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A system-theoretic framework for a wide class of systems I: Applications to numerical analysis

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Abstract

A system-theoretic framework is proposed, which allows the study of hybrid uncertain systems, which do not satisfy the so-called "semigroup property." Characterizations of the notion of robust global asymptotic output stability (RGAOS) are given. Based on the provided characterizations, the qualitative behavior of hybrid systems obtained by time-discretization of systems of ordinary differential equations with a globally asymptotically stable equilibrium point, is studied.

Keywords: Hybrid systems; Robust global asymptotic stability

1. Introduction

In this work a system-theoretic framework is proposed, which allows the study of the qualitative properties of the solutions of uncertain hybrid systems which do not necessarily satisfy the "semigroup property." Since a complete stability theory for such systems and Lyapunov characterizations of the stability notions are absent from the literature, this work aims to provide stability notions and characterizations in analogy to those applied to systems that satisfy the classical "semigroup property." Moreover, in order to motivate our work, we consider important problems where the results of the present work can be directly applied: to finite-dimensional systems that operate under sampled-data feedback control and to systems obtained by time-discretization of systems of ordinary differential equations. Finally, it is shown how the proposed

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framework can be used in order to address certain problems of numerical analysis and convert them into feedback stabilization problems. Essential characterizations of external stability notions and applications to the problem of robustness of sampled-data feedback are provided in a companion paper.

Given a pair of sets $D \subseteq \Re^l$ and $U \subseteq \Re^m$ closed set with $0 \in U$, a positive function $h : \Re^+ \times \Re^n \times U \times D \to (0, r]$ which is bounded by certain constant $r > 0$ and a triplet of vector fields $f : \Re^+ \times \Re^+ \times \Re^n \times \Re^n \times U \times U \times D \times D \to \Re^n$, $H : \Re^+ \times \Re^n \times U \to \Re^p$, $R : \Re^+ \times \Re^n \times \Re^n \times U \times U \times D \times D \to \Re^n$, we consider the hybrid system that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ and for each pair of measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$ the piecewise absolutely continuous function $t \to x(t) \in \Re^n$, produced by the following algorithm:

Step i:

(1) Given τ_i and $x(\tau_i)$, calculate τ_{i+1} using the equation $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$.
(2) Compute the state trajectory $x(t)$, $t \in [\tau_i, \tau_{i+1})$, as the solution of the differential equation $\dot{x}(t) = f(t, \tau_i, x(t), x(\tau_i), u(t), u(\tau_i), d(t), d(\tau_i))$.
(3) Calculate $x(\tau_{i+1})$, using the equation $x(\tau_{i+1}) = R(\tau_i, \lim_{t\to\tau_{i+1}^-} x(t), x(\tau_i), u(\tau_{i+1}), u(\tau_i), d(\tau_{i+1}), d(\tau_i))$.
(4) Compute the output trajectory $Y(t)$, $t \in [\tau_i, \tau_{i+1}]$, using the equation $Y(t) = H(t, x(t), u(t))$.

For $i = 0$ we take $\tau_0 = t_0$ and $x(\tau_0) = x_0$ (initial condition). Schematically, we write

$$\begin{aligned}
&\dot{x}(t) = f\big(t, \tau_i, x(t), x(\tau_i), u(t), u(\tau_i), d(t), d(\tau_i)\big), \quad t \in [\tau_i, \tau_{i+1}),\\
&\tau_0 = t_0, \qquad \tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i)\big), \quad i = 0, 1, \ldots,\\
&x(\tau_{i+1}) = R\Big(\tau_i, \lim_{t\to\tau_{i+1}^-} x(t), x(\tau_i), u(\tau_{i+1}), u(\tau_i), d(\tau_{i+1}), d(\tau_i)\Big),\\
&Y(t) = H\big(t, x(t), u(t)\big)
\end{aligned} \tag{1.1}$$

with initial condition $x(t_0) = x_0$.

A system-theoretic framework is proposed in the present paper, which allows the study of hybrid systems of the form (1.1) under the following hypotheses:

(H1) $f(t, \tau, x, x_0, u, u_0, d, d_0)$ is measurable with respect to $t \geqslant 0$, continuous with respect to $(x, d, u) \in \Re^n \times D \times U$ and such that for every bounded $S \subset \Re^+ \times \Re^+ \times \Re^n \times \Re^n \times U \times U$ there exists constant $L \geqslant 0$ such that

$$\begin{aligned}
&(x - y)'\big(f(t, \tau, x, x_0, u, u_0, d, d_0) - f(t, \tau, y, x_0, u, u_0, d, d_0)\big) \leqslant L|x - y|^2\\
&\quad \forall (t, \tau, x, x_0, u, d, d_0) \in S \times D \times D, \ \forall (t, \tau, y, x_0, v, d, d_0) \in S \times D \times D.
\end{aligned} \tag{1.2a}$$

(H2) There exist functions $\gamma \in K^+$, $a \in K_\infty$, such that

$$\begin{aligned}
&\big|f(t, \tau, x, x_0, u, u_0, d, d_0)\big| \leqslant \gamma(t) a\big(|x| + |x_0| + |u| + |u_0|\big)\\
&\quad \forall (\tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n, \ \forall t \geqslant \tau,
\end{aligned} \tag{1.2b}$$

$$\begin{aligned}
&\big|R(t, x, x_0, u, u_0, d, d_0)\big| \leqslant \gamma(t) a\big(|x| + |x_0| + |u| + |u_0|\big)\\
&\quad \forall (t, u, u_0, d, d_0, x, x_0) \in \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n.
\end{aligned} \tag{1.2c}$$

(H3) $H : \Re^+ \times \Re^n \times U \to \Re^p$ is a continuous map with $H(t, 0, 0) = 0$ for all $t \geqslant 0$.

(H4) There exists a positive, continuous and bounded function $h_l : \Re^+ \times \Re^n \times U \to (0, r]$ and a partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$, i.e., an increasing sequence of times with $T_0 = 0$ and $T_i \to +\infty$ such that

$$\begin{aligned} &h(t, x, u, d) \geqslant \min\{p_\pi(t) - t, h_l(t, x, u)\} \\ &\quad \forall (t, x, u, d) \in \Re^+ \times \Re^n \times U \times D, \end{aligned} \tag{1.2d}$$

where $p_\pi(t) := \min\{T \in \pi;\ t < T\}$.

Systems of the form (1.1) under hypotheses (H1)–(H4) arise frequently in certain applications in mathematical control theory and numerical analysis. We mention here two important applications.

1.1. Application of "sampled-data" feedback

For example, consider the finite-dimensional continuous-time control system $\dot{x}(t) = f(t, x(t), v(t))$, where $x(t) \in \Re^n$, $v(t) \in \Re^m$ and the vector field $f : \Re^+ \times \Re^n \times \Re^m \to \Re^n$ is continuous, locally Lipschitz in $x \in \Re^n$. Suppose that there exists a family of measurable and locally bounded controls $t \to v(t, t_0, x_0)$ parameterized by $(t_0, x_0) \in \Re^+ \times \Re^n$ with the following property: for every $(t_0, x_0) \in \Re^+ \times \Re^n$ the unique solution of $\dot{x}(t) = f(t, x(t), v(t, t_0, x_0))$ with initial condition $x(t_0) = x_0$ exists for all $t \geqslant t_0$ and satisfies $\lim_{t \to +\infty} x(t) = 0 \in \Re^n$. Then application of the measurable and locally bounded controls $t \to v(t, t_0, x_0)$ on the interval $[t_0, t_0 + h(t_0, x_0))$, where $h : \Re^+ \times \Re^n \to (0, r]$ is a positive function bounded by certain constant $r > 0$, gives the control system that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ and for each measurable and locally bounded inputs $u : \Re^+ \to \Re^m$, $e : \Re^+ \to \Re^n$ the absolutely continuous function $[t_0, +\infty) \ni t \to x(t) \in \Re^n$ that satisfies a.e. the differential equation

$$\begin{aligned} &\dot{x}(t) = f\big(t, x(t), v\big(t, \tau_i, x(\tau_i) + e(\tau_i)\big) + u(t)\big), \quad t \in [\tau_i, \tau_{i+1}), \\ &\tau_0 = t_0, \qquad \tau_{i+1} = \tau_i + h\big(\tau_i, x(\tau_i) + e(\tau_i)\big), \quad i = 0, 1, \ldots, \\ &Y(t) = H\big(t, x(t)\big) \end{aligned} \tag{1.3}$$

with initial condition $x(t_0) = x_0$. In this case the measurable and locally bounded inputs $u : \Re^+ \to \Re^m$ and $e : \Re^+ \to \Re^n$ represent the control actuator error and the measurement error, respectively. Sampled-data feedback of this form has been considered in [2,6,10,11,21,24,27]. Particularly, Theorem 9.3.1 in [6] provides links to the classical results in [2,27]. Moreover, control systems under a hybrid feedback law with asynchronous switching rules (as given in [22]) can be modeled as systems of the form (1.3).

1.2. Numerical solutions of ordinary differential equations

For example, consider the finite-dimensional continuous-time dynamical system $\dot{x}(t) = f(t, x(t))$, where $x(t) \in \Re^n$. Let $\pi = \{T_i\}_{i=0}^{\infty}$ a partition of $\Re^+$, i.e., an increasing sequence of times with $T_0 = 0$ and $T_i \to +\infty$ and define $p_\pi(t) := \min\{T \in \pi;\ t < T\}$. Consider the explicit Euler discretization scheme with state-dependent (adaptive) time step $\Re^+ \times \Re^n \ni (t, x) \to h(t, x) > 0$ that produces for each $(t_0, x_0) \in \Re^+ \times \Re^n$ the absolutely continuous function $[t_0, +\infty) \ni t \to x(t) \in \Re^n$ (Euler arc) that satisfies the evolution equation

$$\dot{x}(t) = f\big(\tau_i, x(\tau_i)\big), \quad t \in [\tau_i, \tau_{i+1}),$$
$$\tau_0 = t_0, \qquad \tau_{i+1} = \min\big\{p_\pi(\tau_i), \tau_i + h\big(\tau_i, x(\tau_i)\big)\big\}, \quad i = 0, 1, \ldots, \tag{1.4}$$

with initial condition $x(t_0) = x_0$. The stability properties of the explicit Euler method of discretization are studied in [7,17,30].

An important feature of systems of the form (1.1) under hypotheses (H1)–(H4) is that they do not satisfy the "semigroup property": for example, the solution $x(t)$ of (1.1) with initial condition $x(t_0) = x_0$ does not coincide (in general) for $t \geqslant t_1 > t_0$ with the solution $\tilde{x}(t)$ of (1.1) with initial condition $\tilde{x}(t_1) = x(t_1)$ corresponding to the same measurable and locally bounded inputs $u : \Re^+ \to U$ and $d : \Re^+ \to D$. Thus, from a mathematical point of view, they cannot be considered as systems in the sense given in [16,25]. This feature has important consequences, since the researcher cannot use the tools developed by systems theory and mathematical control theory. In the present paper we relax the notion of a system so that the "semigroup property" does not hold in a strict sense and show that systems of the form (1.1) satisfy the "relaxed" definition. Moreover, the modification introduced allows the results obtained in [16] to hold. Thus we are in a position to develop a complete stability theory, which covers systems of the form (1.1) as well as systems which satisfy the classical "semigroup property."

