) < \delta$, where $\delta = \frac{1}{2} \min \left\{ \frac{1}{G(R_1 + R_2)}; \frac{\varepsilon}{P(R_1)(1 + R_2) + M(R_1)R_2} \right\} > 0$, it holds

that $\sum_{k=1}^N |V(b_k, T_r(b_k)x) - V(a_k, T_r(a_k)x)| < \varepsilon$. The proof is complete. \triangleleft

3rd Auxiliary Result: Estimates with differentiable initial conditions hold for continuous initial conditions as well

The following lemma extends the result presented in [29] and shows that appropriate estimates of the solutions of systems (2.1) hold globally. The proof of the following lemma is similar to the proof of Proposition 2 in [29].

Lemma A.3: *Consider system (2.1) under hypotheses (H1–4). Suppose that there exist mappings $\beta_1 : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$, $\beta_2 : \mathfrak{R}^+ \times \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A \rightarrow \mathfrak{R}$, where $A \subseteq M_D$, with the following properties:*

- (i) *for each $(t, t_0, d) \in \mathfrak{R}^+ \times \mathfrak{R}^+ \times A$, the mappings $x \rightarrow \beta_1(t, x)$, $x \rightarrow \beta_2(t, t_0, x, d)$ are continuous,*
- (ii) *there exists a continuous function $M : \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ such that*

$$\begin{aligned} & \sup\{\beta_2(t_0 + \xi, t_0, x_0, d); \\ & \xi \in [0, T], x_0 \in C^0([-r, 0]; \mathfrak{R}^n), \\ & \|x_0\|_r \leq s, t_0 \in [0, T], d \in A\} \leq M(T, s) \end{aligned}$$

- (iii) *for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^1([-r, 0]; \mathfrak{R}^n) \times A$ the solution $x(t)$ of (2.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $d \in A$ satisfies:*

$$\beta_1(t, T_r(t)x) \leq \beta_2(t, t_0, x_0, d), \quad \forall t \geq t_0 \quad (\text{A.3})$$

Moreover, suppose that one of the following properties holds:

- (iv) $c(T, s) := \sup\{\|T_r(t_0 + \xi)x\|_r; \xi \in [0, T], x_0 \in C^0([-r, 0]; \mathfrak{R}^n), \|x_0\|_r \leq s, t_0 \in [0, T], d \in A\} < +\infty$
- (v) *there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq \beta_1(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$*

Then for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$ the solution $x(t)$ of (2.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $d \in A$ exists for all $t \geq t_0$ and satisfies (A.3).

Proof: We distinguish the following cases:

- (a) Property (iv) holds. The proof will be made by contradiction. Suppose on the contrary that there exists $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$ and $t_1 > t_0$ such that the solution $x(t)$ of (2.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $d \in A$ satisfies:

$$\beta_1(t_1, T_r(t_1)x) > \beta_2(t_1, t_0, x_0, d)$$

Using (2.5) and property (iv) we obtain for all $\tilde{x}_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x_0 - \tilde{x}_0\|_r \leq 1$:

$$\begin{aligned} & \|T_r(t_1)x - T_r(t_1)\tilde{x}\|_r \\ & \leq G\|x_0 - \tilde{x}_0\|_r \exp(\tilde{L}(t_1, \bar{c})(t_1 - t_0)) \end{aligned} \quad (\text{A.4})$$

where $\tilde{x}(t)$ denotes the solution of (2.1) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $d \in A$ and $\bar{c} = 2c(t_1, \|x_0\|_r + 1)$

Let $\varepsilon := \beta_1(t_1, T_r(t_1)x) - \beta_2(t_1, t_0, x_0, d) > 0$. Using property (iv), (A.4), density of $C^1([-r, 0]; \mathfrak{R}^n)$ in $C^0([-r, 0]; \mathfrak{R}^n)$, continuity of the mappings $x \rightarrow \beta_1(t_1, x)$, $x \rightarrow \beta_2(t_1, t_0, x, d)$, we conclude that there exists $\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ such that:

$$\begin{aligned} & \|x_0 - \tilde{x}_0\|_r \leq 1; |\beta_2(t_1, t_0, x_0, d) - \beta_2(t_1, t_0, \tilde{x}_0, d)| \\ & \leq \frac{\varepsilon}{2}; |\beta_1(t_1, T_r(t_1)x) - \beta_1(t_1, T_r(t_1)\tilde{x})| \leq \frac{\varepsilon}{2} \end{aligned}$$

where $\tilde{x}(t)$ denotes the solution of (2.1) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $d \in A$. Combining property (iii) for $\tilde{x}(t)$ with the above inequalities and the definition of ε we obtain $\beta_1(t_1, T_r(t_1)x) > \beta_1(t_1, T_r(t_1)\tilde{x})$, a contradiction.

(b) Property (v) holds. It suffices to show that property (iv) holds. Since there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq \beta_1(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, it follows that from property (iii) that for every $(t_0, \tilde{x}_0, d) \in \mathfrak{R}^+ \times C^1([-r, 0]; \mathfrak{R}^n) \times A$ the solution $\tilde{x}(t)$ of (2.1) with initial condition $T_r(t_0)\tilde{x} = \tilde{x}_0$ corresponding to input $d \in A$ satisfies:

$$a(\mu(t)|\tilde{x}(t)|) \leq R + \beta_2(t, t_0, \tilde{x}_0, d), \quad \forall t \geq t_0$$

Moreover, making use of property (ii) and the above inequality, we obtain that for every $(t_0, \tilde{x}_0, d) \in \mathfrak{R}^+ \times C^1([-r, 0]; \mathfrak{R}^n) \times A$ the solution $\tilde{x}(t)$ of (2.1) with initial condition $T_r(t_0)\tilde{x} = \tilde{x}_0$ corresponding to input $d \in A$ satisfies:

$$\begin{aligned} & \|T_r(t)\tilde{x}\|_r \leq \|\tilde{x}_0\|_r + 1 \\ & + \max_{0 \leq \tau \leq t} \left[\frac{1}{\mu(\tau)} a^{-1}(R + M(\tau, \|\tilde{x}_0\|_r)) \right], \quad \forall t \geq t_0 \end{aligned} \quad (\text{A.5})$$

