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A system-theoretic framework for a wide class of systems II: Input-to-output stability

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Abstract

In this work characterizations of the notion of Weighted Input-to-Output Stability (WIOS) for a wide class of systems with disturbances are given. Particularly, for systems with continuous dependence of the solution on the initial state and the input, the WIOS property is shown to be equivalent to robust forward completeness from the input and robust global asymptotic output stability for the corresponding input-free system.

Keywords: Hybrid systems; Input-to-output stability

1. Introduction

Let $f:\Re^n \times U \to \Re^n$, where $U \subseteq \Re^m$ with $0 \in U$, a locally Lipschitz vector field with $f(0,0)=0$ and consider the solution $x(t)$ of the initial value problem

$$\begin{aligned}&\dot{x}(t) = f\big(x(t), u(t)\big),\\ &x(t) \in \Re^n, \quad u(t) \in U, \qquad (1.1)\end{aligned}$$

corresponding to some measurable and locally bounded input $u:\Re^+ \to U$, with initial condition $x(0)=x_0 \in \Re^n$. It is generally known that the 0-GAS property, i.e., global asymptotic stability of the equilibrium point $0 \in \Re^n$ for the unperturbed system

$$\dot{x}(t) = f\big(x(t), 0\big) \qquad (1.2)$$

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does not guarantee that the solution of (1.1) satisfies $\lim_{t\to+\infty} |x(t)| = 0$ for all measurable and locally bounded inputs $u : \Re^+ \to U$ with $\lim_{t\to+\infty} |u(t)| = 0$ (CICS property—Converging Input Converging State). Moreover, global asymptotic stability of the equilibrium point $0 \in \Re^n$ for the unperturbed system (1.2) (0-GAS property) does not guarantee that the solution of (1.1) remains bounded for all measurable and bounded inputs $u : \Re^+ \to U$ (BIBS property—Bounded Input Bounded State). These important robustness properties (CICS and BIBS properties) are satisfied if system (1.1) satisfies the Input-to-State Stability (ISS) property from the input u. The notion of ISS was given by E.D. Sontag for finite-dimensional systems in [25] and was proved to be useful in many areas of mathematics (Mathematical Control Theory, Dynamical Systems Theory). Various sufficient conditions for ISS were given in [26,27,29] and extensions to continuous-time systems with outputs and discrete-time systems were considered in [7–9,31,32]. Recently, non-uniform notions of ISS that guarantee the CICS and BIBS properties for time-varying systems were given in [18].

It should be emphasized that the 0-GAS property does not even guarantee the existence of a positive continuous function $\delta : \Re^+ \to (0, +\infty)$ such that the solution of (1.1) satisfies $\lim_{t\to+\infty} |x(t)| = 0$ for all measurable and locally bounded inputs $u : \Re^+ \to U$ with $\lim_{t\to+\infty} \delta(t)|u(t)| = 0$ (CWICS property—Converging Weighted Input Converging State). Moreover, the 0-GAS property does not guarantee the existence of a positive continuous function $\delta : \Re^+ \to (0, +\infty)$ such that the solution of (1.1) remains bounded for all measurable and locally bounded inputs $u : \Re^+ \to U$ with $\sup_{t\geqslant 0} \delta(t)|u(t)| < +\infty$ (BWIBS property—Bounded Weighted Input Bounded State). These important robustness properties (CWICS and BWIBS properties) are satisfied if system (1.1) satisfies the non-uniform in time Input-to-State Stability (ISS) property from the input u(see [12–15]).

A recent result in [13] showed that the 0-GAS property combined with forward completeness of system (1.1) for all measurable and locally bounded inputs $u : \Re^+ \to U$ guarantees the non-uniform in time ISS property from the input u (and consequently the CWICS and BWIBS properties are also satisfied).

In the present work we study the problem of finding necessary and sufficient conditions for the CWICS and BWIBS properties. Particularly, we extend the validity of the result showed in [13] to time-varying systems with outputs and we show for a very wide class of systems (that includes certain hybrid systems, systems described by retarded functional differential equations and systems described by partial differential equations) that:

> "If the unperturbed system is globally asymptotically stable and the system has the property of forward completeness to all allowed inputs $u : \Re^+ \to U$ then the system satisfies the CWICS and BWIBS properties."

The proof of the above general result is completely different from the proof given in [13] for systems described by ordinary differential equations. Here, we exploit the system theoretic framework presented in [16] and the solution of the system is related to the solution of an infinite-dimensional discrete-time system (abstract discretization). Recent results developed for infinite-dimensional discrete-time systems make possible the proof of the result under very general hypotheses. Essentially, it is shown that the CWICS and BWIBS properties are consequences of three requirements: (i) 0-GAS property, (ii) forward completeness, and (iii) continuity with respect to the initial state and input. It is expected that the discovery of sufficient conditions for the CWICS and BWIBS properties for a wide class of systems with outputs will motivate similar advances to the advances which were triggered by the ISS property for finite-dimensional systems.

It should be emphasized that stronger robustness results (e.g., BIBS and CICS properties) demand stronger hypotheses (e.g., the hypotheses used in [26,27] for finite-dimensional systems described by ordinary differential equations) in conjunction with the 0-GAS property. The link which connects the CWICS and BWIBS properties with the CICS and BIBS properties is yet to be found. This will be the subject of future research.

The structure of this paper is as follows. In Section 2 the definitions of the stability notions used in this paper are provided. In Section 3, the main results are presented and proved. Section 4 contains certain robustness results for sampled-data feedback control systems with uniform sampling rate. Finally, the conclusions of the paper as well as a brief list with some open problems are provided in Section 5.

Notations. Throughout this paper we adopt the following notations:

* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose.
* We denote by $[R]$ the integer part of the real number R, i.e., the greatest integer, which is less than or equal to R.
* We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. For definitions of classes K, K_∞, KL see [17].
* By $\| \|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. Let $U \subseteq \mathcal{X}$ with $0 \in U$. By $B_U[0,r] := \{u \in U;\ \|u\|_{\mathcal{U}} \leqslant r\}$ we denote the closed sphere in $U \subseteq \mathcal{X}$ with radius $r \geqslant 0$, centered at $0 \in U$. By $B[0,r]$ we denote the closed sphere with radius $r \geqslant 0$ in $\Re^n$, centered at $0 \in \Re^n$.
* By $\mathcal{M}(U)$ we denote the set of all functions $u : \Re^+ \to U$. By u_0 we denote the identity zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geqslant 0$.
* A partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ is an increasing sequence of times with $T_0 = 0$ and $T_i \to +\infty$. For every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ we define $p_\pi(t) := \min\{T \in \pi;\ t < T\}$. The diameter of the partition is defined as $\sup\{T_{i+1} - T_i;\ i = 0, 1, 2, \ldots\} \leqslant +\infty$.

2. Definitions and preliminary results

The definition of a control system with outputs was given in [16], inspired from the definitions in [11,28]. However, in this work a "weaker" version is adopted, which allows important classes of systems (hybrid systems) to be considered as control systems with outputs. Moreover, we focus on continuous-time systems for reasons that are explained below.

Definition 2.1. A control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs consists of

(i) a set U (control set) which is a subset of a normed linear space $\mathcal{U}$ with $0 \in U$ and a set $M_U \subseteq \mathcal{M}(U)$ (allowable control inputs) which contains at least the identity zero input $u_0 \in M_U$ (i.e., the input that satisfies $u_0(t) = 0 \in U$ for all $t \geqslant 0$),
(ii) a set D (disturbance set) and a set $M_D \subseteq \mathcal{M}(D)$, which is called the "set of allowable disturbances,"
(iii) a pair of normed linear spaces $\mathcal{X}, \mathcal{Y}$ called the "state space" and the "output space," respectively,

(iv) a continuous map $H:\Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ that maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, called the "output map,"
(v) and the map $\phi: A_\phi \to \mathcal{X}$ where $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$, called the "transition map," which has the following properties:

(1) *Existence*: For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t] \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$.
(2) *Identity property*: For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t)$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *Weak semigroup property*: For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ there exists a set $\pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$ and a constant $r > 0$, such that for each $t \geqslant t_0$ with $(t, t_0, x_0, u, d) \in A_\phi$:
   (a) $(\tau, t_0, x_0, u, d) \in A_\phi$ for all $\tau \in [t_0, t]$,
   (b) $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t] \cap \pi(t_0, x_0, u, d)$,
   (c) if $(t + r, t_0, x_0, u, d) \in A_\phi$, then it holds that $\pi(t_0, x_0, u, d) \cap [t, t + r] \neq \emptyset$,
   (d) for all $\tau \in \pi(t_0, x_0, u, d)$ with $(\tau, t_0, x_0, u, d) \in A_\phi$ we have $\pi(\tau, \phi(\tau, t_0, x_0, u, d), u, d) = \pi(t_0, x_0, u, d) \cap [\tau, +\infty)$.

Let $T > 0$. A control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs is called T*-periodic*, if:

(a) $H(t + T, x, u) = H(t, x, u)$ for all $(t, x, u) \in \Re^+ \times \mathcal{X} \times U$,
(b) for every $(u, d) \in M_U \times M_D$ and integer k there exist inputs $P_{kT}u \in M_U$, $P_{kT}d \in M_D$ with $(P_{kT}u)(t) = u(t + kT)$ and $(P_{kT}d)(t) = d(t + kT)$ for all $t + kT \geqslant 0$,
(c) for each $(t, t_0, x_0, u, d) \in A_\phi$ with $t \geqslant t_0$ and for each integer k with $t_0 - kT \geqslant 0$ it follows that $(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d) \in A_\phi$, $\pi(t_0 - kT, x_0, P_{kT}u, P_{kT}d) = \bigcup_{\tau \in \pi(t_0, x_0, u, d)} \{\tau - kT\}$ and $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$.