The obtained results are applied in systems obtained by solving numerically systems of ordinary differential equations. The qualitative behavior of the solutions of systems, which are obtained via time-discretization from continuous-time finite-dimensional systems, was the subject of intensive research during the last years. The existence of discretization methods that conserve invariants of the corresponding continuous-time system is studied in [8]. The questions concerning the relation between the attracting sets of the continuous-time ("original") system and its numerical approximation are answered in [7,30]. Both monographs present results that apply to discretization methods with fixed time step. Adaptive discretization schemes or discretization schemes with step-size control are also used in the literature (see [23]). In the present work we consider the implicit Euler method and it is shown that for an autonomous continuous-time system with a globally asymptotically stable equilibrium point, the implicit Euler method applied to an equivalent system (which has been extracted through an appropriate change of coordinates) produces a system of the form (1.1) with a globally asymptotically stable equilibrium point (Theorem 4.1). This implication is important for numerical analysis. The proof of this result uses the stability theory developed in this work and a major theorem proved in [5] for autonomous continuous-time finite-dimensional systems. The proposed method of discretization can be applied in straightforward way for the simulation of the solutions of closed-loop triangular systems under feedback (see Examples 4.3 and 4.4).

The structure of this paper is as follows: in Section 2 the definition of the notion of a control system and definitions of important classes of systems are provided. It is shown that system (1.1) under hypotheses (H1)–(H4) is a control system with outputs that satisfies important properties. In Section 3, the stability theory for control systems with outputs is developed, by extending the results contained in [16]. In Section 4 we consider the application of the stability theory developed in Section 3 to systems obtained by time discretization of systems described by ordinary differential equations. The conclusions of the paper and some final remarks are provided in Section 5.

Notations. Throughout this paper we adopt the following notations:

* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose.

* We denote by $[R]$ the integer part of the real number R, i.e., the greatest integer, which is less than or equal to R.
* $\mathcal{E}$ denotes the class of non-negative C^0 functions $\mu:\Re^+ \to \Re^+$, for which it holds: $\int_0^{+\infty} \mu(t)\,dt < +\infty$ and $\lim_{t\to+\infty} \mu(t) = 0$.
* We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho:\Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. For definitions of classes K, K_∞, KL see [18].
* By $\| \|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. Let $U \subseteq \mathcal{X}$ with $0 \in U$. By $B_U[0,r] := \{u \in U;\ \|u\|_{\mathcal{U}} \leqslant r\}$ we denote the closed sphere in $U \subseteq \mathcal{X}$ with radius $r \geqslant 0$, centered at $0 \in U$. By $B[0,r]$ we denote the closed sphere with radius $r \geqslant 0$ in $\Re^n$, centered at $0 \in \Re^n$.
* By $\mathcal{M}(U)$ we denote the set of all functions $u:\Re^+ \to U$. By u_0 we denote the identity zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geqslant 0$.
* A partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ is an increasing sequence of times with $T_0 = 0$ and $T_i \to +\infty$. For every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ we define $p_\pi(t) := \min\{T \in \pi;\ t < T\}$.

2. Control systems with outputs and equilibrium points

The definition of a control system with outputs was given in [16], inspired from the definitions in [14,25]. However, in this work a "relaxed" version is adopted, which allows important classes of systems (hybrid systems) to be considered as control systems with outputs. Moreover, we focus on continuous-time systems for reasons that are explained below.

Definition 2.1. A control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs consists of

(i) a set U (control set) which is a subset of a normed linear space $\mathcal{U}$ with $0 \in U$ and a set $M_U \subseteq \mathcal{M}(U)$ (allowable control inputs) which contains at least the identity zero input $u_0 \in M_U$ (i.e., the input that satisfies $u_0(t) = 0 \in U$ for all $t \geqslant 0$),
(ii) a set D (disturbance set) and a set $M_D \subseteq \mathcal{M}(D)$, which is called the "set of allowable disturbances,"
(iii) a pair of normed linear spaces $\mathcal{X}, \mathcal{Y}$ called the "state space" and the "output space," respectively,
(iv) a continuous map $H:\Re^+ \times \mathcal{X} \times U \to \mathcal{Y}$ that maps bounded sets of $\Re^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, called the "output map," and
(v) the map $\phi: A_\phi \to \mathcal{X}$, where $A_\phi \subseteq \Re^+ \times \Re^+ \times \mathcal{X} \times M_U \times M_D$, called the "transition map," which has the following properties:
   (1) *Existence*. For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t] \times (t_0, x_0, u, d) \subseteq A_\phi$.
   (2) *Identity property*. For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
   (3) *Causality*. For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t)$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ and $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
   (4) *Weak semigroup property*. For each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ there exists a set $\pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$ and a constant $r > 0$, such that for each $t \geqslant t_0$ with $(t, t_0, x_0, u, d) \in A_\phi$:
      (a) $(\tau, t_0, x_0, u, d) \in A_\phi$ for all $\tau \in [t_0, t]$,

(b) $\phi(t,\tau,\phi(\tau,t_0,x_0,u,d),u,d)=\phi(t,t_0,x_0,u,d)$ for all $\tau\in[t_0,t]\cap\pi(t_0,x_0,u,d)$,
(c) if $(t+r,t_0,x_0,u,d)\in A_\phi$, then it holds that $\pi(t_0,x_0,u,d)\cap[t,t+r]\neq\emptyset$,
(d) for all $\tau\in\pi(t_0,x_0,u,d)$ with $(\tau,t_0,x_0,u,d)\in A_\phi$ we have $\pi(\tau,\phi(\tau,t_0,x_0,u,d),u,d)=\pi(t_0,x_0,u,d)\cap[\tau,+\infty)$.

Let $T>0$. A control system $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ with outputs is called *T-periodic*, if:

(a) $H(t+T,x,u)=H(t,x,u)$ for all $(t,x,u)\in\Re^+\times\mathcal{X}\times U$,
(b) for every $(u,d)\in M_U\times M_D$ and integer k there exist inputs $P_{kT}u\in M_U$, $P_{kT}d\in M_D$ with $(P_{kT}u)(t)=u(t+kT)$ and $(P_{kT}d)(t)=d(t+kT)$ for all $t+kT\geqslant 0$,
(c) for each $(t,t_0,x_0,u,d)\in A_\phi$ with $t\geqslant t_0$ and for each integer k with $t_0-kT\geqslant 0$ it follows that $(t-kT,t_0-kT,x_0,P_{kT}u,P_{kT}d)\in A_\phi$ and $\pi(t_0-kT,x_0,P_{kT}u,P_{kT}d)=\bigcup_{\tau\in\pi(t_0,x_0,u,d)}\{\tau-kT\}$ with $\phi(t,t_0,x_0,u,d)=\phi(t-kT,t_0-kT,x_0,P_{kT}u,P_{kT}d)$.

A control system $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ with outputs is called *time-invariant* or *autonomous*, if:

(a) the output map is independent of t, i.e., $H(t,x,u)\equiv H(x,u)$,
(b) for every $(\theta,u,d)\in\Re\times M_U\times M_D$ there exist inputs $P_\theta u\in M_U$, $P_\theta d\in M_D$ with $(P_\theta u)(t)=u(t+\theta)$ and $(P_\theta d)(t)=d(t+\theta)$ for all $t+\theta\geqslant 0$,
(c) for each $(t,t_0,x_0,u,d)\in A_\phi$ with $t\geqslant t_0$ and for each $\theta\in(-\infty,t_0]$ it follows that $(t-\theta,t_0-\theta,x_0,P_\theta u,P_\theta d)\in A_\phi$ and $\pi(t_0-\theta,x_0,P_\theta u,P_\theta d)=\bigcup_{\tau\in\pi(t_0,x_0,u,d)}\{\tau-\theta\}$ with $\phi(t,t_0,x_0,u,d)=\phi(t-\theta,t_0-\theta,x_0,P_\theta u,P_\theta d)$.

Remark 2.2.

(a) Notice that Definition 2.1 allows us to consider all discrete-time systems as continuous-time systems with $\pi(t_0,x_0,u,d):=Z^+\cap[t_0,+\infty)$, where Z^+ denotes the set of non-negative integers and $\phi(t,t_0,x_0,u,d)=\phi([t],t_0,x_0,u,d)$, where $[t]$ denotes the integer part of $t\geqslant t_0$.
(b) The difference between the present definition of a control system with outputs and [16, Definition 2.1] lies in property (4) (semigroup property). In the above definition we do not require that for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$, there exists $t>t_0$ such that $[t_0,t)\subseteq\pi(t_0,x_0,u,d)$ (in contrast with [16, Definition 2.1]; the classical "semigroup property"). This modification allows us to study important classes of systems, which were excluded by [16, Definition 2.1]. The following example illustrates this point.
(c) It should be emphasized that all systems, which satisfy the classical "semigroup property," (namely, for all $(t_0,x_0,u,d)\in\Re^+\times\mathcal{X}\times M_U\times M_D$, there exists $t>t_0$ such that $[t_0,t)\subseteq\pi(t_0,x_0,u,d)$), are automatically control systems with outputs in the sense of Definition 2.1.

Example 2.3. Consider system (1.1) under hypotheses (H1)–(H4). Clearly, under hypothesis (1.2a), for each pair of measurable and locally bounded inputs $u:\Re^+\to U$ and $d:\Re^+\to D$ and for each $(t_0,x_0)\in\Re^+\times\Re^n$ the piecewise absolutely continuous function $t\to x(t)\in\Re^n$ that satisfies (1.1) with initial condition $x(t_0)=x_0$ is unique and the system (1.1) is a control system $\Sigma:=(\mathcal{X},\mathcal{Y},M_U,M_D,\phi,H)$ with outputs in the sense of Definition 2.1 of the present paper. Particularly, we have $\mathcal{X}=\Re^n$, $\mathcal{Y}=\Re^p$, $\mathcal{U}=\Re^m$ and M_U, M_D the sets of measurable and locally bounded inputs $u:\Re^+ \to U$ and $d:\Re^+ \to D$, respectively. The set $\pi(t_0, x_0, u, d) \subseteq [t_0, +\infty)$ involved in the weak semigroup property consists of the sequence $\pi = \{\tau_i\}_{i=0}^{\infty}$ generated by the recursive relation $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 0, 1, \ldots,$ with $\tau_0 = t_0$. Since $h:\Re^+ \times \Re^n \times U \times D \to (0, r]$ is bounded by the constant $r > 0$, it follows that property (4) of Definition 2.1 holds. Notice that the control system (1.1) fails to satisfy the classical semigroup property. Consequently, the control system (1.1) does not meet the requirements of [16, Definition 2.1].

