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# Nonuniform in time state estimation of dynamic systems

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**Abstract**

In this paper it is shown that, if a time-varying uncertain system is robustly completely detectable, then there exists an estimator for this system, i.e. the state vector of the system can be estimated asymptotically. If the time-varying uncertain system is robustly completely observable, it is shown that there exists an estimator for this system with assignable rate of convergence of the error. Moreover, specialized constructions are developed for the special class of triangular systems.
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*Keywords:* Observability; Detectability; Time-varying feedback; Time-varying systems

## 1. Introduction

One of the biggest challenges of Mathematical Control Theory has been the problem of constructing state observers for nonlinear systems. This problem has attracted a lot of attention in the literature in the past decades (see, for example, [1–6,8–10,15,17,19–25,29,30,32]).

It should be noticed that the problem of design of observers for nonlinear systems has been approached from different research directions. Tsinias in [29,30] provided sufficient conditions for the design of nonlinear global time-invariant observers based on Lyapunov-like characterizations of observability and detectability, which can be verified easily for a special class of nonlinear systems. The works of Gauthier, Kupka and others [8–10] have provided semi-global solutions to the observer problem for systems with analytical dynamics based on a high-gain strategy. The case of observable systems with analytic dynamics and the solvability of a series solution methodology to the observer problem with assignable exponential rate of convergence in transformed coordinates has been considered initially in [17] and later in [20,21]. A transformed coordinates approach for a limited class of systems with smooth dynamics has provided local solutions to the observer problem in [19,25]. On the other hand, a global solution to the observer problem is provided in [1] under the assumptions of Output-to-State Stability and Global Complete Observability. Observers with delays have been considered in [6] for special classes of nonlinear systems and time-varying observers for linear time-varying systems, which guarantee nonuniform in time convergence, have been considered in [32].

The present paper provides new results regarding the nonlinear state estimation problem, referring to a broad class of systems (time-varying uncertain nonlinear systems, which of course include autonomous systems as a special case), under minimal regularity conditions (local Lipschitz continuity for the dynamics and simple continuity for the output map) and easily verifiable observability assumptions, leading to global solutions to the observer problem with assignable rate of convergence of the error. A preliminary version of this paper concerning systems without uncertainties was given in [16].

To fix the ideas, consider a time-varying nonlinear forward complete system of the form

$$\dot{x}(t) = f(t, x(t)); \quad y(t) = h(t, x(t)) \\ x \in \Re^n,\ t \geq 0, y \in \Re. \tag{1.1}$$

The goal is to construct a state observer for (1.1), which will be a dynamic system of the form

$$\dot{z}(t) = k(t, z(t), y(t)); \quad \bar{x}(t) = \Psi(t, z(t), y(t)) \\ z \in \Re^m,\ t \geq 0,\ \bar{x} \in \Re^n \tag{1.2}$$

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where $k : \Re^+ \times \Re^m \times \Re \to \Re^m$, $\Psi : \Re^+ \times \Re^m \times \Re \to \Re^n$ are time-varying mappings. System (1.2) will be a state observer for (1.1), if the series connection of (1.1) followed by (1.2) satisfies the following two properties:

(a) the **Global Convergence Property,** i.e. for every set of initial conditions $(x(t_0), z(t_0)) = (x_0, z_0) \in \Re^n \times \Re^m$, the solution satisfies $\lim_{t\to+\infty} \phi(t)\,|x(t) - \bar{x}(t)| = 0$, $\forall (t_0, x_0, z_0) \in \Re^+ \times \Re^n \times \Re^m$ for certain smooth function $\phi : \Re^+ \to [1, +\infty)$,
(b) the **Consistent Initialization Property**, i.e. for every $(t_0, x_0) \in \Re^+ \times \Re^n$ there exists $z_0 \in \Re^m$ such that the solution $(x(\cdot), z(\cdot))$ of (1.1) with (1.2), initiated from $(x_0, z_0) \in \Re^n \times \Re^m$ at time $t_0 \geq 0$, satisfies $x(t) = \bar{x}(t)$, for all $t \geq t_0$.

Note that, when the global convergence property is satisfied but the consistent initialization property is not, system (1.2) cannot be called an observer, but it will still be capable of asymptotically estimating the states. To describe this situation, the term "estimator" will be used instead of "observer". Particularly, it is known that dynamic output stabilization methods are based on estimates of the state vector of the system and for such purposes the property of consistent initialization is not essential (see [7,28]). Also, note that the function $\phi(t)$ governs the rate of decay of the estimation error and it is desirable to be able to assign it in the construction of an observer or estimator. In what follows, the terms "$\phi$-observer" and "$\phi$-estimator" explicitly indicate the corresponding $\phi$-function that governs the rate of decay of the error (the idea of assigning the rate of decay was exploited in [12,31] for feedback stabilization purposes).

A special class of systems that will receive special attention is the class of triangular time-varying nonlinear forward complete systems of the form

$$\begin{aligned}
&\dot{x}_i(t) = a(t, x_1(t))x_{i+1}(t) + \varphi_i(t, x_1(t)), \quad i = 1, \ldots, n-1\\
&\dot{x}_n(t) = f(t, x(t))\\
&y(t) = x_1(t)\\
&x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n,\ t \geq 0
\end{aligned} \tag{1.3}$$

where $a : \Re^+ \times \Re \to \Re$, $f : \Re^+ \times \Re^n \to \Re$, $\varphi_i : \Re^+ \times \Re \to \Re\ (i = 1, \ldots n-1)$ are locally Lipschitz mappings with $f(t, 0) = \varphi_1(t, 0) = \cdots = \varphi_{n-1}(t, 0) = 0$ for all $t \geq 0$. For this class of systems, explicit constructions will be developed leading to an observer with assignable rate of decay of the error Theorem 3.3.

It should be emphasized that the property of consistent initialization cannot in general be satisfied if the original system is uncertain, i.e. its dynamics contain unknown parameters. However, the notion of the estimator is generally applicable even in the presence of uncertainty. The analysis and results to be presented in this paper also cover the case of uncertain forward complete nonlinear time-varying systems of the form

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), d(t)); \quad y(t) = h(t, x(t))\\
&x \in \Re^n, \quad t \geq 0,\ d(t) \in D,\ y \in \Re
\end{aligned} \tag{1.4}$$

where $D \subset \Re^l$ is a compact set. Under appropriate robust complete observability assumptions for (1.4) and for every given non-decreasing function $\phi : \Re^+ \to [1, +\infty)$, the global convergence property can be guaranteed (Theorem 3.1). Specialized results will be obtained for the triangular uncertain case

$$\begin{aligned}
&\dot{x}_i(t) = a(t, x_1(t))x_{i+1}(t) + \varphi_i(t, x_1(t)),\\
&\quad i = 1, \ldots, n-1\\
&\dot{x}_n(t) = f(t, x(t), d(t))\\
&y(t) = x_1(t)\\
&x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n, \quad t \geq 0,\ d(t) \in D
\end{aligned} \tag{1.5}$$

where $D \subset \Re^l$ is a compact set, $a : \Re^+ \times \Re \to \Re$, $f : \Re^+ \times \Re^n \times D \to \Re$, $\varphi_i : \Re^+ \times \Re \to \Re\ (i = 1, \ldots n-1)$ are locally Lipschitz mappings with $f(t, 0, d) = \varphi_1(t, 0) = \cdots = \varphi_{n-1}(t, 0) = 0$ for all $t \geq 0$ and $d \in D$ (Corollary 3.2).

It must be emphasized that the notions of complete detectability and complete observability (Robust Complete Observability/Detectability) that will be used in this work generalize the corresponding notion of Uniform Complete Observability presented in [28] for autonomous systems, as well as similar notions given in [10]. In particular, for disturbance-free systems with analytical output maps and dynamics, the notion of Robust Complete Observability used in the present work coincides with the notion of Uniform Complete/Infinitesimal Observability of [10], for which appropriate test conditions are available.

### 1.1. Notation

* By $M_D$ we denote the set of all measurable functions from $\Re^+$ to $D$, where $D \subset \Re^m$ is a given compact set.
* By $C^j(A)(C^j(A\ ;\ \Omega))$, where $j \geq 0$ is a non-negative integer, we denote the class of functions (taking values in $\Omega$) that have continuous derivatives of order $j$ on $A$. $\mathcal{L}^\infty(A\ ;\ B)(\mathcal{L}^\infty_{\text{loc}}(A\ ;\ B))$ denotes the set of all measurable functions $u : A \to B$ that are (locally) essentially bounded on $A$.
* For $x \in \Re^n$, $x'$ denotes its transpose and $|x|$ its usual Euclidean norm.
* By $B[x, r]$ where $x \in \Re^n$ and $r \geq 0$, we denote the closed sphere in $\Re^n$ of radius $r$, centered at $x \in \Re^n$.
* $x(t) = x(t, t_0, x_0; d)$ denotes the unique solution of (1.4) at time $t \geq t_0$ that corresponds to some input $d(\cdot) \in M_D$, initiated from $x_0 \in \Re^n$ at time $t_0 \geq 0$.
* For the definition of the classes $K$, $K_\infty$, see [18]. By $KL$ we denote the set of all continuous functions $\sigma = \sigma(s, t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class $K$; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is nonincreasing with $\lim_{t\to+\infty} \sigma(s, t) = 0$.
* The saturation function is defined on $\Re$ as $\text{sat}(x) := \begin{cases} x & \text{if } |x| < 1 \\ x/|x| & \text{if } |x| \geq 1. \end{cases}$

## 2. Basic notions

In this section we provide definitions that play a key role in the proofs of the main results of the paper.

