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On the Liapunov–Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equations☆

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ABSTRACT

The input-to-state stability of time-invariant systems described by coupled differential and difference equations with multiple noncommensurate and distributed time delays is investigated in this paper. Such equations include neutral functional differential equations in Hale's form (which model, for instance, partial element equivalent circuits) and describe lossless propagation phenomena occurring in thermal, hydraulic and electrical engineering. A general methodology for systematically studying the input-to-state stability, by means of Liapunov–Krasovskii functionals, with respect to measurable and locally essentially bounded inputs, is provided. The technical problem concerning the absolute continuity of the functional evaluated at the solution has been studied and solved by introducing the hypothesis that the functional is locally Lipschitz. Computationally checkable LMI conditions are provided for the linear case. It is proved that a linear neutral system in Hale's form with stable difference operator is input-to-state stable if and only if the trivial solution in the unforced case is asymptotically stable. A nonlinear example taken from the literature, concerning an electrical device, is reported, showing the effectiveness of the proposed methodology.

1. Introduction

In this paper systems described by coupled delay differential and difference equations forced by measurable, locally essentially bounded inputs, are considered. The importance in engineering applications of the systems here considered is well known (see Niculescu (2001) and Rasvan and Niculescu (2002) and the references therein). It is assumed that the functionals involved in the dynamics and the input are such that the Carathéodory conditions are verified. The notion of ISS, given in Sontag (1989), has had a great impact on the study of nonlinear delay-free systems and we are confident that this notion will have a great impact also for delayed systems. For instance, the concept of input-to-state stability for coupled delay differential and difference systems here introduced can be used when studying the internal dynamics of recently studied nonlinear delay control systems (see Germani, Manes, and Pepe (2000, 2003)), or when studying the behavior of lossless transmission lines with forcing inputs (see Rasvan and Niculescu (2002)). In the seminal paper Teel (1998), the notion of input-to-state stability has been generalized to systems described by nonlinear retarded functional differential equations and sufficient conditions are stated in the setting of Liapunov–Razumikhin methodology. In the paper (Pepe & Jiang, 2006) the input-to-state stability and the integral input-to-state stability (see Angeli, Sontag, and Wang (2000) and references therein) from a perspective of Liapunov–Krasovskii functionals for systems described by retarded functional differential equations are addressed. In the paper (Pepe, 2007a) the input-to-state stability of systems described by neutral functional differential equations in Hale's form, with linear difference operator, is studied and a Liapunov–Krasovskii methodology is presented. The paper (Rasvan & Niculescu, 2002) focuses on forced oscillations for lossless propagation systems, described by linear coupled differential and difference equations with scalar, globally Lipschitz, nonlinear perturbations, depending on a linear combination of the unknown variables which on the left-hand side of the mathematical model appear differentiated. A single delay, not involving these unknown variables, is considered. The input function, which appears in both the differential and the difference equations of the model, is supposed to be piece-wise continuous and bounded. Computationally checkable LMI conditions, involving the Lipschitz constant, which yield the exponential stability of the solution are provided.

In this paper, we take a step further to study the input-to-state stability property for a class of systems described by

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coupled delay differential and difference equations. Particularly, we present for the first time the Liapunov–Krasovskii methodology for systematically studying the input-to-state stability of general nonlinear infinite-dimensional systems of this kind, with multiple noncommensurate and distributed time delays which may affect all variables in the model, and with the forcing input, measurable and locally essentially bounded, appearing in the differential equation of the model, see Section 4. The input here does not appear in the difference equation, otherwise the very weak hypothesis on such input, which may well describe unknown disturbances, would not allow the continuity (not even piecewise) of the solution, which would be just measurable. This point is very critical (see the technical problems for the correct use of Liapunov–Krasovskii functionals addressed in Pepe (2007a,b)) and deserves further deep investigations, which are beyond the aims of this paper. The studied class of systems includes general neutral systems in Hale's form with nonlinear difference operator (for instance, systems described by Eq. (19), in the time-invariant case, in Pepe (2007a)). We then show, in Section 5, that the proposed methodology leads to computationally checkable LMI conditions upon specification of our systems into the context of linear systems. It is proved that a linear neutral system with stable difference operator is input-to-state stable if and only if the trivial solution of the unforced system is asymptotically stable. In Section 6, a nonlinear example concerning an electrical device, taken from the past literature, is worked out in detail to illustrate the effectiveness of the approach advocated in the paper.

2. Preliminaries

The symbol $\bar{R}$ indicates the extended real line $[-\infty, +\infty]$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. I_j is the identity matrix of dimension j, 0_j is a zero square matrix of dimension j, $0_{i,j}$ is a zero matrix in $R^{i\times j}$, i, j positive integers. A function $u : R^+ \to R^m$, m positive integer, is said to be *essentially bounded* if $\operatorname{ess\,sup}_{t\geq 0} |u(t)| < \infty$. We indicate the essential supremum norm of an essentially bounded function with the symbol $\|\cdot\|_\infty$. For given times $0 \leq T_1 < T_2$, we indicate with $u_{[T_1,T_2)} : [0, +\infty) \to R^m$ the function given by $u_{[T_1,T_2)}(t) = u(t)$ for all $t \in [T_1, T_2)$ and equal to 0 elsewhere. An input $u : R^+ \to R^m$ is said to be *locally essentially bounded* if, for any $T > 0$, $u_{[0,T)}$ is essentially bounded. A function $w : [0, b) \to R$, $0 < b \leq +\infty$, is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0, c]$, $0 < c < b$. For a real $\Delta > 0$, a positive integer n, $L_2([-\Delta, 0]; R^n)$ is the Hilbert space of square Lebesgue integrable functions mapping $[-\Delta, 0]$ into R^n, $C([-\Delta, 0], R^n)$ is the space of continuous functions mapping the interval $[-\Delta, 0]$ into R^n, endowed with the supremum norm. $C^1([-\Delta, 0], R^n)$ is the space of continuously differentiable functions mapping the interval $[-\Delta, 0]$ into R^n. $W^{1,\infty}([-\Delta, 0]; R^n)$ is the space of absolutely continuous functions mapping the interval $[-\Delta, 0]$ into R^n with essentially bounded derivative, endowed with the same norm of $C([-\Delta, 0], R^n)$. For a given function $s : [-\Delta, b) \to R^n$, $0 < b \leq +\infty$, the function $s_t : [-\Delta, 0] \to R^n$, $t \in [0, b)$, is defined as $s_t(\tau) = s(t + \tau)$, $\tau \in [-\Delta, 0]$. Let us here recall that a function $\gamma : R^+ \to R^+$ is: of class K if it is zero at zero, continuous and strictly increasing; of class K_∞ if it is of class K and it is unbounded; of class L if it is continuous, nonincreasing and converges to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \to R^+$ is of class KL if it is of class K in the first argument and is of class L in the second argument. Here an N_a functional is any functional defined in the product space $C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n)$, with d, n positive integers, and taking values in R^+, such that, for suitable functions γ_a and $\bar{\gamma}_a$ of class K_∞, the following inequalities hold for any $\phi \in C([-\Delta, 0]; R^d)$ and any $\psi \in C([-\Delta, 0]; R^n)$:

$$\gamma_a(|\phi(0)|) \leq N_a\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right) \leq \bar{\gamma}_a(\|\phi\|_\infty + \|\psi\|_\infty). \tag{1}$$

For instance, the N_2 norm (see Pepe and Verriest (2003)) in the product space $C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n)$ is an N_a functional.

