On the Observer Problem for Discrete-Time Control Systems

Iasson Karafyllis and Costas Kravaris

Abstract—**This paper studies the construction of observers for nonlinear time-varying discrete-time systems in a general context, where a certain function of the states must be estimated. Appropriate notions of robust complete observability are proposed, under which a constructive proof of existence of an observer is developed. Moreover, a "transitive observer property" is proven, according to which a state observer can be generated as the series connection of two observers. The analysis and the results are developed in general normed linear spaces, to cover both finite-dimensional and infinite-dimensional systems.**

Index Terms—**Discrete-time systems, observer, state estimation.**

I. Introduction

THE observer design problem for nonlinear discrete-time systems with or without inputs has attracted the interest of many researchers and important results on the nonlinear observer design problem can be found in [1], [5], [6], [8], [12], [15], [16], [19], and [26]. The observer design problem for the linear case for finite-dimensional discrete-time systems is now well understood (see the corresponding results provided in [22] and the references therein), but remains still an open problem for infinite-dimensional systems. Results concerning the dead-beat observer synthesis problem were provided in [25]. In [19], a Newton iteration approach was used for the solution of a set of nonlinear algebraic equations, which provided estimates of the states. The applications of solutions to the nonlinear observer design problem for discrete-time systems are widespread (see, for instance, [11], [14], and [17] for applications to the output feedback stabilization problem). Finally, it should be emphasized that the nonlinear observer design problem for the discrete-time case is closely related to the corresponding problem in the continuous-time case when sampling is introduced (see, for instance, [2] and [9]) as well as with the observability problem for discrete-time systems (see, for instance, [18], [22], and [23]).

In this paper, the focus will be on the functional observer design problem, where the objective is to estimate a given nonlinear function of the state vector, instead of the entire state vector. Functional observers have been studied extensively for continuous-time linear systems (see [7] and the references therein), and have been applied to the problem of output feedback controller synthesis, where the function to be estimated is the state feedback function.

In order to provide an informal introduction to the general ideas of this paper, let us consider a finite-dimensional discrete-time system of the form

$$\begin{aligned} x(k+1) &= f(k, x(k), u(k)) \\ y(k) &= h(k, x(k)) \\ x(k) &\in \Re^n, y(k) \in \Re^l, u(k) \in \Re^m, k \in Z^+ \end{aligned} \tag{1}$$

where $y(k)$ represents the measurement. Moreover, consider another output

$$Y(k) = \theta(k, x(k)) \tag{2}$$

where $\theta : Z^+ \times \Re^n \rightarrow \Re^l$ a given function of the states, and the objective is to design a functional observer to estimate the output Y, driven by the measurement of y.

In order to study the functional observer design problem, appropriate notions of (functional) observability must be introduced and, subsequently, the existence of a functional observer must be established. Motivated by problems of inferential control, where the dynamic system can be subject to unknown external disturbances, notions of robust complete observability will be introduced in this paper, for the case where the right hand side of (1) also depends on unknown disturbances $d(k)$. Moreover, in order to make the theoretical formulation more general, the analysis will be performed with the inputs, states and outputs belonging to subsets of general normed linear spaces (not necessarily finite-dimensional). Even though finite-dimensional systems are intended to be the primary target of the present work, results will be applicable to infinite-dimensional systems, which are currently attracting interest in the literature (see [3] and [20]).

The development of the solution of the functional observer problem is constructive, leading to a set of conditions that determine the functional observer. In addition to inferential control applications, the proposed functional observer can be useful for the construction of a regular full-state observer. To illustrate this point, consider the following simple example:

$$\begin{aligned} x_1(k+1) &= f_1(x(k)) \\ x_2(k+1) &= f_2(x_1(k), f_1(x(k))) \\ x_3(k+1) &= a x_3(k) + f_3(x_1(k), x_2(k)) \\ x(k) &= (x_1(k), x_2(k), x_3(k))' \in \Re^3 \end{aligned} \tag{3}$$