**Definition 2.1.** We denote by $K^+$ the class of $C^0$ functions $\phi : \Re^+ \to [1, +\infty)$ and we denote by $K^* \subset K^+$ the class of nondecreasing $C^\infty$ functions, which belong to $K^+$ and satisfy

$$\lim_{t\to+\infty} \dot{\phi}(t)\phi^{-2}(t) = 0. \quad (2.1)$$

For example the functions $\phi(t) = 1$, $\phi(t) = 1 + t$, $\phi(t) = \exp(t)$ all belong to the class $K^*$. The proof of Lemma 2.2 in [12] actually shows an important property for this class of functions: for every function $\phi$ of class $K^+$, there exists a function $\tilde{\phi}$ of class $K^*$, such that: $\phi(t) \le \tilde{\phi}(t)$ for all $t \ge 0$. Lemma 2.2 in [12] is stated for smooth nondecreasing functions but only continuity of $\phi$ is utilized in the proof of the lemma and the assumption that $\phi$ is nondecreasing is not needed (since we can always replace $\phi$ by the nondecreasing continuous function $\bar{\phi}(t) := \max_{0\le\tau\le t} \phi(\tau)$). We next give the notion of Robust Forward Completeness, which was introduced in [14] for uncertain dynamic systems. Consider the system (1.4), where $D \subset \Re^l$ is a compact set and the mappings $f : \Re^+ \times \Re^n \times D \to \Re^n$, $h : \Re^+ \times \Re^n \to \Re$ with $f(t, 0, d) = 0$, $h(t, 0) = 0$ for all $(t, d) \in \Re^+ \times D$, satisfy the following conditions:

(1) The functions $f(t, x, d)$, $h(t, x)$ are continuous.
(2) The function $f(t, x, d)$ is locally Lipschitz with respect to $x$, uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every compact subset $S$ of $\Re^n$, there exists a constant $L \ge 0$ such that

$$|f(t, x, d) - f(t, y, d)| \le L\,|x - y| \quad \forall t \in I,\ \forall(x;\ y) \in S \times S,\ \forall d \in D.$$

Let us denote by $x(t) = x(t, t_0, x_0; d)$ the unique solution of (1.4) at time $t$ that corresponds to input $d \in M_D$, with initial condition $x(t_0) = x_0$ and let $y(t) := h(t, x(t, t_0, x_0; d))$.

**Definition 2.2.** We say that (1.4) is **Robustly Forward Complete (RFC)** if for every $T \ge 0$, $r \ge 0$ it holds that

$$\sup\{|x(t_0 + s)|\,;\,|x_0| \le r, t_0 \in [0, T], s \in [0, T], d(\cdot) \in M_D\} < +\infty.$$

The following proposition clarifies the consequences of the notion of Robust Forward Completeness and provides estimates of the solutions. Its proof can be found in [14].

**Proposition 2.3** (*Lemma 2.3 in [14]*). *Consider system (1.4) with $d \in D$ as input. System (1.4) is RFC if and only if there exist functions $\mu \in K^+$, $a \in K_\infty$ such that for every input $d(\cdot) \in M_D$ and for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.4) corresponding to $d(\cdot)$ and initiated from $x_0$ at time $t_0$ exists for all $t \ge t_0$ and satisfies*

$$|x(t)| \le \mu(t)\, a\,(|x_0|)\,, \quad \forall t \ge t_0. \quad (2.2)$$

The notions of Robust Complete Observability (RCO) and Robust Complete Detectability (RCD) for time-varying systems are given next. The definitions given here directly extend the corresponding notions given in [28], concerning autonomous systems, as well as similar notions given in [10] for autonomous systems with analytic dynamics.

**Definition 2.4.** Consider system (1.4) with $h \in C^1(\Re^+ \times \Re^n\ ;\ \Re)$ and $h(t, 0) = 0$ for all $t \ge 0$. Suppose that (1.4) is RFC. Let $a : \Re^+ \times \Re \to \Re$ be a locally Lipschitz function with

$$\inf\left\{ a(t, y)\ ;\ (t, y) \in \Re^+ \times \Re \right\} > 0. \quad (2.3)$$

Let $m \ge 0$ be an integer and let $\varphi_i : \Re^+ \times \Re \to \Re$ ($i = 1, \ldots, m$) with $\varphi_i(t, 0) = 0$ for all $t \ge 0$ be locally Lipschitz functions with the property that the family of functions, defined recursively below:

$$y_0(t, x) = h(t, x) \quad (2.4a)$$

$$y_i(t, x) := \frac{1}{a(t, h(t, x))} \times \left\{ \frac{\partial y_{i-1}}{\partial t}(t, x) + \frac{\partial y_{i-1}}{\partial x}(t, x) f(t, x, d) - \varphi_i(t, h(t, x)) \right\},$$
$$i = 1, \ldots, m \quad (2.4b)$$

are all independent of $d \in D$ and of class $C^1(\Re^+ \times \Re^n\ ;\ \Re)$. We denote by $Y(t, x)$ the following mapping:

$$Y(t, x) := (y_1(t, x), \ldots, y_m(t, x)). \quad (2.5)$$

We say that a function $\theta \in C^0(\Re^+ \times \Re^n\ ;\ \Re^l)$ with $\theta(\cdot, 0) = 0$ is **Robustly Completely Observable (RCO) with respect to** (1.4) if there exists a function $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\ ;\ \Re^l)$ with $\Psi(t, 0, 0) = 0$ for all $t \ge 0$ such that

$$\theta(t, x) = \Psi(t, h(t, x), Y(t, x)), \quad \forall (t, x) \in \Re^+ \times \Re^n. \quad (2.6)$$

We say that **system** (1.4) **is RCO** if the function $\theta(t, x) = x$ is RCO with respect to (1.4).

We say that a function $\theta \in C^0(\Re^+ \times \Re^n\ ;\ \Re^l)$ with $\theta(\cdot, 0) = 0$ is **Robustly Completely Detectable (RCD) with respect to** (1.4) if there exists a function $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\ ;\ \Re^l)$ with $\Psi(t, 0, 0) = 0$ for all $t \ge 0$, functions $\sigma \in KL$, $\beta \in K^+$ such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$ and $d(\cdot) \in M_D$ the solution $x(t)$ of (1.4) with initial condition $x(t_0) = x_0$ and corresponding to $d(\cdot) \in M_D$ satisfies

$$|\theta(t, x(t)) - \Psi(t, h(t, x(t)), Y(t, x(t)))| \le \sigma\left(\beta(t_0)\,|x_0|\,,\ t - t_0\right), \quad \forall t \ge t_0. \quad (2.7)$$

We say that system (1.4) is RCD if the function $\theta(t, x) = x$ is RCD with respect to (1.4).

**Remark 2.5.** (a) If system (1.4) is RCO, then every function $\theta \in C^0(\Re^+ \times \Re^n\ ;\ \Re^l)$ with $\theta(\cdot, 0) = 0$ is RCO with respect to (1.4).
(b) For a linear system $\dot{x} = A(t)x$, $x \in \Re^n$, $y = h(t)x$, where the matrices $A(t) \in \Re^{n\times n}$ and $h(t) \in \Re^{1\times n}$ have real analytic entries, RCO is equivalent to observability (see pages 279–280 in [26]).

(c) Notice that by virtue of definition (2.4a) and (2.4b), for every input $d(\cdot) \in M_D$ and for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.4) corresponding to $d(\cdot)$ and initiated from $x_0$ at time $t_0$, satisfies the following relations:

$$\dot{y}_i(t) = a(t, y(t))y_{i+1}(t) + \varphi_{i+1}(t, y(t)), \quad \forall t \geq t_0,\ i = 0, \ldots, m-1$$

where $y_i(t) := y_i(t, x(t))$ $(i = 0, \ldots, m)$ and $y(t) = h(t, x(t))$. Thus the functions $\{\varphi_i\}$ $i = 1, \ldots, m$, play the role of "output injection", used in the literature for the construction of observers with linear error dynamics (see [19,25] and the references therein).

(d) The problem of establishing sufficient conditions for RCO of a time-varying system is an open problem. However, the study of this problem is beyond the scope of the present work. In the present work our starting point is to assume RCO and the emphasis is placed on the design of an observer/estimator for such a system.

(e) It should be clear that if $a \in C^0(\Re^+ \times \Re\,;\,\Re)$ satisfies (2.3), then systems (1.3) and (1.5) are RCO under the assumption of RFC. Indeed, (2.4)–(2.6) (with $\theta(t, x) = x$) hold with $m = n-1$, $y_i(t, x) := x_{i+1}$ for $i = 1, \ldots, n-1$, $Y(t, x) := (x_2, \ldots, x_n) \in \Re^{n-1}$ and $\Psi(t, y, w) := (y, w)$ for all $(t, y, w) \in \Re^+ \times \Re \times \Re^{n-1}$.

The following examples show that the notions of RCO and RCD allow us to consider uncertain systems with unobservable linearization.

**Example 2.6.** In this example we show that the single-output system

$$\begin{aligned} &\dot{x}_1 = x_1 + x_2^3; \quad \dot{x}_2 = -x_1 x_2^2 + d(t)x_2 \\ &y = x_1, \quad x = (x_1, x_2)' \in \Re^2, \quad d(\cdot) \in M_{[-1,1]} \end{aligned} \tag{2.8}$$

is RCO. First, we show that system (2.8) is RFC. Notice that for every $(t_0, x_0) \in \Re^+ \times \Re^2$ and $d(\cdot) \in M_{[-1,1]}$, the solution $x(t) = (x_1(t), x_2(t))$ of system (2.8) corresponding to $d(\cdot) \in M_{[-1,1]}$ with initial condition $(x_1(t_0), x_2(t_0)) = x_0$, satisfies the estimate

$$|x(t)| \leq \exp(t)|x_0|, \quad \forall t \geq t_0 \tag{2.9}$$

and consequently system (2.8) is RFC. Inequality (2.9) follows from the evaluation of the time derivative of the function $V(x_1, x_2) = x_1^2 + x_2^2$ along the trajectories of (2.8). Specifically, we obtain $\dot{V} \leq 2V$ and inequality (2.9) is an immediate consequence. Moreover, notice that (2.4)–(2.6) (with $\theta(t, x) = x$) hold with:

$$\begin{aligned} &m = 1,\ \Psi(t, y, w) := \begin{pmatrix} y \\ \operatorname{sgn}(w)\ |w|^{\frac{1}{3}} \end{pmatrix}, \\ &Y(t, x) = y_1(t, x) := x_2^3, \\ &\varphi_1(t, y) = y, \quad a(t, y) \equiv 1. \end{aligned} \tag{2.10}$$