We claim that estimate (A.5) holds for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$. Notice that this claim implies directly that property (iv) holds with

$$c(T, s) \leq s + 1 + \frac{1}{\min_{0 \leq \tau \leq 2T} \mu(\tau)} a^{-1} \left(R + \max_{\substack{0 \leq x \leq 2s, \\ 0 \leq \tau \leq 2T}} M(\tau, s) \right).$$

The proof of the claim will be made by contradiction. Suppose on the contrary that there exist $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$ and $t_1 > t_0$ such that the

solution $x(t)$ of (2.1) with initial condition $T_r(t_0)x = x_0$ corresponding to input $d \in A$ satisfies:

$$\begin{aligned} & \|T_r(t_1)x\|_r > \|x_0\|_r + 1 \\ & + \max_{0 \leq \tau \leq t_1} \left[\frac{1}{\mu(\tau)} a^{-1}(R + M(\tau, \|x_0\|_r)) \right], \quad \forall t \geq t_0 \end{aligned} \quad (\text{A.6})$$

Let $B := \sup_{t_0 \leq \tau \leq t_1} \|T_r(\tau)x\| < +\infty$. Using (2.5) and (A.5), it follows that (A.4) holds for all $\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ with $\|x_0 - \tilde{x}_0\|_r \leq 1$ and $\bar{c} = B + \|x_0\|_r + 2 + \max_{\left\{ \frac{a^{-1}(R+M(t,s))}{\mu(t)}; 0 \leq s \leq \|x_0\|_r + 1, t_0 \leq t \leq t_1 \right\}}$, where $\tilde{x}(t)$ denotes the solution of (2.1) with initial condition $T_r(t_0)\tilde{x} = \tilde{x}_0$ corresponding to input $d \in A$. Let $\varepsilon := \|T_r(t_1)x\|_r - \|x_0\|_r - 1 - \max_{0 \leq \tau \leq t_1} \left[\frac{1}{\mu(\tau)} a^{-1}(R + M(\tau, \|x_0\|_r)) \right] > 0$. Using (A.4), density of $C^1([-r, 0]; \mathfrak{R}^n)$ in $C^0([-r, 0]; \mathfrak{R}^n)$ and continuity of the mapping $x \rightarrow g(x) := \|x\|_r + 1 + \max_{0 \leq \tau \leq t_1} \left[\frac{1}{\mu(\tau)} a^{-1}(R + M(\tau, \|x\|_r)) \right]$, we may conclude that there exists $\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ such that

$$\begin{aligned} & \|x_0 - \tilde{x}_0\|_r \leq 1; |g(x_0) - g(\tilde{x}_0)| \\ & \leq \frac{\varepsilon}{2} \|\|T_r(t_1)x\|_r - \|T_r(t_1)\tilde{x}\|_r\| \leq \frac{\varepsilon}{2} \end{aligned}$$

where $\tilde{x}(t)$ denotes the solution of (2.1) with initial condition $T_r(t_0)\tilde{x} = \tilde{x}_0$ corresponding to input $d \in A$. Combining (A.5) for $\tilde{x}(t)$ with the above inequalities and the definition of ε we obtain $\|T_r(t_1)x\|_r > \|T_r(t_1)\tilde{x}\|_r$, a contradiction. The proof is complete. \triangleleft

We are now in a position to present the proofs of the main results of the present work.

Proof of Theorem 4.1: Implications (a) \Rightarrow (b), (d) \Rightarrow (c), (c) \Rightarrow (e) are obvious. Thus we are left with the proof of implications (b) \Rightarrow (d), (c) \Rightarrow (a) and (e) \Rightarrow (b).

Proof of (b) \Rightarrow (d): The proof of this implication is based on the methodology presented in [1] for finite-dimensional systems as well as the methodologies followed in [13,18,26].

Since (2.1) is non-uniformly in time RGAOS with disturbances $d \in \tilde{M}_D$, there exist functions $\sigma \in KL$, $\beta \in K^+$ such that estimate (3.1) holds for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D$ and $t \geq t_0$. Moreover, by recalling Proposition 7 in [31] there exist functions \tilde{a}_1, \tilde{a}_2 of class K_∞ , such that the KL function $\sigma(s, t)$ is dominated by $\tilde{a}_1^{-1}(\exp(-2t)\tilde{a}_2(s))$. Thus, by taking into account estimate (3.1), we have:

$$\begin{aligned} & \tilde{a}_1(\|H(t, \phi(t, t_0, x_0; d))\|_\gamma) \\ & \leq \exp(-2(t - t_0))\tilde{a}_2(\beta(t_0)\|x_0\|_r), \quad \forall t \geq t_0 \geq 0, \\ & x_0 \in C^0([-r, 0]; \mathfrak{R}^n), \quad d \in \tilde{M}_D \end{aligned} \quad (\text{A.7})$$

Without loss of generality we may assume that $\tilde{a}_1 \in K_\infty$ is globally Lipschitz on \mathfrak{R}^+ with unit Lipschitz constant, namely, $|\tilde{a}_1(s_1) - \tilde{a}_1(s_2)| \leq |s_1 - s_2|$ for all $s_1, s_2 \geq 0$. To see this notice that we can always replace $\tilde{a}_1 \in K_\infty$ by the function $\bar{a}_1(s) := \inf\{\min\{\frac{1}{2}y, \tilde{a}_1(y)\} + |y - s|; y \geq 0\}$, which is of class K_∞ , globally Lipschitz on \mathfrak{R}^+ with unit Lipschitz constant and satisfies $\bar{a}_1(s) \leq \tilde{a}_1(s)$. Moreover, without loss of generality we may assume that $\beta \in K^+$ is non-decreasing.