with measured output

$$y(k) = x_1(k) \tag{4}$$

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where $a \in \Re$ is a constant with $|a| < 1$ and $f_i, i = 1, 2, 3$ are continuous functions that vanish at $x = 0$. This is a finite-dimensional autonomous discrete-time system possessing a special structure that could be exploited for the construction of a state observer. Suppose for the moment that, instead of (4)

$$Y(k) = (x_1(k), x_2(k)) \tag{5}$$

was the measured output. Then, the third equation of (3) could be simulated online, forming a reduced-order open-loop observer for the system

$$\begin{aligned} z(k+1) &= az(k) + f_3(Y(k)) \\ \bar{x}(k) &= (Y(k), z(k)). \end{aligned} \tag{6}$$

Then, the estimation error $e(k) = \bar{x}(k) - x(k)$ satisfies $e(k+1) = ae(k)$ for all $t \geq 0$; therefore, we have exponential convergence of the estimates.

Now, since the measurement is actually $y(k) = x_1(k)$ and not $Y(k) = (x_1(k), x_2(k))$, we must find a way to estimate x_2. Considering the second equation of (3), we see that the right hand side can be immediately calculated from $y(k)$ and $y(k-1)$. This leads to the following deadbeat observer for x_2 (for details, see Proposition 14 and Example 19):

$$\bar{x}_2(k) = f_2(y(k-1), y(k)) \qquad \forall k \geq 1.$$

Equivalently, the auxiliary output (5) can be estimated from the measured output (4) from the measured output (4) via the following functional observer:

$$\begin{aligned} w(k+1) &= y(k) \\ \bar{Y}(k) &= (y(k), f_2(w(k), y(k))). \end{aligned} \tag{7}$$

Intuitively, a state observer for the original system should be able to be constructed by combining the functional observer (7), followed by the state observer (6) that was constructed on the basis of the auxiliary output (5)

$$\begin{aligned} w(k+1) &= y(k) \\ z(k+1) &= az(k) + f_3(y(k), f_2(w(k), y(k))) \\ \bar{x}(k) &= (y(k), f_2(w(k), y(k)), z(k)). \end{aligned}$$

It turns out that the above system is a global observer for (3), with exponential convergence of the state estimates. Moreover, it is the product of a simple and intuitively meaningful construction. If a general, all-purpose nonlinear design method were to be applied, it would lead to an unnecessarily complicated observer, since it would not account for the special structure of the system.

One of the key contributions of the present paper will be the theoretical justification of the foregoing two-step process for the design of a nonlinear state observer. In particular, referring to system (1), when a full-state observer is available

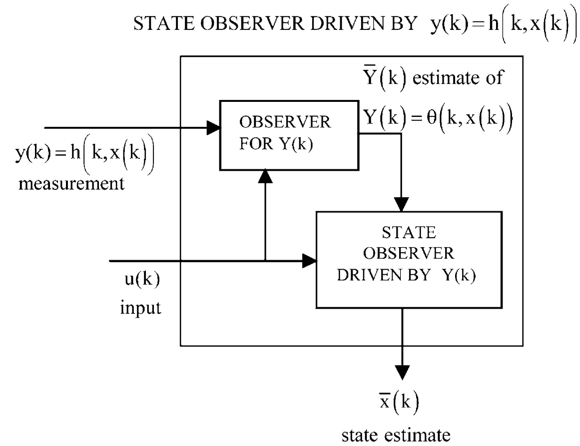

Fig. 1. Transitive observer property.

for the output map $Y = \theta(k, x)$ (through any available design method from the literature), and the proposed functional observer is constructed for the estimation of $Y = \theta(k, x)$ from the measured output $y = h(k, x)$, we show that their series connection generates a full-state observer from the measurement $y = h(k, x)$. We call this important property the "transitive observer property" (TOP) (see Fig. 1). This property can facilitate the observer design, since certain output maps may be more convenient for the design of an observer than the actual measured outputs.