Hence, it follows that system (2.8) is RCO. ◁

**Example 2.7.** In this example we show that the function $\theta(t, x) := x_1 + x_2 + x_3$ is RCD with respect to the following the single-output system

$$\begin{aligned} &\dot{x}_1 = x_1 + x_2^3; \quad \dot{x}_2 = -x_1 x_2^2 + d(t)x_2; \\ &\dot{x}_3 = -(1 + |d(t)|)\,x_3 \\ &y = x_1; \quad (x_1, x_2, x_3) \in \Re^3, \quad d(\cdot) \in M_{[-1,1]} \end{aligned} \tag{2.11}$$

is RCD. First, we show that system (2.8) is RFC. Notice that for every $(t_0, x_0) \in \Re^+ \times \Re^3$ and $d(\cdot) \in M_{[-1,1]}$, the solution $x(t) = (x_1(t), x_2(t), x_3(t))$ of system (2.11) corresponding to $d(\cdot) \in M_{[-1,1]}$ with initial condition $(x_1(t_0), x_2(t_0), x_3(t_0)) = x_0$, satisfies the estimate

$$\begin{aligned} &|(x_1(t), x_2(t))| \leq \exp(t)\,|x_0| \quad \text{and} \\ &|x_3(t)| \leq \exp\left(-(t - t_0)\right)|x_3(t_0)|, \quad \forall t \geq t_0 \end{aligned} \tag{2.12a}$$

and consequently system (2.11) is RFC. Inequalities (2.12a) follow from the evaluation of the time derivative of the functions $V_1(x) = x_1^2 + x_2^2$ and $V_2(x) = x_3^2$ along the trajectories of (2.11). Specifically, we obtain $\dot{V}_1 \leq 2V_1$ $\dot{V}_2 \leq -2V_2$ and inequalities (2.12a) are immediate consequences. Moreover, notice that (2.4), (2.5) and (2.7) with $\theta(t, x) := x_1 + x_2 + x_3$ hold for the following selections:

$$\begin{aligned} &m = 1, \quad \Psi(t, y, w) := y + \operatorname{sgn}(w)\ |w|^{\frac{1}{3}}, \\ &Y(t, x) = y_1(t, x) := x_2^3, \quad \varphi_1(t, y) = y, \quad a(t, y) \equiv 1. \end{aligned}$$

Particularly, inequality (2.7) follows from the above definitions and inequalities (2.12a), which imply that for every $(t_0, x_0) \in \Re^+ \times \Re^3$ and $d(\cdot) \in M_{[-1,1]}$, the solution $x(t) = (x_1(t), x_2(t), x_3(t))$ of system (2.11) corresponding to $d(\cdot) \in M_{[-1,1]}$ with initial condition $(x_1(t_0), x_2(t_0), x_3(t_0)) = x_0$, satisfies the estimate:

$$\begin{aligned} &|\theta(t, x(t)) - \Psi(t, x_1(t), Y(t, x(t)))| \\ &= |x_3(t)| \leq \exp\left(-(t - t_0)\right)|x_3(t_0)|, \quad \forall t \geq t_0. \end{aligned} \tag{2.12b}$$

Consequently, inequality (2.7) holds with $\sigma(s, t) := s\ \exp(-t)$ and $\beta(t) \equiv 1$. Hence, it follows that system (2.11) is RCD. ◁

The notions of $\phi$-Estimator and $\phi$-Observer are crucial for the present work. We emphasize that an estimator is not necessarily an observer since it does not necessarily satisfy the consistent initialization property (see [15]).

**Definition 2.8.** Let $\phi \in K^+$ and $\theta \in C^0(\Re^+ \times \Re^n\,;\,\Re^l)$ with $\theta(\cdot, 0) = 0$. Consider system (1.4) and suppose that it is RFC. The system

$$\begin{aligned} &\dot{z}(t) = k(t, z(t), y(t)); \quad \bar{\theta}(t) = \Psi(t, y(t), z(t)) \\ &z \in \Re^m,\ t \geq 0,\ \bar{\theta} \in \Re^l \end{aligned} \tag{2.13}$$

where $k \in C^0(\Re^+ \times \Re^m \times \Re\,;\,\Re^m)$ with $k(t, 0, 0) = 0$, the map $\tilde{k}(t, z, x) := k(t, z, h(t, x))$ is locally Lipschitz with respect to $(x, z) \in \Re^n \times \Re^m$ and $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\,;\,\Re^l)$ with $\Psi(t, 0, 0) = 0$ for all $t \geq 0$, is called a **$\phi$-Estimator** for $\theta$ with respect to (1.4) if system (1.4) with (2.13) is RFC and there exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $(x_0, z_0) \in \Re^n \times \Re^m$, $t_0 \geq 0$, $d(\cdot) \in M_D$, the unique solution $(x(\cdot), z(\cdot))$ of system (1.4) with (2.13) initiated from

$(x_0, z_0) \in \Re^n \times \Re^m$ at time $t_0 \geq 0$ and corresponding to $d(\cdot) \in M_D$, satisfies the following estimate:

$$\phi(t)\left|\bar{\theta}(t) - \theta(t, x(t))\right| \leq \sigma\left(\beta(t_0)\left|(x_0, z_0)\right|, t - t_0\right), \quad \forall t \geq t_0. \tag{2.14}$$

System (2.13) is called a $\phi$-**Estimator** for system (1.4) if $\theta(t, x) := x$. If $\phi(t) \equiv 1$, then (2.13) is simply called an **Estimator** for $\theta$ with respect to (1.4). In any case, the continuous map $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\ ;\ \Re^l)$ is called the reconstruction map of the ($\phi$-) Estimator for $\theta$ with respect to (1.4).

**Definition 2.9.** Let $\phi \in K^+$ and consider system (1.4). Suppose that (1.2) is a $\phi$-estimator for the identity function $\theta(t, x) \equiv x$ with respect to (1.4) and that (1.2) satisfies the **Consistent Initialization Property**, i.e. for every $(t_0, x_0) \in \Re^+ \times \Re^n$ there exists $z_0 \in \Re^m$ such that the solution $(x(\cdot), z(\cdot))$ of system (1.4) with (1.2) initiated from $(x_0, z_0) \in \Re^n \times \Re^m$ at time $t_0 \geq 0$ and corresponding to arbitrary $d(\cdot) \in M_D$, satisfies

$$x(t) = \Psi(t, y(t), z(t)), \quad \forall t \geq t_0. \tag{2.15}$$

Then we say that system (1.2) is a global $\phi$-**Observer** for (1.4), or that the global $\phi$-observer problem for (1.4) is solvable. If $\phi(t) \equiv 1$ then we say that system (1.2) is a global observer for (1.4).

**Remark 2.10.** Necessary and sufficient conditions for the existence of a global observer for (1.4) with identity reconstruction map, i.e. $z \in \Re^n$ and $\Psi(t, y, z) \equiv z$, are given in [15], by exploiting the notion of the Observer Lyapunov Function (OLF).

**Remark 2.11.** If system (2.13) is an estimator for $\theta$ with respect to (1.4) (i.e. the case of uncertain dynamic system) then the following system:

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), d(t)); \quad \dot{z}(t) = k(t, z(t), h(t, x(t))) \\
&Y(t) = \Psi(t, h(t, x(t)), z(t)) - \theta(t, x(t)) \\
&(x, z) \in \Re^n \times \Re^m, \quad t \geq 0,\ Y \in \Re^l, \quad d(\cdot) \in M_D
\end{aligned}$$

is **non-uniformly in time Robustly Globally Asymptotically Output Stable** (RGAOS, see [14]). Moreover, there exists an estimator for $\theta$ with respect to (1.4) if and only if there exists a function $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\ ;\ \Re^l)$ with $\Psi(t, 0, 0) = 0$ for all $t \geq 0$ such that the **Robust Output Feedback Stabilization problem** (ROFS problem, see [15]) with measured output $y = (h(t, x), z)$ and stabilized output $Y = \Psi(t, z, h(t, x)) - \theta(t, x)$ is globally solvable for the system

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), d(t)); \quad \dot{z}(t) = v(t) \\
&y(t) = (h(t, x(t)), z(t)); \\
&\quad Y(t) = \Psi(t, h(t, x(t)), z(t)) - \theta(t, x(t)) \\
&(x, z) \in \Re^n \times \Re^m,\ t \geq 0,\ v \in \Re^m, \\
&\quad Y \in \Re^l,\ d(\cdot) \in M_D,\ y \in \Re \times \Re^m.
\end{aligned}$$

Consequently, by virtue of Proposition 2.6 in [15], if system (2.13) is an estimator for $\theta$ with respect to (1.4) then there exist functions $V \in C^1(\Re^+ \times \Re^n \times \Re^m\ ;\ \Re^+)$, $a_1, a_2 \in K_\infty$, $\beta, \mu \in K^+$, such that the following inequalities hold for all $(t, z, x, d) \in \Re^+ \times \Re^m \times \Re^n \times D$:

$$\begin{aligned}
&a_1\left(\left|\Psi(t, h(t, x), z) - \theta(t, x)\right|\right) + a_1\left(\mu(t)\left|(z, x)\right|\right) \\
&\quad \leq V(t, z, x) \leq a_2\left(\beta(t)\left|(z, x)\right|\right) \\
&\frac{\partial V}{\partial t}(t, z, x) + \frac{\partial V}{\partial x}(t, z, x) f(t, x, d) \\
&\quad + \frac{\partial V}{\partial z}(t, z, x) k(t, z, h(t, x)) \leq -V(t, z, x).
\end{aligned}$$

## 3. Main results and examples

We are now in a position to state our main results. Our first main result states that there exists an estimator for a RCD function. Moreover, it is possible to assign the convergence rate under the hypothesis of RCO.

**Theorem 3.1.** *If the function $\theta \in C^0(\Re^+ \times \Re^n\ ;\ \Re^l)$ with $\theta(\cdot, 0) = 0$ is RCO with respect to (1.4) then for every $\phi \in K^+$ there exists a $\phi$-estimator for $\theta$ with respect to (1.4). If the function $\theta \in C^0(\Re^+ \times \Re^n\ ;\ \Re^l)$ is RCD then there exists an estimator for $\theta$ with respect to (1.4). Particularly, there exist an integer $m \geq 0$, locally Lipschitz functions $\varphi_i : \Re^+ \times \Re \to \Re$ $(i = 1, \ldots, m)$, $a : \Re^+ \times \Re \to \Re$, functions $\Psi \in C^0(\Re^+ \times \Re \times \Re^m\ ;\ \Re^l)$, $q \in K^*$, a vector $k \in \Re^{m+1}$ and a constant $R > 0$ such that the following system:*

$$\begin{aligned}
&\dot{z}_i = a(t, y) z_{i+1} + \varphi_{i+1}(t, y) + v_i \quad i = 0, \ldots, m-1 \\
&\dot{z}_m = v_m \\
&\bar{\theta} := \Psi(t, y, z_1, \ldots, z_m); \quad z := (z_0, \ldots, z_m) \in \Re^{m+1}, \\
&v := (v_0, \ldots, v_m) \in \Re^{m+1}
\end{aligned} \tag{3.1a}$$

*with*

$$v = a(t, y)\ \mathrm{diag}(Rq(t), R^2 q^2(t), \ldots, R^{m+1} q^{m+1}(t)) k\,(z_0 - y) \tag{3.1b}$$