3. The system equations

Let us consider a system described by the following nonlinear coupled delay differential and difference equations (see Fridman (2002), Germani et al. (2003), Hale and Martinez Amores (1977), Niculescu (2001), Pepe (2005, 2007a), Pepe and Verriest (2003) and Rasvan and Niculescu (2002))

$$\begin{aligned} &\dot{\xi}(t) = f(\xi_t, x_t, u(t)), \quad t \geq 0, \text{ a.e.},\\ &x(t) = g(\xi_t, x_t), \quad t \geq 0 \end{aligned} \tag{2}$$

$$\xi(\tau) = \xi_0(\tau), \qquad x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0], \tag{3}$$

where: $t \in [0, +\infty)$; $\xi(t) \in R^d$; $x(t) \in R^n$; n, d are positive integers; $u(t) \in R^m$ is the measurable locally essentially bounded input function, m is a positive integer; ξ_0 and x_0 are functions in $C([-\Delta, 0]; R^d)$ and $C([-\Delta, 0]; R^n)$, respectively; $\Delta > 0$ is the maximum involved delay; f is a locally Lipschitz continuous functional mapping $C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \times R^m$ into R^d, independent of the second argument at zero (see Definition 5.1, p. 281 in Hale and Lunel (1993)); g is a locally Lipschitz continuous functional mapping $C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n)$ into R^n, independent of the second argument at 0. We assume that $f(0, 0, 0) = 0$, and $g(0, 0) = 0$, thus ensuring that $\xi(t) = 0, x(t) = 0$, for every $t \geq 0$, is the solution of system (2) and (3) corresponding to zero initial conditions and zero input (i.e. the trivial solution). Note that the difference equation in (2) holds at 0 too, that is the initial conditions satisfy the matching condition $x_0(0) = g(\xi_0, x_0)$. Let us now introduce the following hypotheses involving the functionals f, g, u.

($\mathrm{H_{p1}}$) For any (ϕ, ψ, v) in the space

$W^{1,\infty}([-\Delta, 0]; R^d) \times W^{1,\infty}([-\Delta, 0]; R^n) \times R^m$,

it happens that

$$\limsup_{h\to 0^+} \frac{g(\phi_h, \psi_h) - g(\phi, \psi)}{h} \in R^n, \tag{4}$$

where, for $0 < h < \Delta$: $\phi_h \in W^{1,\infty}([-\Delta, 0]; R^d)$ is given by

$$\phi_h(s) = \begin{cases}\phi(s + h), & s \in [-\Delta, -h],\\ \phi(0) + v(s + h), & s \in (-h, 0];\end{cases} \tag{5}$$

$\psi_h \in W^{1,\infty}([-\Delta, 0]; R^n)$ is given by

$$\psi_h(s) = \begin{cases}\psi(s + h), & s \in [-\Delta, -h],\\ \psi(0), & s \in (-h, 0];\end{cases} \tag{6}$$

moreover the functional

$$(\phi, \psi, v) \to \limsup_{h\to 0^+} \frac{g(\phi_h, \psi_h) - g(\phi, \psi)}{h} \tag{7}$$

is bounded on bounded sets $U \subset W^{1,\infty}([-\Delta, 0]; R^d) \times W^{1,\infty}([-\Delta, 0]; R^n) \times R^m$;

($\mathrm{H_{p2}}$) for any continuous function $s : [-\Delta, +\infty) \to R^n$, the functional $F : C([-\Delta, 0]; R^d) \times R^+ \to R^d$, defined as

$$F(\phi, t) = f(\phi, s_t, u(t)), \tag{8}$$

is bounded on any bounded set $U \subset C([-\Delta, 0]; R^d) \times R^+$, and satisfies the modified Carathéodory conditions in $C([-\Delta, 0]; R^d) \times R^+$ (see Section 2.4, p. 100 in Kolmanovskii and Myshkis (1999)). ■

Remark 1. By applying the existence and uniqueness of solutions theorems for time delay systems to system (2), from the hypothesis H_{p2} it follows that system (2) admits a unique solution on a right maximal time interval $[0, b)$, $0 < b \le +\infty$, with $\xi(t)$ componentwise locally absolutely continuous and $x(t)$ continuous. Moreover, if $b < +\infty$, then $\xi(t)$ is unbounded in $[0, b)$ (see Section 2.6, p. 58 in Hale and Lunel (1993) and Sections 2.2 and 2.4, p. 96, 100 in Kolmanovskii and Myshkis (1999)). ■

As is well known, given a continuous functional

$$V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+, \tag{9}$$

the upper right-hand Dini derivative of the time function $w(t) = V\left(\begin{bmatrix}\xi_t \\ x_t\end{bmatrix}\right)$, with $\begin{bmatrix}\xi(t) \\ x(t)\end{bmatrix}$ solution of (2), is given, for $t \ge 0$, by

$$D^+w(t) = \limsup_{h\to 0^+} \frac{V\left(\begin{bmatrix}\xi_{t+h} \\ x_{t+h}\end{bmatrix}\right) - V\left(\begin{bmatrix}\xi_t \\ x_t\end{bmatrix}\right)}{h}. \tag{10}$$

Definition 2 (*See Driver (1962) and Pepe (2007a)*). Let $V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+$ be a continuous functional. The upper right-hand Dini derivative

$$D^+V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \times R^m \to \bar{R} \tag{11}$$

of the functional V is defined, for $\phi \in C([-\Delta, 0]; R^d)$, $\psi \in C([-\Delta, 0]; R^n)$, $v \in R^m$, as

$$D^+V\left(\begin{bmatrix}\phi \\ \psi\end{bmatrix}, v\right) = \limsup_{h\to 0^+} \frac{1}{h}\left(V\left(\begin{bmatrix}\phi_h^\star \\ \psi_h^\star\end{bmatrix}\right) - V\left(\begin{bmatrix}\phi \\ \psi\end{bmatrix}\right)\right), \tag{12}$$

where, for $0 < h < \Delta$, $0 < \theta \le h$: $\phi_\theta^\star \in C([-\Delta, 0]; R^d)$ is given by

$$\phi_\theta^\star(s) = \begin{cases}\phi(s+\theta), & s \in [-\Delta, -\theta], \\ \phi(0) + f(\phi, \psi, v)(s+\theta), & s \in (-\theta, 0];\end{cases}$$

$\psi_\theta^{\star\star} \in C([-\Delta, 0]; R^n)$ is given by

$$\psi_\theta^{\star\star}(s) = \begin{cases}\psi(s+\theta), & s \in [-\Delta, -\theta], \\ \psi(0), & s \in (-\theta, 0];\end{cases} \tag{13}$$

$\psi_h^\star \in C([-\Delta, 0]; R^n)$ is given by

$$\psi_h^\star(s) = \begin{cases}\psi(s+h), & s \in [-\Delta, -h], \\ g(\phi_{s+h}^\star, \psi_{s+h}^{\star\star}), & s \in (-h, 0].\end{cases} \tag{14}$$