Remark 2.2.

(a) Notice that Definition 2.1 allows us to consider all discrete-time systems as continuous-time systems with $\pi(t_0, x_0, u, d) := Z^+ \cap [t_0, +\infty)$, where Z^+ denotes the set of non-negative integers and $\phi(t, t_0, x_0, u, d) = \phi([t], t_0, x_0, u, d)$, where $[t]$ denotes the integer part of $t \geqslant t_0$.
(b) The difference between the present definition of a control system with outputs and Definition 2.1 in [14] lies in property (4) (semigroup property). In the above definition we do not require that for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ there exists $t > t_0$ such that $[t_0, t) \subseteq \pi(t_0, x_0, u, d)$ (in contrast with Definition 2.1 in [14]). This modification allows us to study important classes of systems, which were excluded by Definition 2.1 in [14] (see [16], where it is shown that a wide class of hybrid systems satisfies the requirements of the present definition of a control system with outputs).

We next provide definitions of some important classes of systems with outputs (see also [16]).

Definition 2.3. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs. We say that system Σ

(i) has the "*Boundedness-Implies-Continuation*" (*BIC*) property if for each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in [t_0, +\infty]$, such that $[t_0, t_{\max}) \times \{(t_0, x_0, u, d)\} \subseteq A_\phi$ and for all $t \geqslant t_{\max}$ it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$;

(ii) is *forward complete* if for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, $(t, t_0, x_0, u, d) \in A_\phi$ for all $t \geqslant t_0$. Clearly, every forward complete control system has the BIC property;

(iii) is simply *robustly forward complete* (*RFC*) if it has the BIC property and for every $r \geqslant 0$, $T \geqslant 0$, it holds that

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u_0, d)\bigr\|_{\mathcal{X}};\ s \in [0, T],\ \|x_0\|_{\mathcal{X}} \leqslant r,\ t_0 \in [0, T],\ d \in M_D\bigr\} < +\infty;$$

(iv) is *robustly forward complete* (*RFC*) from the input $u \in M_U$ if it has the BIC property and for every $r \geqslant 0$, $T \geqslant 0$, it holds that

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u, d)\bigr\|_{\mathcal{X}};\ u \in \mathcal{M}\bigl(B_U[0, r]\bigr) \cap M_U,\ s \in [0, T],\ \|x_0\|_{\mathcal{X}} \leqslant r,\ t_0 \in [0, T],\ d \in M_D\bigr\} < +\infty.$$

It is clear that Definition 2.3 provides additional properties for the set $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$ of Definition 2.1. For example, if system Σ satisfies the BIC property then for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $t_{\max} = t_{\max}(t_0, x_0, u, d) > t_0$ such that

$$A_\phi = \bigcup_{(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D} [t_0, t_{\max}) \times \bigl\{(t_0, x_0, u, d)\bigr\}.$$

Moreover, if the system is forward complete then

$$A_\phi = \biggl(\bigcup_{t_0 \in \Re^+} [t_0, +\infty) \times \{t_0\}\biggr) \times \mathcal{X} \times M_U \times M_D.$$

The following definition clarifies the notion of an equilibrium point for control systems (see also [16]).

Definition 2.4. Consider a continuous-time control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ and suppose that $H(t, 0, 0) = 0$ for all $t \geqslant 0$. We say that $0 \in \mathcal{X}$ is a *robust equilibrium point* for Σ if

(i) for every $(t, t_0, d) \in \Re^+ \times \Re^+ \times M_D$ with $t \geqslant t_0$ it holds that $\phi(t, t_0, 0, u_0, d) = 0$,

(ii) for every $\varepsilon > 0$, $T, h \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x) \in [0, T] \times \mathcal{X}$, $\tau \in [t_0, t_0 + h]$ with $\|x\|_{\mathcal{X}} < \delta$ it holds that $(\tau, t_0, x, u_0, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\bigl\{\bigl\|\phi(\tau, t_0, x, u_0, d)\bigr\|_{\mathcal{X}};\ d \in M_D,\ \tau \in [t_0, t_0 + h],\ t_0 \in [0, T]\bigr\} < \varepsilon.$$

We say that $0 \in \mathcal{X}$ is a *robust equilibrium point from the input $u \in M_U$* for Σ if $0 \in \mathcal{X}$ is a robust equilibrium point for Σ and

(iii) for every $\varepsilon > 0$, $T, h \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x, u) \in [0, T] \times \mathcal{X} \times M_U, \tau \in [t_0, t_0 + h]$ with $\|x\|_{\mathcal{X}} + \sup_{t \geqslant 0} \|u(t)\|_{\mathcal{U}} < \delta$ it holds that $(\tau, t_0, x, u, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\big\{\big\|\phi(\tau, t_0, x, u, d)\big\|_{\mathcal{X}};\ d \in M_D,\ \tau \in [t_0, t_0 + h],\ t_0 \in [0, T]\big\} < \varepsilon.$$

The notion of Robust Global Asymptotic Output Stability (RGAOS) is an internal stability property, i.e., is applied only when the control input is identically zero (input-free case). Essential properties of RGAOS are developed in [14,16] for control systems with outputs. Particularly, Lemmas 3.3–3.5 and Theorem 3.6 in [16] hold (in [14] the notion of RGAOS was given by the name non-uniform in time Robust Global Asymptotic Output Stability).

Definition 2.5. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs that has the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. We say that Σ is *Robustly Globally Asymptotically Output Stable* (*RGAOS*) if Σ is RFC and the following properties hold:

(P1) Σ is *Robustly Lagrange Output Stable*, i.e., for every $\varepsilon > 0$, $T \geqslant 0$, it holds that

$$\sup\big\{\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}};\ t \geqslant t_0,\ \|x_0\|_{\mathcal{X}} \leqslant \varepsilon,\ t_0 \in [0, T],\ d \in M_D\big\} < +\infty$$

(*Robust Lagrange Output Stability*).

(P2) Σ is *Robustly Lyapunov Output Stable*, i.e., for every $\varepsilon > 0$ and $T \geqslant 0$ there exists $\delta := \delta(\varepsilon, T) > 0$ such that:

$$\|x_0\|_{\mathcal{X}} \leqslant \delta, \quad t_0 \in [0, T] \quad \Rightarrow \quad \big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \leqslant \varepsilon, \quad \forall t \geqslant t_0,\ \forall d \in M_D$$

(*Robust Lyapunov Output Stability*).

(P3) Σ satisfies the *Robust Output Attractivity Property*, i.e., for every $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists $\tau := \tau(\varepsilon, T, R) \geqslant 0$, such that:

$$\|x_0\|_{\mathcal{X}} \leqslant R, \quad t_0 \in [0, T] \quad \Rightarrow \quad \big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \leqslant \varepsilon, \quad \forall t \geqslant t_0 + \tau,\ \forall d \in M_D.$$

Moreover, if there exists $a \in K_\infty$ such that $a(\|x\|_{\mathcal{X}}) \leqslant \|H(t, x, 0)\|_{\mathcal{Y}}$ for all $(t, x) \in \Re^+ \times \mathcal{X}$, then we say that Σ is *Robustly Globally Asymptotically Stable* (*RGAS*).

In contrast to RGAOS, the notion of Weighted Input-to-Output Stability (WIOS) is an external stability property, i.e., it can be applied to cases where the input is not identically zero. The definition of the WIOS property is given next.

Definition 2.6. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. We say that Σ satisfies the *Weighted Input-to-Output Stability property* (*WIOS*) from the input $u \in M_U$ if Σ is RFC from the input $u \in M_U$ and there exist functions $\sigma \in KL$, $\beta, \gamma \in K^+$, $\rho \in K_\infty$ such that the following estimate holds for all $u \in M_U$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$:

$$\big\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\big\|_{\mathcal{Y}} \leqslant \max\Big\{\sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}},t-t_0\big),\ \sup_{\tau\in[t_0,t]}\sigma\big(\beta(\tau)\rho\big(\gamma(\tau)\big\|u(\tau)\big\|_{\mathcal{U}}\big),t-\tau\big)\Big\}. \tag{2.1}$$

Moreover, if there exists $a\in K_\infty$ such that $a(\|x\|_{\mathcal{X}})\leqslant\|H(t,x,u)\|_{\mathcal{Y}}$ for all $(t,x,u)\in\Re^+\times\mathcal{X}\times U$ then we say that Σ satisfies the *Weighted Input-to-State Stability property* (*WISS*) from the input $u\in M_U$.

The following lemma must be compared to Lemma 1.1 in [4, p. 131] and Proposition 3.2 in [10]. It shows that for periodic systems estimate (2.1) is equivalent to a simpler estimate.

Lemma 2.7. *Suppose that $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ is T-periodic. If Σ satisfies the WIOS property from the input $u\in M_U$, then there exist functions $\sigma\in KL$, $\gamma\in K^+$ and $\rho\in K_\infty$ such that estimate* (2.1) *holds for all $(t_0,x_0,d,u)\in\Re^+\times\mathcal{X}\times M_D\times M_U$ and $t\geqslant t_0$ with $\beta(t)\equiv 1$.*

Proof. The proof is based on the following observation: if $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ is T-periodic then for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$ it holds that $\phi(t,t_0,x_0,u,d)=\phi(t-kT,t_0-kT,x_0,P_{kT}u,P_{kT}d)$ and

$$H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big) = H\big(t-kT,\phi(t-kT,t_0-kT,x_0,P_{kT}u,P_{kT}d),(P_{kT}u)(t-kT)\big),$$

where $k:=[\frac{t_0}{T}]$ denotes the integer part of $\frac{t_0}{T}$ and the inputs $P_{kT}u\in M_U$, $P_{kT}d\in M_D$ are defined in Definition 2.1.