The structure of the paper is as follows: In Section II. we review fundamental properties and bounds for the solutions of difference equations with inputs. The results of Section II, will enable the derivation of convergence estimates in subsequent sections. In Section III, definitions for the notion of observer and functional observer for general time-varying nonlinear discrete-time systems with inputs are provided. The proposed observer notions are weaker than existing notions in the literature and make more intuitive sense, as will be seen through an illustrative example. The proposed observer notions will play a critical role in establishing the transitive observer property, which would not hold under more stringent specifications for the observer. Moreover, in Section III, the definition of the notion of complete observability is given. In Section IV, we provide the main results of this paper on the construction of functional observers and their application to the design of full-state observers, as well as finite-dimensional and infinite-dimensional examples illustrating the main results. Finally, in Section V, some concluding remarks are presented.

Notations:

- By $\| \|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. By $| |$ we denote the euclidean norm of $\Re^n$.
- We denote by Z^+ the set of non-negative integers and by $\Re^+$ the set of non-negative real numbers.
- Let $\mathcal{U}$ a normed linear space, $U \subseteq \mathcal{U}$ a non-empty set, $u \in U \subseteq \mathcal{U}$ and $r \geq 0$. We denote by $B_U[u, r]$ the closed ball in $U \subseteq \mathcal{U}$ of radius $r \geq 0$ centered at $u \in U$, i.e., $B_U[u, r] := \{v \in U; \|v - u\|_{\mathcal{U}} \leq r\}$.
- We denote by $CU(Z^+ \times A; \Theta)$, where A is a subset of the normed linear space $\mathcal{X}$ and Θ is a normed linear space, the class of continuous maps $\Psi : Z^+ \times A \to \Theta$ with

the properties that: i) for every pair of bounded sets $I \subset Z^+, O \subset A$ the image set $\Psi(I \times O)$ is bounded, and ii) $\Psi(k,0) = 0$ for all $k \in Z^+$.

- We denote by M_U the set of sequences with values in the set U.
- For definitions of classes K, K_∞, KL see [13]. K^+ denotes the class of positive functions $a : Z^+ \to (0,+\infty)$.
- x' denotes the transpose of the vector $x \in \Re^n$ and A' denotes the transpose of the matrix $A \in \Re^{n\times n}$.

The following convention will be adopted throughout this paper: The Cartesian product of two normed linear spaces $C := \mathcal{X} \times \mathcal{Y}$ will be considered to be endowed with the norm $\|(x,y)\|_C := \sqrt{\|x\|_{\mathcal{X}}^2 + \|y\|_{\mathcal{Y}}^2}$, unless stated otherwise.

II. Fundamental Properties of the Solutions of Difference Equations

Consider the following discrete-time system:

$$x(k+1) = f(k, d(k), x(k), u(k))$$
$$x(k) \in \mathcal{X}, d(k) \in D, u(k) \in U \subseteq \mathcal{U}, k \in Z^+ \quad (8)$$
$$y(k) = h(k, x(k)), y(k) \in \mathcal{Y} \quad (9)$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{U}$ is a triplet of normed linear spaces, D is the set of disturbances (or time-varying parameters), $U \subseteq \mathcal{U}$ is the set of inputs with $0 \in U$ and $f : Z^+ \times D \times \mathcal{X} \times U \to \mathcal{X}, h : Z^+ \times \mathcal{X} \to \mathcal{Y}$ with $f(k,d,0,0) = 0$ and $h(k,0) = 0$ for all $(k,d) \in Z^+ \times D$. Let $x(k, k_0, x_0, u, d)$ denote the solution of (8) at time $k \geq k_0$ with initial condition $x(k_0) = x_0 \in \mathcal{X}$ corresponding to inputs (sequences) $(u,d) \in M_U \times M_D$.

Notions of robust forward completeness and robust equilibrium point were introduced in [10] for a wide class of systems (that encompasses discrete-time systems). In what follows, these notions will be reviewed, since they will be needed in subsequent developments. In particular, the following definitions are [10, Defs. 2.2 and 2.3], specialized to the case of discrete-time systems (8).