In the following, it will be useful to consider the input-to-state stability of (only) the difference part of system (2), $x(t) = g(\xi_t, x_t)$ (see Pepe, Jiang, and Fridman (2008)). A continuous-time difference equation can be re-written, in many ways, as a discrete-time equation on a suitably chosen Banach space (see Germani et al. (2003) and Pepe (2003), for an explicit expression of the discrete-time system in the case of discrete delays). For instance, there exists a suitable function $\hat{G}$ by which the difference equation can be transformed into the discrete-time system in the Banach space $C([-\Delta, 0]; R^n)$

$$\hat{\chi}(k+1) = \hat{G}(\hat{\chi}(k), \hat{\zeta}(k)), \quad k = 0, 1, \ldots, \tag{15}$$

where $\hat{\chi}(k)(\tau) = x(k\delta + \tau)$, $\hat{\zeta}(k)(\tau) = \begin{bmatrix}\xi((k+1)\delta + \tau) \\ \xi(k\delta + \tau)\end{bmatrix}$, $\tau \in [-\Delta, 0]$, $0 < \delta \le a$ ($[-a, 0]$ being the interval of independence of the functional g with respect to the second argument) is such that, for a suitable integer $r > 1$, $\Delta = r\delta$, $\frac{\Delta}{r} \le a < \frac{\Delta}{r-1}$. Moreover, setting $\chi(k) = \hat{\chi}(rk)$, $\zeta(k)(\tau) = \begin{bmatrix}\xi^T(r\delta(k+1) + \tau) & \xi^T(r\delta k + \tau)\end{bmatrix}^T$, $\tau \in [-\Delta, 0]$, a suitable functional G exists such that the following discrete-time system can be used with Liapunov methodologies for ISS problems (see Jiang and Wang (2001) and Karafyllis (2006a,b))

$$\chi(k+1) = G(\chi(k), \zeta(k)), \quad k = 0, 1, \ldots. \tag{16}$$

If one considers $\zeta(k)$ as an input, then by Theorem 3 in Pepe et al. (2008), one can check if system (16) is ISS with respect to such an input. In the following sections, the ISS of system (16) will be meant with respect to ζ.

4. The Liapunov–Krasovskii theorem for ISS

Definition 3 (*See Sontag (1989)*). The system described by the equations (2) is said to be input-to-state stable (ISS) if there exist a KL function β and a K function γ such that, for any continuous initial state $\begin{bmatrix}\xi_0 \\ x_0\end{bmatrix}$ (satisfying the matching condition) and any measurable locally essentially bounded input u, the solution exists for all $t \ge 0$ and furthermore it satisfies

$$\left\|\begin{bmatrix}\xi(t) \\ x(t)\end{bmatrix}\right\| \le \beta(\|\xi_0\|_\infty + \|x_0\|_\infty, t) + \gamma(\|u_{[0,t)}\|_\infty). \tag{17}$$

Let us here recall that a continuous functional $V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+$ is locally Lipschitz with respect to the norm of the uniform topology if $\forall \begin{bmatrix}\phi \\ \psi\end{bmatrix} \in C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+$ there exist a neighborhood $U_{\phi,\psi}$ of $\begin{bmatrix}\phi \\ \psi\end{bmatrix}$ and a constant $L_{\phi,\psi}$ such that the inequality $|V(y_1, z_1) - V(y_2, z_2)| \le L_{\phi,\psi}(\|y_1 - y_2\|_\infty + \|z_1 - z_2\|_\infty)$ holds $\forall \begin{bmatrix}y_1 \\ z_1\end{bmatrix}, \begin{bmatrix}y_2 \\ z_2\end{bmatrix} \in U_{\phi,\psi}$.

Lemma 4 (*See Pepe (2007a)*). *Let $V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+$ be a continuous functional, locally Lipschitz with respect to the norm of the uniform topology. Let the function $w : [0, b) \to R^+$ be defined as $w(t) = V(\xi_t, x_t)$, where (ξ_t, x_t) is the solution of system (2) and (3) on a right maximal time interval $[0, b)$, $0 < b \le +\infty$.*

Then $D^+w(t) = D^+V(\xi_t, x_t, u(t))$, $t \in [0, b)$, a.e.

Next two lemmas are needed in order to solve the problems concerning the absolute continuity on the time domain of a continuous functional evaluated at the solution (see Pepe (2007b), as far as systems described by retarded functional differential equations are concerned).

Lemma 5. *Let $V : C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n) \to R^+$ be a continuous functional, locally Lipschitz with respect to the norm of the uniform topology. Let the initial conditions in (3) belong to $W^{1,\infty}([-\Delta, 0]; R^d) \times W^{1,\infty}([-\Delta, 0]; R^n)$. Let the function $w : [0, b) \to R^+$ be defined as $w(t) = V(\xi_t, x_t)$, where (ξ_t, x_t) is the solution of system (2) and (3) on a right maximal time interval $[0, b)$, $0 < b \le +\infty$.*
Then the function $t \to w(t)$ is locally absolutely continuous in $[0, b)$.

Proof. We have to prove that, for any given $c \in (0, b)$, the function w is absolutely continuous in $[0, c]$. Let $c \in (0, b)$ be arbitrarily given. Let $[-a, 0]$, $0 < a < \Delta$, be the interval of independence of the functional g with respect to the second argument. A step procedure is here used. The first step is as follows. Let $\epsilon = \min\{a, c\}$. From the hypotheses on the functional g in H_{p_1}, and from Theorem 2 in Pepe (2007a), it follows that, in $[0, \epsilon]$, the part $x(t)$ of the solution is almost everywhere differentiable with essentially bounded derivative. Therefore $x_t \in W^{1,\infty}([-\Delta, 0]; R^n)$, $t \in [0, \epsilon]$. It follows that, for any $t_1 < t_2$ in $[0, \epsilon]$ the following equalities–inequalities hold

$$\|x_{t_2}-x_{t_1}\|_\infty \le \sup_{\tau\in[-\Delta,0]} |x(t_2+\tau)-x(t_1+\tau)| = \sup_{\tau\in[-\Delta,0]} \left|x(-\Delta)+\int_{-\Delta}^{t_2+\tau}\dot{x}(\theta)\mathrm{d}\theta - x(-\Delta) - \int_{-\Delta}^{t_1+\tau}\dot{x}(\theta)\mathrm{d}\theta\right| = \sup_{\tau\in[-\Delta,0]}\left|\int_{t_1+\tau}^{t_2+\tau}\dot{x}(\theta)\mathrm{d}\theta\right| \le \operatorname{ess}\sup_{\tau\in[-\Delta,\epsilon]}|\dot{x}(\tau)|(t_2-t_1). \quad (18)$$

Analogously, the part ξ_t of the solution belongs to $W^{1,\infty}([-\Delta,0];R^d)$, $t\in[0,\epsilon]$ too, and the inequality holds

$$\|\xi_{t_2}-\xi_{t_1}\|_\infty \le \operatorname{ess}\sup_{\tau\in[-\Delta,\epsilon]}|\dot{\xi}(\tau)|(t_2-t_1). \quad (19)$$

It follows that the function $t\to(\xi_t,x_t)$ is absolutely continuous in $[0,\epsilon]$. Since the functional V is locally Lipschitz, it follows that it is uniformly Lipschitz on the compact set $S_\epsilon=\{(\xi_t,x_t),\, t\in[0,\epsilon]\}$. It follows that the function $t\to w(t)$ is absolutely continuous in $[0,\epsilon]$. If $c>\epsilon$, the reasoning is repeated in the same way in $[ka,\min\{(k+1)a,c\}]$, $k=1,2,\ldots,$ until k is such that $(k+1)a>c$. Therefore, the function $t\to w(t)$ is absolutely continuous in $[0,c]$. ■

Lemma 6. *The following statements are equivalent:*
(1) *The system described by (2) with continuous initial conditions is ISS;*
(2) *The system described by (2) with initial conditions in $W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n)$ is ISS.*

Proof. The implication 1 => 2 is obvious. Let us prove the implication 2 => 1. So, let us hypothesize that for any initial conditions in $W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n)$ and any measurable locally essentially bounded input, the inequality (17) holds for the corresponding solution $(\xi(t),x(t))$.

Claim: The inequality (17) holds also in the case of (simply) continuous initial conditions.