Since (2.1) is Robustly Forward Complete (RFC), by virtue of Lemma 3.5 in [15], there exist functions $\mu \in K^+$, $a \in K_\infty$, such that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$ we have:

$$\|\phi(t, t_0, x_0; d)\|_r \leq \mu(t)a(\|x_0\|_r), \quad \forall t \geq t_0 \quad (\text{A.8})$$

Moreover, without loss of generality we may assume that $\mu \in K^+$ is non-decreasing. Making use of (2.5) and (A.8), we obtain the following property for the solution of (2.1):

$$\begin{aligned} & \|H(t, \phi(t, t_0, x; d)) - H(t, \phi(t, t_0, y; d))\|_\gamma \\ & \leq B(t, \|x\|_r + \|y\|_r) \exp(\bar{L}(t, \|x\|_r + \|y\|_r)(t - t_0)) \\ & \|x - y\|_r \text{ for all } t \geq t_0 \text{ and } (t_0, x, y, d) \in \mathfrak{R}^+ \\ & \times C^0([-r, 0]; \mathfrak{R}^n) \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D \end{aligned} \quad (\text{A.9})$$

where

$$\begin{aligned} \bar{L}(t, s) & := \tilde{L}(t, 2\mu(t)a(s)); \\ B(t, s) & := GL_H(t, 2\mu(t)a(s)) \end{aligned}$$

and $\tilde{L}(\cdot)$, $L_H(\cdot)$ are the functions involved in (2.5) and (2.3), respectively. Furthermore, hypothesis (H2) implies that the mapping

$$\zeta(s) := \sup\{|f(t, x, d)| : t + \|x\|_r \leq s, d \in D\}, \quad s \geq 0$$

is finite-valued and non-decreasing. Since $x(t) = x(0) + \int_{t_0}^t f(\tau, T_r(\tau)x, d(\tau))d\tau$, using the definition above in conjunction with (A.8) we obtain:

$$\begin{aligned} |x(t) - x(0)| & \leq (t - t_0)G_1(t, \|x\|_r) \\ G_1(t, s) & := \zeta(t + \mu(t)a(s)) \end{aligned}$$

and consequently

$$\begin{aligned} & \|\phi(t, t_0, x; d) - x\|_r \leq (t - t_0)G_1(t, \|x\|_r) \\ & + G_2(x, t - t_0) \text{ for all } t \geq t_0 \text{ and } (t_0, x, d) \in \mathfrak{R}^+ \\ & \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D \end{aligned} \quad (\text{A.10})$$

where the functional

$$\begin{aligned} G_2(x, h) & := \sup\{|x(0) - x(\theta)|; \theta \in [-\min\{h, r\}, 0]\} \\ & + \begin{cases} 0 & \text{if } h \geq r \\ \sup\{|x(\theta + h) - x(\theta)|; \theta \in [-r, -h]\} & \text{if } 0 \leq h < r \end{cases} \end{aligned}$$

is defined for all $(x, h) \in C^0([-r, 0]; \mathfrak{R}^n) \times \mathfrak{R}^+$. Notice that $\lim_{h \rightarrow 0^+} G_2(x, h) = 0$ for all $x \in C^0([-r, 0]; \mathfrak{R}^n)$ and consequently for every $\varepsilon > 0$, $R \geq 0$, $x \in C^0([-r, 0]; \mathfrak{R}^n)$, there exists $T(\varepsilon, R, x) > 0$ such that:

$$\begin{aligned} t_0 \leq t \leq t_0 + T(\varepsilon, R, x) & \Rightarrow \|\phi(t, t_0, x; d) - x\|_r \\ & \leq \varepsilon, \text{ for all } (t_0, x, d) \in [0, R] \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D \end{aligned} \quad (\text{A.11})$$

We define for all $q \in N$:

$$\begin{aligned} U_q(t, x) & := \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, x; d))\|_\gamma) - q^{-1}\} \\ & \times \exp((\tau - t)) : \tau \geq t, d \in \tilde{M}_D\} \end{aligned} \quad (\text{A.12})$$

Clearly, we obtain from estimate (A.7) and definition (A.12):

$$\begin{aligned} & \max\{0, \tilde{a}_1(\|H(t, x)\|_\gamma) - q^{-1}\} \\ & \leq U_q(t, x) \leq \tilde{a}_2(\beta(t)\|x\|_r), \quad \forall (t, x, q) \in \mathfrak{R}^+ \\ & \times C^0([-r, 0]; \mathfrak{R}^n) \times N \end{aligned} \quad (\text{A.13})$$

Moreover, by virtue of definition (A.12) we obtain for all $(h, t, x, d, q) \in \mathfrak{R}^+ \times \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D \times N$:

$$U_q(t + h, \phi(t + h, t, x; d)) \leq \exp(-h)U_q(t, x) \quad (\text{A.14})$$

By virtue of estimate (A.7) it follows that for every $(q, R) \in N \times \mathfrak{R}^+$, $\tau \geq t + \tilde{T}(R, q)$, $(t, d) \in [0, R] \times \tilde{M}_D$, and $x \in C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x\|_r \leq R$, it holds: $\tilde{a}_1(\|H(\tau, \phi(\tau, t, x; d))\|_\gamma) \leq \exp(-2(\tau - t))\tilde{a}_2(\beta(t)\|x\|_r) \leq q^{-1}$, where

$$\tilde{T}(R, q) := \max\left\{0, \frac{1}{2} \log(1 + q\tilde{a}_2(\beta(R)R))\right\} \quad (\text{A.15})$$

Thus, by virtue of definitions (A.12), (A.15), we conclude that:

$$\begin{aligned} U_q(t, x) & = \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, x; d))\|_\gamma) - q^{-1}\} \\ & \times \exp((\tau - t)) : t \leq \tau \leq t + \xi, d \in \tilde{M}_D, \\ & \forall \xi \geq \tilde{T}(\max\{t, \|x\|_r\}, q)\} \end{aligned} \quad (\text{A.16})$$