Since $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ satisfies the WIOS property from the input $u\in M_U$, there exist functions $\sigma\in KL$, $\beta,\gamma\in K^+$, $\rho\in K_\infty$ such that (2.1) holds for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$ and $t\geqslant t_0$. Consequently, it follows that the following estimate holds for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$ and $t\geqslant t_0$:

$$\big\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\big\|_{\mathcal{Y}} \leqslant \max\Big\{\sigma\big(\beta(t_0-kT)\|x_0\|_{\mathcal{X}},t-t_0\big),\ \sup_{\tau\in[t_0-kT,t-kT]}\sigma\big(\beta(\tau)\rho\big(\gamma(\tau)\big\|(P_{kT}u)(\tau)\big\|_{\mathcal{U}}\big),t-kT-\tau\big)\Big\}.$$

Setting $\tau=s-kT$ and since $0\leqslant t_0-[\frac{t_0}{T}]T<T$, for all $t_0\geqslant 0$, we obtain

$$\big\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\big\|_{\mathcal{Y}} \leqslant \max\Big\{\tilde\sigma\big(\|x_0\|_{\mathcal{X}},t-t_0\big),\ \sup_{s\in[t_0,t]}\tilde\sigma\big(\beta(s-kT)\rho\big(\gamma(s-kT)\big\|(P_{kT}u)(s-kT)\big\|_{\mathcal{U}}\big),t-s\big)\Big\}, \tag{2.2}$$

where $\tilde\sigma(s,t):=\sigma((1+r)s,t)$ and $r:=\max\{\beta(t);\ 0\leqslant t\leqslant T\}$. By virtue of Lemma 2.3 in [12], there exist functions $\tilde\gamma\in K^+$ and $\tilde\rho\in K_\infty$ such that $\beta(t)\rho(\gamma(t)s)\leqslant\tilde\rho(\tilde\gamma(t)s)$ for all $t,s\geqslant 0$. The previous observation in conjunction with estimate (3.8) and the identity $(P_{kT}u)(s-kT)=u(s)$ for all $s\geqslant 0$, implies that the following estimate holds for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$ and $t\geqslant t_0$:

$$\begin{aligned}&\left\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\right\|_{\mathcal{Y}}\\&\quad\leqslant\max\Big\{\tilde{\sigma}\big(\|x_0\|_{\mathcal{X}},t-t_0\big),\sup_{s\in[t_0,t]}\tilde{\sigma}\big(\tilde{\rho}\big(\tilde{\gamma}(s-kT)\left\|u(s)\right\|_{\mathcal{U}}\big),t-s\big)\Big\}.\end{aligned}\tag{2.3}$$

Setting $\tilde{\gamma}(t):=\max\{\gamma(s);\ s\in[0,t]\}$, we obtain from (2.3) that estimate (2.1) holds with $\beta(t)\equiv 1$ and $\tilde{\sigma}\in KL$, $\tilde{\rho}\in K_\infty$, $\tilde{\gamma}\in K^+$ in place of $\sigma\in KL$, $\rho\in K_\infty$, $\gamma\in K^+$, respectively. The proof is complete. □

We next provide necessary conditions for the WIOS property.

Lemma 2.8 *(Necessary Conditions for the WIOS property). Suppose that* $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ *satisfies the WIOS property; particularly there exist functions* $\sigma\in KL$, $\beta,\gamma\in K^+$, $\rho\in K_\infty$ *such that estimate* (2.1) *holds for all* $u\in M_U$, $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ *and* $t\geqslant t_0$. *Then there exist functions* $\zeta\in K_\infty$, $\delta\in K^+$ *such that the following estimate holds for all* $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$ *and* $t\geqslant t_0$:

$$\begin{aligned}&\left\|H\big(t,\phi(t,t_0,x_0,u,d),u(t)\big)\right\|_{\mathcal{Y}}\\&\quad\leqslant\max\Big\{\sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}},t-t_0\big),\sup_{t_0\leqslant\tau\leqslant t}\zeta\big(\delta(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\big)\Big\}\end{aligned}\tag{2.4}$$

(*Sontag-like estimate*).

Proof. Suppose that $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ satisfies the WIOS property and there exist functions $\sigma\in KL$, $\beta,\gamma\in K^+$, $\rho\in K_\infty$ such that estimate (2.1) holds for all $u\in M_U$, $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ and $t\geqslant t_0$. By virtue of Lemma 2.3 in [12], there exist functions $\delta\in K^+$ and $\tilde{\rho}\in K_\infty$ such that $\beta(t)\rho(\gamma(t)s)\leqslant\tilde{\rho}(\delta(t)s)$ for all $t,s\geqslant 0$. Let $\zeta\in K_\infty$ such that $\zeta(s)\geqslant\sigma(\tilde{\rho}(s),0)$ for all $s\geqslant 0$. Estimate (2.4) follows directly from estimate (2.1) and the previous definitions. □

Remark 2.9. We call estimate (2.4) "a Sontag-like estimate," because E.D. Sontag formulated ISS in [25] for finite-dimensional continuous-time systems using an estimate of the form

$$\left\|\phi(t,t_0,x_0,u,d)\right\|_{\mathcal{X}}\leqslant\max\Big\{\sigma\big(\|x_0\|_{\mathcal{X}},t-t_0\big),\sup_{t_0\leqslant\tau\leqslant t}\zeta\big(\left\|u(\tau)\right\|_{\mathcal{U}}\big)\Big\}.\tag{2.5}$$

Moreover, Sontag and Wang formulated IOS in [31,32] for continuous-time finite-dimensional systems using an estimate of the form (2.4) with $\beta(t)\equiv\delta(t)\equiv 1$. On the other hand, estimates of the form (2.1) ("fading memory estimates") were first used by Praly and Wang in [22] for the formulation of exp-ISS and by L. Grune in [3] for the formulation of Input-to-State Dynamical Stability (ISDS) with $H(t,x)=x$, $\beta(t)\equiv\gamma(t)\equiv 1$, which was proved to be qualitatively equivalent with (2.5) for finite-dimensional continuous-time systems.

Lemma 2.10 *(Necessary Conditions for the WIOS property). Consider system* $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ *and suppose that* Σ *is RFC from the input* $u\in M_U$ *and there exist functions* $\sigma\in KL$, $\beta,\delta\in K^+$, $\zeta\in K_\infty$ *such that estimate* (2.4) *holds for all* $u\in M_U$, $(t_0,x_0,d)\in\Re^+\times\mathcal{X}\times M_D$ *and* $t\geqslant t_0$. *Then the output trajectory* $t\to H(t,\phi(t,t_0,x_0,u,d),u(t))$ *is bounded for all inputs* $u\in M_U$ *with* $\sup_{t\geqslant 0}\delta(t)\|u(t)\|_{\mathcal{U}}<+\infty$ (*Bounded Weighted Input Bounded Output property*) *and* $\lim_{t\to+\infty}H(t,\phi(t,t_0,x_0,u,d),u(t))=0$ *for all inputs* $u\in M_{\mathcal{U}}$ *with* $\lim_{t\to+\infty}\delta(t)\|u(t)\|_{\mathcal{U}}=0$ (*Converging Weighted Input Converging Output property*).

Proof. The fact that the output trajectory $t \to H(t, \phi(t, t_0, x_0, u, d), u(t))$ is bounded for all inputs $u \in M_U$ with $\sup_{t \geqslant 0} \delta(t) \|u(t)\|_{\mathcal{U}} < +\infty$ is an immediate consequence of estimate (2.4). Next we show that $\lim_{t \to +\infty} H(t, \phi(t, t_0, x_0, u, d), u(t)) = 0$ for all inputs $u \in M_U$ with $\lim_{t \to +\infty} \delta(t) \|u(t)\|_{\mathcal{U}} = 0$ (Converging Weighted Input Converging Output property). Let arbitrary $\varepsilon > 0$, $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and input $u \in M_U$ with $\lim_{t \to +\infty} \delta(t) \|u(t)\|_{\mathcal{U}} = 0$. Clearly, there exists $T > t_0$ such that $\zeta(\delta(t) \|u(t)\|_{\mathcal{U}}) \leqslant \varepsilon$ for all $t \geqslant T$. By virtue of the weak semigroup property for Σ and since $(T + r, t_0, x_0, u, d) \in A_\phi$, where r is the constant involved in the weak semigroup property for Σ, there exists $\tilde{T} \in \pi(t_0, x_0, u, d) \cap [T, T + r] \neq \emptyset$ with $\phi(t, \tilde{T}, \phi(\tilde{T}, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $t \geqslant \tilde{T}$. Inequality (2.4) in conjunction with the fact that $\zeta(\delta(t) \|u(t)\|_{\mathcal{U}}) \leqslant \varepsilon$ for all $t \geqslant \tilde{T}$ implies for all $t \geqslant \tilde{T}$:

$$
\begin{aligned}
&\left\| H\big(t, \phi(t, t_0, x_0, u, d), u(t)\big) \right\|_{\mathcal{Y}} \\
&\quad = \left\| H\big(t, \phi\big(t, \tilde{T}, \phi(\tilde{T}, t_0, x_0, u, d), u, d\big), u(t)\big) \right\|_{\mathcal{Y}} \\
&\quad \leqslant \max\Big\{ \sigma\big(\beta(\tilde{T}) \left\| \phi(\tilde{T}, t_0, x_0, u, d) \right\|_{\mathcal{X}}, t - \tilde{T}\big), \sup_{\tilde{T} \leqslant \tau \leqslant t} \zeta\big(\delta(\tau) \left\| u(\tau) \right\|_{\mathcal{U}}\big) \Big\} \\
&\quad \leqslant \max\big\{ \sigma\big(\beta(\tilde{T}) \left\| \phi(\tilde{T}, t_0, x_0, u, d) \right\|_{\mathcal{X}}, t - \tilde{T}\big), \varepsilon \big\}.
\end{aligned}
$$