To prove the claim, by contradiction, let us suppose that the inequality (17) does not hold for certain continuous initial conditions $(\bar{\xi}_0,\bar{x}_0)$ (satisfying $\bar{x}_0(0)=g(\bar{\xi}_0,\bar{x}_0)$). So, there exist a measurable locally essentially bounded input, $\bar{u}$, and a time $t_1\ge 0$, such that the following inequality holds for the corresponding solution $(\bar{\xi}(t),\bar{x}(t))$:

$$\left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\| > \beta(\|\bar{\xi}_0\|_\infty+\|\bar{x}_0\|_\infty,t_1)+\gamma(\|\bar{u}_{[0,t_1)}\|_\infty). \quad (20)$$

Let $\epsilon>0$ such that the following inequality holds

$$\left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\| > \beta(\|\bar{\xi}_0\|_\infty+\|\bar{x}_0\|_\infty+\epsilon,t_1)+\gamma(\|\bar{u}_{[0,t_1)}\|_\infty)+\epsilon. \quad (21)$$

Let $[-a,0]$, $a>0$, be the interval of independence of the functionals f and g with respect to the second arguments.

Let η be an arbitrarily chosen positive real. Since $C^1([-\Delta,0];R^d)\times C^1([-\Delta,0];R^n)$ is dense in $C([-\Delta,0];R^d)\times C([-\Delta,0];R^n)$, there exists a function $\begin{bmatrix}y\\z\end{bmatrix}$ in $C^1([-\Delta,0];R^d)\times C^1([-\Delta,0];R^n)$ such that $\|y-\bar{\xi}_0\|_\infty+\|z-\bar{x}_0\|_\infty<\frac{\eta}{6}$, and $|g(y,z)-g(\bar{\xi}_0,\bar{x}_0)|<\frac{\eta}{6}$. Let h be a positive real such that $a>h>0$, $h\sup_{s\in[-\Delta,0]}|\dot{z}(s)|<\frac{\eta}{6}$ and $|g(\bar{\xi}_0,\bar{x}_0)-z(-h)|<\frac{\eta}{6}$. Let $p\in W^{1,\infty}([-\Delta,0];R^n)$ be defined as

$$p(s)=\begin{cases} z(s) & s\in[-\Delta,-h)\\ -z(-h)\dfrac{s}{h}+g(y,z)\dfrac{s+h}{h} & s\in[-h,0]. \end{cases} \quad (22)$$

Since

$$\sup_{s\in[-h,0]}|z(s)-p(s)|\le h\sup_{s\in[-h,0]}|\dot{z}(s)|+|g(y,z)-z(-h)|, \quad (23)$$

it follows that $\|y-\bar{\xi}_0\|_\infty+\|p-\bar{x}_0\|_\infty<\eta$. Therefore, there exists a sequence of functions $\begin{bmatrix}\bar{\xi}_0^j\\ \bar{x}_0^j\end{bmatrix}\in W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n)$ (the superscript $j=0,1,\ldots$ is the index of the term in the sequence) satisfying the matching condition $\bar{x}_0^j(0)=g(\bar{\xi}_0^j,\bar{x}_0^j)$, such that

$$\lim_{j\to+\infty}\|\bar{\xi}_0^j-\bar{\xi}_0\|_\infty+\|\bar{x}_0^j-\bar{x}_0\|_\infty=0. \quad (24)$$

Let $(\bar{\xi}^j(t),\bar{x}^j(t))$ be the solution corresponding to the initial conditions $(\bar{\xi}_0^j,\bar{x}_0^j)$ and to the input $\bar{u}$. Let l be the first nonnegative integer such that $(l+1)a>t_1$. By Theorem 2.2, p. 43, in Hale and Lunel (1993) (see also Section 2.6, p. 58 in Hale and Lunel (1993)), there exist $l+2$ positive reals $\delta_0<\delta_1<\cdots<\delta_l<\delta_{l+1}=\epsilon$, such that, if $\|\bar{\xi}_{ia}^j-\bar{\xi}_{ia}\|_\infty+\|\bar{x}_{ia}^j-\bar{x}_{ia}\|_\infty<\delta_i$, then

$$\sup_{\tau\in[ia,\min\{(i+1)a,t_1\}]}\|\bar{\xi}^j(\tau)-\bar{\xi}(\tau)\|_\infty+\|\bar{x}^j(\tau)-\bar{x}(\tau)\|_\infty<\delta_{i+1}, \quad (25)$$

$i=0,1,\ldots,l$. Let $\bar{j}$ such that $\|\bar{\xi}_0^{\bar{j}}-\bar{\xi}_0\|_\infty+\|\bar{x}_0^{\bar{j}}-\bar{x}_0\|_\infty<\delta_0$. Then, the following inequality holds

$$\left\|\begin{bmatrix}\bar{\xi}^{\bar{j}}(t)\\ \bar{x}^{\bar{j}}(t)\end{bmatrix}-\begin{bmatrix}\bar{\xi}(t)\\ \bar{x}(t)\end{bmatrix}\right\|<\epsilon,\quad t\in[-\Delta,t_1]. \quad (26)$$

From (21) and (26) and from (17) it follows that

$$\left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\| \le \left\|\begin{bmatrix}\bar{\xi}^{\bar{j}}(t_1)\\ \bar{x}^{\bar{j}}(t_1)\end{bmatrix}\right\|+\epsilon \le \beta\left(\|\bar{\xi}_0^{\bar{j}}\|_\infty+\|\bar{x}_0^{\bar{j}}\|_\infty,t_1\right) + \gamma(\|\bar{u}_{[0,t_1)}\|_\infty)+\epsilon \le \beta(\|\bar{\xi}_0\|_\infty+\|\bar{x}_0\|_\infty+\epsilon,t_1) + \gamma(\|\bar{u}_{[0,t_1)}\|_\infty)+\epsilon < \left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\|. \quad (27)$$

Therefore, if (20) were true, the contradiction would follow $\left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\| < \left\|\begin{bmatrix}\bar{\xi}(t_1)\\ \bar{x}(t_1)\end{bmatrix}\right\|$. ■

Theorem 7. *Let the system described by the Eq. (16) be ISS. Let there exist a locally Lipschitz functional $V : C([-\Delta,0];R^d)\times C([-\Delta,0];R^n)\to R^+$, functions α_1, α_2 of class K_∞, and functions α_3, ρ of class K, such that, with a suitable N_a functional, the following hypotheses are satisfied:*

$$(\mathrm{H}_1)\begin{cases}\alpha_1(|\phi(0)|)\le V\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\le\alpha_2\left(N_a\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\right),\\ \forall\begin{bmatrix}\phi\\ \psi\end{bmatrix}\in W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n)\\ \text{with } \psi(0)=g(\phi,\psi);\end{cases}$$

$$(\mathrm{H}_2)\begin{cases}D^+V\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix},u\right)\le-\alpha_3\left(N_a\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\right),\\ \forall\begin{bmatrix}\phi\\ \psi\end{bmatrix}\in W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n),\\ u\in R^m:\psi(0)=g(\phi,\psi),\quad N_a\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\ge\rho(|u|).\end{cases}$$

Then, system (2) is ISS.

Proof. Let us consider system (2). By Lemma 6, we suppose that the initial conditions belong to $W^{1,\infty}([-\Delta,0];R^d)\times W^{1,\infty}([-\Delta,0];R^n)$. Let $\begin{bmatrix}\xi(t)\\ x(t)\end{bmatrix}$ be the solution on $[0,b)$, $0<b\le+\infty$. Let $w(t)=V\left(\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}\right)$. From Lemma 4, $D^+w(t)=D^+V\left(\begin{bmatrix}\xi_t\\ x_t\end{bmatrix},u(t)\right)$, *a.e.* From Lemma 5 it follows that the function $t\to w(t)$ is locally absolutely continuous. Therefore, we can adopt here the well-known reasoning used in the main Theorem in Sontag (1989). Let the input $u(t)$ be such that $\operatorname{ess}\sup_{t\ge0}|u(t)|=v$, for a suitable $v\in R^+$.