If $h(\tau + T, x, u, d) = h(\tau, x, u, d)$, $f(t + T, \tau + T, x, x_0, u, u_0, d, d_0) = f(t, \tau, x, x_0, u, u_0, d, d_0)$, $R(\tau + T, x, x_0, u, u_0, d, d_0) = R(\tau, x, x_0, u, u_0, d, d_0)$ and $H(t + T, x, u) = H(t, x, u)$ for certain $T > 0$ and for $(t, \tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n$ with $t \geqslant \tau$, then system (1.1) is *T-periodic*. Moreover, if $h(\tau, x, u, d) = h(x, u, d)$, $f(t, \tau, x, x_0, u, u_0, d, d_0) = f(t - \tau, x, x_0, u, u_0, d, d_0)$, $R(\tau, x, x_0, u, u_0, d, d_0) = R(x, x_0, u, u_0, d, d_0)$ and $H(t, x, u) = H(x, u)$ for $(t, \tau, u, u_0, d, d_0, x, x_0) \in \Re^+ \times \Re^+ \times U \times U \times D \times D \times \Re^n \times \Re^n$ with $t \geqslant \tau$ then system (1.1) is autonomous. □

We next give definitions of some important classes of control systems.

Definition 2.4. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs. We say that system Σ

(i) has the *Boundedness-Implies-Continuation* (BIC) property if for each $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in [t_0, +\infty]$, such that $[t_0, t_{\max}) \times (t_0, x_0, u, d) \subseteq A_\phi$ and for all $t \geqslant t_{\max}$ it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$,

(ii) is *forward complete* if for every $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, $(t, t_0, x_0, u, d) \in A_\phi$ for all $t \geqslant t_0$. Clearly, every forward complete control system has the BIC property,

(iii) is simply *robustly forward complete* (RFC) if it has the BIC property and for every $R \geqslant 0$, $T \geqslant 0$, it holds that

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u_0, d)\bigr\|_{\mathcal{X}};\ s \in [0, T],\ \|x_0\|_{\mathcal{X}} \leqslant R,\ t_0 \in [0, T],\\ d \in M_D\bigr\} < +\infty,$$

(iv) is *robustly forward complete* (RFC) *from the input* $u \in M_U$ if it has the BIC property and for every $R \geqslant 0$, $T \geqslant 0$, it holds that

$$\sup\bigl\{\bigl\|\phi(t_0 + s, t_0, x_0, u, d)\bigr\|_{\mathcal{X}};\ u \in M\bigl(B_U[0, R]\bigr) \cap M_U,\ s \in [0, T],\ \|x_0\|_{\mathcal{X}} \leqslant R,\\ t_0 \in [0, T],\ d \in M_D\bigr\} < +\infty.$$

The BIC property is a property that depends on the kind of the system rather than the system itself and can be verified for wide classes of systems. In [16] it is shown that the BIC property is satisfied by systems described by ordinary differential equations (finite-dimensional) as well as systems described by retarded functional differential equations. The following proposition shows that the BIC property holds for system (1.1) under hypotheses (H1)–(H4).

Proposition 2.5. *System* (1.1) *under hypotheses* (H1)–(H4) *has the BIC property.*

Proof. Standard arguments from the theory of existence of solutions of ordinary differential equations (see, for instance, [3]) show that if $x(t)$ is defined for some $t > t_0$ then there exists $\varepsilon > 0$ such that the solution $x(\tau)$ is also defined for $\tau \in [t, t + \varepsilon)$. Thus for each $(t_0, x_0, u, d) \in \Re^+ \times \Re^n \times M_U \times M_D$, there exists a maximal existence time, i.e., there exists $t_{\max} \in (t_0, +\infty]$, for which the solution $x(t)$ of (1.1) is defined on $[t_0, t_{\max})$ and cannot be continued further.

We next show that if $t_{\max} < +\infty$ then the solution $x(t)$ of (1.1) cannot be bounded on $[t_0, t_{\max})$. Consequently, system (1.1) has the BIC property. The proof of this implication depends on the following claim.

Claim 1. *Let $s > 0$. Every infinite sequence $\{\tau_i\}$, $i = 0, 1, \ldots,$ with $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$, $i = 0, 1, \ldots,$ and $\tau_0 \geqslant 0$ satisfies $\tau_i \to +\infty$, where $p_\pi(t) := \min\{T \in \pi;\ t < T\}$ and $\pi = \{T_i\}_{i=0}^{\infty}$ is the partition involved in hypothesis* (H4).

Proof of Claim 1. Since $p_\pi(\tau_i) > \tau_i$, it follows that $\{\tau_i\}$, $i = 0, 1, \ldots,$ is an increasing sequence. Consequently, we have $\tau_i \to \sup\{\tau_i,\ i = 0, 1, \ldots\}$. Suppose that $\sup\{\tau_i,\ i = 0, 1, \ldots\} < +\infty$. In this case there exist $N, \tilde{N} > 0$ such that $T_N, T_{N-1} \in \pi$ and $T_{N-1} < \tau_i < T_N$, $i = \tilde{N}, \tilde{N} + 1, \ldots.$ Thus we obtain $p_\pi(\tau_i) = T_N$, $i = \tilde{N}, \tilde{N} + 1, \ldots,$ and consequently $\tau_{i+1} \geqslant \min\{T_N, \tau_i + s\}$, $i = \tilde{N}, \tilde{N} + 1, \ldots.$ Clearly, since $\tau_{i+1} < T_N$, $i = \tilde{N}, \tilde{N} + 1, \ldots,$ it follows that we must have $\tau_i + s = \min\{T_N, \tau_i + s\}$, $i = \tilde{N}, \tilde{N} + 1, \ldots,$ and this implies $\tau_{i+1} \geqslant \tau_i + s$, $i = \tilde{N}, \tilde{N} + 1, \ldots.$ Thus we obtain $\tau_i \geqslant T_{N-1} + (i - \tilde{N})s$, $i = \tilde{N}, \tilde{N} + 1, \ldots,$ which shows that $\sup\{\tau_i,\ i = 0, 1, \ldots\} = +\infty$, a contradiction. The proof of the claim is complete. □

We are now ready to show the required implication. Suppose that $t_{\max} < +\infty$. Let $\{t_0 = \tau_0, \tau_1, \ldots\}$ the sequence of times satisfying $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 0, 1, 2, \ldots,$ and $R := \sup\{|u(t)|;\ t \in [t_0, t_{\max}]\}$. We consider the following cases:

(1) The cardinal number of the set $\{\tau_0, \tau_1, \ldots\}$ is finite. Standard arguments from the theory of existence of solutions of ordinary differential equations show that in this case we have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$.

(2) The cardinal number of the set $\{\tau_0, \tau_1, \ldots\}$ is infinite. In this case we have $\sup\{\tau_0, \tau_1, \ldots\} \leqslant t_{\max} + r$. However, if $x(t)$ is bounded (say $x(t) \in B[0, \rho]$ for some $\rho > 0$) then we may define $s := \min\{h_l(t, x, u);\ (t, x) \in [t_0, t_{\max} + r] \times B[0, \rho] \times B_U[0, R]\}$. Since the set $[t_0, t_{\max} + r] \times B[0, \rho] \times B_U[0, R]$ is compact ($U \subseteq \Re^m$ is closed) and h_l is continuous, we have $s > 0$. Moreover, by virtue of (1.2d), we have $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$, $i = 0, 1, \ldots,$ with $\tau_0 \geqslant 0$. It follows from Claim 1 that $\tau_i \to +\infty$, which contradicts the fact that $\sup\{\tau_0, \tau_1, \ldots\} \leqslant t_{\max} + r < +\infty$. Thus we conclude that the solution $x(t)$ of (1.1) is not bounded.

In any case the hypothesis $t_{\max} < +\infty$ leads to the conclusion that the solution $x(t)$ of (1.1) is not bounded, which shows that system (1.1) has the BIC property. □

The following definition clarifies the notion of an equilibrium point for control systems with outputs in the sense of Definition 2.1.

Definition 2.6. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ and suppose that $H(t, 0, 0) = 0$ for all $t \geqslant 0$. We say that $0 \in \mathcal{X}$ is a *robust equilibrium point* for Σ if

(i) for every $(t, t_0, d) \in \Re^+ \times \Re^+ \times M_D$ with $t \geqslant t_0$ it holds that $\phi(t, t_0, 0, u_0, d) = 0$,

(ii) for every $\varepsilon > 0$, $T, h \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x) \in [0, T] \times \mathcal{X}$, $\tau \in [t_0, t_0 + h]$ with $\|x\|_{\mathcal{X}} < \delta$ it holds that $(\tau, t_0, x, u_0, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\left\{\left\|\phi(\tau, t_0, x, u_0, d)\right\|_{\mathcal{X}};\ d \in M_D,\ \tau \in [t_0, t_0 + h],\ t_0 \in [0, T]\right\} < \varepsilon.$$

We say that $0 \in \mathcal{X}$ is a *robust equilibrium point from the input $u \in M_U$ for Σ* if $0 \in \mathcal{X}$ is a robust equilibrium point for Σ and

(iii) for every $\varepsilon > 0$, $T, h \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0, x, u) \in [0, T] \times \mathcal{X} \times M_U$, $\tau \in [t_0, t_0 + h]$ with $\|x\|_{\mathcal{X}} + \sup_{t \geqslant 0} \|u(t)\|_{\mathcal{U}} < \delta$ it holds that $(\tau, t_0, x, u, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\left\{\left\|\phi(\tau, t_0, x, u, d)\right\|_{\mathcal{X}};\ d \in M_D,\ \tau \in [t_0, t_0 + h],\ t_0 \in [0, T]\right\} < \varepsilon.$$

The following proposition guarantees that system (1.1) under hypotheses (H1)–(H4) has a robust equilibrium point.

Proposition 2.7. $0 \in \Re^n$ *is a robust equilibrium point from the input $u \in M_U$ for system* (1.1) *under hypotheses* (H1)–(H4).

Proof. Since $f(t, \tau, 0, 0, 0, 0, d, d_0) = 0$, $R(\tau, 0, 0, 0, 0, d, d_0) = 0$ for all $(\tau, d, d_0) \in \Re^+ \times D \times D$, $t \geqslant \tau$ and $H(t, 0, 0) = 0$ for all $t \geqslant 0$, it follows that property (i) of Definition 2.6 is automatically satisfied. It suffices to show that for every $\varepsilon > 0$, $T, T' \in \Re^+$ there exists $\delta := \delta(\varepsilon, T, T') > 0$ such that for all $(t_0, x_0, u, d) \in [0, T] \times \Re^n \times M_U \times M_D$, $t \in [t_0, t_0 + T']$ with $|x_0| + \sup_{t \geqslant 0} |u(t)| < \delta$ it holds that the solution $x(t)$ of (1.1) with initial condition $x(t_0) = x_0$ corresponding to inputs $(u, d) \in M_U \times M_D$ exists and satisfies $\sup\{|x(t)|;\ d \in M_D,\ t \in [t_0,\ t_0 + T'],\ t_0 \in [0, T]\} < \varepsilon$.