*is a ($\phi$-) estimator for $\theta$ with respect to (1.4).*

If Theorem 3.1 is specialized to the triangular uncertain case (1.5) (see Remark 2.5(e)) we obtain the following corollary:

**Corollary 3.2.** *Consider system (1.5), where $D \subset \Re^l$ is a compact set, $a : \Re^+ \times \Re \to \Re$, $f : \Re^+ \times \Re^n \times D \to \Re$, $\varphi_i : \Re^+ \times \Re \to \Re$ $(i = 1, \ldots, n-1)$ are locally Lipschitz mappings with $f(t, 0, d) = \varphi_1(t, 0) = \cdots = \varphi_{n-1}(t, 0) = 0$ for all $t \geq 0$ and $d \in D$. Suppose that (2.3) holds and that (1.5) is RFC. Then for every $\phi \in K^+$ there exists a $\phi$-estimator for (1.5). Particularly, for every $\phi \in K^+$, there exist a function $q \in K^*$, a vector $k \in \Re^n$ and a constant $R > 0$ such that the following system:*

$$\begin{aligned}
&\dot{z}_i = a(t, y) z_{i+1} + \varphi_i(t, y) + v_i \quad i = 1, \ldots, n-1 \\
&\dot{z}_n = v_n \\
&\bar{x} := (y, z_2, \ldots, z_n)'; \quad z := (z_1, \ldots, z_n) \in \Re^n, \\
&\quad v := (v_1, \ldots, v_n) \in \Re^n
\end{aligned} \tag{3.2a}$$

*with*

$$v = a(t, y)\,\mathrm{diag}(Rq(t), R^2q^2(t), \ldots, R^nq^n(t))k\,(z_1 - y) \tag{3.2b}$$

*is a $\phi$-estimator for (1.5).*

Our second main result deals with the solvability of the global $\phi$-observer problem for the triangular case (1.3).

**Theorem 3.3.** *Consider system (1.3), where $a : \Re^+ \times \Re \to \Re$, $f : \Re^+ \times \Re^n \to \Re$, $\varphi_i : \Re^+ \times \Re \to \Re$ $(i = 1, \ldots, n-1)$ are locally Lipschitz mappings with $f(t, 0) = \varphi_1(t, 0) = \cdots = \varphi_{n-1}(t, 0) = 0$ for all $t \geq 0$. Suppose that (2.3) holds and that (1.3) is RFC. Then the global $\phi$-observer problem for (1.3) is solvable for all $\phi \in K^+$. Particularly, for every $\phi \in K^+$ there exist functions $q \in K^*$, $\tilde{\beta} \in K^+$, a vector $k \in \Re^n$ and a constant $R > 0$ such that the following system:*

$$\begin{aligned}
\dot{z}_i &= a(t, y)z_{i+1} + \varphi_i(t, y) + v_i \quad i = 1, \ldots, n-1\\
\dot{z}_n &= \tilde{\beta}(t)\,(1 + \exp(t)\,|w|)\,\mathrm{sat}\left(\frac{f(t, y, z_2, \ldots, z_n)}{\tilde{\beta}(t)(1 + \exp(t)\,|w|)}\right) + v_n\\
\dot{w} &= -w\\
\bar{x} &:= (y, z_2, \ldots, z_n)',\ z := (z_1, \ldots, z_n) \in \Re^n,\\
&\quad v := (v_1, \ldots, v_n) \in \Re^n,\ w \in \Re
\end{aligned} \tag{3.3}$$

*with $v$ given by (3.2b) is a global $\phi$-observer for (1.3).*

Although Theorem 3.3 deals with the triangular case (1.3), it should be noted that there is a wide class of systems of the form (1.1) that can be transformed (via an appropriate change of coordinates) to the triangular case (1.3). The reader should notice that the constructed observer (3.3) for system (1.3) is a high-gain type observer (with increasing time-varying gains). High-gain type observers were also considered in [8–10] for autonomous systems with analytic dynamics.

The proofs of the main results depend on the two following facts. Their proofs can be found at the Appendix.

**Fact 1.** *For each pair of $C^0$ functions $a : \Re^+ \to \Re$, $b : \Re^+ \to \Re^+$ that satisfy $\int_0^{+\infty} a(t)\mathrm{d}t = +\infty$, $\lim_{t\to+\infty} \frac{b(t)}{a(t)} = M \geq 0$, $a(t) > 0$ for all $t \geq T$ and for certain $T \geq 0$, there exist constants $K_1, K_2 > 0$ with the following property: if $y : \Re^+ \to \Re^+$ is an absolutely continuous function that satisfies the following differential inequality a.e. for $t \in [t_0, t_1]$:*

$$\dot{y}(t) \leq -a(t)y(t) + b(t) \tag{3.4}$$

*then the following inequality holds:*

$$y(t) \leq K_1 y(t_0) + K_2, \quad \forall t \in [t_0, t_1]. \tag{3.5}$$

**Fact 2.** *Suppose that $\Psi \in C^0(\Re^+ \times \Re^m\ ;\ \Re^n)$. Then there exist functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$ such that for every $(t, x, y) \in \Re^+ \times \Re^m \times \Re^m$, it holds that:*

$$\begin{aligned}
&|\Psi(t, x) - \Psi(t, y)|\\
&\quad \leq a_1\,(\beta(t)\,|x - y|) + a_1\,(a_2(|y|)\ |x - y|)\,.
\end{aligned} \tag{3.6}$$

The following facts will be used for the proof of the main results: Fact 3 is a direct consequence of Lemma 3.3 and Lemma 3.5 in [14], Fact 4 is a direct consequence of Corollary 10 and Remark 11 in [27] and Fact 5 is a direct consequence of Lemma 3.2 in [13] (see also Lemma 2.3 in [11]).

**Fact 3** (*Lemma 3.3, Lemma 3.5 in [14]*). *Consider system (1.4), where $h : \Re^+ \times \Re^n \to \Re^k$ is a $C^0$ function with $h(t, 0) = 0$ for all $t \geq 0$. Suppose that (1.4) is RFC. Moreover, suppose that for every $\varepsilon > 0, T \geq 0$ and $R \geq 0$, there exists a $\tau := \tau\,(\varepsilon, T, R) \geq 0$, such that*

$$\begin{aligned}
&|x_0| \leq R,\ t_0 \in [0, T] \Rightarrow |y(t)| \leq \varepsilon, \quad \forall t \geq t_0 + \tau,\\
&\quad \forall d(\cdot) \in M_D \quad \textit{(Robust Global Output Attractivity)}
\end{aligned}$$

*where $x(t) = x(t, t_0, x_0; d)$ denotes the unique solution of (1.4) at time $t$ that corresponds to input $d \in M_D$, with initial condition $x(t_0) = x_0$ and $y(t) := h(t, x(t, t_0, x_0; d))$. Then there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that for every $t_0 \geq 0$ and $x_0 \in \Re^n$ it holds that $|y(t)| \leq \sigma\,(\beta(t_0)\,|x_0|\,,\ t - t_0)$, for all $d(\cdot) \in M_D$ and $t \geq t_0$.*

**Fact 4.** *For every $p \in K_\infty$ there exists a function $\kappa \in K_\infty$ such that $p(r\,s) \leq \kappa(r)\,\kappa(s)$, for all $r, s \geq 0$.*

**Fact 5.** *Let $D \subset \Re^l$ be a compact set and let $f : \Re^+ \times \Re^n \times D \to \Re^m$ be a continuous mapping with $f(t, 0, d) = 0$ for all $(t, d) \in \Re^+ \times D$. There exist functions $\zeta \in K_\infty$ and $\beta \in K^+$ such that $|f(t, x, d)| \leq \zeta\,(\beta(t)\,|x|)$ for all $(t, x, d) \in \Re^+ \times \Re^n \times D$.*

The construction of the observer/estimator for our main results is achieved by means of the following lemma, which deals with the robust stabilization problem of a special class of linear uncertain systems.

**Lemma 3.4.** *Consider the system:*

$$\begin{aligned}
\dot{x}_i &= a(t, \theta)x_{i+1} + v_i \quad i = 1, \ldots, n-1\\
\dot{x}_n &= v_n + u
\end{aligned} \tag{3.7}$$

*where $x = (x_1, \ldots, x_n) \in \Re^n$ is the state, $v = (v_1, \ldots, v_n) \in \Re^n$ is the input, $\theta(t) \in \Theta \subseteq \Re^m$ is the vector of time-varying parameters, $u \in \Re$ and $a \in C^0(\Re^+ \times \Theta\,;\,\Re)$ is a mapping that satisfies $\inf\left\{a(t, \theta)\ ;\ (t, \theta) \in \Re^+ \times \Theta\right\} > 0$.*

*Then for every $\phi \in K^+$ there exist $q \in K^*, \rho \in K^+$, a vector $k \in \Re^n$ and constants $\gamma, R, M > 0$ such that for every $(t_0, \theta, x_0, u) \in \Re^+ \times C^0(\Re^+\,;\ \Theta) \times \Re^n \times L^\infty_{\mathrm{loc}}(\Re^+\,;\,\Re)$ the solution of system (3.7) with*

$$v = a(t, \theta)\,\mathrm{diag}\left(Rq(t), R^2q^2(t), \ldots, R^nq^n(t)\right)k\,x_1 \tag{3.8}$$

*initial condition $x(t_0) = x_0$ and corresponding to inputs $(\theta, u) \in C^0(\Re^+\,;\ \Theta) \times L^\infty_{\mathrm{loc}}(\Re^+\,;\,\Re)$ satisfies the following estimate for all $t \geq t_0$:*

$$\begin{aligned}
&\phi(t)\,|x(t)| \leq \rho(t_0)\exp\left(-\gamma\,(t - t_0)\right)\,|x_0|\\
&\quad + M \sup_{\tau\in[t_0,t]}\left(\frac{|u(\tau)|}{\phi(\tau)}\right).
\end{aligned} \tag{3.9}$$