Equality (A.16) implies the following inequalities for all $t \in [0, R]$, and $(x, y) \in C^0([-r, 0]; \mathfrak{R}^n) \times C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x\|_r \leq R, \|y\|_r \leq R$:

$$\begin{aligned}
& |U_q(t, y) - U_q(t, x)| = \\
& |\sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, y; d))\|_\gamma) - q^{-1}\} \exp((\tau - t)) : t \leq \tau \leq t + \tilde{T}(R, q), d \in \tilde{M}_D\} \\
& \quad - \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t, x; d))\|_\gamma) - q^{-1}\} \exp((\tau - t)) : t \leq \tau \leq t + \tilde{T}(R, q), d \in \tilde{M}_D\}| \\
& \leq \sup\{\exp((\tau - t)) |\tilde{a}_1(\|H(\tau, \phi(\tau, t, y; d))\|_\gamma) - \tilde{a}_1(\|H(\tau, \phi(\tau, t, x; d))\|_\gamma)| : t \leq \tau \leq t + \tilde{T}(R, q), d \in \tilde{M}_D\} \\
& \leq \sup\{\exp((\tau - t)) \|H(\tau, \phi(\tau, t, y; d)) - H(\tau, \phi(\tau, t, x; d))\|_\gamma : t \leq \tau \leq t + \tilde{T}(R, q), d \in \tilde{M}_D\}
\end{aligned} \tag{A.17}$$

Notice that in the above inequalities we have used the facts that the functions $\max\{0, s - q^{-1}\}$ and $\tilde{a}_1(s)$ are globally Lipschitz on \mathfrak{R}^+ with unit Lipschitz constant. From (A.9) and (A.17) we deduce for all $t \in [0, R]$, and $(x, y) \in C^0([-r, 0]; \mathfrak{R}^n) \times C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x\|_r \leq R, \|y\|_r \leq R$:

$$|U_q(t, y) - U_q(t, x)| \leq G_3(R, q) \|y - x\|_r \tag{A.18}$$

where

$$\begin{aligned}
G_3(R, q) &:= B(R + \tilde{T}(R, q), 2R) \\
&\exp(\tilde{T}(R, q)(1 + \tilde{L}(R + \tilde{T}(R, q), 2R)))
\end{aligned} \tag{A.19}$$

Next, we establish continuity with respect to t on $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$. Let $R \geq 0$, $q \in N$ arbitrary, $t_1, t_2 \in [0, R]$ with $t_1 \leq t_2$, and $x \in C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x\|_r \leq R$. Clearly, we have for all $d \in \tilde{M}_D$:

$$\begin{aligned}
& |U_q(t_1, x) - U_q(t_2, x)| \\
& \leq (1 - \exp(-(t_2 - t_1))) U_q(t_1, x) \\
& \quad + |\exp(-(t_2 - t_1)) U_q(t_1, x) - U_q(t_2, \phi(t_2, t_1, x; d))| \\
& \quad + |U_q(t_2, \phi(t_2, t_1, x; d)) - U_q(t_2, x)|
\end{aligned}$$

By virtue of (A.10), (A.11), (A.14), (A.18) and the above inequality we obtain for all $t_1, t_2 \in [0, R]$ with

$t_1 \leq t_2 \leq t_1 + T(1, R, x)$ (where $T(\varepsilon, R, x) > 0$ is involved in (A.11)) and $d \in \tilde{M}_D$:

$$\begin{aligned}
& |U_q(t_1, x) - U_q(t_2, x)| \\
& \leq (t_2 - t_1) U_q(t_1, x) + \exp(-(t_2 - t_1)) U_q(t_1, x) \\
& \quad - U_q(t_2, \phi(t_2, t_1, x; d)) + G_3(R + 1, q) \\
& \quad \times [G_2(x, t_2 - t_1) + (t_2 - t_1) G_1(R, R)]
\end{aligned} \tag{A.20}$$

Definition (A.12) implies that for every $\varepsilon > 0$, there exists $d_\varepsilon \in \tilde{M}_D$ with the following property:

$$\begin{aligned}
& U_q(t_1, x) - \varepsilon \\
& \leq \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \\
& \quad \times \exp((\tau - t_1)); \tau \geq t_1\} \leq U_q(t_1, x)
\end{aligned} \tag{A.21}$$

Thus using definition (A.12) we obtain:

$$\begin{aligned}
& \exp(-(t_2 - t_1)) U_q(t_1, x) - U_q(t_2, \phi(t_2, t_1, x; d_\varepsilon)) \\
& \leq \max\{A_q(t_1, t_2, x), B_q(t_1, t_2, x)\} \\
& \quad - B_q(t_1, t_2, x) + \varepsilon \exp(-(t_2 - t_1))
\end{aligned} \tag{A.22}$$

where

$$\begin{aligned}
A_q(t_1, t_2, x) &:= \\
& \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \\
& \quad \times \exp((\tau - t_2)); t_2 \geq \tau \geq t_1\} \\
B_q(t_1, t_2, x) &:= \\
& \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \\
& \quad \times \exp((\tau - t_2)); \tau \geq t_2\}
\end{aligned} \tag{A.23}$$

Since the functions $\max\{0, s - q^{-1}\}$ and $\tilde{a}_1(s)$ are globally Lipschitz on \mathfrak{R}^+ with unit Lipschitz constant, we obtain:

$$\begin{aligned}
& A_q(t_1, t_2, x) - B_q(t_1, t_2, x) \\
& \leq \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \exp((\tau - t_2)); t_2 \geq \tau \geq t_1\} \\
& \quad - \max\{0, \tilde{a}_1(\|H(t_2, \phi(t_2, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \\
& \leq \sup\{\max\{0, \tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\}; t_2 \geq \tau \geq t_1\} - \max\{0, \tilde{a}_1(\|H(t_2, \phi(t_2, t_1, x; d_\varepsilon))\|_\gamma) - q^{-1}\} \\
& \leq \sup\{|\tilde{a}_1(\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon))\|_\gamma) - \tilde{a}_1(\|H(t_2, \phi(t_2, t_1, x; d_\varepsilon))\|_\gamma)|; t_2 \geq \tau \geq t_1\} \\
& \leq \sup\{\|H(\tau, \phi(\tau, t_1, x; d_\varepsilon)) - H(t_2, \phi(t_2, t_1, x; d_\varepsilon))\|_\gamma; t_2 \geq \tau \geq t_1\}
\end{aligned} \tag{A.24}$$