Claim 2. *For every $\varepsilon > 0$, $T > 0$ there exists $\delta := \delta(\varepsilon, T) > 0$, such that if $|x(t_0)| + \sup_{t \geqslant 0} |u(t)| < \delta$ then the unique solution of* (1.1) *starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, \tau_1]$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_1]$, where $\tau_1 = t_0 + h(t_0, x(t_0), u(t_0), d(t_0))$.*

Proof of Claim 2. Let $L > 0$ the constant that satisfies (1.2a) for the compact set $S := [0, T + r] \times [0, T + r] \times B[0, \varepsilon] \times B[0, \varepsilon] \times B_U[0, \varepsilon] \times B_U[0, \varepsilon]$. It follows from (1.2a)–(1.2b) that the following inequality holds for all $x, x_0 \in B[0, \varepsilon]$, $u, u_0 \in B_U[0, \varepsilon]$, $\tau \in [0, T]$, $t \in [\tau, \tau + r]$ and $d, d_0 \in D$:

$$x' f(t, \tau, x, x_0, u, u_0, d, d_0) \leqslant (L + M)\big(|x|^2 + a^2\big(|x_0| + |u| + |u_0|\big)\big), \tag{2.1}$$

where $M := \frac{1 + \max\{\gamma^2(t);\ t \in [0, T + r]\}}{2}$ and $\gamma \in K^+$, $a \in K_\infty$ are the functions involved in (1.2b). Let $\rho > 0$ the unique solution of the equation:

$$\varepsilon_1^2 = 4 \exp\big(2(L + M)r\big)\big(2\rho^2 + a^2(3\rho)\big), \tag{2.2}$$

where $r > 0$ is the upper bound for h and

$$\varepsilon_1 := \min\left\{\varepsilon,\ \frac{1}{4} a^{-1}\left(\frac{\varepsilon}{\max\{\gamma(t);\ t \in [0, T + r]\}}\right)\right\} > 0. \tag{2.3}$$

Define

$$\delta = \min\left\{\frac{\varepsilon_1}{2}; \rho\right\}. \tag{2.4}$$

Let arbitrary $(x(t_0), u, d) \in \Re^n \times M_U \times M_D$ with $|x(t_0)| + \sup_{t \geqslant 0} |u(t)| < \delta$ and consider the unique solution $x(t) \in \Re^n$ of (1.1) starting from $x(t_0)$ and corresponding to input $(u, d) \in M_U \times M_D$. Since (2.4) implies $\delta < \varepsilon_1$, it follows that $|x(t_0)| < \varepsilon_1$. Next we show that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, \tau_1)$. The proof will be made by contradiction. Suppose that there exists $t_1 \in (t_0, \tau_1)$ with $|x(t_1)| \geqslant \varepsilon_1$. Let t_ε the maximal time in the interval $[t_0, t_1]$ such that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, t_\varepsilon)$. By virtue of continuity of the solution with respect to time on the interval $[t_0, \tau_1)$, the maximal time t_ε is well defined. By continuity of the solution with respect to time we must have $|x(t_\varepsilon)| = \varepsilon_1$. On the other hand, inequality (2.1) in conjunction with the fact $|x(t_0)| + \sup_{t \geqslant 0} |u(t)| < \delta$ and definition (2.4) implies $\frac{d}{dt}|x(t)|^2 \leqslant 2(L + M)(|x(t)|^2 + a^2(3\rho))$ for almost all $t \in [t_0, t_\varepsilon]$. The previous differential inequality, in conjunction with (2.2), the fact $t_\varepsilon < \tau_1 \leqslant t_0 + r$ and inequality $|x(t_0)| < \rho$, directly implies that $|x(t_\varepsilon)| \leqslant \frac{\varepsilon_1}{2} < \varepsilon_1$, which contradicts $|x(t_\varepsilon)| = \varepsilon_1$. We conclude that $|x(t)| < \varepsilon_1$ for all $t \in [t_0, \tau_1)$.

By virtue of uniform continuity of the solution on the interval $[t_0, \tau_1)$ (notice that by (1.2b) $\dot{x}(t)$ is bounded on $[t_0, \tau_1)$), it follows that the limit $\lim_{t \to \tau_1^-} x(t)$ exists and satisfies $|\lim_{t \to \tau_1^-} x(t)| \leqslant \varepsilon_1$. Using (1.2c) in conjunction with (2.3), (2.4), the facts $t_0 < \tau_1 \leqslant t_0 + r$, $|x(t_0)| + \sup_{t \geqslant 0} |u(t)| < \delta$ and $|\lim_{t \to \tau_1^-} x(t)| \leqslant \varepsilon_1$, we conclude that $|x(\tau_1)| < \varepsilon$. The previous inequality combined with the facts $\varepsilon_1 \leqslant \varepsilon$ and $|x(t)| < \varepsilon_1$ for all $t \in [t_0, \tau_1)$, implies that $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_1]$. Consequently, Claim 2 is proved. □

Using induction, the fact $\tau_i \leqslant t_0 + ir$ for all non-negative integers i (where $r > 0$ is the upper bound for h) and Claim 2, we may conclude that the following claim holds.

Claim 3. *For every $\varepsilon > 0$, $T > 0$, $N > 0$ integer, there exists $\delta := \delta(\varepsilon, T, N) > 0$, such that if $|x(t_0)| + \sup_{t \geqslant 0} |u(t)| < \delta$ then the unique solution of* (1.1) *starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, \tau_N]$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_N]$, where $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 1, \ldots, N - 1$, with $\tau_0 = t_0$.*

Consider the continuous function h_l involved in hypothesis (H4). For this function we have the following claim.

Claim 4. *For every $\varepsilon > 0$, $T > 0$, there exists integer $N := N(\varepsilon, T) > 0$, such that $\tau_N > T$, where $s := \min\{h_l(t, x, u);\ (t, x, u) \in [0, T] \times B[0, \varepsilon] \times B_U[0, \varepsilon]\}$ and the sequence $\{\tau_i\}_{i=0}^N$ satisfies $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N - 1$, with arbitrary initial condition $\tau_0 \geqslant 0$, where $p_\pi(t) := \min\{T \in \pi;\ t < T\}$ and $\pi = \{T_i\}_{i=0}^\infty$ is the partition involved in hypothesis* (H4).

Proof of Claim 4. Let arbitrary $\varepsilon > 0$, $T > 0$. Since the set $[0, T] \times B[0, \varepsilon] \times B_U[0, \varepsilon]$ is compact ($U \subseteq \Re^m$ is closed) and h_l is continuous, we have $s > 0$. Consider the infinite sequence $\{y_i\}_{i=0}^\infty$ which satisfies $y_{i+1} = \min\{p_\pi(y_i), y_i + s\}$, $i = 1, 2, \ldots$, with $y_0 = 0$. By virtue of Claim 1 showed in the proof of Proposition 2.5 we have $y_i \to +\infty$ and consequently for every $T > 0$, there exists integer $N > 0$, such that $y_N > T$. Let arbitrary $\tau_0 \geqslant 0$ and arbitrary sequence $\{\tau_i\}_{i=0}^N$ that satisfies $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N - 1$. By virtue of Theorem 1.6.1 in [19] (Comparison principle) we have $\tau_i \geqslant y_i$, $i = 1, \ldots, N$, which implies $\tau_N > T$. The proof of the claim is complete. □

We are now ready to show the required property. Let arbitrary $\varepsilon > 0$, $T, T' \in \Re^+$. Claim 4 implies that there exists integer $N := N(\varepsilon, T + T' + r) > 0$, such that $\tau_N > T + T' + r$, where $s := \min\{h_l(t, x, u);\ (t, x, u) \in [0, T + T' + r] \times B[0, \varepsilon] \times B_U[0, \varepsilon]\}$ and the sequence $\{\tau_i\}_{i=0}^{N}$ satisfies $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$, $i = 1, \ldots, N-1$, with arbitrary initial condition $\tau_0 \geqslant 0$. On the other hand, by virtue of Claim 3, there exists $\delta := \delta(\varepsilon, T, N) > 0$, such that if $|x(t_0)| + \sup_{t\geqslant 0} |u(t)| < \delta$ then the unique solution of (1.1) starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, \tau_N]$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_N]$, where $\tau_{i+1} = \tau_i + h(\tau_i, x(\tau_i), u(\tau_i), d(\tau_i))$, $i = 1, \ldots, N-1$, with $\tau_0 = t_0$. Hypothesis (H4) implies that the sequence $\{\tau_i\}_{i=0}^{N}$ satisfies the inequality $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$ for all integers i for which $\tau_i \leqslant T + T' + r$. If we assume that $\tau_N \leqslant T + T'$ then we obtain a contradiction and thus we conclude that $\tau_N > T + T'$. It follows that if $|x(t_0)| + \sup_{t\geqslant 0} |u(t)| < \delta$ then the unique solution of (1.1) starting from $x(t_0)$ at time $t_0 \in [0, T]$ and corresponding to inputs $(u, d) \in M_U \times M_D$ exists for all $t \in [t_0, t_0 + T']$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, t_0 + T']$. □

3. Stability notions for control systems with outputs

The notions robust global asymptotic stability is given in [16] for a wide class of control systems. For reasons of completeness we repeat the definition here for the class of systems allowed by Definition 2.1.

Definition 3.1. Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs that has the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. We say that Σ is *robustly globally asymptotically output stable* (RGAOS) if Σ is RFC and the following properties hold:

(P1) Σ is *robustly Lagrange output stable*, i.e., for every $\varepsilon > 0$, $T \geqslant 0$, it holds that

$$\sup\big\{\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}};\ t \geqslant t_0,\ \|x_0\|_{\mathcal{X}} \leqslant \varepsilon,\ t_0 \in [0, T],\ d \in M_D\big\} < +\infty$$

(robust Lagrange output stability).

(P2) Σ is *robustly Lyapunov output stable*, i.e., for every $\varepsilon > 0$ and $T \geqslant 0$ there exists a $\delta := \delta(\varepsilon, T) > 0$ such that

$$\|x_0\|_{\mathcal{X}} \leqslant \delta, \quad t_0 \in [0, T] \quad \Rightarrow \quad \big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \leqslant \varepsilon \quad \forall t \geqslant t_0,\ \forall d \in M_D$$

(robust Lyapunov output stability).

(P3) Σ satisfies the *robust output attractivity property*, i.e., for every $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geqslant 0$, such that

$$\|x_0\|_{\mathcal{X}} \leqslant R, \quad t_0 \in [0, T] \quad \Rightarrow \quad \big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \leqslant \varepsilon \quad \forall t \geqslant t_0 + \tau,\ \forall d \in M_D.$$

Moreover, if there exists $a \in K_\infty$ such that $a(\|x\|_{\mathcal{X}}) \leqslant \|H(t, x, 0)\|_{\mathcal{Y}}$ for all $(t, x) \in \Re^+ \times \mathcal{X}$, then we say that Σ is *robustly globally asymptotically stable* (RGAS).

It should be emphasized that the results contained in [16] are not affected by the modification of the semigroup property introduced in this work. Particularly, Lemmas 3.3–3.5 and Theorem 3.6 in [16] hold (in [16] the notion of RGAOS was given by the name non-uniform in time robust global asymptotic output stability). For reader's convenience, we mention two important estimates for RGAOS and RFC:

- if system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RGAOS, then there exist functions $\sigma \in KL$, $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$:

$$\left\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big) \right\|_{\mathcal{Y}} \leqslant \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big), \tag{3.1}$$

- system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RFC from the input $u \in M_U$ if and only if there exist functions $\mu \in K^+$, $a \in K_\infty$ and a constant $R \geqslant 0$ such that the following estimate holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$:

$$\left\| \phi(t, t_0, x_0, u, d) \right\|_{\mathcal{X}} \leqslant \mu(t) a\Big(R + \|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} \left\| u(\tau) \right\|_{\mathcal{U}}\Big) \quad \forall t \geqslant t_0. \tag{3.2}$$

The result of Lemma 3.5 in [16] can be strengthened under the following hypothesis:

(A1) For every $(u, \lambda) \in M_U \times \Re^+$, we have $\tilde{u} \in M_U$, where $\tilde{u}$ is the input that satisfies $\tilde{u}(t) = \lambda u(t)$ for all $t \geqslant 0$.