Definition 1: Let u_0 denote the identity zero input, i.e., $u_0(k) = 0 \in U$ for all $k \in Z^+$. We say that system (8) is ***robustly forward complete (RFC)*** if for every $r \geq 0, T \in Z^+$, it holds that the first equation at the bottom of the page holds. We say that system (8) is ***robustly forward complete (RFC)*** **from the input $u \in M_U$** if for every $r \geq 0, T \in Z^+$, it holds that the second equation at the bottom of the page holds.

Definition 2: Consider system (8) and let u_0 denote the identity zero input, i.e., $u_0(k) = 0 \in U$ for all $k \in Z^+$. We say that $0 \in \mathcal{X}$ is a ***robust equilibrium point*** for (8) if for every $\varepsilon > 0, T, h \in Z^+$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that the third equation at the bottom of the page holds.

The following lemma is a specialization of [10, Lemma 3.5] to the case of discrete-time systems (8). It provides a characterization of the RFC property.

Lemma 3: Consider system (8) and let u_0 denote the identity zero input, i.e., $u_0(k) = 0 \in U$ for all $k \in Z^+$.

i) (8) is RFC from the input $u \in M_U$ if and only if there exist functions $\mu \in K^+, a \in K_\infty$ and a constant $R \geq 0$ such that the following estimate holds for all $u \in M_U, (k_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$ and $k \geq k_0$:

$$\|x(k, k_0, x_0, u, d)\|_{\mathcal{X}} \leq \mu(k) a\left(R + \|x_0\|_{\mathcal{X}} + \sup_{l \in \{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}}\right). \quad (10)$$

ii) Equation (8) is RFC, if and only if, there exist functions $\mu \in K^+, a \in K_\infty$ and a constant $R \geq 0$, such that for every $d \in M_D, (k_0, x_0) \in Z^+ \times \mathcal{X}$, we have

$$\|x(k, k_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \mu(k) a(\|x_0\|_{\mathcal{X}} + R) \qquad \forall k \geq k_0. \quad (11)$$

Moreover, if $0 \in \mathcal{X}$ is a robust equilibrium point for (8), then inequality (11) holds with $R = 0$.

Our main assumption concerning system (8) is as follows.

H1) There exist functions $a \in K_\infty, \gamma \in K^+$ such that $\|f(k,d,x,u)\|_{\mathcal{X}} \leq a(\gamma(k)\|x\|_{\mathcal{X}}) + a(\gamma(k)\|u\|_{\mathcal{U}})$, for all $(k,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U$.

The following fact is an important implication of hypothesis **H1)** to discrete-time systems (8):

Fact I: There exist functions $\mu \in K^+, a \in K_\infty$ and a constant $R \geq 0$ such that for every $(k_0, x_0, d, u) \in Z^+ \times \mathcal{X} \times$

$$\sup\left\{\|x(k_0+l, k_0, x_0, u_0, d)\|_{\mathcal{X}};\ \begin{matrix} l \in \{0,\ldots,T\}, \|x_0\|_{\mathcal{X}} \leq r \\ k_0 \in \{0,\ldots,T\}, d \in M_D \end{matrix}\right\} < +\infty$$

$$\sup\left\{\|x(k_0+l, k_0, x_0, u, d)\|_{\mathcal{X}};\ \begin{matrix} u \in M_{B_U[0,r]}, l \in \{0,\ldots,T\} \\ \|x_0\|_{\mathcal{X}} \leq r, k_0 \in \{0,\ldots,T\}, d \in M_D \end{matrix}\right\} < +\infty$$

$$\sup\left\{\|x(k, k_0, x_0, u_0, d)\|_{\mathcal{X}};\ \begin{matrix} \|x_0\|_{\mathcal{X}} < \delta, k \in \{k_0,\ldots,k_0+h\} \\ d \in M_D, k_0 \in \{0,\ldots,T\} \end{matrix}\right\} < \varepsilon$$

$M_D \times M_U$, the corresponding solution $x(k,k_0,x_0,u,d)$ of (8) satisfies the following estimate for all $k \geq k_0$:

$$\|x(k,k_0,x_0,u,d)\|_{\mathcal{X}} \leq \mu(k) a\left(R + \|x_0\|_{\mathcal{X}} + \sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}}\right). \quad (12)$$