The estimate above implies that $\|H(t, \phi(t, t_0, x_0, u, d), u(t))\|_{\mathcal{Y}} \leqslant \varepsilon$ for $t \geqslant \tilde{T}$ sufficiently large. Since $\varepsilon > 0$ is arbitrary we conclude that $\lim_{\to +\infty} H(t, \phi(t, t_0, x_0, u, d), u(t)) = 0$. □

3. Main results

Consider a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs $H : \Re^+ \times \mathcal{X} \to Y$, which is Robustly Forward Complete from the input $u \in M_U$ and satisfies the following hypotheses:

(A1) There exists a partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ with finite diameter such that:
  - (i) for every sequence $d_i \in M_D$, $u_i \in M_U$, the functions $d(t) := d_i(t)$ and $u(t) := u_i(t)$, $T_i \leqslant t < T_{i+1}$, belong to M_D and M_U, respectively;
  - (ii) for each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ it holds that $\pi \cap (t_0, +\infty) \subset \pi(t_0, x_0, u, d)$, where $\pi(t_0, x_0, u, d)$ is the set involved in property (4) of Definition 2.1 (weak semigroup property);
  - (iii) for each bounded set $S \subset \mathcal{X}$ and for every $(T, R, \varepsilon) \in \Re^+ \times \Re^+ \times (0, +\infty)$ there exists $\delta > 0$ such that $\sup\{\|\phi(\tau, t_0, x, u, d) - \phi(\tau, t_0, x_0, v, d)\|_{\mathcal{X}};\ d \in M_D, \tau \in [t_0, p_\pi(t_0)]\} < \varepsilon$ for all $t_0 \in [0, T] \cap \pi$, $x, x_0 \in S$, $u, v \in \mathcal{M}(B_U[0, R]) \cap M_U$ with $\|x - x_0\|_{\mathcal{X}} + \sup_{t_0 \leqslant \tau \leqslant p(t_0)} \|u(\tau) - v(\tau)\|_{\mathcal{U}} < \delta$.

(A2) The subset $MU \subseteq M_U$ of inputs $u \in M_U$ with $\sup_{t \geqslant 0} \|u(t)\|_{\mathcal{U}} < +\infty$, is a normed linear space with norm $\|u\|_{MU} := \sup_{t \geqslant 0} \|u(t)\|_{\mathcal{U}}$.

(A3) *Complete Continuity of the Output Map*: For every pair of bounded sets $I \subset \Re^+$, $S \subset \mathcal{X}$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|H(t, x) - H(t, x_0)\|_{\mathcal{Y}} < \varepsilon$, for all $t \in I$, $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{X}} < \delta$.

Hypotheses (A2) and (A3) are standard assumptions, which can be verified easily for a very wide class of systems. Hypothesis (A1) is a technical hypothesis that guarantees two properties:

(a) continuity of the transition map with respect to the external input and the initial state,
(b) the set of times that satisfies the semigroup property, i.e., the set $\pi(t_0, x_0, u, d)$, contains all members of the partition $\pi = \{T_i\}_{i=0}^{\infty}$ greater than t_0. Thus the partition $\pi = \{T_i\}_{i=0}^{\infty}$ can be used in order to discretize time for all initial conditions and inputs. Notice that this requirement is automatically satisfied if $\pi(t_0, x_0, u, d) = [t_0, +\infty)$.

The following section and the examples of this section show that hypotheses (A1)–(A3) are satisfied by a wide class of systems. Let us remark one case where hypotheses (A1)–(A3) are (generally) not satisfied; the case of a finite-dimensional control system under sampled-data feedback which is either discontinuous or the sampling times are affected by noise. If the sampled-data feedback law is discontinuous then (in general) the solution does not depend continuously on the initial condition and consequently property (iii) of hypothesis (A1) does not hold. If the sampling times are affected by noise then there is no partition $\pi = \{T_i\}_{i=0}^{\infty}$ which satisfies property (ii) of hypothesis (A1).

For systems satisfying hypotheses (A1)–(A3), our main result provides equivalent characterizations for the WIOS property. Furthermore, it shows that for the WIOS property the "fading memory" estimate (2.1) is qualitatively equivalent to the "Sontag-like" estimate (2.4).

Theorem 3.1 *(Necessary and sufficient conditions for the WIOS property). Let the system* $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$*, which is RFC from the input* $u \in M_U$ *and satisfies hypotheses* (A1)–(A3)*. Moreover, suppose that* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$ *for* Σ*. The following statements are equivalent*:

(i) Σ *satisfies the WIOS property from the input* $u \in M_U$*,*
(ii) *there exist functions* $\zeta \in K_\infty$*,* $\delta \in K^+$ *such that estimate* (2.4) *holds for all* $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ *and* $t \geqslant t_0$*,*
(iii) Σ *is RGAOS.*

Proof. Notice that the implications (i) ⇒ (ii) and (ii) ⇒ (iii) are immediate. Thus we are left with the proof of the implication (iii) ⇒ (i).

By virtue of Lemma 3.2 in [16] and since (i) hypothesis (A2) implies that $\tilde{u} \in M_U$ for all $(u, \lambda) \in M_U \times \Re^+$, where $\tilde{u}$ is the input that satisfies $\tilde{u}(t) = \lambda u(t)$ for all $t \geqslant 0$, (ii) Σ is RFC from the input $u \in M_U$, and (iii) $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ, there exist functions $\mu \in K^+$, $a \in K_\infty$ such that the following estimate holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$:

$$\left\|\phi(t, t_0, x_0, u, d)\right\|_{\mathcal{X}} \leqslant \mu(t) a\Bigl(\|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} \left\|u(\tau)\right\|_{\mathcal{U}}\Bigr), \quad \forall t \geqslant t_0. \tag{3.1}$$

By virtue of Lemma 3.4 in [14] and since $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RGAOS, we guarantee the existence of functions $\sigma \in KL$, $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$:

$$\left\|H\bigl(t, \phi(t, t_0, x_0, u_0, d), 0\bigr)\right\|_{\mathcal{Y}} \leqslant \sigma\bigl(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\bigr). \tag{3.2}$$

Let $\pi = \{T_i\}_{i=0}^{\infty}$ the partition of $\Re^+$ for which hypothesis (A1) is satisfied and let $r > 0$ its diameter, i.e., $r := \sup\{T_{i+1} - T_i;\ i = 0, 1, 2, \ldots\} < +\infty$.

The proof is divided into two parts:

Part 1: We apply an abstract discretization technique, which provides an infinite-dimensional discrete-time system, which satisfies the non-uniform in time IOS property (in the sense described in [15]).

Part 2: The solution of the discrete-time system obtained from the Part 1 is related to the solution of Σ and we show that estimate (2.1) holds for all bounded inputs $u \in MU$.

Since estimate (2.1) holds for all bounded inputs $u \in MU$, we conclude that Σ satisfies WIOS property.

Part 1: Abstract discretization

Define $\mathcal{FX}$ the normed linear space of bounded functions $x:[0,r] \to \mathcal{X}$ with norm $\|x\|_{\mathcal{FX}} := \sup_{\theta \in [0,r]} \|x(\theta)\|_{\mathcal{X}}$. Let CM_D, CMU denote the set of sequences with values in M_D and MU, respectively. Define for $(i,x,d,u) \in Z^+ \times \mathcal{FX} \times M_D \times M_U$, where $x = \{x(\theta) \in \mathcal{X};\ \theta \in [0,r]\} \in \mathcal{FX}$,

$$f(i,d,x,u) := \begin{cases} \phi(\theta - r + T_{i+1}, T_i, x(r), u, d), & \theta \in [r - T_{i+1} + T_i, r], \\ 0, & \theta \in [0, r - T_{i+1} + T_i], \end{cases} \in \mathcal{FX}, \quad (3.3)$$

a map, which by virtue of (3.1) is well defined and satisfies

$$\big\|f(i,d,x,u)\big\|_{\mathcal{FX}} \leqslant \tilde{\mu}(i) a\Big(\|x\|_{\mathcal{FX}} + \sup_{T_i \leqslant s \leqslant T_{i+1}} \big\|u(s)\big\|_{\mathcal{U}}\Big),$$
$$\forall (i,x,d,u) \in Z^+ \times FX \times M_D \times M_U, \quad (3.4)$$

where $\tilde{\mu}(i) := \max_{T_i \leqslant t \leqslant T_{i+1}} \mu(t)$. By virtue of hypothesis (A1) and inequality (3.4) it follows that f is completely continuous, i.e., satisfies the following hypothesis:

(H1) For every bounded sets $S \subset \mathcal{FX} \times MU$, $I \subset Z^+$ and for every $\varepsilon > 0$, the set $f(I \times M_D \times S)$ is bounded and there exists $\delta > 0$ such that $\sup\{\|f(i,d,x,u) - f(i,d,x_0,v)\|_{\mathcal{X}};\ d \in M_D\} < \varepsilon$, for all $i \in I$, $(x,u) \in S$, $(x_0,v) \in S$ with $\|x - x_0\|_{\mathcal{FX}} + \|u - v\|_{MU} < \delta$. Moreover, it holds that $f(i,d,0,0) = 0$ for all $(i,d) \in Z^+ \times M_D$.