From H_1, H_2 it follows that there exist a function β_1 of class KL and a function γ_1 of class K such that the inequality

$$|\xi(t)| \leq \beta_1 \left(N_a \left(\begin{bmatrix} \xi_0 \\ x_0 \end{bmatrix} \right), t \right) + \gamma_1(v) \tag{28}$$

holds $\forall t \in [0, b)$. From inequalities (1), taking into account Remark 1, it follows that the solution $\begin{bmatrix} \xi(t) \\ x(t) \end{bmatrix}$ exists $\forall t \geq 0$ and the following inequality holds

$$|\xi(t)| \leq \beta_1 \left(\bar{\gamma}_a \left(\|\xi_0\|_\infty + \|x_0\|_\infty \right), t \right) + \gamma_1(v), \quad t \geq 0. \tag{29}$$

From the hypothesis that system (16) is ISS, it follows that there exist a function β_2 of class KL and a function γ_2 of class K such that the following inequality holds

$$|x(t)| \leq \beta_2(\|x_0\|_\infty, t) + \gamma_2(\sup_{\tau \in [-\Delta, t)} |\xi(\tau)|), \quad t \geq 0. \tag{30}$$

From (29) and (30), it follows that the inequalities hold

$$\begin{aligned} &\|\xi_t\|_\infty \leq \|\xi_0\|_\infty + \beta_1 \left(\bar{\gamma}_a \left(\|\xi_0\|_\infty + \|x_0\|_\infty \right), 0 \right) + \gamma_1(v), \\ &\|x_t\|_\infty \leq \|x_0\|_\infty + \beta_2(\|x_0\|_\infty, 0) \\ &\qquad + \gamma_2(2\|\xi_0\|_\infty + 2\beta_1(\bar{\gamma}_a(\|\xi_0\|_\infty + \|x_0\|_\infty), 0)) \\ &\qquad + \gamma_2(2\gamma_1(v)), \quad \forall t \geq 0; \end{aligned} \tag{31}$$

$$\begin{aligned} |x(t)| \leq\ & \beta_2(\|x_{\bar{t}}\|_\infty, t - \bar{t}) + \gamma_2(\beta_1(\bar{\gamma}_a(\|\xi_0\|_\infty + \|x_0\|_\infty), \bar{t} - \Delta) \\ & + \gamma_1(v)), \quad \forall t, \bar{t} : t \geq \bar{t} \geq \Delta; \end{aligned} \tag{32}$$

Setting in (32) $\bar{t} = \frac{1}{2}t, t \geq 2\Delta$, and substituting $\|x_{\bar{t}}\|_\infty$ with the r.h.s. of the second inequality in (31), it follows that (29), (31), (32) yield the ISS inequality (17).

Remark 8. Whether the ISS of the system described by the Eq. (16) is necessary for the ISS of the overall system (2) is to be investigated. Here we point out that, for the following example (case of the neutral system 1.6 in Hale and Lunel (2002), rewritten in the coupled form)

$$\begin{aligned} \dot{\xi}(t) &= -\xi(t) + x(t-1) + u(t), \\ x(t) &= \xi(t) - x(t-1), \end{aligned} \tag{33}$$

it happens that the system described by the corresponding equation (16) is not ISS (it is not asymptotically stable), but the overall system is globally asymptotically stable and, for any given constant input $u(t) = v$, $v \in R$, the following equalities hold: $\lim_{t \to +\infty} x(t) = v$, $\lim_{t \to +\infty} \xi(t) = 2v$. Moreover, in all the performed simulations, bounded inputs yield bounded state variables.

Remark 9. If in Theorem 7 the hypothesis that the system described by the Eq. (16) is ISS is not introduced, and, instead of the hypothesis H_1, the following hypothesis

$$\alpha_1 \left(N_a \left(\begin{bmatrix} \phi \\ \psi \end{bmatrix} \right) \right) \leq V \left(\begin{bmatrix} \phi \\ \psi \end{bmatrix} \right) \leq \alpha_2 \left(N_a \left(\begin{bmatrix} \phi \\ \psi \end{bmatrix} \right) \right),$$
$$\forall \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in C([-\Delta, 0]; R^d) \times C([-\Delta, 0]; R^n)$$

is introduced, then, the following results hold: the solution of (2) exists $\forall t \geq 0$; the input-state inequality

$$N_a \left(\begin{bmatrix} \xi_t \\ x_t \end{bmatrix} \right) \leq \beta \left(N_a \left(\begin{bmatrix} \xi_0 \\ x_0 \end{bmatrix} \right), t \right) + \gamma(\|u_{[0,t)}\|_\infty) \tag{34}$$

holds $\forall t \geq 0$, for suitable functions β of class KL and γ of class K. For instance, if the N_2 norm (by which the critical derivation of $x(t)$ can be avoided) is used as N_a functional, then one may achieve an input-state inequality which involves the L_2 norm of x_t (see the results for the L_2-Stability given in Pepe (2005) and Pepe and Verriest (2003)).

Remark 10. The overall system (2) cannot be regarded as a cascade of two subsystems. Actually it consists of the interconnection of a differential system and a difference one (the variable ξ of the differential part forces the difference part and the variable x of the difference part forces the differential part). If the hypotheses H_1, H_2 are satisfied, then the differential part of the system is ISS with respect to the input u (when the initial condition $x_0 = 0$, see the inequality (29)), though forced by the variable $x(t) \neq 0, t \geq 0$, too (this is a key point, note that the hypotheses H_1, H_2 involve the overall system equations). Only when this result is achieved, then the overall system could be regarded, at the aim of studying the ISS, as a cascade of ISS systems, provided that the hypothesis that the difference part of the system is ISS with respect to the input ξ (first variable) is introduced.

Remark 11. The N_a functional is introduced in order to yield as much generality as possible for Theorem 7. For instance, the N_a functional may be a seminorm. This allows also a lot of freedom in the choice of the Liapunov–Krasovskii functional. As an illustrating example, let us consider the following system described by scalar coupled delay differential and difference equations

$$\begin{aligned} \dot{\xi}(t) &= -\xi(t) + (1 + x^2(t - \Delta))(-\xi^3(t) + u(t)) \\ x(t) &= \frac{1}{2}x(t - \Delta) + \xi(t)\xi^2(t - \Delta). \end{aligned} \tag{35}$$

In this case the ISS can be proved by the functional $V\left(\begin{bmatrix} \phi \\ \psi \end{bmatrix}\right) = \phi^2(0)$. The N_a functional defined as $N_a\left(\begin{bmatrix} \phi \\ \psi \end{bmatrix}\right) = |\phi(0)|$ can be used. Note that the second variable is not involved at all. The function ρ of class K defined as $\rho(|u|) = |u|^{\frac{1}{3}}$ can be used.