To prove Fact I, notice that by virtue of Lemma 3 it suffices to show that system (8) under hypothesis (H1) is Robustly Forward Complete from the input $u \in M_U$. This follows by considering arbitrary $r \geq 0, T \in Z^+$, then defining recursively the sequence of sets in $\mathcal{X}$ by $A(l) := f([0,2T] \times D \times A(l-1) \times B_U[0,r])$ for $l = 1,\ldots,T$, where $A(0) := \{x \in \mathcal{X}; \|x\|_{\mathcal{X}} \leq r\}$, which are bounded by virtue of hypothesis **H1)** and finally noticing that the equation at the bottom of the page holds.

Our main assumption concerning the input set U is as follows.

H2) U is a positive cone, i.e., for all $u \in U$ and $\lambda \geq 0$ it follows that $(\lambda u) \in U$.

The following lemma shows that under hypothesis **H2)**, it is possible to obtain sharper estimates than estimate (10) for the solutions of (8).

Lemma 4: Suppose that hypotheses (H1-2) are fulfilled for system (8). Then there exist functions $\mu \in K^+, a \in K_\infty$ such that for every $(k_0,x_0,d,u) \in Z^+ \times \mathcal{X} \times M_D \times M_U$, the corresponding solution $x(k)$ of (8) with $x(k_0) = x_0$ satisfies estimate (12) with $R = 0$, i.e.,

$$\|x(k)\|_{\mathcal{X}} \leq \mu(k) a\left(\|x_0\|_{\mathcal{X}} + \sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}}\right) \qquad \forall k \geq k_0. \quad (13)$$

Proof: Consider the following discrete-time dynamical system:

$$\begin{aligned} x(k+1) &= \tilde{f}(k,\tilde{d}(k),x(k),z(k)) \\ z(k+1) &= z(k) \\ (x(k),z(k)) &\in C, \tilde{d}(k) \in \tilde{D}, k \in Z^+ \end{aligned} \quad (14)$$

where $\tilde{d} := (d,v), \tilde{D} = D \times V, V := B_U[0,1], \tilde{f}(k,\tilde{d},x,z) := f(k,d,x,|z|v), C := \mathcal{X} \times \Re$ with norm $\|(x,z)\|_C := (\|x\|_{\mathcal{X}}^2 + |z|^2)^{(1/2)}$. Notice that by virtue of hypothesis **H1)** we know that there exist functions $a \in K_\infty, \gamma \in K^+$ such that $\|f(k,d,x,u)\|_{\mathcal{X}} \leq a(\gamma(k)\|x\|_{\mathcal{X}}) + a(\gamma(k)\|u\|_{\mathcal{U}})$, for all $(k,x,d,u) \in Z^+ \times \mathcal{X} \times D \times U$, which directly implies the following inequality:

$$\begin{aligned} &\|\tilde{f}(k,\tilde{d},x,z),z)\|_C \leq \tilde{a}(\tilde{\gamma}(k)\|(x,z)\|_C) \\ &\text{for all } (k,x,z,\tilde{d}) \in Z^+ \times \mathcal{X} \times \Re \times \tilde{D} \end{aligned} \quad (15)$$

where $\tilde{a}(s) := 2a(s) + s \in K_\infty$ and $\tilde{\gamma}(k) := \gamma(k) + 1 \in K^+$. The fact that system (14) is robustly forward complete can be shown in exactly the same way as in the proof of Fact I, by exploiting (15).