Define $\mathcal{FY}$ the normed linear space of bounded functions $Y:[0,r] \to \mathcal{Y}$ with norm $\|Y\|_{\mathcal{FY}} := \sup_{\theta \in [0,r]} \|Y(\theta)\|_{\mathcal{Y}}$. Let the output map defined for $(i,x) \in Z^+ \times \mathcal{FX}$,

$$\tilde{H}(i,x) := H\big(\max\{0, T_i - r + \theta\}, x(\theta)\big), \quad \theta \in [0,r]. \quad (3.5)$$

By virtue of hypothesis (A3) and the fact that the continuous map $H:\Re^+ \times \mathcal{X} \to \mathcal{Y}$ maps bounded sets of $\Re^+ \times \mathcal{X}$ into bounded sets of $\mathcal{Y}$, it follows that $\tilde{H}$ is completely continuous, i.e., satisfies the following hypothesis:

(H2) For every pair of bounded sets $I \subset Z^+$, $S \subset \mathcal{FX}$ and for every $\varepsilon > 0$ the set $\tilde{H}(I \times S)$ is bounded and there exists $\delta > 0$ such that $\|\tilde{H}(i,x) - \tilde{H}(i,x_0)\|_{\mathcal{FY}} < \varepsilon$, for all $i \in I$, $x, x_0 \in S$ with $\|x - x_0\|_{\mathcal{FX}} < \delta$. Moreover, it holds that $\tilde{H}(i,0) = 0$ for all $t \in Z^+$.

Next consider the discrete-time system:

$$x_{i+1} = f(i, d_i, x_i, u_i),$$
$$Y_i = \tilde{H}(i, x_i),$$
$$(i, x_i, Y_i, d_i, u_i) \in Z^+ \times \mathcal{FX} \times \mathcal{FY} \times M_D \times MU. \tag{3.6}$$

Using induction, hypothesis (A1)(ii) and the weak semigroup property for Σ it can be easily shown that for every sequence $\{d_i\}_{i=0}^{\infty} \in CM_D$ the solution of (3.6) with initial condition $x_{i_0} = x_0 \in \mathcal{FX}$ and corresponding to input $\{d_i\}_{i=0}^{\infty} \in CM_D$ and $u_i \equiv 0$ for all $i \geqslant i_0$ satisfies:

$$x_i = \begin{cases} \phi(\theta - r + T_i, T_{i_0}, x_0(r), u_0, Pd), & \theta \in [r - T_i + T_{i-1}, r], \\ 0, & \theta \in [0, r - T_i + T_{i-1}], \end{cases} \quad \text{for all } i \geqslant i_0 + 1, \tag{3.7}$$

where $Pd : \Re^+ \to D$ with $Pd(t) := d_i(t)$ for all $T_i \leqslant t < T_{i+1}$, $i = 0, 1, 2, \ldots$, which by virtue of hypothesis (A1) belongs to M_D. Moreover, using (3.2) in conjunction with (3.7) we obtain

$$\|Y_i\|_{\mathcal{FY}} \leqslant \sigma\big(\beta(T_{i_0})\|x_{i_0}\|_{\mathcal{FX}}, \max\{0; T_i - T_{i_0} - r\}\big) \quad \text{for all } i \geqslant i_0 + 1. \tag{3.8}$$

Estimate (3.8) implies that system (3.6) implies the Robust Output Attractivity Property given in [15]. Thus system (3.6) is non-uniformly in time RGAOS as explained in [15]. Consequently, by virtue of Proposition 3.3 in [15] and properties (H1)–(H2) above, we conclude that the discrete-time system (3.6) satisfies the non-uniform in time IOS property in the sense of [15].

Since the discrete-time system (3.6) satisfies the non-uniform in time IOS property in the sense of [15], there exist functions $\tilde{\sigma} \in KL$, $\tilde{\beta}, \tilde{\gamma} \in K^+$ and $\tilde{\rho} \in K_\infty$ such that for all $\{u_i \in MU\}_{i=0}^{\infty}$, $(i_0, x_0, \{d_i\}_{i=0}^{\infty}) \in Z^+ \times \mathcal{FX} \times CM_D$ the following estimate holds for all $i \geqslant i_0$ for the solution x_i of (3.6) with initial condition $x_{i_0} = x_0$ and corresponding to inputs $\{u_i\}_{i=0}^{\infty} \in CMU$, $\{d_i\}_{i=0}^{\infty} \in CM_D$:

$$\big\|\tilde{H}(i, x_i)\big\|_{\mathcal{FY}} \leqslant \max\Big\{\tilde{\sigma}\big(\tilde{\beta}(i_0)\|x_0\|_{\mathcal{FX}}, i - i_0\big), \sup_{i_0 \leqslant j \leqslant i} \tilde{\sigma}\Big(\tilde{\beta}(j)\tilde{\rho}\Big(\tilde{\gamma}(j)\sup_{\tau \geqslant 0}\big\|u_j(\tau)\big\|_{\mathcal{U}}\Big), i - j\Big)\Big\}. \tag{3.9}$$

Part 2: Proof of estimate (2.1)

Notice that definition (3.3) of the evolution map f of the discrete-time system shows that the solution x_i of (3.6) with arbitrary initial condition $x_{i_0} = x_0$ and corresponding to arbitrary inputs $\{u_i\}_{i=0}^{\infty} \in CMU$, $\{d_i\}_{i=0}^{\infty} \in CM_D$ coincides for all $i \geqslant i_0 + 1$ with the solution with arbitrary initial condition x_{i_0} that satisfies $x_{i_0}(r) = x_0(r)$ corresponding to inputs $\{d_i\}_{i=0}^{\infty} \in CM_D$ and $\{\tilde{u}_i\}_{i=0}^{\infty} \in CMU$ with $\tilde{u}_j(t) = u_j(t)$ for all $t \in [T_j, T_{j+1}]$ and $j = i_0, i_0 + 1, \ldots, i - 1$. Thus by virtue of (3.9), it follows that for all $\{u_i \in MU\}_{i=0}^{\infty}$, $(i_0, x_0, \{d_i\}_{i=0}^{\infty}) \in Z^+ \times \mathcal{FX} \times CM_D$ the following estimate holds for all $i \geqslant i_0 + 1$ for the solution x_i of (3.6) with initial condition $x_{i_0} = x_0$ and corresponding to inputs $\{u_i\}_{i=0}^{\infty} \in CMU$, $\{d_i\}_{i=0}^{\infty} \in CM_D$:

$$\big\|\tilde{H}(i, x_i)\big\|_{\mathcal{FY}} \leqslant \max\Big\{\tilde{\sigma}\big(\tilde{\beta}(i_0)\big\|x_0(r)\big\|_{\mathcal{X}}, i - i_0\big), \sup_{i_0 \leqslant j \leqslant i-1} \tilde{\sigma}\Big(\tilde{\beta}(j)\tilde{\rho}\Big(\tilde{\gamma}(j)\sup_{T_j \leqslant s \leqslant T_{j+1}}\big\|u_j(s)\big\|_{\mathcal{U}}\Big), i - j\Big)\Big\}. \tag{3.10}$$

Let arbitrary $(d, u) \in M_D \times MU$, $(t_0, x_0) \in \Re^+ \times \mathcal{X}$ and consider the transition map $\phi(t, t_0, x_0, u, d)$ of Σ. The definitions (3.3) and (3.5) of the evolution and output maps f, $\tilde{H}$ of the discrete-time system (3.6) imply that the transition map $\phi(t, t_0, x_0, u, d)$ of Σ satisfies the following identities for all $i \geqslant i_0 + 1$:

$$x(t) = \phi(t, t_0, x_0, u, d) = x_i(t + r - T_i) \quad \text{for all } t \in [T_{i-1}, T_i],$$
$$Y(t) = H\big(t, \phi(t, t_0, x_0, u, d)\big) = Y_i(t + r - T_i) \quad \text{for all } t \in [T_{i-1}, T_i],$$

where $T_{i_0} = p_\pi(t_0)$, $p_\pi(t) := \min\{T \in \pi;\ t < T\}$ and x_i denotes the solution of (3.6) with arbitrary initial condition $x_{i_0} \in \mathcal{FX}$ that satisfies $x_{i_0}(r) = \phi(T_{i_0}, t_0, x_0, u, d)$ and corresponding to the constant inputs $\{u_i \equiv u\}_{i=0}^{\infty} \in CMU$, $\{d_i \equiv d\}_{i=0}^{\infty} \in CM_D$. Combining the above identities with (3.10) we obtain for all $i \geqslant i_0 + 1$,

$$\sup_{T_{i-1} \leqslant t \leqslant T_i} \|Y(t)\|_{\mathcal{Y}} \leqslant \max\Big\{\tilde{\sigma}\big(\tilde{\beta}(i_0)\|x_{i_0}(r)\|_{\mathcal{X}}, i - i_0\big), \sup_{i_0 \leqslant j \leqslant i-1} \tilde{\sigma}\Big(\tilde{\beta}(j)\tilde{\rho}\Big(\tilde{\gamma}(j) \sup_{T_j \leqslant s \leqslant T_{j+1}} \|u(s)\|_{\mathcal{U}}\Big), i - j\Big)\Big\}. \tag{3.11}$$

Let $\bar{\beta}, \bar{\gamma} \in K^+$ non-decreasing functions, which satisfy $\bar{\beta}(T_{j-1}) \geqslant \tilde{\beta}(j)$, $\bar{\gamma}(T_j) \geqslant \tilde{\gamma}(j)$ for all integers $j \geqslant 1$. Estimate (3.11) in conjunction with the trivial inequalities $T_{i_0} - r \leqslant t_0 < T_{i_0}$ and the causality property for Σ (which shows that $Y(t)$ depends only on the values of $u \in MU$ in the interval $[t_0, t]$) implies (notice that without loss of generality we may assume that $\sigma(s, t)$ is of class K_∞ for each $t \geqslant 0$):

$$\|Y(t)\|_{\mathcal{Y}} \leqslant \max\Big\{\tilde{\sigma}\Big(\bar{\beta}(t_0)\|x(T_{i_0})\|_{\mathcal{X}}, \frac{t - t_0 - r}{r}\Big), \sup_{t_0 \leqslant \tau \leqslant t} \tilde{\sigma}\Big(\bar{\beta}(\tau)\tilde{\rho}\big(\bar{\gamma}(\tau)\|u(\tau)\|_{\mathcal{U}}\big), \frac{t - \tau}{r}\Big)\Big\}, \quad \forall t \geqslant t_0 + r. \tag{3.12}$$