Notice that by virtue of (2.3), (A.4) and (A.5), we obtain for all $\tau \in [t_1, t_2]$ with $t_1 \leq t_2 \leq t_1 + T(1, R, x), t_1, t_2 \in [0, R]$:

$$\begin{aligned} & \|H(\tau, \phi(\tau, t_1, x; d_\varepsilon)) - H(t_2, \phi(t_2, t_1, x; d_\varepsilon))\|_Y \\ & \leq \|H(\tau, \phi(\tau, t_1, x; d_\varepsilon)) - H(t_1, x)\|_Y \\ & \quad + \|H(t_2, \phi(t_2, t_1, x; d_\varepsilon)) - H(t_1, x)\|_Y \\ & \leq 2(t_2 - t_1)L_H(R, 2R + 2)(1 + G_1(R, R)) \\ & \quad + 2L_H(R, 2R + 2) \sup\{G_2(x, h); h \in [0, t_2 - t_1]\} \end{aligned} \tag{A.25}$$

Distinguishing the cases $A_q(t_1, t_2, x) \geq B_q(t_1, t_2, x)$ and $A_q(t_1, t_2, x) \leq B_q(t_1, t_2, x)$ it follows from (A.22), (A.24), and (A.25) that:

$$\begin{aligned} & \exp(-(t_2 - t_1))U_q(t_1, x) - U_q(t_2, \phi(t_2, t_1, x; d_\varepsilon)) \leq \\ & \leq 2(t_2 - t_1)L_H(R, 2R + 2)(1 + G_1(R, R)) \\ & \quad + 2L_H(R, 2R + 2) \sup\{G_2(x, h); h \in [0, t_2 - t_1]\} + \varepsilon \end{aligned}$$

Combining the previous inequality with (A.20) and the right hand side of (A.13), we obtain:

$$\begin{aligned} & |U_q(t_1, x) - U_q(t_2, x)| \\ & \leq (t_2 - t_1)(\tilde{a}_2(\beta(R)R) + G_1(R, R)G_3(R + 1, q) \\ & \quad + 2L_H(R, 2R + 2)(1 + G_1(R, R))) \\ & \quad + (2L_H(R, 2R + 2) + G_3(R + 1, q)) \\ & \quad \times \sup\{G_2(x, h); h \in [0, t_2 - t_1]\} + \varepsilon \end{aligned} \tag{A.26}$$

Since (A.26) holds for all $\varepsilon > 0, R \geq 0, q \in N, x \in C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x\|_r \leq R$ and $t_1, t_2 \in [0, R]$ with $t_1 \leq t_2 \leq t_1 + T(1, R, x)$, it follows that:

$$\begin{aligned} & |U_q(t_1, x) - U_q(t_2, x)| \\ & \leq G_4(R, q)|t_2 - t_1| \\ & \quad + \sup\{G_2(x, h); h \in [0, |t_2 - t_1|]\} \text{ for all } R \geq 0, \\ & \quad q \in N, x \in C^0([-r, 0]; \mathfrak{R}^n) \text{ with } \|x\|_r \leq R \\ & \quad \text{and } t_1, t_2 \in [0, R] \text{ with } |t_2 - t_1| \leq T(1, R, x) \end{aligned} \tag{A.27}$$

where $G_4(R, q) := \tilde{a}_2(\beta(R)R) + (1 + G_1(R, R))G_3(R + 1, q) + 2L_H(R, 2R + 2)(1 + G_1(R, R))$. Finally, we define:

$$V(t, x) := \sum_{q=1}^{\infty} \frac{2^{-q}U_q(t, x)}{1 + G_3(q, q) + G_4(q, q)} \tag{A.28}$$

Inequality (A.13) in conjunction with definition (A.28) implies (4.1) with $a_2 = \tilde{a}_2$ and $a_1(s) := \sum_{q=1}^{\infty} \frac{2^{-q} \max\{0, \tilde{a}_1(s) - q^{-1}\}}{1 + G_3(q, q) + G_4(q, q)}$, which is a function of class K_∞ . Moreover, by virtue of definition (A.28) and inequality (A.14) we obtain for all $(h, t, x, d) \in \mathfrak{R}^+ \times \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times \tilde{M}_D$:

$$V(t + h, \phi(t + h, t, x; d)) \leq \exp(-h)V(t, x) \tag{A.29}$$

Next define

$$M(R) := 1 + \sum_{q=1}^{[R]+1} \frac{2^{-q}G_3(R, q)}{1 + G_3(q, q) + G_4(q, q)} \tag{A.30}$$

which is a positive non-decreasing function. Using (A.18) and definition (A.28) as well as the fact $G_3(R, q) \leq G_3(q, q)$ for $q > R$, we conclude that property (P1) of Definition 3.6 holds. Let $d \in D$ and define $\tilde{d}(t) \equiv d$. Definition (3.3) and inequality (A.29) imply that for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ we get:

$$\begin{aligned} V^0(t, x; f(t, x, d)) & := \limsup_{\substack{h \rightarrow 0^+ \\ y \rightarrow 0, y \in C^0([-r, 0]; \mathfrak{R}^n)}} \frac{V(t + h, E_h(x; f(t, x, d)) + hy) - V(t, x)}{h} \\ & \leq \limsup_{h \rightarrow 0^+} \frac{V(t + h, \phi(t + h, t, x; \tilde{d})) - V(t, x)}{h} \\ & \quad + \limsup_{\substack{h \rightarrow 0^+ \\ y \rightarrow 0, y \in C^0([-r, 0]; \mathfrak{R}^n)}} \frac{V(t + h, E_h(x; f(t, x, d)) + hy) - V(t + h, \phi(t + h, t, x; \tilde{d}))}{h} \\ & \leq -V(t, x) + \limsup_{\substack{h \rightarrow 0^+ \\ y \rightarrow 0, y \in C^0([-r, 0]; \mathfrak{R}^n)}} \frac{V(t + h, E_h(x; f(t, x, d)) + hy) - V(t + h, \phi(t + h, t, x; \tilde{d}))}{h} \end{aligned}$$