**Proof.** Without loss of generality we may assume that the given function of our problem $\phi \in K^+$ is continuously differentiable (if $\phi \in K^+$ is not continuously differentiable we may replace it by a continuously differentiable function $\tilde{\phi} \in K^+$which satisfies $\tilde{\phi}(t) \geq \phi(t)$ for all $t \geq 0$). Let $q \in K^*$ a function that satisfies

$$q(t) \geq \left|\dot{\phi}(t)\right| \phi^{-1}(t) + \phi^2(t), \quad \forall t \geq 0 \tag{3.10}$$

where $\phi \in K^+$ is the given function of our problem. Let $A := \left\{ a_{i,j} \ : \ i, j = 1, \ldots, n \right\}$ with $a_{i,j} := 1$ for $i = 1, \ldots, n-1$, $j = i+1$, $a_{i,j} = 0$ for $i = 1, \ldots, n$, $j \neq i+1$ and let $c' = (1, 0, \ldots, 0) \in \Re^n$. There exist a vector $k = (k_1, \ldots, k_n)' \in \Re^n$, constants $\mu, K_1, K_2 > 0$ and a positive definite symmetric matrix $P \in \Re^{n\times n}$, such that

$$P(A + kc') + (A + kc')'P \leq -\mu P \tag{3.11}$$

$$K_1 I \leq P \leq K_2 I \tag{3.12}$$

where $I \in \Re^{n\times n}$ denotes the identity matrix. Let

$$l := \inf\left\{ a(t, \theta) \ ; \ (t, \theta) \in \Re^+ \times \Theta \right\} > 0. \tag{3.13}$$

Define:

$$R := \max\left\{1 \ ; \ \frac{8\sqrt{n}K_2}{\mu K_1 l}\right\}. \tag{3.14}$$

Consider the time-varying transformation

$$R^{i-1}q^{i-1}(t)y_i = \phi(t)x_i, \quad i = 1, \ldots, n. \tag{3.15}$$

Define:

$$\tilde{F}_i(t, y_i) := \frac{\dot{\phi}(t)}{R\phi(t)q(t)} y_i, \quad i = 1, \ldots, n. \tag{3.16}$$

It follows from (3.10), (3.15) and (3.16) that the following inequalities hold for all $t \geq 0$ and $y_i \in \Re$:

$$\left|\tilde{F}_i(t, y_i)\right| \leq \frac{\mu\, l}{8\sqrt{n}} \frac{K_1}{K_2} \left|y_i\right|, \quad i = 2, \ldots, n. \tag{3.17}$$

For every input $(\theta, u) \in C^0(\Re^+ ; \Theta) \times \mathcal{L}^\infty_{\text{loc}}(\Re^+ ; \Re)$ the solution of the closed-loop system (3.7) with (3.8) is described in $y$-coordinates by the following system of differential equations:

$$\begin{aligned}\dot{y} &= a(t, \theta)Rq(t)(A + kc')y + Rq(t)\tilde{F}(t, y)\\ &\quad -\dot{q}(t)q^{-1}(t)B\, y + \phi(t)R^{1-n}q^{1-n}(t)bu\end{aligned} \tag{3.18}$$

where $y = (y_1, \ldots, y_n) \in \Re^n$, $B := \text{diag}\,(0, 1, \ldots, n-1)$, $\tilde{F}(t, y) := \left(\tilde{F}_1(t, y_1)\,,\ \tilde{F}_2(t, y_2)\,,\ \ldots\,,\ \tilde{F}_n(t, y_n)\right)'$ (defined by (3.16) and $b := (0, \ldots, 0, 1)'$.

Let arbitrary $(t, \theta, x_0, u) \in \Re^+ \times C^0(\Re^+ ; \Theta) \times \Re^n \times L^\infty_{\text{loc}}(\Re^+ ; \Re)$ and consider the solution $x(t)$ of the closed-loop system (3.7) with (3.8), initial condition $x(t_0) = x_0$ and corresponding to inputs $(\theta, u) \in C^0(\Re^+ ; \Theta) \times L^\infty_{\text{loc}}(\Re^+ ; \Re)$. Clearly, for the solution $x(t)$ there exists a maximal existence time $t_{\max} > t_0$ such that the solution is defined on $[t_0, t_{\max})$ and cannot be further continued. Define the function $V(t) = y'(t)Py(t)$, where $y(t)$ is defined by the transformation (3.15). By virtue of (3.11)–(3.14), (3.17) and (3.18), it follows that the derivative of $V(t)$ satisfies the following inequality a.e. for $t \in [t_0, t_{\max})$:

$$\begin{aligned}\dot{V}(t) &\leq -\mu\, l\, Rq(t)V(t) + 2Rq(t)K_2 \left|y(t)\right| \left|\tilde{F}(t, y(t))\right|\\ &\quad + 2K_2 \left|y(t)\right|^2 \frac{\dot{q}(t)}{q(t)} \left|B\right|\\ &\quad + 2K_2 \left|y(t)\right| \frac{\phi(t)}{R^{n-1}q^{n-1}(t)} \left|u(t)\right|\\ &\leq -Rq(t) \left(\frac{\mu}{4}l - \frac{2K_2}{K_1 R}(n-1)\frac{\dot{q}(t)}{q^2(t)}\right) V(t)\\ &\quad + \frac{4K_2^2}{\mu\, K_1\, l} Rq(t) \frac{\phi^2(t) \left|u(t)\right|^2}{R^{2n}q^{2n}(t)}.\end{aligned}$$

The above differential inequality in conjunction with (3.12) and inequality $|y| \leq \phi(t)\,|x| \leq R^{n-1}q^{n-1}(t)\,|y|$ (which is a direct implication of definition (3.15), implies that the following estimate for all $t \in [t_0, t_{\max})$:

$$\begin{aligned}\phi^2(t)\left|x(t)\right|^2 &\leq \frac{K_2}{K_1} R^{2(n-1)}q^{2(n-1)}(t) \left(\frac{q(t)}{q(t_0)}\right)^{\frac{2K_2(n-1)}{K_1}}\\ &\quad \times \exp\left(-\frac{\mu Rl}{4}\int_{t_0}^t q(s)\mathrm{d}s\right)\phi^2(t_0)\ \left|x_0\right|^2\\ &\quad + \frac{4K_2^2}{\mu\, K_1^2 l}\int_{t_0}^t \left(\frac{q(t)}{q(\tau)}\right)^{2(n-1)\left(1+\frac{K_2}{K_1}\right)} \exp\left(-\frac{\mu Rl}{4}\int_\tau^t q(s)\mathrm{d}s\right)\\ &\quad \times \frac{\phi^2(\tau)\left|u(\tau)\right|^2}{Rq(\tau)}\mathrm{d}\tau.\end{aligned}$$

The above estimate implies that the solution $x(t)$ of the closed-loop system (3.7) with (3.8), initial condition $x(t_0) = x_0$ and corresponding to inputs $(\theta, u) \in C^0(\Re^+ ; \Theta) \times L^\infty_{\text{loc}}(\Re^+ ; \Re)$ exists for all $t \geq t_0$ (i.e., $t_{\max} = +\infty$). Define

$$I(t) := \frac{16K_2^2}{\mu^2\, K_1^2 R^2 l^2}\int_0^t \left(\frac{q(t)}{q(\tau)}\right)^{2a} \frac{J(\tau)}{J(t)}\frac{\mu\, R\, l}{4}q(\tau)\mathrm{d}\tau \tag{3.19a}$$

$$J(t) := \exp\left(\frac{\mu\, Rl}{4}\int_0^t q(s)\mathrm{d}s\right) \tag{3.19b}$$

$$a := (n-1)\left(1 + \frac{K_2}{K_1}\right). \tag{3.19c}$$

Clearly, the estimate given above for the solution $x(t)$ in conjunction with Eq. (3.19) implies the following estimate for all $t \geq t_0$:

$$\begin{aligned}\phi(t)\left|x(t)\right| &\leq \left(\frac{K_2}{K_1}\right)^{\frac{1}{2}} R^{n-1}q^{n-1}(t_0)\left(\frac{q(t)}{q(t_0)}\right)^a\\ &\quad \times \exp\left(-\frac{\mu\, r}{8}l\int_{t_0}^t q(s)\mathrm{d}s\right)\phi(t_0)\left|x_0\right| + (I(t))^{\frac{1}{2}}\\ &\quad \times \sup_{\tau\in[t_0,t]}\left(\frac{\phi(\tau)\left|u(\tau)\right|}{q(\tau)}\right).\end{aligned} \tag{3.20a}$$

Notice that definition (3.19b) implies the equality $\int_0^t (q(\tau))^{-2a} J(\tau)\frac{\mu\, R\, l}{4}q(\tau)\mathrm{d}\tau = \int_0^t (q(\tau))^{-2a}\,\dot{J}(\tau)\mathrm{d}\tau$. Using the previous equality and integrating by parts the integral in (3.19), we obtain

$$I(t) = \frac{16K_2^2}{\mu^2 K_1^2 R^2 l^2}(1 + 2ag(t)) \tag{3.20b}$$

where

$$g(t) := \int_0^t \left(\frac{q(t)}{q(\tau)}\right)^{2a} \exp\left(-\frac{\mu R}{4} l \int_\tau^t q(s)\mathrm{d}s\right) \frac{\dot{q}(\tau)}{q(\tau)}\mathrm{d}\tau. \tag{3.21}$$

Definition (3.21) implies that $g(t)$ satisfies the following differential equation:

$$\dot{g}(t) = -q(t)\left(\frac{\mu R}{4} l - 2a\dot{q}(t)q^{-2}(t)\right) g(t) + \dot{q}(t)q^{-1}(t). \tag{3.22}$$

Since $q \in K^*$ it follows from (2.1):

$$\lim_{t\to+\infty} \dot{q}(t)q^{-2}(t) = 0. \tag{3.23}$$

Fact 1 in conjunction with (3.23) implies that there exists a constant $G > 0$ such that $g(t) \le G$ for all $t \ge 0$. It follows from estimate (3.20a) and inequalities (3.10) and (3.20b) that there exist a constant $M_2 > 0$ such that

$$\phi(t)|x(t)| \le M_1 q^{n-1}(t_0)\left(\frac{q(t)}{q(t_0)}\right)^a \exp\left(-2\gamma \int_{t_0}^t q(s)\mathrm{d}s\right) \times \phi(t_0)|x_0| + M_2 \sup_{\tau\in[t_0,t]}\left(\frac{|u(\tau)|}{\phi(\tau)}\right) \tag{3.24}$$

for all $t \ge t_0$ with $M_1 := \left(\frac{K_2}{K_1}\right)^{\frac{1}{2}} R^{n-1}$ and $\gamma := \frac{\mu R}{16} l$. Next consider the function

$$L(t) := q^a(t)\exp\left(-\gamma \int_0^t q(s)\mathrm{d}s\right). \tag{3.25}$$

It is clear that $L(t)$ satisfies the differential equation $\dot{L}(t) = -q(t)\left(\gamma - a\frac{\dot{q}(t)}{q^2(t)}\right) L(t)$. By virtue of (3.23) and Fact 1, it follows that there exists a constant $K > 0$ such that $L(t) \le K$ for all $t \ge 0$. Combining estimate (3.24) with definition (3.25) and using the fact that $q(t) \ge 1$ for all $t \ge 0$, we obtain

$$\phi(t)|x(t)| \le M_1 K \phi(t_0)\exp\left(\gamma \int_0^{t_0} q(s)\mathrm{d}s\right) \times \exp(-\gamma(t - t_0))|x_0| + M_2 \sup_{\tau\in[t_0,t]}\left(\frac{|u(\tau)|}{\phi(\tau)}\right). \tag{3.26}$$

It is clear from (3.26) that estimate (3.9) holds with $M := M_2$ and $\rho(t) := M_1 K\phi(t)\exp\left(\gamma \int_0^t q(s)\mathrm{d}s\right)$. ◁

We are now in a position to prove the main results of the present work.