We next show that $0 \in \mathcal{X} \times \Re$ is a robust equilibrium point for (14). It suffices to show that for every $\varepsilon > 0, N \in Z^+$ and $T \in Z^+$ there exists $\delta := \delta(\varepsilon,N,T) \in (0,\varepsilon]$ such that

$$\begin{aligned} &\|(x_0,z_0)\|_C \leq \delta, k_0 \in \{0,\ldots,T\} \\ &\Rightarrow \sup\{\|(x(k_0+l,k_0,x_0,z_0;\tilde{d}),z_0)\|_C; \\ &l \in \{0,\ldots,N\}, \tilde{d} \in M_{\tilde{D}}\} \leq \varepsilon \end{aligned}$$

where $x(k,k_0,x_0,z_0;\tilde{d})$ denotes the x-component of the unique solution of (14) initiated from $(x_0,z_0) \in C$ at time $k_0 \in Z^+$ and corresponding to $\tilde{d} \in M_{\tilde{D}}$ (notice that the z-component of the unique solution of (14) satisfies $z(k) = z_0$ for all $k \geq k_0$). We prove this fact by induction on $N \in Z^+$. First notice that the fact holds for $N = 0$ (by selecting $\delta(\varepsilon,0,T) = \varepsilon$). We next assume that the fact holds for some $N \in Z^+$ and we prove it for the next integer $N+1$. In order to have $\|(x(k_0+N+1,k_0,x_0,z_0;\tilde{d}),z_0)\|_C \leq \varepsilon$, by virtue of inequality (15) it suffices to have $\|(x(k_0+N,k_0,x_0,z_0;\tilde{d}),z_0)\|_C \leq \rho(\varepsilon,N,T)$, where $\rho(\varepsilon,N,T) := (\tilde{a}^{-1}(\varepsilon))/(\max\{\tilde{\gamma}(k); 0 \leq k \leq T+N\})$. It follows that the selection $\delta(\varepsilon,N+1,T) := \min\{\delta(\varepsilon,N,T), \delta(\rho(\varepsilon,N,T),N,T)\} > 0$ guarantees that

$$\begin{aligned} &\sup\{\|(x(k_0+l,k_0,x_0,z_0;\tilde{d}),z_0)\|_C; \\ &0 \leq l \leq N, \tilde{d} \in M_{\tilde{D}}\} \\ &\leq \varepsilon \leq \min\{\varepsilon, \rho(\varepsilon,N,T)\} \end{aligned}$$

and $\|(x(k_0+N+1,k_0,x_0,z_0;\tilde{d}),z_0)\|_C \leq \varepsilon$, for all $\|(x_0,z_0)\|_C \leq \delta, k_0 \in \{0,\ldots,T\}$.

It follows from Lemma 3 that there exist functions $\mu \in K^+, a \in K_\infty$ such that for every $(x_0,z_0) \in \mathcal{X} \times \Re, \tilde{d} \in M_{\tilde{D}}, k_0 \in Z^+$, the solution $(x(k),z(k))$ of (14) with initial condition $(x(k_0),z(k_0)) = (x_0,z_0)$ and corresponding to input $\tilde{d} \in M_{\tilde{D}}$ satisfies

$$\|(x(k),z(k))\|_C \leq \mu(k) a(\|x_0\|_{\mathcal{X}} + |z_0|) \qquad \forall k \geq k_0. \quad (16)$$

Finally, notice that that for every $x_0 \in \mathcal{X}, k_0 \in Z^+, k \in Z^+$ with $k \geq k_0, (u,d) \in M_U \times M_D$, the unique solution $x(k)$ of (8), with initial condition $x(k_0) = x_0$ and corresponding to $(u,d) \in M_U \times M_D$ coincides on $\{k_0,\ldots,k\}$ with the component $x(k)$ of the solution of (14) with initial condition $(x(k_0),z(k_0)) = (x_0, \sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}})$ and corresponding to $(d,v) \in M_{\tilde{D}}$, where

$$v(\tau) := \frac{u(\tau)}{\sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}}} \in V, \quad \text{if } \sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_{\mathcal{U}} > 0 \text{ and } \tau \in \{k_0,\ldots,k\}$$

and

$$v(\tau) = 0 \in V, \text{ if otherwise.}$$

$$\left\{x(k_0+l,k_0,x_0;u,d); \begin{array}{c} \|x_0\|_{\mathcal{X}} \leq r, k_0 \in \{0,\ldots,T\} \\ l \in \{0,\ldots,T\}, d \in M_D, u \in M_{B_U[0,r]} \end{array}\right\} \subseteq A(l), \qquad \text{for all } l = 0,\ldots,T$$

This observation, in conjunction with inequality (16) gives the desired estimate (13). The proof is complete. ◁

Our main assumption concerning the output map of system (8), (9) is as follows.