Lemma 3.2. *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property under hypothesis* (A1). *Suppose that Σ is RFC from the input $u \in M_U$ and that* $0 \in \mathcal{X}$ *is a robust equilibrium point from the input $u \in M_U$ for Σ. Then there exist functions $\mu \in K^+$, $a \in K_\infty$ such that estimate* (3.2) *holds for all $(t_0, x_0, d, u) \in \Re^+ \times \mathcal{X} \times M_D \times M_U$ with* $R = 0$.

Proof. Consider the control system $\tilde{\Sigma} := (\tilde{\mathcal{X}}, \mathcal{Y}, M_{\{0\}}, M_{\tilde{D}}, \tilde{\phi}, H)$, where $\tilde{\mathcal{X}} = \mathcal{X} \times \Re$ with norm $\|(x, z)\|_{\tilde{\mathcal{X}}} := (\|x\|_{\mathcal{X}}^2 + |z|^2)^{1/2}$, $\tilde{D} = D \times V$, $V := U \cap B_{\mathcal{U}}[0, 1]$, $M_{\tilde{D}} = M_D \times M_V$, $\tilde{\phi}(t, t_0, x_0, z_0, 0, (d, v)) := (\phi(t, t_0, x_0, |z_0|v, d), z_0)$. It can be verified immediately (using the facts that Σ is RFC from the input $u \in M_U$ and that $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ) that $\tilde{\Sigma}$ is RFC and that $0 \in \tilde{\mathcal{X}}$ is a robust equilibrium point for $\tilde{\Sigma}$.

It follows from Lemma 3.5 in [16] that there exist functions $\mu \in K^+$, $a \in K_\infty$ such that for every $(t_0, x_0, z_0, d, v) \in \Re^+ \times \mathcal{X} \times \Re \times M_D \times M_V$, we have:

$$\left\| \tilde{\phi}\big(t, t_0, x_0, z_0, 0, (d, v)\big) \right\|_{\tilde{\mathcal{X}}} \leqslant \mu(t) a\big(\|x_0\|_{\mathcal{X}} + |z_0|\big) \quad \forall t \geqslant t_0. \tag{3.3}$$

Finally, notice that for every $(t, t_0, x_0, d, u) \in \Re^+ \times \Re^+ \times \mathcal{X} \times M_D \times M_U$ with $t \geqslant t_0$, we have $\phi(\tau, t_0, x_0, u, d) := \phi(\tau, t_0, x_0, |z_0|v, d)$ for all $\tau \in [t_0, t]$, where $z_0 = \sup_{\tau \in [t_0, t]} \|u(\tau)\|_U$ and the input $v \in M_V$ is defined by the following relations:

$$v(\tau) := \frac{u(\tau)}{\sup_{s \in [t_0, t]} \|u(s)\|_{\mathcal{U}}} \in V \quad \text{if } \sup_{\tau \in [t_0, t]} \left\| u(\tau) \right\|_{\mathcal{U}} > 0$$

and

$$v(\tau) = 0 \in V \quad \text{if } \sup_{\tau \in [t_0, t]} \left\| u(\tau) \right\|_{\mathcal{U}} = 0.$$

The above observation, in conjunction with inequality (3.3), gives the desired estimate (3.2) with $R = 0$. $\square$

The following corollary is an immediate consequence of Proposition 2.7 and Lemma 3.2.

Corollary 3.3. *Suppose that for every $(u, \lambda) \in U \times \Re^+$, it holds that $(\lambda u) \in U$. System* (1.1) *under hypotheses* (H1)–(H4) *is RFC from the input $u \in M_U$ if and only if there exist functions $\mu \in K^+$, $a \in K_\infty$ such that the following estimate holds for all $(t_0, x_0, d, u) \in \Re^+ \times \Re^n \times M_D \times M_U$ for the solution $x(t)$ of* (1.1) *with initial condition $x(t_0) = x_0$ corresponding to inputs $(d, u) \in M_D \times M_U$:*

$$\left|x(t)\right| \leqslant \mu(t) a\Bigl(|x_0| + \sup_{t_0 \leqslant \tau \leqslant t} \left|u(\tau)\right|\Bigr) \quad \forall t \geqslant t_0. \tag{3.4}$$

We next provide the definition of uniform robust global asymptotic output stability, in terms of KL functions, which is completely analogous to the finite-dimensional case (see [18,20,28,29]). It is clear that such a definition is equivalent to a $\delta - \varepsilon$ definition (analogous to Definition 3.1).

Definition 3.4. Suppose that the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs is RGAOS and there exist $\sigma \in KL$ such that estimate (3.1) holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$ with $\beta(t) \equiv 1$. Then we say that Σ is *uniformly robustly globally asymptotically output stable* (URGAOS).

The following lemma must be compared to Lemma 1.1 in [9, p. 131] and Proposition 3.2 in [13]. It shows that for periodic systems RGAOS is equivalent to URGAOS.

Lemma 3.5. *Suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is T-periodic. If Σ is non-uniformly in time RGAOS, then Σ is URGAOS.*

Proof. The proof is based on the following observation: if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is T-periodic then for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$ it holds that $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$ and $H(t, \phi(t, t_0, x_0, u, d), u(t)) = H(t - kT, \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d), (P_{kT}u)(t - kT))$, where $k := [t_0/T]$ denotes the integer part of t_0/T and the inputs $P_{kT}u \in M_U$, $P_{kT}d \in M_D$ are defined in Definition 2.1.

Since $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RGAOS, there exist functions $\sigma \in KL$, $\beta \in K^+$ such that (3.1) holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$. Consequently, it follows that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$:

$$\left\| H\bigl(t, \phi(t, t_0, x_0, u_0, d), 0\bigr)\right\|_{\mathcal{Y}} \leqslant \sigma\left(\beta\left(t_0 - \left[\frac{t_0}{T}\right]T\right)\|x_0\|_{\mathcal{X}}, t - t_0\right).$$

Since $0 \leqslant t_0 - \left[\frac{t_0}{T}\right]T < T$, for all $t_0 \geqslant 0$, it follows that the following estimate holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $t \geqslant t_0$:

$$\left\| H\bigl(t, \phi(t, t_0, x_0, u_0, d), 0\bigr)\right\|_{\mathcal{Y}} \leqslant \tilde{\sigma}\bigl(\|x_0\|_{\mathcal{X}}, t - t_0\bigr),$$

where $\tilde{\sigma}(s, t) := \sigma(rs, t)$ and $r := \max\{\beta(t);\ 0 \leqslant t \leqslant T\}$. The previous estimate in conjunction with Definition 3.4 implies that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is URGAOS. The proof is complete. □

One of the most important tools for establishing RGAOS for a control system is the Lyapunov functional. The following theorem shows that the existence of a Lyapunov functional is a necessary and sufficient condition for RGAOS for systems that satisfy the following hypothesis:

(A2) There exists a constant $r > 0$, a continuous, bounded, positive function $h : \Re^+ \times \Re^+ \to (0, r]$ and a partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it holds that $\pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r] \neq \emptyset$, where $b(t, \rho) := \min\{p_\pi(t), t + h(t, \rho)\}$.

Remark 3.6.

(a) Hypothesis (A2) holds automatically for the case of the classical semigroup property; namely if for all $(t_0, x_0, u, d) \in \Re^+ \times \mathcal{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t) \subseteq \pi(t_0, x_0, u, d)$. Hence, property (A2) is satisfied by systems described by ordinary differential equations (finite-dimensional) as well as systems described by retarded functional differential equations. Moreover, hypothesis (H4) guarantees that hypothesis (A2) holds for system (1.1) under hypotheses (H1)–(H4) if system (1.1) is RFC (with $h(t, \rho) := \min_{|x| \leqslant \rho} h_l(t, x, 0)$).
(b) Notice that hypothesis (A2) guarantees the existence of $\tau \in (t_0, t_0 + r]$ such that $\tau \in \pi(t_0, x_0, u_0, d)$ for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$. Thus for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, it follows that the set $\pi(t_0, x_0, u_0, d) \setminus \{t_0\}$ cannot be empty.

Theorem 3.7 *(Lyapunov functionals). Suppose that the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs satisfies hypothesis* (A2) *and the BIC property and $0 \in \mathcal{X}$ is a robust equilibrium point for Σ. System Σ is RGAOS if and only if there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\beta, \gamma, \mu \in K^+$ with $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$, such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ there exists $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$, where $b(t, \rho) := \min\{p_\pi(t), t + h(t, \rho)\}$ is the function involved in hypothesis* (A2)*, with $(\tau, t_0, x_0, u_0, d) \in A_\phi$ and the following properties*:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \leqslant V(t_0, x_0) \leqslant a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \forall t \in [t_0, \tau], \tag{3.5a}$$

$$V\big(\tau, \phi(\tau, t_0, x_0, u_0, d)\big) \leqslant \eta\big(\tau, t_0, V(t_0, x_0)\big), \tag{3.5b}$$

where $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial value problem:

$$\dot{\eta} = -\gamma(t)\rho(\eta) + \gamma(t)\varphi\left(\int_0^t \gamma(s)\,ds\right), \qquad \eta(t_0) = \eta_0 \geqslant 0. \tag{3.5c}$$

Particularly,

(a) *if system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RGAOS then there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\mu, \beta \in K^+$, $a_1, a_2 \in K_\infty$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$ properties* (3.5a)–(3.5c) *are satisfied with $\eta(t, t_0, s) := \exp(-(t - t_0))s$, $\rho(s) := s$, $\gamma(t) \equiv 1$ and $\varphi(t) \equiv 0$,*
(b) *if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$ and if for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ we have in addition $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leqslant V(t_0, x_0) \leqslant a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, where $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$ the time for which* (3.5a)–(3.5b) *hold, then Σ is URGAOS.*

Proof. Suppose first that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RGAOS. Then by virtue of statement (ii) of Theorem 3.6 in [16] there exist functions $\mu, \beta \in K^+$, $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$, we have:

$$\begin{aligned} &\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}} \\ &\quad \leqslant \sigma\big(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big) \quad \forall t \geqslant t_0. \end{aligned} \tag{3.6}$$

Moreover, by recalling Proposition 7 in [26] there exist functions a_1, a_2 of class K_∞, such that the KL function $\sigma(s,t)$ is dominated by $a_1^{-1}(\exp(-2t)a_2(s))$. Combining the previous observations with estimate (3.6) we obtain the following estimate that holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$:

$$\begin{aligned} &a_1\big(\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ &\quad \leqslant \exp\big(-2(t - t_0)\big)a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) \quad \forall t \geqslant t_0. \end{aligned} \tag{3.7}$$

We define for all $(t_0, x_0) \in \Re^+ \times \mathcal{X}$:

$$\begin{aligned} V(t_0, x_0) := \sup\big\{&\exp(t - t_0)a_1\big(\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} \\ &+ \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big);\ t \geqslant t_0,\ d \in M_D\big\}. \end{aligned} \tag{3.8}$$

It is immediate to verify that definition (3.8) in conjunction with estimate (3.7) guarantees that inequality (3.5a) holds for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$. Moreover, definition (3.8) guarantees that inequality (3.5b) holds with $\eta(t, t_0, s) := \exp(-(t - t_0))s$ for all $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and $\tau \in \pi(t_0, x_0, u_0, d)$ (and consequently, (3.5c) holds for $\rho(s) := s$, $\gamma(t) \equiv 1$ and $\varphi(t) \equiv 0$).