**Proof of Theorem 3.1.** Let $\phi \in K^+$ be given. Clearly, the hypotheses of Theorem 3.1 guarantee the existence of a function $\Psi \in C^0(\Re^+ \times \Re \times \Re^m ; \Re^l)$ with $\Psi(t, 0, 0) = 0$ for all $t \ge 0$ such that

$$\theta(t, x) = \Psi(t, h(t, x), Y(t, x)), \quad \forall (t, x) \in \Re^+ \times \Re^n \tag{3.27}$$

for the case of RCO or there exist functions $\sigma \in KL$, $\beta \in K^+$ such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$ and $d(\cdot) \in M_D$ the solution $x(t)$ of (1.4) with initial condition $x(t_0) = x_0$ and corresponding to $d(\cdot) \in M_D$ satisfies

$$|\theta(t, x(t)) - \Psi(t, h(t, x(t)), Y(t, x(t)))| \le \sigma(\beta(t_0)|x_0|, t - t_0), \quad \forall t \ge t_0 \tag{3.28}$$

for the case of RCD, where in both cases the mappings $Y(t, x)$, $y_i(t, x)$ $(i = 0, \ldots, m)$, $a : \Re^+ \times \Re \to \Re$, $\{\varphi_i\}$, $i = 1, \ldots, m$ are defined by (2.4), (2.5) and satisfy (2.3).

Clearly, definitions (2.4a), (2.4b) and (2.5) guarantee that $Y(t, 0) = 0$ for all $t \ge 0$. Using Fact 5 in conjunction with the continuity of $f, a, h, Y$ and compactness of $D \subset \Re^l$, we obtain functions $p \in K_\infty$ and $\beta \in K^+$ such that

$$|h(t, x)| + |Y(t, x)| + \left|\frac{\partial y_m}{\partial t}(t, x) + \frac{\partial y_m}{\partial x}(t, x) f(t, x, d)\right| \le p(\beta(t)|x|), \quad \forall (t, x, d) \in \Re^+ \times \Re^n \times D. \tag{3.29}$$

Since system (1.4) is RFC, by virtue of Proposition 2.3, there exist functions $\tilde{a} \in K_\infty$ and $\mu \in K^+$ such that for every input $d(\cdot) \in M_D$ and for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.4) corresponding to $d(\cdot)$ and initiated from $x_0$ at time $t_0$ exists for all $t \ge t_0$ and satisfies

$$|x(t)| \le \mu(t)\,\tilde{a}(|x_0|), \quad \forall t \ge t_0. \tag{3.30}$$

Moreover, by virtue of Fact II, there exist functions $a_1, a_2 \in K_\infty$, $\gamma \in K^+$ such that for every $(t, y, z, w) \in \Re^+ \times \Re \times \Re^m \times \Re^m$, it holds that

$$|\Psi(t, y, z) - \Psi(t, y, w)| \le a_1(\gamma(t)|z - w|) + a_1(a_2(|y| + |w|)|z - w|). \tag{3.31}$$

Notice that for every input $d(\cdot) \in M_D$ and for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the components of the mapping $(y_1(t), \ldots, y_m(t)) = Y(t) = Y(t, x(t)) := (y_1(t, x(t)), \ldots, y_m(t, x(t))) \in \Re^m$, where $x(t)$ denotes the unique solution of (1.4) corresponding to $d(\cdot)$ and initiated from $x_0$ at time $t_0$, satisfy the following relations:

$$\dot{y}_i(t) = a(t, y(t))y_{i+1}(t) + \varphi_{i+1}(t, y(t)), \quad \forall t \ge t_0, \quad i = 0, \ldots, m-1 \tag{3.32}$$

where $y_0(t) = y(t) = h(t, x(t))$. By virtue of Fact IV, there exists a function $\kappa \in K_\infty$ such that

$$p(rs) + a_1(rs) + a_2(rs) \le \kappa(r)\kappa(s), \quad \forall r, s \ge 0 \tag{3.33}$$

where $p \in K_\infty$ is the function involved in (3.29) and $a_1, a_2 \in K_\infty$ are the functions involved in (3.31). It follows by (3.29), (3.30), (3.32) and (3.33) that for every input $d(\cdot) \in M_D$ and for every $(t_0, x_0) \in \Re^+ \times \Re^n$, we have

$$|y(t)| + |Y(t, x(t))| \le \tilde{\beta}(t)\kappa(\tilde{a}(|x_0|)), \quad \forall t \ge t_0 \tag{3.34a}$$

$$|\dot{y}_m(t)| \le \tilde{\beta}(t)\kappa(\tilde{a}(|x_0|)), \quad \text{a.e. for } t \ge t_0 \tag{3.34b}$$

where $\tilde{\beta} \in K^+$ is a function that satisfies $\tilde{\beta}(t) \ge \kappa(\beta(t)\mu(t))$ for all $t \ge 0$. Let $\tilde{\phi} \in K^+$ a function that satisfies:

$$\tilde{\phi}(t) \ge \tilde{\beta}(t) + \frac{\gamma(t) + \kappa(\tilde{\beta}(t))}{\kappa^{-1}\left(\frac{\exp(-t)}{\phi(t)}\right)}, \quad \forall t \ge 0 \tag{3.35}$$

where $\phi \in K^+$ is the given function of our problem and $\gamma \in K^+$ is the function involved in (3.31). Let $q \in K^*$, $k \in \Re^{m+1}$, $R > 0$ (to be selected) and consider the solution of (3.1) with initial condition $z(t_0)$. Define $e(t) := (e_1(t), \ldots, e_{m+1}(t))' := z(t) - (y(t), Y(t))'$. It follows from (3.1) and (3.32) that the following differential equations hold:

$$\dot{e}_i(t) = a(t, y(t))e_{i+1}(t) + v_i(t), \quad i = 1, \ldots, m \tag{3.36a}$$

$$\dot{e}_{m+1}(t) = -\dot{y}_m(t) + v_{m+1} \tag{3.36b}$$

where

$$v = a(t, y(t)) \times \text{diag}\left(Rq(t), R^2q^2(t), \ldots, R^{m+1}q^{m+1}(t)\right) k\, e_1. \tag{3.36c}$$

By virtue of (2.3), (3.36) and Lemma 3.4, there exist $q \in K^*$, $\rho \in K^+$, a vector $k \in \Re^{m+1}$ and constants $R, M > 0$ such that for every $(t_0, z(t_0), \dot{y}_m) \in \Re^+ \times \Re^{m+1} \times L^\infty_{\text{loc}}(\Re^+; \Re)$ the solution of system (3.1) with initial condition $z(t_0)$ and corresponding to input $\dot{y}_m \in L^\infty_{\text{loc}}(\Re^+; \Re)$ satisfies the estimate for all $t \geq t_0$:

$$\tilde{\phi}(t)\, |z(t) - (y(t), Y(t, x(t)))| \leq \rho(t_0)\, |z(t_0) - (y(t_0), Y(t_0, x(t_0)))| + M \sup_{\tau\in[t_0,t]} \left(\frac{|\dot{y}_m(\tau)|}{\tilde{\phi}(\tau)}\right). \tag{3.37}$$

Clearly, inequalities (3.29), (3.34b), (3.35) in conjunction with (3.37) imply that:

$$\tilde{\phi}(t)\, |z(t) - (y(t), Y(t, x(t)))| \leq \rho(t_0)\, |z(t_0)| + \rho(t_0) p(\beta(t_0)\, |x(t_0)|) + M\, \kappa\, (\tilde{a}(|x(t_0)|)), \quad \forall t \geq t_0. \tag{3.38}$$

Making use of inequalities (3.31), (3.33), (3.34a), (3.35) and (3.38) we conclude

$$\phi(t)\, |\Psi(t, y(t), z_1(t), \ldots, z_m(t)) - \Psi(t, y(t), Y(t, x(t)))| \leq \exp(-(t - t_0))\omega\, (\lambda(t_0)\, |(x(t_0), z(t_0))|), \quad \forall t \geq t_0 \tag{3.39}$$

for appropriate functions $\omega \in K_\infty$ and $\lambda \in K^+$. Notice that, by virtue of (3.30), (3.34a) and (3.38), system (1.4) with (3.1) is RFC. Inequality (3.39) in conjunction with (3.27) or (3.28), proves that system (3.1) is a ($\phi$-) estimator for $\theta$ with respect to (1.4). The proof is complete. ◁

**Proof of Theorem 3.3.** Let $\phi \in K^+$ be arbitrary. Since system (1.3) is RFC, by virtue of Proposition 2.3, there exist functions $\tilde{a} \in K_\infty$ and $\mu \in K^+$ such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.3) initiated from $x_0$ at time $t_0$ exists for all $t \geq t_0$ and satisfies (3.30). Using Fact 5 in conjunction with the continuity of $f$, we obtain functions $p \in K_\infty$ and $\beta \in K^+$ such that

$$|f(t, x)| \leq p\, (\beta(t)\, |x|), \quad \forall (t, x) \in \Re^+ \times \Re^n. \tag{3.40}$$

By virtue of (3.30) and (3.40) and Fact 4, there exist functions $\kappa \in K_\infty$, $\tilde{\beta} \in K^+$ such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.3) initiated from $x_0$ at time $t_0$ satisfies

$$|f(t, x(t))| \leq \tilde{\beta}(t)\, \kappa\, (|x_0|), \quad \forall t \geq t_0. \tag{3.41}$$