Let $R \geq \max\{t, \|x\|_r\}$. Definition (3.2) and property (A.11) imply that $t+h \leq R+1$, $\|\phi(t+h, t, x; \tilde{d})\|_r \leq R+1$, $\|E_h(x; f(t, x, d)) + hy\|_r \leq R+1$ for h and $\|y\|_r$ sufficiently small. Using property (P1) of Definition 3.6 and previous inequalities we obtain:

$$V^0(t, x; f(t, x, d)) \leq -V(t, x) + M(R+1) \limsup_{h \rightarrow 0^+} \frac{\|E_h(x; f(t, x, d)) - \phi(t+h, t, x; \tilde{d})\|_r}{h}$$

We set $\phi(t+h, t, x; \tilde{d}) = x(t+h+\theta)$; $\theta \in [-r, 0]$. Notice that $\phi(t+h, t, x; \tilde{d}) - E_h(x; f(t, x, d)) = hy_h$, where

$$y_h := \begin{cases} \frac{\theta+h}{h} \left(\frac{x(t+\theta+h) - x(t)}{\theta+h} - f(t, x, d) \right) & \text{for } -h < \theta \leq 0 \\ 0 & \text{for } -r \leq \theta \leq -h \end{cases}$$

with $\|y_h\|_r \leq \sup\left\{ \left| \frac{x(t+s) - x(t)}{s} - f(t, x, d) \right|; 0 < s \leq h \right\}$.

Since $\lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h} = f(t, x, d)$, we obtain that $y_h \rightarrow 0$ as $h \rightarrow 0^+$. Hence, we obtain

$$\limsup_{h \rightarrow 0^+} \frac{\|E_h(x; f(t, x, d)) - \phi(t+h, t, x; \tilde{d})\|_r}{h} = 0$$

and consequently (4.2) holds with $\gamma(t) \equiv 1$ and $\rho(s) := s$.

Finally, we establish continuity of V with respect to t on $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ and property (P2) of Definition 3.6. Notice that by virtue of (A.27) and the fact $G_4(R, q) \leq G_4(q, q)$ for $q > R$, we obtain:

$$\begin{aligned} |V(t_1, x) - V(t_2, x)| &\leq P(R) [|t_2 - t_1| \\ &+ \sup\{G_2(x, h); h \in [0, |t_2 - t_1|]\}] \\ &\text{for all } R \geq 0, x \in C^0([-r, 0]; \mathfrak{R}^n) \text{ with } \|x\|_r \\ &\leq R \text{ and } t_1, t_2 \in [0, R] \text{ with } |t_2 - t_1| \leq T(1, R, x) \end{aligned} \tag{A.31}$$

where

$$P(R) := 1 + \sum_{q=1}^{\lfloor R \rfloor + 1} \frac{2^{-q} G_4(R, q)}{1 + G_3(q, q) + G_4(q, q)}$$

is a positive non-decreasing function. Inequality (A.31) in conjunction with the fact that $\lim_{h \rightarrow 0^+} G_2(x, h) = 0$ for all $x \in C^0([-r, 0]; \mathfrak{R}^n)$, establishes continuity of V with respect to t on $\mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$. Moreover, for every absolutely continuous function $x : [-r, 0] \rightarrow \mathfrak{R}^n$ with $\|x\|_r \leq R$ and essentially bounded derivative, it holds that $\sup\{G_2(x, h); h \in [0, |t_2 - t_1|]\} \leq |t_2 - t_1| \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)|$

for $|t_2 - t_1| \leq r$. It follows from (A.10), (A.11) and

the previous inequality that $T(1, R, x) \geq \frac{r}{(1+r)(1+G_1(R, R) + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)|)}$. Property (P2) of Definition 3.6 with $G(R) := \frac{1+r}{r}(1 + G_1(R, R) + R)$ is a direct consequence of (A.31) and the two previous inequalities.

Proof of (c) \Rightarrow (a):

Case 1: (2.1) is RFC

Consider a solution $x(t)$ of (2.1) under hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$. By virtue of Lemma A.2, for every $T \in (t_0, +\infty)$, the mapping $[t_0, T] \ni t \rightarrow V(t, T_r(t)x)$ is absolutely continuous.

It follows from (4.2) and Lemma A.1 that $\frac{d}{dt}(V(t, T_r(t)x)) \leq -\gamma(t)\rho(V(t, T_r(t)x))$ a.e. on $[t_0, +\infty)$. The previous differential inequality in conjunction with the comparison lemma in [22] and Lemma 4.4 in [26] shows that there exists $\sigma \in KL$ such that

$$V(t, T_r(t)x) \leq \sigma \left(V(t_0, x_0), \int_{t_0}^t \gamma(s) ds \right) \text{ for all } t \geq t_0 \tag{A.32}$$

It follows from Lemma A.3 that the solution $x(t)$ of (2.1) under hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ satisfies (A.32) for all $t \geq t_0$. Next, we distinguish the following cases:

- (1) If (4.1) holds, then properties (P1–3) of Definition 3.2 are direct consequences of (A.32), (4.1) and the fact that $\int_0^{+\infty} \gamma(t) dt = +\infty$.
- (2) If (4.6) holds, then (A.32) implies the following estimate:

$$\begin{aligned} &|h(t, x(t))| \\ &\leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r), \int_{t_0}^t \gamma(s) ds \right) \right), \\ &\forall t \geq t_0 \end{aligned}$$

Since $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ is continuous with $h(t, 0) = 0$ for all $t \geq -r$, it follows from Lemma 3.2 in [15] that there exist functions $\zeta \in K_\infty$ and $\delta \in K^+$ such that:

$$|h(t-r, x)| \leq \zeta(\delta(t)|x|), \forall (t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$$