Conversely, suppose that there exist mappings $V : \Re^+ \times \mathcal{X} \to \Re^+$, $\mu, \beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)\,dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2, a \in K_\infty$ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ with $\rho(s) < s$ for all $s > 0$, such that inequalities (3.5a)–(3.5b) hold with $\eta(t, t_0, \eta_0)$ being the unique solution of the initial value problem (3.5c). By virtue of Lemma 3.3 in [16], it suffices to show that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is RFC and satisfies the robust output attractivity property (property (P3) of Definition 3.1). Notice that Lemma 5.2 in [15] implies that there exist a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that the following inequalities are satisfied for all $t_0 \geqslant 0$:

$$0 \leqslant \eta(t, t_0, \eta_0) \leqslant \sigma\big(\eta_0 + M, q(t, t_0)\big) \quad \forall t \geqslant t_0,\ \forall \eta_0 \geqslant 0, \tag{3.9a}$$

where $q(t, t_0) := \int_{t_0}^t \gamma(s)\,ds$. Furthermore, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, it follows from Lemma 4.4 in [20] that there exists $\sigma(\cdot) \in KL$ such that the following inequalities are satisfied for all $t_0 \geqslant 0$:

$$0 \leqslant \eta(t, t_0, \eta_0) \leqslant \sigma(\eta_0, t - t_0) \quad \forall t \geqslant t_0,\ \forall \eta_0 \geqslant 0. \tag{3.9b}$$

Let arbitrary $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ and let $\tau_0 = t_0$. We consider the sequence $x_i = \phi(\tau_i, \tau_{i-1}, x_{i-1}, u_0, d)$, $i \geqslant 1$, and $\tau_i \in \pi(\tau_{i-1}, x_{i-1}, u_0, d) \cap [b(\tau_{i-1}, \|x_{i-1}\|_{\mathcal{X}}), \tau_{i-1} + r]$, $i \geqslant 1$, for which (3.5a)–(3.5b) hold with (τ_{i-1}, x_{i-1}) in place of (t_0, x_0). By virtue of the weak semigroup property (property (4) of Definition 2.1) we have $\tau_i \in \pi(t_0, x_0, u_0, d)$, $\tau_i \leqslant t_0 + ir$ and $x_i = \phi(\tau_i, t_0, x_0, u_0, d)$ for all $i \geqslant 1$. The semigroup property for $\eta(t, t_0, \eta_0)$ in conjunction with inequality (3.5b) and trivial induction arguments imply that

$$\begin{aligned} &a_1\big(\big\| H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \leqslant V(\tau_{i-1}, x_{i-1}) \\ &\quad \leqslant \eta\big(\tau_{i-1}, t_0, V(t_0, x_0)\big) \quad \text{for all } i \geqslant 1 \text{ and } t \in [\tau_{i-1}, \tau_i]. \end{aligned} \tag{3.10}$$

Combining (3.5a) and (3.9a) with (3.10) and using the fact that $\max\{t_0, t-r\} \leqslant \tau_{i-1}$ for $t \in [\tau_{i-1}, \tau_i]$, $i \geqslant 1$, we obtain for all $i \geqslant 1$ and $t \in [t_0, \tau_i]$:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ \leqslant \sigma\big(a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) + M, q\big(\max\{t_0, t-r\}, t_0\big)\big) \tag{3.11}$$

which directly implies

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}} + \mu(t)\big\|\phi(t, t_0, x_0, u_0, d)\big\|_{\mathcal{X}}\big) \\ \leqslant \sigma\big(a_2\big(\beta(t_0)\|x_0\|_{\mathcal{X}}\big) + M, 0\big) \quad \text{for all } i \geqslant 1 \text{ and } t \in [t_0, \tau_i]. \tag{3.12}$$

We next show that $(t, t_0, x_0, u_0, d) \in A_\phi$ for all $t \geqslant t_0$. By virtue of estimate (3.12) and the BIC property, it suffices to show that $\tau_i \to +\infty$. Let arbitrary $T > 0$. Since the set $[0, T+r] \times [0, \varepsilon]$ is compact, where $\varepsilon := \frac{a_1^{-1}(\sigma(a_2(\beta(t_0)\|x_0\|_{\mathcal{X}})+M,0))}{\min\{\mu(t);\ t\in[0,T+r]\}}$ and h is continuous, we have $s := \min\{h(t, \rho);\ (t, \rho) \in [0, T+r] \times [0, \varepsilon]\} > 0$. Consider the infinite sequence $\{y_i\}_{i=0}^{\infty}$ which satisfies $y_{i+1} = \min\{p_\pi(y_i), y_i + s\}$, $i = 1, 2, \ldots$, with $y_0 = 0$. By virtue of Claim 1 showed in the proof of Proposition 2.5 we have $y_i \to +\infty$ and consequently for every $T > 0$, there exists integer $N > 0$, such that $y_N > T + r$. Clearly, the sequence $\{\tau_i\}_{i=0}^{\infty}$ satisfies $\tau_{i+1} \geqslant \min\{p_\pi(\tau_i), \tau_i + s\}$ for all integers i for which $\tau_i \leqslant T + r$. By virtue of Theorem 1.6.1 in [19] (Comparison principle) we have $\tau_i \geqslant y_i$, for all integers i for which $\tau_i \leqslant T + r$, which implies $\tau_N > T$.

It follows that estimates (3.11) and (3.12) hold for all $t \geqslant t_0$. Robust forward completeness is an immediate consequence of (3.12) and the robust output attractivity property (property (P3) of Definition 3.1) is an immediate consequence of estimate (3.11).

Notice that if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and if for every $(t_0, x_0, d) \in \Re^+ \times \mathcal{X} \times M_D$ we have in addition $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leqslant V(t_0, x_0) \leqslant a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, then using (3.9b) instead of (3.9a), we obtain in addition the following estimate for all $t \geqslant t_0$:

$$a_1\big(\big\|H\big(t, \phi(t, t_0, x_0, u_0, d), 0\big)\big\|_{\mathcal{Y}}\big) \leqslant \sigma\big(a_2\big(\|x_0\|_{\mathcal{X}}\big), \max\{0, t - t_0 - r\}\big).$$

The above estimate directly implies that Σ is URGAOS. The proof is complete. □

4. Applications to numerical analysis

The relation between the qualitative behavior of the numerical solutions of initial value problems described by systems of ordinary differential equations and the qualitative behavior of the "actual" solution, is a well-known problem in numerical analysis and the fundamental work of Dahlquist is now part of numerical analysis textbooks (see, for instance, [23] as well as [7,19, 30] for an exposition to the numerical problem from a "difference equation" point of view). For numerical discretization schemes, the important questions of order and local discretization error, consistency and numerical stability have been studied extensively (see, for instance, the references in [19,30]) and have been related to the qualitative behavior of the numerical solutions for linear systems. For non-linear systems the questions concerning the relation between the attracting sets of the continuous-time ("original") system and its numerical approximation are answered in [7,30]. Both monographs present results that apply to discretization methods with fixed time step.

In the present work we assume that the following autonomous finite-dimensional (non-linear) system

$$\dot{z}(t) = f\big(z(t), d(t)\big), \quad z(t) \in \Re^n,\ d(t) \in D, \qquad Y = z, \tag{4.1}$$

is URGAS, where $D \subset \Re^l$ is a compact set, M_D the set of measurable and locally bounded inputs $d:\Re^+ \to D$, $f:\Re^n \times D \to \Re^n$ is a continuous vector field with $f(0,d)=0$ for all $d \in D$, locally Lipschitz for $z \neq 0$ uniformly in $d \in D$. Let a homeomorphism $\Phi:\Re^n \to \Re^n$ with $\Phi(0)=0$, which is C^1 on $\Re^n$ and consider the numerical approximation of (4.1) with variable integration step size under the change of coordinates $x=\Phi(z)$:

$$\begin{aligned}
&\dot{x}(t) = F\big(h_i, x(\tau_i), d(\tau_i)\big), \quad t \in [\tau_i, \tau_{i+1}),\\
&\tau_0 = t_0, \qquad \tau_{i+1} = \tau_i + h_i,\\
&h_i = \min\big\{p_\pi(\tau_i) - \tau_i;\ \exp\big(-u(\tau_i)\big)\big\},\\
&Y(t) = \Phi^{-1}\big(x(t)\big),
\end{aligned} \tag{4.2}$$

where $\pi = \{T_i\}_{i=0}^{\infty}$ is a partition of $\Re^+$, $F:\Re^+ \times \Re^n \times D \to \Re^n$ is a (not necessarily continuous) vector field with $F(h,0,d)=0$ for all $(h,d) \in \Re^+ \times D$, $\lim_{h\to 0^+} F(h,\Phi(z),d) = D\Phi(z)f(z,d)$ for all $(z,d) \in \Re^n \times D$. Notice that the condition $\lim_{h\to 0^+} F(h,\Phi(z),d) = D\Phi(z)f(z,d)$ is a consistency condition for the numerical scheme applied to (4.1) under the coordinate change $x=\Phi(z)$. Clearly, for every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$, system (4.2) is a hybrid system of the form (1.1) with $U := [0,+\infty) \subset \mathcal{U} := \Re$, $R(\tau,x,x_0,u,u_0,d,d_0) := x$, $f(t,\tau,x,x_0,u,u_0,d,d_0) := F(h,x_0,d_0)$, $H(t,x) := \Phi^{-1}(x)$ and $h(\tau,x_0,u_0,d_0) := \min\{p_\pi(\tau)-\tau;\ \exp(-u_0)\}$ for all $(t,\tau,x,x_0,u,u_0,d,d_0) \in \Re^+ \times \Re^+ \times \Re^n \times \Re^n \times U \times U \times D \times D$. Notice that system (4.2) fails to be autonomous. However, if the partition $\pi = \{T_i\}_{i=0}^{\infty}$ is periodic (e.g., $\pi = \{iT\}_{i=0}^{\infty}$ for certain $T>0$), then system (4.2) is periodic. Furthermore, notice that hypotheses (H1), (H3) and (H4) are automatically satisfied for system (4.2). If in addition there exists $a \in K_\infty$ such that $|F(h,x,d)| \leqslant a(|x|)$ for all $(h,x,d) \in \Re^+ \times \Re^n \times D$ with $h \leqslant 1$, then hypothesis (H2) is satisfied as well.