Without loss of generality, we may assume that the function $\mu \in K^+$ involved in (3.1) is non-decreasing. Inequality (3.1) in conjunction with (3.12) implies the following estimate for all $t \geqslant t_0 + r$:

$$\|Y(t)\|_{\mathcal{Y}} \leqslant \max\Big\{\tilde{\sigma}\Big(\bar{\beta}(t_0)\mu(t_0 + r)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0 \leqslant \tau \leqslant t} \|u(\tau)\|_{\mathcal{U}}\Big), \frac{t - t_0 - r}{r}\Big), \sup_{t_0 \leqslant \tau \leqslant t} \tilde{\sigma}\Big(\bar{\beta}(\tau)\tilde{\rho}\big(\bar{\gamma}(\tau)\|u(\tau)\|_{\mathcal{U}}\big), \frac{t - \tau}{r}\Big)\Big\}. \tag{3.13}$$

Since the output map $H : \Re^+ \times \mathcal{X} \to \mathcal{Y}$ satisfies

(i) for every bounded set $S \subset \Re^+ \times \mathcal{X}$ the set $H(S)$ is bounded,
(ii) $H(t, 0) = 0$ for all $t \geqslant 0$,
(iii) for every $\varepsilon > 0$, $t \geqslant 0$ there exists $\delta := \delta(\varepsilon, t) > 0$ such that $\sup\{\|H(\tau, x)\|_{\mathcal{Y}};\ \tau \geqslant 0, |\tau - t| + \|x\|_{\mathcal{X}} < \delta\} < \varepsilon$ (immediate consequence of hypothesis (A3)),

it follows from Lemma 3.3 in [14] that there exists a pair of functions $\zeta \in K_\infty$ and $\delta \in K^+$ such that

$$\|H(t, x)\|_{\mathcal{Y}} \leqslant \zeta\big(\delta(t)\|x\|_{\mathcal{X}}\big), \quad \forall (t, x) \in \Re^+ \times \mathcal{X}. \tag{3.14}$$

Without loss of generality, we may assume that the function $\delta \in K^+$ involved in (3.14) is non-decreasing. Combining the above inequality with estimate (3.1), we obtain

$$\left\|Y(t)\right\|_{\mathcal{Y}} \leqslant \zeta\Big(\delta(t_0+r)\mu(t_0+r)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\leqslant\tau\leqslant t}\left\|u(\tau)\right\|_{\mathcal{U}}\Big)\Big), \quad \forall t \in [t_0, t_0+r]. \tag{3.15}$$

Defining

$$\omega(s,t) := \exp(r-t)\zeta(s) + \begin{cases} \tilde{\sigma}\big(s, \frac{t-r}{r}\big) & \text{if } t > r, \\ \exp(r-t)\tilde{\sigma}(s,0) & \text{if } 0 \leqslant t \leqslant r, \end{cases} \tag{3.16a}$$

$$\hat{\beta}(t) := \bar{\beta}(t)\big(1+\mu(t+r)\big) + \delta(t+r)\mu(t+r) \tag{3.16b}$$

and combining estimates (3.13), (3.15) we obtain the following estimate for all $t \geqslant t_0$:

$$\begin{aligned}\left\|Y(t)\right\|_{\mathcal{Y}} \leqslant \max\Big\{&\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\leqslant\tau\leqslant t}\|u(\tau)\|_{\mathcal{U}}\Big), t-t_0\Big), \\ &\sup_{t_0\leqslant\tau\leqslant t}\omega\big(\hat{\beta}(\tau)\tilde{\rho}\big(\bar{\gamma}(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}\big), t-\tau\big)\Big\}.\end{aligned} \tag{3.17}$$

Lemma 2.3 in [12] implies the existence of a pair of functions $\tilde{a} \in K_\infty$ and $q \in K^+$ being non-decreasing such that $\hat{\beta}(t)a(2s) \leqslant \tilde{a}(q(t)s)$ for all $t, s \geqslant 0$. Since the following inequality holds for all $t \geqslant t_0$:

$$\begin{aligned}&\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\leqslant\tau\leqslant t}\left\|u(\tau)\right\|_{\mathcal{U}}\Big), t-t_0\Big) \\ &\quad\leqslant \max\Big\{\omega\Big(\hat{\beta}(t_0)a\big(2\|x_0\|_{\mathcal{X}}\big), t-t_0\Big), \omega\Big(\hat{\beta}(t_0)a\Big(2\sup_{t_0\leqslant\tau\leqslant t}\left\|u(\tau)\right\|_{\mathcal{U}}\Big), t-t_0\Big)\Big\},\end{aligned} \tag{3.18}$$

and $\hat{\beta}(t)a(2s) \leqslant \tilde{a}(q(t)s)$ for all $t, s \geqslant 0$ with $q \in K^+$ being non-decreasing we obtain from (3.18):

$$\begin{aligned}&\omega\Big(\hat{\beta}(t_0)a\Big(\|x_0\|_{\mathcal{X}} + \sup_{t_0\leqslant\tau\leqslant t}\left\|u(\tau)\right\|_{\mathcal{U}}\Big), t-t_0\Big) \\ &\quad\leqslant \max\Big\{\bar{\omega}\big(q(t_0)\|x_0\|_{\mathcal{X}}, t-t_0\big), \sup_{t_0\leqslant\tau\leqslant t}\bar{\omega}\big(q(\tau)\left\|u(\tau)\right\|_{\mathcal{U}}, t-\tau\big)\Big\},\end{aligned} \tag{3.19}$$

where $\bar{\omega}(s,t) := \omega(\tilde{a}(s), t)$. Finally, let $\beta(t) := 1 + q(t) + \hat{\beta}(t)$, $\sigma(s,t) := \bar{\omega}(s,t) + \omega(s,t)$, $\rho(s) := s + \tilde{\rho}(s)$, $\gamma(t) := \bar{\gamma}(t) + q(t)$. The previous definitions in conjunction with (3.17) and (3.19) imply estimate (2.1).

The proof is complete. □

The first example shows that finite-dimensional systems described by ordinary differential equations are systems for which Theorem 3.1 can be applied.

Example 3.2. Let $U \subseteq \Re^m$ be a subspace and let $D \subseteq \Re^l$ be a compact set. Every pair of continuous mappings $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$, $H : \Re^+ \times \Re^n \to \Re^k$, with $H(t,0) = 0$, $f(t,0,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$ and such that the vector field $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$, satisfies the following Lipschitz condition:

The function $f(t,x,u,d)$ is locally Lipschitz with respect to (x,u), uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n \times U$, there exists a constant $L \geqslant 0$ such that

$$\begin{aligned}&\bigl|f(t,x,u,d)-f(t,y,v,d)\bigr| \leqslant L\bigl|(x-y,u-v)\bigr|,\\ &\quad \forall t \in I,\ \forall (x,u;y,v) \in S \times S,\ \forall d \in D,\end{aligned}$$

defines a continuous-time control system $\Sigma := (\Re^n, \Re^k, M_U, M_D, \phi, H)$ with outputs and the BIC property, by the evolution equation:

$$\begin{aligned}&\dot{x}(t) = f\bigl(t, x(t), u(t), d(t)\bigr),\\ &Y(t) = H\bigl(t, x(t)\bigr).\end{aligned}$$

This fact is an immediate consequence of Proposition 3.7.2 in [28]. In this case $M_U \subseteq \mathcal{M}(U)$ is the set of measurable and locally essentially bounded mappings $u : \Re^+ \to U$ and $M_D \subseteq \mathcal{M}(D)$ is the set of measurable and locally essentially bounded mappings $d : \Re^+ \to D$. Notice that $\Sigma := (\Re^n, \Re^k, M_U, M_D, \phi, H)$ satisfies the "classical" semigroup property. Moreover, notice that Theorem 2.6 in [17] guarantees that hypothesis (A1) holds and that $0 \in \Re^n$ is a robust equilibrium point from the input $u \in M_U$ for Σ. Hypotheses (A2)–(A3) hold as well and consequently, we conclude that Theorem 3.1 holds for $\Sigma := (\Re^n, \Re^k, M_U, M_D, \phi, H)$.

The following example is an immediate consequence of Theorems 2.2 and 3.2 in [4], concerning continuous dependence on initial conditions and continuation of solutions of retarded functional differential equations, respectively.