Without loss of generality we assume that $\tilde{\beta} \in K^+$ is non-decreasing. Let $q \in K^*$, $k \in \Re^n$, $R > 0$ (to be selected) and consider the solution $(x(t), z(t), w(t))$of (1.3) and (3.3) with (3.2b) with initial condition $(x(t_0), z(t_0), w(t_0)) = (x_0, z_0, w_0)$ for arbitrary $(t_0, x_0, z_0, w_0) \in \Re^+ \times \Re^n \times \Re^n \times \Re$. Clearly, there exists $t_{\max} \in (t_0, +\infty]$ such that the solution $(x(t), z(t), w(t))$of (1.3) and (3.3) with (3.2b) exists for $t \in [t_0, t_{\max})$ and cannot be further continued. Note that $\exp(t)w(t) = \exp(t_0)w_0$ for all $t \in [t_0, t_{\max})$. The previous equality in conjunction with (3.41) gives

$$|f(t, x(t))| \leq \tilde{\beta}(t)(1 + \exp(t)\, |w(t)|), \quad \forall t \in [t_0, t_{\max}), \text{ provided that } \kappa\, (|x_0|) \leq 1 + \exp(t_0)\, |w_0|\,. \tag{3.42}$$

Define:

$$\tilde{\phi}(t) := \exp(t)\phi(t) + \tilde{\beta}(t) \tag{3.43}$$

and $e(t) := z(t) - x(t)$. It follows from (1.3), (3.3) and (3.2b) that the following differential equations hold:

$$\dot{e}_i(t) = a(t, y(t))e_{i+1}(t) + v_i(t), \quad i = 1, \ldots, n-1 \tag{3.44a}$$

$$\dot{e}_n(t) = \tilde{\beta}(t)\,(1 + \exp(t)\, |w(t)|) \times \text{sat}\left(\frac{f(t, y(t), z_2(t), \ldots, z_n(t))}{\tilde{\beta}(t)(1 + \exp(t)\, |w(t)|)}\right) - f(t, x(t)) + v_n(t) \tag{3.44b}$$

where

$$v = a(t, y)\, \text{diag}\left(Rq(t), R^2q^2(t), \ldots, R^nq^n(t)\right) k\, e_1. \tag{3.44c}$$

By virtue of (2.3) and Lemma 3.4, there exist $q \in K^*$, $\rho \in K^+$, a vector $k \in \Re^n$ and constants $R, M > 0$ such that the following estimate holds for all $t \in [t_0, t_{\max})$:

$$\tilde{\phi}(t)\, |z(t) - x(t)| \leq \rho(t_0)\, |z_0 - x_0| + M \times \sup_{t_0\leq\tau\leq t} \left|\frac{f(\tau, x(\tau))}{\tilde{\phi}(\tau)} - \frac{\tilde{\beta}(\tau)(1 + \exp(\tau)w(\tau))}{\tilde{\phi}(\tau)} \times \text{sat}\left(\frac{f(\tau, y(\tau), z_1(\tau), \ldots, z_m(\tau))}{\tilde{\beta}(\tau)(1 + \exp(\tau)w(\tau))}\right)\right|. \tag{3.45}$$

Estimate (3.45) combined with (3.41), (3.43) and the fact that $\exp(t)w(t) = \exp(t_0)w_0$ for all $t \in [t_0, t_{\max})$, implies that the following estimate holds for all $t \in [t_0, t_{\max})$:

$$\phi(t)\, |z(t) - x(t)| \leq \exp(-t)\, [\rho(t_0)|z_0 - x_0| + M\kappa(\, |x_0|) + M\, (1 + \exp(t_0)|w_0|)]. \tag{3.46}$$

Estimates (3.30) and (3.46), in conjunction with the fact that $\exp(t)w(t) = \exp(t_0)w_0$ for all $t \geq t_0$, show that system (1.3) with (3.3) and (3.2b) is RFC. Consequently, $t_{\max} = +\infty$. Moreover, notice that, if $z_0 = x_0$ and $\kappa\, (|x_0|) \leq |w_0|$, then it follows from (3.42) that $\bar{x}(t) = x(t)$ for all $t \geq t_0$ (Consistent Initialization Property). Indeed, using (3.42), the reader can verify that the solution of system (1.3) with (3.3) and (3.2b) with $z_0 = x_0$ and $\kappa\, (|x_0|) \leq |w_0|$ satisfies $z(t) = x(t)$ for all $t \geq t_0$ (by simply substituting $z(t) = x(t)$ in the differential equations (3.3) with (3.2b)).

It follows from Fact 3, that since:

(i) estimate (3.46) holds (which guarantees Robust Global Output Attractivity for the output $Y := \phi(t)\ (z - x)$),
(ii) system (1.3) with (3.3) and (3.2b) is RFC,
(iii) the point $(x, z, w) = (0, 0, 0)$ is the equilibrium point of (1.3) with (3.3) and (3.2b),
(iv) the right-hand sides of differential (1.3) with (3.3) and (3.2b) are locally Lipschitz,

then there exist functions $\sigma \in KL$ and $b \in K^+$ such that for every $t_0 \geq 0$ and $(x_0, z_0, w_0) \in \Re^n \times \Re^n \times \Re$ the following estimate holds:

$$\phi(t)\ |z(t) - x(t)| \leq \sigma\left(b(t_0)\,|(x_0, z_0, w_0)|\,,\ t - t_0\right), \quad \forall t \geq t_0. \tag{3.47}$$

Estimate (3.47) implies that (3.3), (3.2b) is a $\phi$-estimator for the function $\theta(t, x) \equiv x$ with respect to (1.3). Since (3.3), (3.2b) is a $\phi$-estimator for system (1.3), which satisfies the consistent initialization property and since $\phi \in K^+$ is arbitrary, we conclude that the global $\phi$-observer problem for (1.3) is solvable. ◁

The following illustrating examples show the applicability of Theorem 3.1 to nonlinear uncertain systems.

**Example 3.5.** Consider again the RCO system (2.8), which was studied in Example 2.6. Notice that by virtue of (2.9) we have

$$|y(t)| + |Y(t, x(t))| + |\dot{y}_1(t, x(t))| \leq 3\exp(5t)\left(|x_0| + |x_0|^3 + |x_0|^5\right), \quad \forall t \geq t_0 \tag{3.48}$$

where the mapping $Y(t, x) = y_1(t, x) := x_2^3$ is defined in (2.10). Making use of the inequality $|\mathrm{sgn}(x)|x|^{\frac{1}{3}} - \mathrm{sgn}(y)|y|^{\frac{1}{3}}| \leq 2|x - y|^{\frac{1}{3}}$, which holds for all $x, y \in \Re$, we obtain that:

$$|\Psi(t, y, z) - \Psi(t, y, w)| \leq 2\ |z - w|^{\frac{1}{3}}, \quad \forall (t;\ y, z, w) \in \Re^+ \times \Re^3 \tag{3.49}$$

where $\Psi$ is defined in (2.10). Using Lemma 3.4 for $\phi(t) = \exp(5t)$ we guarantee the existence of a function $\rho \in K^+$ and constants $M, R > 0$ such that for every $(t_0, z_0, x_0, d) \in \Re^+ \times \Re^2 \times \Re^2 \times M_D$, the solution of (2.8) with

$$\begin{aligned} \dot{z}_0 &= y(t) + z_1 - 12R\ \exp(10t)(z_0 - y(t)); \\ \dot{z}_1 &= -72R^2 \exp(20t)(z_0 - y(t)) \end{aligned} \tag{3.50}$$

$$\bar{x} = \Psi(t, y, z_1); \quad z := (z_0, z_1) \in \Re^2,\ t \geq 0$$

and initial condition $(z(t_0), x(t_0)) = (z_0, x_0)$, corresponding to input $d \in M_D$, satisfies the following estimate for all $t \geq t_0$:

$$\begin{aligned} &|z(t) - (y(t), Y(t, x(t)))| \\ &\quad \leq \exp(-5t)(\rho(t_0)|z(t_0) - (y(t_0), Y(t_0, x(t_0)))| \\ &\qquad + M\ (|x(t_0)| + |x(t_0)|^3 + |x(t_0)|^5)). \end{aligned} \tag{3.51}$$

The reader should notice that system (3.50) coincides with system (3.1) for $q(t) = \exp(10t)$, $m = 1$ and $\varphi_1(t, y) = y$. Let $a(s) := 2s^{\frac{1}{3}}$. The following inequality follows from (3.49) and estimate (3.51):

$$\begin{aligned} &|x(t) - \Psi(t, y(t), z_2(t))| \\ &\quad \leq \exp(-(t - t_0))a(\rho(t_0)|z(t_0) - (y(t_0), Y(t_0))| \\ &\qquad + M(|x(t_0)| + |x(t_0)|^3 + |x(t_0)|^5)), \quad \forall t \geq t_0. \end{aligned} \tag{3.52}$$

Estimate (3.52) guarantees that system (3.50) is a $\tilde{\phi}$-estimator for system (2.6) with $\tilde{\phi}(t) = \exp(pt)$, $p \in [0, 1)$, i.e. an estimator for the state of system (2.8) which guarantees exponential convergence. ◁

**Example 3.6.** Consider again system (2.11), for which it was shown in Example 2.7 that the function $\theta(t, x) := x_1 + x_2 + x_3$ is RCD with respect to (2.11). Making use of inequalities (2.12a) and (2.12b) and working as in the previous example it can be shown that there exists a constant $R > 0$ such that the following system:

$$\begin{aligned} \dot{z}_0 &= y(t) + z_1 - 12R\ \exp(10t)(z_0 - y(t)); \\ \dot{z}_1 &= -72R^2 \exp(20t)(z_0 - y(t)) \end{aligned} \tag{3.53}$$

$$\bar{\theta} = y + \mathrm{sgn}(z_1)\ |z_1|^{\frac{1}{3}}; \quad z := (z_0, z_1) \in \Re^2,\ t \geq 0$$

is an estimator for $\theta$ with respect to (2.11), which guarantees exponential convergence. Particularly, there exists a function $\rho \in K^+$ and a constant $M > 0$ such that for every $(t_0, z_0, x_0, d) \in \Re^+ \times \Re^2 \times \Re^3 \times M_D$ the solution of (2.11) with (3.53) and initial condition $(z(t_0), x(t_0)) = (z_0, x_0)$ corresponding to input $d \in M_D$ satisfies the following estimate for all $t \geq t_0$:

$$\begin{aligned} &\left|\theta(t, x(t)) - y(t) - \mathrm{sgn}(z_1(t))\ |z_1(t)|^{\frac{1}{3}}\right| \\ &\quad \leq \exp\left(-(t - t_0)\right)\ |x_3(t_0)| \\ &\qquad + \exp(-(t - t_0))a\left(\rho(t_0)\left|z_1(t_0) - x_2^3(t_0)\right|\right. \\ &\qquad \left. + M\left(|x(t_0)| + |x(t_0)|^3 + |x(t_0)|^5\right)\right) \end{aligned}$$

where $a(s) := 2s^{\frac{1}{3}}$. ◁

## 4. Conclusions

In this paper we have given sufficient conditions for the existence of estimators and the solvability of the global observer problem for dynamic systems. It is showed that if a time-varying uncertain system is RCD then there exists an estimator for this system, i.e. we can estimate asymptotically the state vector of the system. Moreover, if a time-varying uncertain system is RCO then there exists an estimator for this system that guarantees convergence of the estimates with “arbitrary fast” rate of convergence. Finally, the obtained results are specialized to the triangular time-varying case, where it is shown that a time-varying observer can be constructed.