In the above framework, the explicit Euler method corresponds to $F(h,x,d) := D\Phi(\Phi^{-1}(x))\,f(\Phi^{-1}(x),d)$, while for the implicit Euler method the vector field $F(h,x,d)$ is defined by the formula $F(h,x,d) := D\Phi(\Phi^{-1}(y))f(\Phi^{-1}(y),d)$, were $y \in \Re^n$ is one of the solutions of the equation:

$$y - x - hD\Phi\big(\Phi^{-1}(y)\big)f\big(\Phi^{-1}(y),d\big) = 0. \tag{4.3}$$

In case that (4.3) admits no solutions, $F(h,x,d)$ may be defined in an arbitrary way. Similarly, the vector field $F(h,x,d)$ may be defined for all Runge–Kutta methods. Notice that *one advantage* of the framework presented in the present work is that variable step sizes can be represented easily, by selecting in an appropriate way the partition $\pi = \{T_i\}_{i=0}^{\infty}$ and the input $u:\Re^+ \to [0,+\infty)$.

We consider whether the step size can be selected appropriately so that the numerical solution of (4.2) has the same qualitative properties (for example, $\lim_{t\to+\infty}|Y(t)|=0$) with the "actual" solution of system (4.1). In terms of the stability theory given in the previous section, we may state this problem in the following way:

(P) Construct a homeomorphism $\Phi:\Re^n \to \Re^n$ with $\Phi(0)=0$, which is C^1 on $\Re^n$ and a continuous function $\varphi:\Re^n \to [0,+\infty)$, such that for each partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ system (4.2) with $u(t) = \varphi(x(t))$ is URGAOS.

Clearly, the solvability of problem (P) is very important, since non-solvability would imply that the numerical solution is useless. Moreover, solvability of problem (P) guarantees that the global discretization error is bounded on the positive semi-axis. Partial answers to problem (P) are given in [19,30], for the disturbance-free case, where the notions of A-stability and B-stability of Runge–Kutta (theta) methods with no coordinate change play an important role to the following cases:

(1) Linear case: $f(x) := Ax$, where the matrix A is Hurwitz.
(2) Dissipative case: $x' f(x) \leqslant -\beta|x|^2$ for all $x \in \Re^n$ and for certain constant $\beta > 0$.

In the above cases $\varphi : \Re^n \to [0, +\infty)$ may be selected to be constant, i.e., $\varphi(x) \equiv r > 0$ (case of fixed step size).

The reader should notice that problem (P) is actually a feedback stabilization problem, where the feedback function $\varphi : \Re^n \to [0, +\infty)$ determines the integration step size. Consequently, a *second advantage* of the framework presented in the present work is that feedback control theory tools (in principle) can be applied in order to stabilize system (4.2), i.e., solve problem (P).

We next state the main result of the section.

Theorem 4.1. *If* $n \neq 4, 5$ *then problem* (P) *is solvable for the implicit Euler method.*

It should be emphasized that the result of Theorem 4.1 is novel even in the disturbance-free case. For the proof of Theorem 4.1 we rely on three results: (i) the major result showed in [5], which guarantees the existence of a change of coordinates for (4.1) such that the transformed system is exponentially stable, (ii) Brouwer's fixed point theorem, and finally (iii) Theorem 3.7 of the previous section. The reason for the restriction of dimension of system (4.1) is closely related to the original Poincaré conjecture as remarked in [5].

Proof of Theorem 4.1. If $n \neq 4, 5$ and (a) $f : \Re^n \times D \to \Re^n$ is a continuous vector field with $f(0, d) = 0$ for all $d \in D$, locally Lipschitz for $z \neq 0$, uniformly in $d \in D$, (b) $D \subset \Re^l$ is a compact set, and (c) system (4.1) is URGAS, then by virtue of the procedure in the proof of Theorem 2 in [5], there exist a positive definite matrix $Q \in \Re^{n\times n}$, a constant $c > 0$ and a homeomorphism $\Phi : \Re^n \to \Re^n$ with $\Phi(0) = 0$, which is C^1 on $\Re^n$ and is a diffeomorphism on $\Re^n \setminus \{0\}$ such that for the system (4.1) under the change of coordinates $x = \Phi(z)$, namely, the finite-dimensional system

$$\dot{x}(t) = \tilde{f}\bigl(x(t), d(t)\bigr), \quad x(t) \in \Re^n, \tag{4.4}$$

where $\tilde{f}(x, d) := D\Phi(\Phi^{-1}(x)) f(\Phi^{-1}(x), d)$, satisfies

$$x' Q \tilde{f}(x, d) \leqslant -cx' Qx \quad \forall (x, d) \in \Re^n \times D. \tag{4.5}$$

Define for each $(R, x) \in \Re^+ \times \Re^n$:

$$a_x(R) := R + \max\bigl\{\bigl|\tilde{f}(y, d) - \tilde{f}(x, d)\bigr|;\ d \in D,\ y \in B_R(x)\bigr\}, \tag{4.6a}$$

$$\gamma(x) := \max\bigl\{\bigl|\tilde{f}(x, d)\bigr|;\ d \in D\bigr\}, \tag{4.6b}$$

where $B_R(x) := \{y \in \Re^n;\ |y - x| \leqslant R\}$. Clearly, by virtue of compactness of the set $D \times B_R(x)$ and upper and lower semi-continuity of the set-valued map $(R, x) \to D \times B_R(x)$, it follows from Theorem 1.4.16 in [1] that the mapping $(R, x) \to a_x(R)$, defined by (4.6a), is continuous and $a_x \in K_\infty$ for each fixed $x \in \Re^n$. We denote by $a_x^{-1} \in K_\infty$ the inverse function of $a_x \in K_\infty$ for each $x \in \Re^n$. Since the mapping $(R, x) \to a_x(R)$ is continuous, we have the mapping $(R,x) \to a_x^{-1}(R)$ is continuous. Similarly, by virtue of compactness of the set D, it follows that the mapping $x \to \gamma(x)$, defined by (4.6b), is continuous. Next define:

$$\varphi(x) := -\log\left(\frac{a_x^{-1}(\gamma(x)+1)}{2\gamma(x)+1}\right). \tag{4.7}$$

Definition (4.7) implies that φ is a continuous function. The fact that φ is non-negative follows from definition (4.6a), which implies $a_x(\gamma(x)+1) \geqslant \gamma(x)+1$ and consequently $a_x^{-1}(\gamma(x)+1) \leqslant \gamma(x)+1$ for all $x \in \Re^n$. The previous inequality in conjunction with definition (4.7) implies

$$\frac{a_x^{-1}(\gamma(x)+1)}{2\gamma(x)+1} \leqslant 1 \quad \forall x \in \Re^n. \tag{4.8}$$

We next establish the following claims for the homeomorphism $\Phi : \Re^n \to \Re^n$ given above and the continuous function $\varphi : \Re^n \to [0,+\infty)$ defined by (4.7):

(1) For each $(h,x,d) \in \Re^+ \times \Re^n \times D$, with $h \leqslant \exp(-\varphi(x))$, the set of all solutions $y \in \Re^n$ of Eq. (4.3), denoted by $G(h,x,d) \subseteq \Re^n$, is non-empty.
(2) For every selection $F(h,x,d) \in G(h,x,d)$ (not necessarily continuous), system (4.2) with $u(t) = \varphi(x(t))$ and $\varphi : \Re^n \to [0,+\infty)$ defined by (4.7) satisfies hypothesis (H2).
(3) For every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$, the quadratic function $V(x) := x'Qx$ satisfies the hypotheses of statement (b) of Theorem 3.7 for system (4.2) with $u(t) = \varphi(x(t))$ and $\varphi : \Re^n \to [0,+\infty)$ defined by (4.7).

It follows from Theorem 3.7 that for every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$, system (4.2) with $u(t) = \varphi(x(t))$ and $\varphi : \Re^n \to [0,+\infty)$ defined by (4.7) is URGAOS.

Proof of the first claim. It suffices to show that for each $(h,x,d) \in \Re^+ \times \Re^n \times D$ with $h \leqslant \exp(-\varphi(x))$ there exists $R > 0$ such that the continuous mapping $P_{h,x,d}(y) := x + h\tilde{f}(y,d)$ maps $B_R(x) := \{y \in \Re^n;\ |y-x| \leqslant R\}$ into $B_R(x)$. It follows from Brouwer's fixed point theorem that for each $(h,x,d) \in \Re^+ \times \Re^n \times D$, there exists at least one solution of Eq. (4.3).

Let $R := a_x^{-1}(\gamma(x)+1)$, where the functions a_x, γ are defined by (4.6a), (4.6b). Clearly, using definitions (4.6a), (4.6b) we have for all $y \in B_R(x)$:

$$\begin{aligned}\left|P_{h,x,d}(y) - x\right| = h\left|\tilde{f}(y,d)\right| \leqslant h\left|\tilde{f}(y,d) - \tilde{f}(x,d)\right| + h\left|\tilde{f}(x,d)\right| &\leqslant h a_x(R) + h\gamma(x) \\ &\leqslant h\big(2\gamma(x)+1\big).\end{aligned}$$

Since $h \leqslant \exp(-\varphi(x))$, where φ is defined by (4.7), we obtain from the above inequality:

$$\left|P_{h,x,d}(y) - x\right| \leqslant a_x^{-1}\big(\gamma(x)+1\big) = R.$$

Thus we have $P_{h,x,d}(y) \in B_R(x)$ for all $y \in B_R(x)$. $\square$

Proof of the second claim. Notice that for all $(h,x,d) \in \Re^+ \times \Re^n \times D$ and every solution $y \in \Re^n$ of (4.3) we have by virtue of (4.5):

$$\begin{aligned}2hx'Q\tilde{f}(y,d) &= 2hy'Q\tilde{f}(y,d) - 2h^2\tilde{f}'(y,d)Q\tilde{f}(y,d) \\ &\leqslant -2hcy'Qy - 2h^2\tilde{f}'(y,d)Q\tilde{f}(y,d).\end{aligned}$$

The previous inequality implies the following estimate for all $(h,x,d) \in \Re^+ \times \Re^n \times D$ and every solution $y \in \Re^n$ of (4.3):

$$y'Qy \leqslant \frac{1}{1+2hc}x'Qx. \tag{4.9}$$

Notice that since $Q \in \Re^{n\times n}$ is positive definite, there exist constants $K_1, K_2 > 0$ such that:

$$K_1|x|^2 \leqslant x'Qx \leqslant K_2|x|^2 \quad \forall x \in \Re^n. \tag{4.10}$$

Let a selection $F(h,x,d) \in G(h,x,d)$ for all $(h,x,d) \in \Re^+ \times \Re^n \times D$ with $h \leqslant \exp(-\varphi(x))$, where $G(h,x,d) \subseteq \Re^n$ denotes the set of solutions $y \in \Re^n$ of (4.3) for each fixed $(h,x,d) \in \Re^+ \times \Re^n \times D$. Clearly, inequalities (4.9), (4.10) imply that the following inequality holds for all $(h,x,d) \in \Re^+ \times \Re^n \times D$ with $h \leqslant \exp(-\varphi(x))$:

$$|y| \leqslant \left(\frac{K_2}{K_1}\right)^{\frac{1}{2}}|x|.$$

It follows from continuity of f, $D\Phi$, Φ^{-1} and definition $F(h,x,d) := D\Phi(\Phi^{-1}(y)) \times f(\Phi^{-1}(y),d)$ that for every selection $F(h,x,d) \in G(h,x,d)$ (not necessarily continuous), system (4.2) with $u(t) = \varphi(x(t))$ and $\varphi : \Re^n \to [0,+\infty)$ defined by (4.7) satisfies hypothesis (H2). □