Combining the two previous inequalities we obtain:

$$\begin{aligned} & \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| \leq \max\{\zeta(\phi(t_0)\|x_0\|_r), \\ & a_1^{-1}(\sigma(a_2(\pi(t_0)\|x_0\|_r), 0))\}, \\ & \text{for all } t \in [t_0, t_0 + r] \\ & \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| \\ & \leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r), \int_{t_0}^{t-r} \gamma(s) ds \right) \right), \\ & \text{for all } t \geq t_0 + r \end{aligned}$$

where $\phi(t) := \beta(t) + \max_{0 \leq \tau \leq t+r} \delta(\tau)$. The above estimates, in conjunction with the facts that $\int_{+\infty} \gamma(t) dt = +\infty$ and $H: \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is 0 equivalent to the finite-dimensional mapping h show that properties (P1–3) of Definition 3.2 hold for system (2.1). Hence, system (2.1) is non-uniformly RGAOS with disturbances $d \in M_D$

Case 2: There exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that

$$\begin{aligned} a(\mu(t)|x(0)|) & \leq V(t, x) + R \text{ for all } (t, x) \in \mathfrak{R}^+ \\ & \times C^0([-r, 0]; \mathfrak{R}^n) \end{aligned} \quad (\text{A.33})$$

Consider a solution of (2.1) under hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$. By virtue of Lemma A.2, for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \rightarrow v(t, T_r(t)x)$ is absolutely continuous. It follows from (4.2) and Lemma A.1 that for every $T \in (t_0, t_{\max})$ it holds that $\frac{d}{dt}(V(t, T_r(t)x)) \leq -\gamma(t)\rho(V(t, T_r(t)x))$ a.e. on $[t_0, T]$. The previous differential inequality in conjunction with the comparison lemma in [22] and Lemma 4.4 in [26] shows that there exists $\sigma \in KL$ such that

$$\begin{aligned} V(t, T_r(t)x) & \leq \sigma \left(V(t_0, x_0), \int_{t_0}^t \gamma(s) ds \right), \\ & \text{for all } t \in [t_0, T] \end{aligned} \quad (\text{A.34})$$

Combining (A.33), (A.34) and (3.2) we obtain:

$$\begin{aligned} & |x(t)| \\ & \leq \frac{1}{\mu(t)} a^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r), \int_{t_0}^t \gamma(s) ds \right) + R \right), \\ & \text{for all } t \in [t_0, T] \end{aligned} \quad (\text{A.35})$$

Estimate (A.35) shows that $t_{\max} = +\infty$ and consequently estimates (A.34), (A.35) hold for all $t \geq t_0$.

It follows from Lemma A.3 that the solution $x(t)$ of (2.1) under hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ satisfies (A.34) and (A.35) for all $t \geq t_0$. Therefore system (2.1) is RFC and estimate (3.1) is a direct consequence of (A.34) and (4.1) (or (4.6)), as in the previous case.

Proof of (e) \Rightarrow (b):

Let arbitrary $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$ and $d \in M_D$ and consider the solution $x(t)$ of (2.1) with initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ corresponding to $d \in M_D$ and defined on $[t_0 - r, +\infty)$. Setting $x(t) := x(t_0 - r)$ for $t \in [t_0 - r - \tau, t_0 - r]$, we may assume that for each time $t \in [t_0, +\infty)$ the unique solution of (2.1) belongs to $C^0([t_0 - r - \tau, t]; \mathfrak{R}^n)$. Moreover, we have $\|T_{r+\tau}(t_0)x\|_{r+\tau} = \|T_r(t_0)x\|_r = \|x_0\|_r$. Since (2.1) is Robustly Forward Complete (RFC), by virtue of Lemma 3.5 in [15], there exist functions $\mu \in K^+$, $a \in K_\infty$, such that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$, estimate (A.8) holds. Without loss of generality we may assume that $\mu \in K^+$ is non-decreasing, so that the following estimate holds:

$$\|T_{r+\tau}(t)x\|_{r+\tau} \leq \mu(t)a(\|x_0\|_r), \forall t \geq t_0 \quad (\text{A.36})$$

Let $V(t) := V(t, T_{r+\tau}(t)x)$, which is a lower semi-continuous function on $[t_0, +\infty)$. Notice that, by virtue of Lemma A.1, we obtain:

$$\begin{aligned} D^+V(t) & \leq V^0(t, T_{r+\tau}(t)x; f(t, T_r(0)T_{r+\tau}(t)x), \\ & d(t)), \text{ for all } t \geq t_0 \end{aligned} \quad (\text{A.37})$$

where $D^+V(t) := \limsup_{h \rightarrow 0^+} \frac{V(t+h, T_r(t+h)x) - V(t, T_r(t)x)}{h}$.

It follows from definition (4.5) that:

$$\text{If } t \geq t_0 + \tau \text{ then } T_{r+\tau}(t)x \in S(t) \quad (\text{A.38})$$

Inequality (A.37) in conjunction with (A.38) and inequality (4.4) gives:

$$\begin{aligned} D^+V(t) & \leq -\gamma(t)\rho(V(t)) + \gamma(t)\mu \left(\int_0^t \gamma(s) ds \right), \\ & \text{for all } t \geq t_0 + \tau \end{aligned} \quad (\text{A.39})$$

Lemma 2.8 in [18], in conjunction with (A.39) and Lemma 5.2 in [13] imply that there exist a function

$\sigma(\cdot) \in KL$ and a constant $R > 0$ such that the following inequality is satisfied:

$$V(t) \leq \sigma \left(V(t_0 + \tau) + R, \int_{t_0 + \tau}^t \gamma(s) ds \right), \quad \forall t \geq t_0 + \tau \tag{A.40}$$

It follows from the right hand-side inequality (4.3), (A.36) and (A.40) that the following estimate holds:

$$V(t) \leq \sigma \left(a_2(\beta(t_0 + \tau)\mu(t_0 + \tau)a(\|x_0\|_r)) + R, \int_{t_0 + \tau}^t \gamma(s) ds \right), \quad \forall t \geq t_0 + \tau \tag{A.41}$$