Example 3.3. Let $U \subseteq \Re^m$ be a subspace and let $D \subseteq \Re^l$ be a compact set. Every pair of completely continuous mappings $f : \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \to \Re^n$, $H : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^k$, where $r > 0$ is a constant, with $H(t,0) = 0$, $f(t,0,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$ and such that the vector field $f : \Re^+ \times C^0([-r,0];\Re^n) \times U \times D \to \Re^n$, satisfies the following Lipschitz condition:

The function $f(t,x,u,d)$ is locally Lipschitz with respect to (x,u), uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every closed and bounded subset S of $C^0([-r,0];\Re^n) \times U$, there exists a constant $L \geqslant 0$ such that

$$\begin{aligned}&\bigl|f(t,x,u,d)-f(t,y,v,d)\bigr| \leqslant L|u-v| + L \max_{\tau \in [-r,0]} \bigl|x(\tau)-y(\tau)\bigr|,\\ &\quad \forall t \in I,\ \forall (x,u;y,v) \in S \times S,\ \forall d \in D,\end{aligned}$$

defines a continuous-time control system $\Sigma := (C^0([-r,0];\Re^n), \Re^k, M_U, M_D, \phi, H)$ with outputs and the BIC property, by the evolution equation:

$$\begin{aligned}&\dot{x}(t) = f\bigl(t, T_r(t)x, u(t), d(t)\bigr),\\ &Y(t) = H\bigl(t, T_r(t)x\bigr),\\ &T_r(t)x = x(t+\theta), \quad \theta \in [-r,0].\end{aligned}$$

In this case M_U and M_D are the sets of measurable and locally bounded functions with values in U and D, respectively. Notice that $\Sigma := (C^0([-r,0];\Re^n), \Re^k, M_U, M_D, \phi, H)$ satisfies the "classical" semigroup property and that $0 \in C^0([-r,0];\Re^n)$ is a robust equilibrium point for Σ. Finally, notice that Theorems 2.2 in [4] guarantees that hypothesis (A1) holds for $\Sigma := (C^0([-r,0];\Re^n), \Re^k, M_U, M_D, \phi, H)$. Hypotheses (A2)–(A3) hold as well and consequently, we conclude that Theorem 3.1 holds for $\Sigma := (C^0([-r,0];\Re^n), \Re^k, M_U, M_D, \phi, H)$.

The following example presents a class of neutral functional evolution equations, for which Theorem 3.1 can be applied in straightforward manner. The example is concerned with the case of continuous time difference equations, which was recently studied in [21]. The importance of continuous time difference equations in applications is explained in [4,21], since such models appear in economics, gas dynamics and lossless propagation models (see references in [21]).

Example 3.4. Let constants $R > r > 0$, a non-empty compact set $D \subset \Re^l$, positive functions $\tau_i : \Re^+ \to [r, R]$, $i = 1, \dots, p$, and a pair of continuous vector fields $f : \Re^+ \times D \times (\Re^n)^p \times \Re^m \to \Re^n$, $H : \Re^+ \times \Re^n \to \Re^k$ with $f(t, d, 0, \dots, 0, 0) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \Re^+ \times D$. Let $\mathcal{X}$ the normed linear space of the bounded functions $x : [-R, 0] \to \Re^n$ with $\|x\|_{\mathcal{X}} := \sup_{\theta\in[-R,0]} |x(\theta)|$, $\mathcal{Y} := \Re^k$, $\mathcal{U} := U = \Re^m$, M_U the set of all locally bounded functions $u : \Re^+ \to \Re^m$ and M_D the set of all functions $d : \Re^+ \to D$. Let $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ and consider the solution $x(t) \in \Re^n$ of the neutral functional evolution equations:

$$\begin{aligned}
&x(t) = f\big(t, d(t), x\big(t - \tau_1(t)\big), \dots, x\big(t - \tau_p(t)\big), u(t)\big),\\
&Y(t) = H\big(t, x(t)\big),\\
&x(t) \in \Re^n, \quad Y(t) \in \Re^k, \quad u(t) \in \Re^m, \quad d(t) \in D, \quad t \geqslant 0,
\end{aligned} \tag{3.20}$$

with initial condition $x(t_0+\theta) = x_0(\theta)$, $\theta \in [-R, 0]$, corresponding to inputs $(u, d) \in M_U \times M_D$. It is clear that (3.20) describes a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and evolution map ϕ defined by $\phi(t, t_0, x_0, u, d) = x(t + \theta)$, $\theta \in [-R, 0]$. Systems described by neutral functional evolution equations of the form (3.20) are considered in [4,21]. Here we show that if system (3.20) with $u(t) \equiv 0$ is RGAOS then system (3.20) satisfies the WIOS property. To this purpose we use Theorem 3.1 and we notice that hypotheses (A2)–(A3) are automatically satisfied for system (3.20). Next we show the following claims.

Claim 1. *System* (3.20) *is RFC from the input* $u \in M_U$.

Let $T, \rho \geqslant 0$ and let arbitrary $(t_0, x_0, u, d) \in [0, T] \times B_{\mathcal{X}}[0, \rho] \times M_U \cap \mathcal{M}(B_U[0, \rho]) \times M_D$. Notice that from (3.20) it follows that $\sup\{|x(t)|;\ t \in [t_0, t_0 + r]\} \leqslant \rho'$, where $\rho' \geqslant 0$ sufficiently large such that $f([0, T+r] \times D \times (B[0, \rho])^p \times B[0, \rho]) \subseteq B[0, \rho']$ (continuity of the vector field $f : \Re^+ \times D \times (\Re^n)^p \times \Re^m \to \Re^n$ in conjunction with the compactness of $D \subset \Re^l$ guarantees the existence of such $\rho' \geqslant 0$). Applying repeatedly the previous observation, we conclude that for all positive integers k there exists $\rho'' \geqslant 0$ such that $\sup\{|x(t)|;\ t \in [t_0, t_0 + kr]\} \leqslant \rho''$. This proves that system (3.20) is RFC from the input $u \in M_U$.

Claim 2. $0 \in \mathcal{X}$ *is a robust equilibrium point from the input* $u \in M_U$ *for system* (3.20).

Let $\varepsilon > 0$, $T \in \Re^+$ arbitrary and let $\delta > 0$ such that $f([0, T+r] \times D \times (B[0, \delta])^p \times B[0, \delta]) \subset B[0, \varepsilon]$ (continuity of the vector field $f : \Re^+ \times D \times (\Re^n)^p \times \Re^m \to \Re^n$ in conjunction with the compactness of $D \subset \Re^l$ and the fact $f(t, d, 0, \dots, 0, 0) = 0$ guarantees the existence of such $\delta > 0$). Let arbitrary $(t_0, x_0, u, d) \in [0, T] \times B_{\mathcal{X}}[0, \delta] \times M_U \cap \mathcal{M}(B_U[0, \delta]) \times M_D$ and notice that from (3.20) it follows that $\sup\{|x(t)|;\ t \in [t_0 - R, t_0 + r]\} < \varepsilon$. Applying repeatedly the previous observation, we conclude that for all positive integers k there exists $\delta' \geqslant 0$ such that $\sup\{|x(t)|;\ t \in [t_0 - R, t_0 + kr]\} < \varepsilon$. This proves that $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for system (3.20).

Claim 3. *System* (3.20) *satisfies hypothesis* (A1).

We set $\pi = \{ir\}_{i=0}^{\infty}$. Clearly, requirements (i) and (ii) of hypothesis (A1) are satisfied (notice that $\pi(t_0, x_0, u, d) = [t_0, +\infty)$). Let arbitrary bounded set $S \subset \mathcal{X}$ and arbitrary $(T, \rho, \varepsilon) \in \Re^+ \times \Re^+ \times (0, +\infty)$. Let $\rho' \geqslant 0$ sufficiently large such that $S \subset B_{\mathcal{X}}[0, \rho']$ and let $\delta > 0$ such that $|f(t, d, x_1, \ldots, x_p, u) - f(t, d, y_1, \ldots, y_p, v)| < \varepsilon$ for all $(t, x_1, \ldots, x_p, u) \in [0, T + r] \times D \times (B[0, \rho'])^p \times B[0, \rho]$, $(t, y_1, \ldots, y_p, v) \in [0, T + r] \times D \times (B[0, \rho'])^p \times B[0, \rho]$ with $\max_{i=1,\ldots,p} |x_i - y_i| + |u - v| < \delta$. The functional evolution equations (3.20) guarantee that for all $t_0 \in [0, T] \cap \pi$, $y_0, x_0 \in S$, $u, v \in \mathcal{M}(B_U[0, \rho]) \cap M_U$ with $\|x - x_0\|_{\mathcal{X}} + \sup_{t_0 \leqslant \tau \leqslant t_0 + r} \|u(\tau) - v(\tau)\|_{\mathcal{U}} < \delta$ the solutions $x(t), y(t)$ of (3.20) with initial conditions $x(t_0 + \theta) = x_0(\theta)$, $\theta \in [-R, 0]$, and $y(t_0 + \theta) = x_0(\theta)$, $\theta \in [-R, 0]$, respectively, corresponding to inputs $(u, d) \in M_U \times M_D$ and $(v, d) \in M_U \times M_D$, respectively, satisfy $\sup\{|x(t) - y(t)|;\ t \in [t_0, t_0 + r]\} < \varepsilon$. This proves requirement (iii) of hypothesis (A1).

Claims 1–3 above in conjunction with Theorem 3.1 show that if system (3.20) with $u(t) \equiv 0$ is RGAOS then system (3.20) satisfies the WIOS property.

4. Applications to sampled-data feedback control

In this section we provide certain robustness results for sampled-data feedback control systems. Sampled-data feedback control has been considered in [1,2,5,6,19,20,23,24,30]. In this section we consider sampled-data feedback control with uniform sampling rate. All statements in this section can be proved in a straightforward way using the results of the previous sections and are left to the reader.