Proof of the third claim. First notice that for every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$, hypothesis (A2) is satisfied for system (4.2) with $h(t,\rho) := \min\{\exp(-\varphi(x));\ |x| \leqslant \rho\}$ and $r := 1$. Since $\Phi : \Re^n \to \Re^n$ is a homeomorphism with $\Phi(0) = 0$, there exists a function $a \in K_\infty$ such that:

$$\left|\Phi^{-1}(x)\right| \leqslant a\big(|x|\big) \quad \forall x \in \Re^n. \tag{4.11}$$

Let $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ and let $\tau = \min\{p_\pi(t_0), t_0 + \varphi(x_0)\}$. Clearly, we have $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + 1]$, where $b(t,\rho) := \min\{p_\pi(t), t + h(t,\rho)\}$ the function involved in hypothesis (A2) with $h(t,\rho) := \min\{\exp(-\varphi(x));\ |x| \leqslant \rho\}$. Notice that the solution of (4.2) on $[t_0, \tau]$ is given by

$$x(t) = \frac{\tau - t}{\tau - t_0}x_0 + \frac{t - t_0}{\tau - t_0}y, \tag{4.12}$$

where $y \in \Re^n$ is one of the solutions of the equations $y - x_0 - hD\Phi(\Phi^{-1}(y))f(\Phi^{-1}(y), d(t_0)) = 0$ with $h = \min\{p(t_0) - t_0, \exp(-\varphi(x_0))\} > 0$. Let $\gamma > 0$ appropriate constant such that $\exp(\gamma h) \leqslant 1 + 2ch$ for all $h \in [0,1]$. Inequality (4.9) and Eq. (4.12) imply that:

$$V\big(x(\tau)\big) \leqslant \exp\big(-\gamma(\tau - t_0)\big)V(x_0), \tag{4.13}$$

where $V(x) := x'Qx$. Thus the function V satisfies inequality (3.5b) of Theorem 3.7 with $\eta(t, t_0, s) := s\exp(-\gamma(t - t_0))$. Consequently, (3.5c) holds with $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$ and $\rho(s) := \gamma s$. Moreover, by virtue of (4.10)–(4.12) we conclude that inequality (3.5a) of Theorem 3.7 holds with $a_2(s) := K_2 s^2$, $\beta(t) = \mu(t) \equiv 1$, $a_1(s) := \frac{1}{2}K_1 \min\big\{\big(a^{-1}\big(\frac{s}{2}\big)\big)^2, \frac{s^2}{4}\big\}$. The proof is complete. □

Remark 4.2.

(a) It should be emphasized that the role of the feedback function $\varphi : \Re^n \to [0,+\infty)$ is to guarantee that the integration step size h is sufficiently small so that Eq. (4.3) admits at least one solution. If Eq. (4.3) happens to be solvable for all $h \geqslant 0$ then $\varphi : \Re^n \to [0,+\infty)$ may be selected to be constant, i.e., $\varphi(x) \equiv r > 0$ (case of fixed step size).

(b) Notice that the procedure in the proof of Theorem 2 in [5], shows that (4.5) holds with $Q = I$ (the identity matrix). However, we chose to write (4.5) for a positive definite symmetric matrix $Q \in \Re^{n\times n}$ since (4.5) becomes less demanding (for example, in Example 4.3 below, if (4.5) was written with $Q = I$ then (4.5) would not hold and an additional linear transformation would be necessary).

(c) Notice that one disadvantage of the proposed (modified) implicit Euler method given by (4.3) is that there is no systematic procedure for the construction of the homeomorphism $\Phi : \Re^n \to \Re^n$ with $\Phi(0) = 0$, which is C^1 on $\Re^n$ and is a diffeomorphism on $\Re^n \setminus \{0\}$ such that the (transformed) system (4.1) under the change of coordinates $x = \Phi(z)$, satisfies (4.4) and (4.5). However, we are in a position to identify a class of non-linear control systems (named systems in strict feedback form), which arise frequently in feedback stabilization problems in mathematical control theory, where the transformation $\Phi : \Re^n \to \Re^n$ can be given explicitly (see Example 4.3) or a systematic procedure for the construction of $\Phi : \Re^n \to \Re^n$ can be applied (backstepping method, see Example 4.4). Since the behavior of the closed-loop system is usually tested numerically, the application of the implicit Euler method given by (4.3) for the transformed system guarantees that the simulation will produce qualitatively correct results.

Example 4.3. Consider the following non-linear finite-dimensional control system:

$$\begin{aligned}
&\dot z_i = f_i(z_1,\ldots,z_i) + z_{i+1}, \quad 1 \leqslant i \leqslant n-1,\\
&\dot z_n = f_n(z_1,\ldots,z_n) + u,\\
&z = (z_1,\ldots,z_n)' \in \Re^n, \qquad u \in \Re,
\end{aligned} \tag{4.14}$$

where each function $f_i : \Re^i \to \Re$ $(i = 1,\ldots,n)$ is of class $C^{n-i}(\Re^i;\Re)$ and satisfies $f_i(0,\ldots,0) = 0$. Systems of the form (4.14) are called triangular systems or systems in strict feedback form (see [4,18]). One way to construct a feedback law $u = k(z)$ such that the origin is a globally asymptotically stable equilibrium point for (4.14) with $u = k(z)$ is the so-called method of feedback linearization (see [12]). The method consists of two steps:

Step 1. Define recursively the functions $\varphi_i : \Re^i \to \Re$ $(i = 1,\ldots,n)$ using the condition:

$$\varphi_i(z_1,\ldots,z_i) := \sum_{j=1}^{i-1}\big(f_j(z_1,\ldots,z_j) + z_{j+1}\big)\frac{\partial\varphi_{i-1}}{\partial z_j}(z_1,\ldots,z_{i-1}), \quad i = 2,\ldots,n. \tag{4.15}$$

Particularly, for $i = 1$, we have:

$$\varphi_1(z_1) := z_1. \tag{4.16}$$

It is clear that the functions defined by (4.15), (4.16) satisfy $\frac{\partial\varphi_i}{\partial z_i} \equiv 1$, $\varphi_i(0,\ldots,0) = 0$ $(i = 1,\ldots,n)$ and consequently the transformation $\Phi : \Re^n \to \Re^n$ defined by

$$\Phi(z) := \big(\varphi_1(z_1) \quad \varphi_2(z_1,z_2) \quad \ldots \quad \varphi_n(z_1,\ldots,z_n)\big)' \tag{4.17}$$

is a global diffeomorphism on $\Re^n$ with $\Phi(0) = 0$.

Step 2. Define the feedback function:

$$\begin{aligned}
k(z) := &-f_n(z_1,\ldots,z_n) - \sum_{j=1}^{n-1}\big(f_j(z_1,\ldots,z_j) + z_{j+1}\big)\frac{\partial\varphi_n}{\partial z_j}(z_1,\ldots,z_n)\\
&- \sum_{j=1}^{n} q_j\varphi_j(z_1,\ldots,z_j),
\end{aligned} \tag{4.18}$$

where q_i $(i = 1, \dots, n)$ are positive numbers such that the polynomial $p(s) := s^n + q_n s^{n-1} + \dots + q_2 s + q_1$ is Hurwitz.

The diffeomorphism (4.17) transforms the closed-loop system (4.14) with $u = k(z)$ under the change of coordinates $x = \Phi(z)$, to the linear system:

$$\dot{x} = Ax \quad \text{with } A := \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ -q_1 & -q_2 & \dots & -q_n \end{bmatrix}, \tag{4.19}$$

where A is a Hurwitz matrix. Thus there exists a positive definite matrix $Q \in \Re^{n\times n}$ and a constant $c > 0$ such that (4.5) holds for the transformed closed-loop system. It follows that problem (P) is solvable for the implicit Euler method based on the diffeomorphism $\Phi : \Re^n \to \Re^n$ defined by (4.15)–(4.17), with constant step size, i.e., for every $r > 0$ and for every partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\Re^+$ the hybrid system:

$$\begin{aligned} &\dot{x}(t) = h_i^{-1}\big((I - h_i A)^{-1} - I\big)x(\tau_i), \quad t \in [\tau_i, \tau_{i+1}), \\ &\tau_0 = t_0, \qquad \tau_{i+1} = \tau_i + h_i, \\ &h_i = \min\{p_\pi(\tau_i) - \tau_i;\ r\}, \\ &Y(t) = \Phi^{-1}\big(x(t)\big) \end{aligned} \tag{4.20}$$

is URGAOS.

Example 4.4. Consider the following non-linear finite-dimensional control system:

$$\begin{aligned} &\dot{z}_i = \sum_{j=1}^{i} \phi_{ij}(d, z_1, \dots, z_i) z_j + g_i(d, z_1, \dots, z_i) z_{i+1}, \quad 1 \leqslant i \leqslant n-1, \\ &\dot{z}_n = \sum_{j=1}^{n} \phi_{nj}(d, z_1, \dots, z_n) + g_n(d, z_1, \dots, z_n) u, \\ &z = (z_1, \dots, z_n)' \in \Re^n, \qquad u \in \Re, \qquad d \in D, \end{aligned} \tag{4.21}$$

where $D \subset \Re^l$ is compact, each function $\phi_{ij} : D \times \Re^i \to \Re$ $(i = 1, \dots, n,\ j = 1, \dots, i)$ is continuous and each function $g_i : D \times \Re^i \to \Re$ $(i = 1, \dots, n)$ is continuous and satisfies $g_i(d, z_1, \dots, z_i) \neq 0$ for all $(d, z_1, \dots, z_i) \in D \times \Re^i$. Systems of the form (4.14) are called triangular systems or systems in strict feedback form (see [4,18]). A feedback law $u = k(z)$ such that the origin is a robustly globally asymptotically stable equilibrium point for (4.21) with $u = k(z)$ is the backstepping method presented in [4]. The method provides a systematic procedure for the construction of a global diffeomorphism on $\Re^n$ with $\Phi(0) = 0$ such that (4.5) holds with $Q = I$ for the transformed closed-loop system (4.21) with $u = k(z)$. It follows that problem (P) is solvable for the implicit Euler method. We conclude that for the triangular case (4.21) the backstepping method is a step-by-step procedure that allows the construction of: (a) a robust feedback stabilizing law, (b) a control Lyapunov function, and (c) a reliable numerical scheme.

5. Conclusions

A system-theoretic framework is proposed in the present paper, which allows the study of hybrid uncertain systems, which do not satisfy the "semigroup property." Characterizations of robust global asymptotic output stability (RGAOS) are given. Based on the provided characterizations, the qualitative behavior of hybrid systems obtained by solving numerically systems of ordinary differential equations is studied. Specifically, the implicit Euler method is considered and it is shown that for an autonomous continuous-time system with a globally asymptotically stable equilibrium point, the implicit Euler method applied to an equivalent system (which has been extracted through an appropriate change of coordinates) produces a hybrid system with a globally asymptotically stable equilibrium point. This implication is important for numerical analysis. The proof of this result uses the stability theory developed in this work and a major theorem proved recently in [5] for autonomous continuous-time finite-dimensional systems.

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