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## Appendix

**Proof of Fact 1.** Clearly, the differential inequality (3.4) implies that:

$$y(t) \le \exp\left(-\int_{t_0}^{t} a(s)\mathrm{d}s\right) y(t_0) + \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} a(s)\mathrm{d}s\right) b(\tau)\mathrm{d}\tau, \quad \forall t \in [t_0, t_1]. \quad \text{(A.1)}$$

Moreover, since $a(t) > 0$ for all $t \ge T$, we obtain for all $t_0 \ge 0$ and $t \ge t_0$:

$$\begin{aligned}\int_{t_0}^{t} a(s)\mathrm{d}s &= \int_{t_0}^{t} |a(s)|\,\mathrm{d}s + 2\int_{t_0}^{t} \min\{0, a(s)\}\mathrm{d}s \\ &\ge 2\int_{t_0}^{t} \min\{0, a(s)\}\mathrm{d}s \ge 2\int_{0}^{t} \min\{0, a(s)\}\mathrm{d}s \\ &\ge 2\int_{0}^{T} \min\{0, a(s)\}\mathrm{d}s.\end{aligned}$$

We define $K_1 := \exp\left(-2\int_0^T \min\{0, a(s)\}\mathrm{d}s\right)$ and the previous inequalities in conjunction with (A.1) give

$$y(t) \le K_1 y(t_0) + \exp\left(-\int_{0}^{t} a(s)\mathrm{d}s\right) \times \int_{0}^{t} \exp\left(\int_{0}^{\tau} a(s)\mathrm{d}s\right) b(\tau)\mathrm{d}\tau, \quad \forall t \in [t_0, t_1]. \quad \text{(A.2)}$$

We define the function $p(t) := \int_0^t \exp\left(\int_0^\tau a(s)\mathrm{d}s\right) b(\tau)\mathrm{d}\tau$. This function is nondecreasing and consequently we either have $p(t) \le K_3$ for some $K_3 > 0$ or $\lim_{t\to+\infty} p(t) = +\infty$. For the first case, inequality (3.5) is implied by (A.2) with $K_2 = K_1 K_3$. For the second case, notice that since $\int_0^{+\infty} a(t)\mathrm{d}t = +\infty$ and $\lim_{t\to+\infty} \frac{b(t)}{a(t)} = M \ge 0$, we can apply L'Hospital's rule for the function $q(t) := p(t)\exp\left(-\int_0^t a(s)\mathrm{d}s\right)$ and obtain that $\lim_{t\to+\infty} q(t) = M$. Thus we may define $K_2 := \sup_{t\ge 0} q(t)$ and (3.5) is implied by inequality (A.2). The proof is complete. ◁

**Proof of Fact 2.** Clearly the function

$$\gamma(r, s) := \sup\{|\Psi(t, x) - \Psi(t, y)| \;;\; |(t, y)| \le r, \; |x - y| \le s\} \quad \text{(A.3)}$$

is continuous, nonnegative and satisfies $\gamma(r, 0) = 0$ for all $r \ge 0$. Consequently, by virtue of Fact 5, there exist functions $\tilde{a} \in K_\infty$, $\tilde{\beta} \in K^+$ being increasing, such that $\gamma(r, s) \le \tilde{a}\left(\tilde{\beta}(r)\, s\right)$ for all $r, s \ge 0$. Definition (A.3), in conjunction with the previous inequality, implies

$$|\Psi(t, x) - \Psi(t, y)| \le \tilde{a}\left(\tilde{\beta}(|(t, y)|)\; |x - y|\right), \quad \forall (t, x, y) \in \Re^+ \times \Re^m \times \Re^m. \quad \text{(A.4)}$$

Since $\tilde{\beta} \in K^+$ is increasing, we have $\tilde{\beta}(|(t, y)|) \le \tilde{\beta}(2t) + \tilde{\beta}(2|y|)$ and using the properties of $K_\infty$ functions we obtain from inequality (A.4)

$$|\Psi(t, x) - \Psi(t, y)| \le \tilde{a}\left(2\tilde{\beta}(2t)\; |x - y|\right) + \tilde{a} \times \left(2\tilde{\beta}(2|y|)\; |x - y|\right), \quad \forall (t, x, y) \in \Re^+ \times \Re^m \times \Re^m. \quad \text{(A.5)}$$

Define $R := 2\tilde{\beta}(0)$, $a_2(s) := s + 2\tilde{\beta}(2s) - R$ and notice $a_2 \in K_\infty$. Using again the properties of $K_\infty$ functions we obtain from inequality (A.5):

$$|\Psi(t, x) - \Psi(t, y)| \le \tilde{a}\left(2\tilde{\beta}(2t)\; |x - y|\right) + \tilde{a}(2R\; |x - y|) + \tilde{a}(2a_2(|y|)\; |x - y|), \quad \forall (t, x, y) \in \Re^+ \times \Re^m \times \Re^m. \quad \text{(A.6)}$$

Inequality (3.6) is directly implied by inequality (A.6) with $\beta(t) := \tilde{\beta}(2t) + R$, $a_1(s) := 2\tilde{a}(2s)$. ◁

## References


[1] A. Astolfi, L. Praly, Global complete observability and output-to-state stability imply the existence of a globally convergent observer, Mathematics of Control, Signals and Systems 18 (2006) 32–65.

[2] Y. Chitour, Time-varying high-gain observers for numerical differentiation, IEEE Transactions on Automatic Control 47 (9) (2002) 1565–1569.

[3] A.M. Dabroom, H.K. Khalil, Discrete-time implementation of high-gain observers for numerical differentiation, International Journal of Control 72 (17) (1999) 1523–1537.

[4] S. Diop, J.W. Grizzle, P.E. Moraal, A. Stefanopoulou, Interpolation and numerical differentiation for observer design, Proceedings of ACC (1994) 1329–1333.

[5] S. Diop, J.W. Grizzle, F. Chaplais, On numerical differentiation algorithms for nonlinear estimation, in: Proceedings of the IEEE Conference on Decision and Control, Sydney, Australia, 2000.

[6] R. Engel, G. Kreisselmeier, A continuous-time observer which converges in finite time, IEEE Transactions on Automatic Control 47 (7) (2002) 1202–1204.

[7] F. Esfandiari, H.K. Khalil, Output feedback stabilization of fully linearizable systems, International Journal of Control 56 (1992) 1007–1037.

[8] J.P. Gauthier, H. Hammouri, S. Othman, A simple observer for nonlinear systems, IEEE Transactions on Automatic Control 37 (6) (1992) 875–880.

[9] J.P. Gauthier, I. Kupka, Observability and observers for nonlinear systems, SIAM Journal on Control and Optimization 32 (4) (1994) 975–994.

[10] J.P. Gauthier, I. Kupka, Deterministic Observation Theory and Applications, Cambridge University Press, 2001.

[11] I. Karafyllis, J. Tsinias, A converse Lyapunov theorem for non-uniform in time global asymptotic stability and its application to feedback stabilization, SIAM Journal on Control and Optimization 42 (3) (2003) 936–965.

[12] I. Karafyllis, Non-uniform stabilization of control systems, IMA Journal of Mathematical Control and Information 19 (4) (2002) 419–444.

[13] I. Karafyllis, The non-uniform in time small-gain theorem for a wide class of control systems with outputs, European Journal of Control 10 (4) (2004) 307–323.

[14] I. Karafyllis, Non-uniform in time Robust global asymptotic output stability, Systems and Control Letters 54 (3) (2005) 181–193.

[15] I. Karafyllis, C. Kravaris, Robust output feedback stabilization and nonlinear observer design, Systems and Control Letters 54 (10) (2005) 925–938.

[16] I. Karafyllis, C. Kravaris, Non-uniform in time state estimation of dynamical systems, in: Proceedings of NOLCOS 2004, Stuttgart, Germany.

[17] N. Kazantzis, C. Kravaris, Nonlinear observer design using Lyapunov's auxiliary theorem, Systems and Control Letters 34 (1998) 241–247.

[18] H.K. Khalil, Nonlinear Systems, 2nd edition, Prentice-Hall, 1996.
[19] A.J. Krener, A. Isidori, Linearization by output injection and nonlinear observers, Systems and Control Letters 3 (1983) 47–52.
[20] A.J. Krener, M. Xiao, Nonlinear observer design in the siegel domain, SIAM Journal Control and Optimization 41 (3) (2001) 932–953.
[21] A.J. Krener, M. Xiao, Observers for linearly unobservable nonlinear systems, Systems and Control Letters 46 (2002) 281–288.
[22] A. Levant, Robust exact differentiation via sliding mode technique, Automatica 34 (3) (1998) 379–384.
[23] A. Levant, Higher-order sliding modes, differentiation and output feedback control, International Journal of Control 76 (2003) 924–941.
[24] F. Plestan, J.W. Grizzle, Synthesis of nonlinear observers via structural analysis and numerical differentiation, in: Proceedings of the European Control Conference, 1999.
[25] W. Respondek, A. Pogromsky, H. Nijmeijer, Time scaling for observer design with linearizable error dynamics, Automatica 40 (2004) 277–285.
[26] E.D. Sontag, Mathematical Control Theory, 2nd edition, Springer-Verlag, New York, 1998.
[27] E.D. Sontag, Comments on integral variants of ISS, Systems and Control Letters 34 (1998) 93–100.
[28] A. Teel, L. Praly, Tools for semiglobal stabilization by partial state and output feedback, SIAM Journal on Control and Optimization 33 (5) (1995) 1443–1488.
[29] J. Tsinias, Observer design for nonlinear systems, Systems and Control Letters 13 (1989) 135–142.
[30] J. Tsinias, Further results on the observer design problem, Systems and Control Letters 14 (1990) 411–418.
[31] J. Tsinias, I. Karafyllis, ISS property for time-varying systems and application to partial-static feedback stabilization and asymptotic tracking, IEEE Transactions on Automatic Control 44 (11) (1999) 2179–2185.
[32] J. Tsinias, Backstepping design for time-varying nonlinear systems with unknown parameters, Systems and Control Letters 39 (4) (2000) 219–227.