Next, we distinguish the following cases:

- 1) if (4.3) holds, then (A.41) in conjunction with (4.3) and Lemma 3.3 in [15] shows that (2.1) is non-uniformly RGAOS with disturbances $d \in \tilde{M}_D$.
- 2) If (4.7) holds, then (A.41) implies the following estimate:

$$|h(t, x(t))| \leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0 + \tau)\mu(t_0 + \tau)a(\|x_0\|_r)) + R, \int_{t_0 + \tau}^t \gamma(s) ds \right) \right), \quad \forall t \geq t_0 + \tau$$

and consequently

$$\sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| \leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0 + \tau)\mu(t_0 + \tau)a(\|x_0\|_r)) + R, \int_{t_0 + \tau}^{t-r} \gamma(s) ds \right) \right), \quad \forall t \geq t_0 + \tau + r$$

The above estimate, in conjunction with the fact that $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow Y$ is equivalent to the finite-dimensional mapping h and Lemma 3.3 in [15] shows that (2.1) is non-uniformly RGAOS with disturbances $d \in \tilde{M}_D$. The proof is complete. \triangleleft

Proof of Theorem 4.2: Implications (a) \Rightarrow (b), (d) \Rightarrow (c), (c) \Rightarrow (e) are obvious. Thus we are left with the proof of implications (b) \Rightarrow (d), (c) \Rightarrow (a) and (e) \Rightarrow (b). The proof of implication of (b) \Rightarrow (d) is exactly the same with that of Theorem 4.1 for the special case of the constant function $\beta(t) \equiv 1$. Moreover, the fact that V is T -periodic (or time-independent) if (2.1) is T -periodic (or autonomous) can be shown in the same way as in [18]. The proof of implication (c) \Rightarrow (a) is exactly the same with the proof of implication (c) \Rightarrow (a) of Theorem 3.5 with the only difference that since

$h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ is continuous and T -periodic with $h(t, 0) = 0$ for all $t \geq -r$, it follows from Lemma 3.2 in [15] implies that there exist a function $\zeta \in K_\infty$ such that:

$$|h(t - r, x)| \leq \zeta(|x|), \quad \forall (t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$$

Finally, the proof of implication (e) \Rightarrow (b) follows the same arguments as the proof of implication (e) \Rightarrow (b) of Theorem 4.1, with the difference that, by virtue of inequalities (4.11a,b), the function $V(t) := V(t, T_{r+\tau}(t)x)$ satisfies the following differential inequalities:

$$D^+ V(t) \leq \beta V(t), \quad \text{for all } t \geq t_0 \tag{A.42}$$

$$D^+ V(t) \leq -\rho(V(t)), \quad \text{for all } t \geq t_0 + \tau \tag{A.43}$$

Thus the comparison lemma in [22], Lemma 4.4 in [26] in conjunction with (A.37) shows that there exists $\sigma \in KL$ such that the following inequality is satisfied:

$$V(t) \leq \sigma(V(t_0 + \tau), t - t_0 - \tau), \quad \forall t \geq t_0 + \tau \tag{A.44}$$

Moreover, differential inequality (A.42) implies $V(t) \leq \exp(\beta(t - t_0))V(t_0)$ for all $t \geq t_0$. Combining the previous estimate with (A.44) we obtain:

$$V(t) \leq \omega(V(t_0), t - t_0), \quad \forall t \geq t_0 \tag{A.45}$$

where $\omega(s, t) := \max\{\exp(\beta\tau)s, \sigma(s, 0)\}$ for $t \in [0, r)$ and $\omega(s, t) := \max\{\exp(r - t)\exp(\beta\tau)s, \sigma(s, t - r)\}$ for $t \geq r$. From this point the proof can be continued in exactly the same way as in the proof of Theorem 4.1. The proof is complete. \triangleleft

Finally, we provide the proof of Lemma 3.5.

Proof of Lemma 3.5: The proof is based on the following observation: if (2.1) is T -periodic then for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times M_D$ it holds that $\phi(t, t_0, x_0; d) = \phi(t - kT, t_0 - kT, x_0; P_{kT}d)$ and $H(t, \phi(t, t_0, x_0; d)) = H(t - kT, \phi(t - kT, t_0 - kT, x_0; P_{kT}d))$, where $k := \lfloor t_0/T \rfloor$ denotes the integer part of t_0/T and

$(P_{kT}d)(t) = d(t+kT)$ for all $t+kT \geq 0$. Notice that if $d \in M_D$ then $P_{kT}d \in M_D$ and if $d \in \tilde{M}_D$ then $P_{kT}d \in \tilde{M}_D$.

Since (2.1) is non-uniformly in time RGAOS, there exist functions $\sigma \in KL$, $\beta \in K^+$ such that (3.1) holds for all $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geq t_0$. Consequently, it follows that the following estimate holds for all $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geq t_0$:

$$\begin{aligned} & \|H(t, \phi(t, t_0, x_0; d))\|_{\mathcal{Y}} \\ & \leq \sigma\left(\beta\left(t_0 - \left\lfloor \frac{t_0}{T} \right\rfloor T\right)\|x_0\|_r, t - t_0\right) \end{aligned}$$

Since $0 \leq t_0 - \left\lfloor \frac{t_0}{T} \right\rfloor T < T$, for all $t_0 \geq 0$, it follows that the following estimate holds for all $(t_0, x_0) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$, $d \in M_D$ (or $d \in \tilde{M}_D$) and $t \geq t_0$:

$$\|H(t, \phi(t, t_0, x_0; d))\|_{\mathcal{Y}} \leq \tilde{\sigma}(\|x_0\|_r, t - t_0)$$

where $\tilde{\sigma}(s, t) := \sigma(Rs, t)$ and $R := \max\{\beta(t); 0 \leq t \leq T\}$. The previous estimate in conjunction with Definition 3.4 implies that (2.1) is URGAOS. The proof is complete. \triangleleft