5. The linear case

As far as the linear case is concerned, let us consider a system described by the following linear equations

$$\begin{aligned} \dot{\xi}(t) &= A_0\xi(t) + \sum_{i=1}^{p} A_i\xi(t - \Delta_i) + \sum_{i=1}^{p} B_i x(t - \Delta_i) + Gu(t), \\ x(t) &= D_0\xi(t) + \sum_{i=1}^{p} D_i\xi(t - \Delta_i) + \sum_{i=1}^{p} C_i x(t - \Delta_i), \end{aligned} \tag{36}$$

where A_0, D_0, A_i, B_i, C_i, D_i, $i = 1, 2, \ldots, p$, G are real matrices of suitable dimension, $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_p$ are the (arbitrary, noncommensurate) delays. The methodology proposed in this paper yields LMI conditions for the input-to-state stability of the general multiple noncommensurate delays case, as shown in the following:

Corollary 12. *Let the LMIs* (1) (2) *in (Corollary 5, Pepe, 2005) be feasible. Then, for any given delays* $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_p$*, the resulting system* (36) *is ISS.*

Proof. The proof is achieved by applying Theorem 7, with the functional (18) (19) in Pepe (2005) and using the N_2 norm as N_a functional. ■

In the paper (Pepe & Jiang, 2006) it is proved for systems described by retarded functional differential equations that the asymptotic stability of the trivial solution in the unforced case is equivalent to the input-to-state stability. In the following an analogous result for linear systems described by neutral functional differential equations in Hale's form (special case of system (36)) is given. It is to be noted that, while for linear retarded functional differential equations studied in Pepe and Jiang (2006) the asymptotic stability of the trivial solution in the unforced case implies the exponential stability, this is not true in general for linear neutral systems, that is the asymptotic stability of the trivial solution does not imply the exponential stability, as shown in the Example 1.6 in Hale and Lunel (2002). The asymptotically stable trivial solution of a linear neutral system in Hale's form is also exponentially stable if the difference operator is (asymptotically) stable, according to Definition 3.1, p. 275, in Hale and Lunel (1993). Let us consider a system described by the following neutral equation in Hale's form with discrete delays (see Chapter 9 in Hale and Lunel (1993))

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}x_t) = \mathcal{L}x_t + Gu(t), \tag{37}$$

where $\mathcal{D} : C([-\Delta, 0]; R^n) \to R^n$ is a linear stable (according to Definition 3.1, p. 275, in Hale and Lunel (1993)) operator defined as

$$\mathcal{D}\phi = \phi(0) - \sum_{i=1}^{p} C_i\phi(-\Delta_i), \qquad \phi \in C([-\Delta, 0]; R^n), \tag{38}$$

$\mathcal{L} : C([-\Delta, 0]; R^n) \to R^n$ is a linear operator defined as

$$\mathcal{L}\phi = A_0\phi(0) + \sum_{i=1}^{p} A_i\phi(-\Delta_i), \qquad \phi \in C([-\Delta, 0]; R^n), \tag{39}$$

$A_0, A_i, C_i, i = 1, 2, \ldots, p, G$ are real matrices of suitable dimension, $\Delta_i, i = 1, 2, \ldots, p$ are nonnegative (arbitrary, noncommensurate) delays.

Theorem 13. *System (37) (with given delays) is ISS (see Pepe, 2007a and Sontag, 1989) if and only if the trivial solution in the unforced case is asymptotically stable.*

Proof. As is well known, the input-to-state stability implies the asymptotic stability of the trivial solution in the unforced case. Let us prove that the asymptotic stability of the trivial solution in the unforced case implies the input-to-state stability. Taking into account that system (37) is a special case of system (2) (see Fridman (2002) and Niculescu (2001) and the references therein), and Lemma 6, let us suppose that the initial conditions belong to $W^{1,\infty}([-\Delta, 0]; R^n)$. Let $T(t)$ be the semigroup associated with the neutral system (37) (see Lemma 2.1, p. 263, in Hale and Lunel (1993)). The following relation holds (see (2.5) in Kappel and Kunisch (1981))

$$x_t = T(t)\phi + \int_{-\Delta}^{0} T(t-s)G_0 Gu(s)\mathrm{d}s, \tag{40}$$

where $G_0(0) = I_n$ and $G_0(\theta) = 0_n, \theta \in [-\Delta, 0)$. Let $q(\lambda) = 0$ and $q_0(\lambda) = 0$ be the characteristic equations associated with system (37) and (37) with zero right-hand side ($\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}x_t) = 0$), respectively (see (2.8), p. 264, in Hale and Lunel (1993)). Let $\alpha_{\mathcal{D}} = \sup\{\mathrm{Re}\ \lambda : q_0(\lambda) = 0\}$. From the (asymptotic) stability of the operator $\mathcal{D}$ it follows that $\alpha_{\mathcal{D}} < 0$ (see Theorem 3.5, p. 275, in Hale and Lunel (1993)). From Rouché Theorem it follows that in the half-plane Re $\lambda > \alpha_{\mathcal{D}}$ the characteristic equation $q(\lambda) = 0$ can only have finitely many roots. As a consequence, from the asymptotic stability of the unforced system (37), it follows that $\sup\{\mathrm{Re}\,\lambda : q(\lambda) = 0\} < 0$ (if it were zero, there would be a root on the imaginary axis for the equation $q(\lambda) = 0$, contradicting the hypothesis of the asymptotic stability of the trivial solution of system (37) in the unforced case). From Corollary 4.1, p. 278 in Hale and Lunel (1993), it follows that for the semigroup $T(t)$ there exist positive reals M, ω such that $\|T(t)\| \le Me^{-\omega t}$, where

$$\|T(t)\| = \sup_{\phi\in C([-\Delta,0];R^n)} \frac{\|T(t)\phi\|_\infty}{\|\phi\|_\infty}. \tag{41}$$

From (40) it follows that

$$|x(t)| \le Me^{-\omega t}\|\phi\|_\infty + \int_0^t Me^{-\omega(t-s)}|G||u(s)|\mathrm{d}s. \tag{42}$$

Therefore, by

$$|x(t)| \le Me^{-\omega t}\|\phi\|_\infty + \frac{M}{\omega}|G| \sup_{s\in[0,t]} |u(s)| \tag{43}$$

the input-to-state stability is proved. ■

By Theorem 13 it follows that the many computationally checkable LMI conditions available in the literature for the delay-dependent and delay-independent asymptotic stability of the trivial solution in the unforced case of linear neutral systems in Hale's form, with stable difference operator (see, for instance, Fridman (2002), Niculescu (2001) and Park and Won (1999) and the references therein), are computationally checkable conditions for the input-to-state stability too.

In the following Corollary a transformation of system (36) into a neutral system in Hale's form similar to the one proposed in Hale and Martinez Amores (1977) is used.

Corollary 14. *Consider system (36) with given delays. Let the operator $\mathcal{D} : C([-\Delta, 0]; R^n) \to R^n$, defined as $\mathcal{D}\phi = \phi(0) - \sum_{i=1}^{p} C_i\phi(-\Delta_i)$, be stable. Let there exist a Hurwitz matrix $H \in R^{n\times n}$ such that the trivial solution of the following neutral system in Hale's form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi(t) \\ \mathcal{D}x_t - \sum_{i=1}^{p} D_i\xi(t-\Delta_i) \end{pmatrix} = \begin{pmatrix} A_0\xi(t) + \sum_{i=1}^{p} A_i\xi(t-\Delta_i) + \sum_{i=1}^{p} B_i x(t-\Delta_i) \\ D_0A_0\xi(t) + \sum_{i=1}^{p} D_0A_i\xi(t-\Delta_i) \\ + \sum_{i=1}^{p} D_0B_i x(t-\Delta_i) \\ + H\left(\mathcal{D}x_t - D_0\xi(t) - \sum_{i=1}^{p} D_i\xi(t-\Delta_i)\right) \end{pmatrix} \tag{44}$$

is asymptotically stable.

Then, system (36) with the given delays is ISS.