Consider the finite-dimensional continuous-time control system

$$\begin{aligned}
&\dot{x}(t) = f\big(t, x(t), v(t)\big),\\
&Y(t) = H\big(t, x(t)\big),\\
&x(t) \in \Re^n, \quad v(t) \in \Re^m, \quad Y(t) \in \Re^k, \quad t \geqslant 0,
\end{aligned} \tag{4.1}$$

where the vector fields $f : \Re^+ \times \Re^n \times \Re^m \to \Re^n$, $H : \Re^+ \times \Re^n \to \Re^k$ are continuous, $f : \Re^+ \times \Re^n \times \Re^m \to \Re^n$ is locally Lipschitz in $(x, v) \in \Re^n \times \Re^m$ with $f(t, 0, 0) = 0$, $H(t, 0) = 0$ for all $t \geqslant 0$. We assume that:

Continuous Output Complete Controllability. There exists a family of measurable and locally bounded controls $t \to v(t, t_0, x_0)$ parameterized by $(t_0, x_0) \in \Re^+ \times \Re^n$, $\mu \in K^+$, $a \in K_\infty$, a continuous function $A : \Re^+ \times \Re^+ \times \Re^+ \to \Re^+$ with $A(t, s, 0) = 0$ for all $t, s \geqslant 0$ and a constant $r > 0$ such that

(i) for every $(t_0, x_0) \in \Re^+ \times \Re^n$ the unique solution of (4.1) with initial condition $x(t_0) = x_0$ and corresponding to input $v(t) = v(t, t_0, x_0)$, exists for all $t \geqslant t_0$ and satisfies $|v(t, t_0, x_0)| + |x(t)| \leqslant \mu(t) a(|x_0|)$ for all $t \geqslant t_0$ and $Y(t) = 0$ for all $t \geqslant t_0 + r$,

(ii) for all $(t_0, x, x_0, y_0) \in \Re^+ \times \Re^n \times \Re^n \times \Re^n$ it holds that

$$\left| f\big(t, x, v(t, t_0, x_0)\big) - f\big(t, x, v(t, t_0, y_0)\big)\right| \leqslant A\big(t, |x| + |x_0| + |y_0|, |x_0 - y_0|\big)$$

for all $t \geqslant t_0$.

We consider the control system $\Sigma := (\Re^n, \Re^k, M_U, M_D, \phi, H)$ that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ and for each locally bounded input $e : \Re^+ \to \Re^n$ the absolutely continuous function $[t_0, +\infty) \ni t \to x(t) \in \Re^n$ that satisfies a.e. the differential equation

$$\begin{aligned} &\dot{x}(t) = f\big(t, x(t), v\big(t, \tau_i, x(\tau_i) + e(\tau_i)\big)\big), \quad t \in [\tau_i, \tau_{i+1}), \\ &\tau_0 = t_0, \qquad \tau_{i+1} = r\left(1 + \left[\frac{\tau_i}{r}\right]\right), \quad i = 0, 1, \ldots, \\ &Y(t) = H\big(t, x(t)\big), \end{aligned} \tag{4.2}$$

with initial condition $x(t_0) = x_0$, where $[\frac{\tau_i}{r}]$ denotes the integer part of the real number $\frac{\tau_i}{r}$. In this case the locally bounded input $e : \Re^+ \to \Re^n$ represents the state measurement error and M_U denotes the set of locally bounded inputs $e : \Re^+ \to U = \Re^n$. The sets D and M_D are irrelevant and thus we may consider $D := \{0\}$.

It should be clear to the reader that system (4.2) satisfies hypotheses (H1)–(H4) given in [16] and thus Propositions 2.5 and 2.7 in [16] indicate that (4.2) defines a control system with outputs in the sense of Definition 2.1. Particularly, system (4.2) satisfies the weak semigroup property with $\pi(t_0, x_0, u, d) := (\pi \cup \{t_0\}) \cap [t_0, +\infty)$, where $\pi := \{0, r, 2r, 3r, \ldots\}$. Moreover, system (4.2) has the Boundedness-Implies-Continuation (BIC) property and $0 \in \Re^n$ is a robust equilibrium point for (4.2) from the locally bounded input $e : \Re^+ \to \Re^n$. The hypothesis of Continuous Output Complete Controllability in conjunction with Lemma 3.3 in [14], guarantees that system (4.2) is RGAOS for the input-free case $e(t) \equiv 0$.

The following hypothesis in addition to (i) and (ii) of Continuous Output Complete Controllability, guarantees that system (4.2) is RFC from the locally bounded input $e : \Re^+ \to \Re^n$:

(iii) for every $(t_0, x_0, y_0) \in \Re^+ \times \Re^n \times \Re^n$ the unique solution of (4.1) with initial condition $x(t_0) = x_0$ and corresponding to input $v(t) = v(t, t_0, y_0)$, exists and satisfies $|x(t)| \leqslant \mu(t) a(|x_0| + |y_0|)$ for all $t \geqslant t_0$.

Moreover, hypothesis (ii) of Continuous Output Complete Controllability, in conjunction with hypothesis (iii) above, shows that all hypotheses (A1–(A3) of Theorem 3.1 are indeed satisfied (particularly, (A1) holds with $\pi := \{0, r, 2r, 3r, \ldots\}$). Thus we obtain the following result:

Corollary 4.1. *Consider system* (4.2) *under hypotheses* (i)–(iii) *stated above. System* (4.2) *satisfies the WIOS property from the input from the input $e : \Re^+ \to \Re^n$; particularly, there exist functions $\sigma \in KL$, $\beta, \gamma \in K^+$, $\rho \in K_\infty$ such that the following estimate holds for every locally bounded input $e : \Re^+ \to \Re^n$, $(t_0, x_0) \in \Re^+ \times \Re^n$ and $t \geqslant t_0$:*

$$\begin{aligned} \left|H\big(t, x(t)\big)\right| \leqslant \max\Big\{&\sigma\big(\beta(t_0)|x_0|, t - t_0\big), \\ &\sup_{\tau \in [t_0, t]} \sigma\big(\beta(\tau)\rho\big(\gamma(\tau)\big|e\big(q(\tau, t_0)\big)\big|\big), t - \tau\big)\Big\}, \end{aligned} \tag{4.3}$$

where $x(t)$ denotes the solution of (4.2) *with initial condition $x(t_0) = x_0$ corresponding to the locally bounded input $e : \Re^+ \to \Re^n$, $q(t, t_0) := \max\{\tau \in \{t_0\} \cup \pi;\ \tau \leqslant t\}$, $\pi := \{0, r, 2r, 3r, \ldots\}$.*

Notice that Theorem 3.1 can guarantee the existence of functions $\sigma \in KL$, $\beta, \gamma \in K^+$, $\rho \in K_\infty$ such that estimate (2.1) holds, namely,

$$\left|H\left(t, x(t)\right)\right| \leqslant \max\left\{\sigma\left(\beta(t_0)|x_0|, t-t_0\right), \sup_{\tau\in[t_0,t]} \sigma\left(\beta(\tau)\rho\left(\gamma(\tau)\left|e(\tau)\right|\right), t-\tau\right)\right\}.$$

Estimate (4.3) is a direct consequence of the above estimate and the observation that the solution $x(t)$ of (4.2) with initial condition $x(t_0) = x_0$ and corresponding to input $e : \Re^+ \to \Re^n$ coincides with the solution of (4.2) initiated from the same initial condition and corresponding to the piecewise constant input $\tilde{e} : \Re^+ \to \Re^n$ defined by

$$\tilde{e}(t) := e(\tau_i) \quad \text{for } t \in [\tau_i, \tau_{i+1}),\ i = 1, 2, \ldots.$$

Corollary 4.1 is an important robustness result, which shows that if the state measurement error $e : \Re^+ \to \Re^n$ converges "sufficiently fast" to zero, then the output of (4.2) actually converges to zero. Thus application of the open-loop controls $t \to v(t, t_0, x_0)$ in a "sampled-data way" (as in system (4.2)) preserves certain robustness properties of usual ordinary feedback control. On the other hand, the application of the open-loop controls $t \to v(t, t_0, x_0)$ in a "direct way" usually is not robust to state measurement errors (see the Introduction in [28]).

5. Conclusions and open problems

In this work characterizations of the notion of Weighted Input-to-Output Stability (WIOS) for a wide class of systems with disturbances are given. The class of systems studied include systems, which do not necessarily satisfy the semigroup property. Particularly, for systems with continuous dependence of the solution on the initial state and the input, the WIOS property is shown to be equivalent to robust forward completeness from the input and robust global asymptotic output stability for the corresponding input-free system (0-GAS property). The obtained results are applied to sampled-data feedback control systems. It is shown that application of the open-loop controls in a "sampled-data way" preserves certain robustness properties featured by usual ordinary feedback control.

It is an open problem to find characterizations for the Uniform Input-to-Output Stability property, i.e., the property that guarantees the Bounded-Input-Bounded-Output and Converging-Input-Converging-Output properties. The Uniform Input-to-Output Stability property may be defined using a "Sontag-like estimate"

$$\left\|H\left(t, \phi(t, t_0, x_0, u, d), u(t)\right)\right\|_{\mathcal{Y}} \leqslant \max\left\{\sigma\left(\|x_0\|_{\mathcal{X}}, t-t_0\right), \sup_{t_0\leqslant\tau\leqslant t} \zeta\left(\left\|u(\tau)\right\|_{\mathcal{U}}\right)\right\}$$

or a "fading memory estimate"

$$\left\|H\left(t, \phi(t, t_0, x_0, u, d), u(t)\right)\right\|_{\mathcal{Y}} \leqslant \max\left\{\sigma\left(\|x_0\|_{\mathcal{X}}, t-t_0\right), \sup_{\tau\in[t_0,t]} \sigma\left(\rho\left(\left\|u(\tau)\right\|_{\mathcal{U}}\right), t-\tau\right)\right\}.$$

It is not known whether "fading memory estimates" are qualitatively equivalent to "Sontag-like estimates" (although this equivalence holds for finite-dimensional systems described by ordinary differential equations with identity output map). Another open problem is to obtain Lyapunov-like characterizations of the WIOS and Uniform IOS properties, which allow the determination of the functions involved in the estimates of the corresponding properties. These topics will be the subject of future research. It is expected that the discovery of sufficient conditions for the WIOS and Uniform IOS properties for a wide class of systems with outputs will motivate similar advances to the advances which were triggered by these properties for finite-dimensional systems.

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