Proof. Since the operator $\mathcal{D}$ is stable, so is the difference operator in the neutral system (44). Since system (44) is asymptotically stable, by applying Theorem 13 it results that system (44) is input-to-state stable with respect to a measurable and locally essentially bounded input $u(t) \in R^m$ appearing on the right-hand side of the equation, in an adding term $\begin{bmatrix} G \\ D_0G \end{bmatrix} u(t)$. Any solution of system (36) is also solution of system (44), with such forcing term. For that, just consider that solutions of (36) coincide with the solutions of system (44), with the forcing term, when the initial conditions satisfy the matching condition $\mathcal{D}x_0 - D_0\xi_0(0) - \sum_{i=1}^{p} D_i\xi_0(-\Delta_i) = 0$. Therefore, the solutions of system (36) satisfy the ISS inequality (17), since this inequality is satisfied by all solutions of system (44) with the forcing term. ■

Remark 15. Corollary 14 allows us to use the many computationally checkable LMIs for delay-dependent and delay-independent asymptotic stability of the trivial solution of linear neutral systems available in the literature (see, for instance, Fridman (2002), Niculescu (2001) and Park and Won (1999) and the references therein) in order to prove the input-to-state stability of system (36). It is worth pointing out that the trivial solution of the scalar lossless propagation coupled equations studied in Section 4.6.6, p. 193, in Niculescu (2001), describing hydraulic and electrical engineering systems, is asymptotically stable if and only if the trivial solution of the corresponding system (44), with a suitable negative real H, is asymptotically stable (for instance, $H = A_0$). That is, when a forcing input is considered, conditions given in Corollary 14 are, in this case, also necessary. ■

6. Illustrative practical example

Let us consider the electrical device containing an LC transmission line given in Rasvan and Niculescu (2002) (see also the example 5.55, p. 213 in Niculescu (2001)), described by the following coupled delay differential and difference equations

$$\dot{\xi}(t) = A\xi(t) + \begin{pmatrix} -\frac{1}{C_1} f_1(\xi_1(t)) \\ 0 \end{pmatrix} + Bx(t-\Delta) + \begin{bmatrix} -1/R_1C_1 \\ 0 \end{bmatrix} E(t), \quad t \geq 0, \ a.e. \tag{45}$$

$$x(t) = D\xi(t) + Fx(t-\Delta), \quad t \geq 0$$
$$\xi(\tau) = \xi_0(\tau), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0],$$

where: $\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \in R^2$; $x(t) \in R^2$; $E(t) \in R$ is the forcing input (measurable, locally essentially bounded); $\Delta = \sqrt{LC}$; $\xi_0, x_0 \in C([-\Delta, 0]; R^2)$;

$$A = \begin{bmatrix} -\dfrac{1+R_1\sqrt{\frac{C}{L}}}{R_1C_1} & 0 \\ 0 & -\dfrac{\sqrt{\frac{C}{L}}}{\left(1+R_2\sqrt{\frac{C}{L}}\right)C_2} \end{bmatrix};$$

$$B = \begin{bmatrix} 0 & 2\dfrac{\sqrt{\frac{C}{L}}}{C_1} \\ 2\dfrac{\sqrt{\frac{C}{L}}}{\left(1+R_2\sqrt{\frac{C}{L}}\right)C_2} & 0 \end{bmatrix}; \tag{46}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{1+R_2\sqrt{\frac{C}{L}}} \end{bmatrix}; \qquad F = \begin{bmatrix} 0 & -1 \\ -\dfrac{1-R_2\sqrt{\frac{C}{L}}}{1+R_2\sqrt{\frac{C}{L}}} & 0 \end{bmatrix};$$

R_1, R_2, C, C_1, C_2, L are (positive real) electrical parameters (resistors, capacitors, inductors); f_1 is a scalar continuous function describing a nonlinear resistor. The matrix F has eigenvalues inside the open unit circle, therefore the linear continuous-time difference part of system (45), with no forcing input ($\xi(t) = 0$), is asymptotically stable. By Theorem 3.5, p. 275, in Hale and Lunel (1993), it follows that the continuous-time difference part of system (45) is ISS. Let us now apply Theorem 7 with the following Liapunov–Krasovskii functional

$$V\left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\right) = \phi_1^{\mathrm{T}}(0)P\phi_1(0) + \int_{-\Delta}^{0} \phi_2^{\mathrm{T}}(\tau)Q(\tau)\phi_2(\tau)\mathrm{d}\tau, \tag{47}$$

where: $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$, $\phi_1 \in C([-\Delta, 0]; R^2)$, $\phi_2 \in C([-\Delta, 0]; R^2)$; $P \in R^{2\times 2}$ is a diagonal positive matrix;

$$Q(\tau) = -\frac{\tau}{\Delta}Q_1 + \frac{\tau+\Delta}{\Delta}Q_2 \quad \tau \in [-\Delta, 0]; \tag{48}$$

$Q_1, Q_2 \in R^{2\times 2}$ are positive symmetric matrices, $Q_2 > Q_1$.

The derivative of such functional is given by

$$\begin{aligned} D^+V&\left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, e\right) \\ &= \eta^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}P + PA + D^{\mathrm{T}}Q_2D & PB + D^{\mathrm{T}}Q_2F \\ B^{\mathrm{T}}P + F^{\mathrm{T}}Q_2D & F^{\mathrm{T}}Q_2F - Q_1 \end{bmatrix} \eta \\ &\quad - \frac{1}{\Delta}\int_{-\Delta}^{0} \phi_2^{\mathrm{T}}(\tau)(Q_2 - Q_1)\phi_2(\tau)\mathrm{d}\tau \\ &\quad - \frac{2}{C_1}P(1,1)\phi_1^1(0)f_1(\phi_1^1(0)) - P(1,1)\frac{2}{R_1C_1}\phi_1^1(0)e, \end{aligned} \tag{49}$$

where: $\eta^{\mathrm{T}} = \begin{bmatrix} \phi_1^{\mathrm{T}}(0) & \phi_2^{\mathrm{T}}(-\Delta) \end{bmatrix}$; $P(1,1)$ is the element first row first column of matrix P, $\phi_1^1(0)$ is the first component of $\phi_1(0)$.

If $\sqrt{\phi_1^{\mathrm{T}}(0)\phi_1(0) + \int_{-\Delta}^{0} \phi_2^{\mathrm{T}}(\tau)\phi_2(\tau)\mathrm{d}\tau} \geq \rho|e|$, with ρ a positive real, then the following inequality holds

$$P(1,1)\frac{2}{R_1C_1}|\phi_1^1(0)e| \leq \frac{1}{\rho}P^2(1,1)\frac{1}{R_1^2C_1^2}|\phi_1^1(0)|^2 + \frac{1}{\rho}\phi_1^{\mathrm{T}}(0)\phi_1(0) + \frac{1}{\rho}\int_{-\Delta}^{0} \phi_2^{\mathrm{T}}(\tau)\phi_2(\tau)\mathrm{d}\tau. \tag{50}$$

Since ρ can be arbitrarily chosen, it follows from Theorem 7 that system (45) is ISS with respect to the (measurable and locally essentially bounded) input E, provided the following conditions hold for suitable positive diagonal matrix $P \in R^{2\times 2}$, positive symmetric matrices $Q_2, Q_1 \in R^{2\times 2}$, $Q_2 > Q_1$, and positive real ω:

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + D^{\mathrm{T}}Q_2D & PB + D^{\mathrm{T}}Q_2F \\ B^{\mathrm{T}}P + F^{\mathrm{T}}Q_2D & F^{\mathrm{T}}Q_2F - Q_1 \end{bmatrix} + \omega \begin{bmatrix} 1 & 0_{1,3} \\ 0_{3,1} & 0_{3,3} \end{bmatrix} < 0; \tag{51}$$

$$\sigma f_1(\sigma) \geq -\frac{C_1\omega}{2P_1(1,1)}\sigma^2, \quad \forall \sigma \in R. \tag{52}$$

Note that here

$$N_a\left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\right) = \sqrt{\phi_1^{\mathrm{T}}(0)\phi_1(0) + \int_{-\Delta}^{0} \phi_2^{\mathrm{T}}(\tau)\phi_2(\tau)\mathrm{d}\tau}, \tag{53}$$

which defines a seminorm, has been chosen. Though an analysis of the electrical parameters value such that the conditions (51) and (52) are verified is beyond the aims of this paper, we show here by a numerical example that the conditions here provided may be less conservative than the ones provided in Rasvan and Niculescu (2002). Let us consider the following value of the parameters $R1 = 10$, $R2 = 500$, $C = C1 = C2 = 0.00001$, $L = 0.01$. With this value of the parameters, the condition (51) is feasible (checked by Matlab), and the condition (52) becomes

$$\sigma f_1(\sigma) \geq -0.084\sigma^2, \quad \forall \sigma \in R. \tag{54}$$

That is, with this value of the parameters, system (45) is ISS with respect to any measurable locally essentially bounded input, for any positive resistor (described by f_1) and for negative resistors which verify (54). Note that the result here given is global, therefore it cannot be achieved by means of first-order approximation methods. With this value of the parameters, the conditions given in the paper (Rasvan and Niculescu (2002), see (14), (17), (18)) provide exponential stability of solutions, with piece-wise continuous bounded inputs, if $\sigma f_1(\sigma) \geq 0.015\,\sigma^2$, $\forall \sigma \in R$, thus not allowing negative resistors and limiting the class of positive resistors. The lower conservativeness of condition (54) is due to an increased number of variables in the LMI (51) with respect to the ones in the LMI (16) provided in Rasvan and Niculescu (2002).

7. Conclusions

In this paper the input-to-state stability of systems described by coupled delay differential and difference equations is studied. The Liapunov–Krasovskii methodology to check such type of stability is provided. Sufficient LMI conditions are provided for the general linear case, and the equivalence of asymptotic stability and input-to-state stability for neutral systems in Hale's form with stable difference operator is proved. A nonlinear system taken from the literature, concerning an electrical device with lossless transmission line, is studied, showing the effectiveness of the methodology.

An interesting topic to be studied is the application of small-gain-type arguments to conclude ISS of the coupled system on the basis of the ISS of each individual subsystem (the differential and the difference ones). Small-gain arguments allow us to consider inputs also in the difference part of the equations. The application of small-gain arguments to the systems here studied will be the topic of forthcoming work.

Links between global exponential stability and ISS are an interesting topic to be investigated too. It is well known that globally exponentially stable (in the unforced case) nonlinear delay-free systems are also input-to-state stable, provided that the function describing the dynamics is globally Lipschitz (see Lemma 4.6, p. 176, in Khalil (1996)). This property holds also for systems described by retarded functional differential equations, provided that the functional describing the dynamics is globally Lipschitz (see Yeganefar, Pepe, and Dambrine (in press)). The proof of such result is obtained by means of the Liapunov–Krasovskii converse Theorem (see Theorem 1.3, p. 210, in Kolmanovskii and Myshkis (1999)). An analogous converse theorem for systems described by coupled delay differential and difference equations is missing in the literature, and should be studied first, in order to investigate the above links for the systems here studied.

Converse Liapunov–Krasovskii Theorems for the robust stability (see Lin, Sontag, and Wang (1996) as far as delay-free systems are concerned) of time-delay systems will be also topic of forthcoming investigation.

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References

Angeli, D., Sontag, E. D., & Wang, Y. (2000). A Characterization of integral input to state stability. *IEEE Transactions on Automatic Control*, *45*, 1082–1097.
Driver, R. D. (1962). Existence and stability of solutions of a delay-differential system. *Archive for Rational Mechanics and Analysis*, (10), 401–426.
Fridman, E. (2002). Stability of linear descriptor systems with delay: A Lyapunov-based approach. *Journal of Mathematical Analysis and Applications*, *273*, 24–44.
Germani, A., Manes, C., & Pepe, P. (2000). Local asymptotic stability for nonlinear state feedback delay systems. *Kybernetika*, *36*(1), 31–42.
Germani, A., Manes, C., & Pepe, P. (2003). Input–output linearization with delay cancellation for nonlinear delay systems: The problem of the internal stability. *International Journal of Robust and Nonlinear Control*, *13*(9), 909–937.
Hale, J. K., & Martinez Amores, P. (1977). Stability in neutral equations. *Journal of Nonlinear Analysis. Theory, Methods and Applications*, *1*(1), 161–172.
Hale, J. K., & Lunel, S. M. V. (1993). *Introduction to functional differential equations*. Springer Verlag.
Hale, J. K., & Lunel, S. M. V. (2002). Strong Stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, *19*, 5–23.
Jiang, Z.-P., & Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, *37*, 857–869.
Kappel, F., & Kunisch, K. (1981). Spline approximations for neutral functional differential equations. *SIAM Journal on Numerical Analysis*, *18*(6), 1058–1080.
Karafyllis, I. (2006a). Lyapunov theorems for systems described by retarded functional differential equations. *Journal of Nonlinear Analysis. Theory, Methods & Applications*, *64*(3), 590–617.
Karafyllis, I. (2006b). Non-uniform in time robust global asymptotic output stability for discrete-time systems. *International Journal of Robust and Nonlinear Control*, *16*(4), 191–214.
Khalil, H. K. (1996). *Nonlinear systems* (third edition). New Jersey: Prentice Hall.
Kolmanovskii, V., & Myshkis, A. (1999). *Introduction to the theory and applications of functional differential equations*. Kluwer Academic Publishers.
Lin, Y., Sontag, E. D., & Wang, Y. (1996). A smooth converse lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, *34*, 124–160.
Niculescu, S.-I. (2001). *Lecture notes in control and information sciences, delay effects on stability, a robust control approach*, Springer.
Park, J. H., & Won, S. (1999). Asymptotic stability of neutral systems with multiple delays. *Journal of Optimization Theory and Applications*, *103*(1), 183–200.
Pepe, P. (2003). The Liapunov's second method for continuous time difference equations. *International Journal of Robust and Nonlinear Control*, *13*(15), 1389–1405.
Pepe, P. (2005). On the asymptotic stability of coupled delay differential and continuous time difference equations. *Automatica*, *41*(1), 107–112.
Pepe, P. (2007a). On Liapunov–Krasovskii functionals under carathéodory conditions. *Automatica*, *43*(4), 701–706.
Pepe, P. (2007b). The problem of the absolute continuity for Liapunov–Krasovskii functionals. *IEEE Transactions on Automatic Control*, *52*(5), 953–957.
Pepe, P., & Jiang, Z.-P. (2006). A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, *55*(12), 1006–1014.
Pepe, P., Jiang, Z.-P., & Fridman, E. (2008). A new Lyapunov–Krasovskii methodology for coupled delay differential and difference equations. *International Journal of Control*, *81*(1), 107–115.
Pepe, P., & Verriest, E. I. (2003). On the stability of coupled delay differential and continuous time difference equations. *IEEE Transactions on Automatic Control*, *48*(8), 1422–1427.
Rasvan, V., & Niculescu, S.-I. (2002). Oscillations in lossless propagation models: A Liapunov–Krasovskii approach. *IMA Journal of Mathematical Control and Information*, *19*, 157–172.
Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, *34*, 435–443.
Teel, A. R. (1998). Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, *43*(7), 960–964.
Yeganefar, N., Pepe, P., & Dambrine, M. (2008). Input-to-state stability of time-delay systems: A link with exponential stability, *IEEE Transactions on Automatic Control* (in press).

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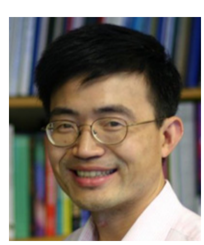

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