H3) The output map $h : Z^+ \times \mathcal{X} \to \mathcal{Y}$ is continuous and for every pair of bounded sets $I \subset Z^+, O \subset \mathcal{X}$ the image set $h(I \times O)$ is bounded and $h(k,0) = 0$ for all $k \in Z^+$, i.e., $h \in CU(Z^+ \times \mathcal{X}; \mathcal{Y})$ (see notations).

The following lemma is an important tool for the derivation of estimates of the solutions of difference equations and will be used extensively in the next section of this paper.

Lemma 5: Suppose that hypotheses **H1)–H3)** are fulfilled for system (8), (9). Then the following statements are equivalent.

i) System (8), (9) satisfies the ***robust output attractivity property***, i.e., for every $\varepsilon > 0, T \in Z^+$ and $R \geq 0$, there exists $\tau := \tau(\varepsilon, T, R) \in Z^+$, such that for every $x_0 \in \mathcal{X}, (u,d) \in M_U \times M_D$, with $\|x_0\|_{\mathcal{X}} + \sup_{l \in Z^+} \|u(l)\|_{\mathcal{U}} \leq R, k_0 \in \{0, \ldots T\}$, the solution $x(k)$ of (8), (9) with initial condition $x(k_0) = x_0$ and corresponding to input $(u,d) \in M_U \times M_D$ satisfies

$$\|h(k, x(k))\|_{\mathcal{Y}} \leq \varepsilon, \ \forall k \geq k_0 + \tau$$

ii) There exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $x_0 \in \mathcal{X}, k_0 \in Z^+, (u,d) \in M_U \times M_D$, the unique solution $x(k)$ of (8), (9), with initial condition $x(k_0) = x_0$ and corresponding to $(u,d) \in M_U \times M_D$, satisfies for all $k \geq k_0$

$$\|h(k, x(k))\|_{\mathcal{Y}} \leq \sigma\left(\beta(k_0)\left(\|x_0\|_{\mathcal{X}} + \sup_{l \in \{k_0, \ldots, k\}} \|u(l)\|_{\mathcal{U}}\right), k - k_0\right). \tag{17}$$

The proof of Lemma 5 relies on characterizations of the notion of non-uniform in time robust global asymptotic output stability (RGAOS), which are provided in [10] for a wide class of systems with outputs (that includes discrete-time systems). For reasons of completeness, we have included all characterizations of non-uniform in time RGAOS based on [10] in the Appendix, appropriately specialized for the case of discrete-time systems (8), (9).

Proof: i) $\Rightarrow$ ii) Notice that for every $(x_0, z_0) \in \mathcal{X} \times \Re, k_0 \in Z^+, (d,v) \in M_{\tilde{D}}$ the component $x(k)$ of the solution of (14) with initial condition $(x(k_0), z(k_0)) = (x_0, z_0)$ and corresponding to $(d,v) \in M_{\tilde{D}}$ coincides on the time set $\{k_0, k_0+1, k_0+2, \ldots\}$ with the unique solution $x(k)$ of (8), with initial condition $x(k_0) = x_0$ and corresponding to $(u,d) \in M_U \times M_D$, where

$$u(k) = |z_0| v(k) \in U$$
$$\sup_{l \in Z^+} \|u(l)\|_{\mathcal{U}} = |z_0|.$$

Notice that, as it is shown in the proof of Lemma 4, system (14) is Robustly Forward Complete and $0 \in \mathcal{X} \times \Re$ is a robust equilibrium point for (14). Consequently, by virtue of Lemmas 22 and 23 in the Appendix, in conjunction with hypothesis i) (which implies the robust output attractivity property for system (14) with output $y(k) = h(k, x(k))$), it follows that there exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $(x_0, z_0) \in \mathcal{X} \times \Re, k_0 \in Z^+, (d,v) \in M_{\tilde{D}}$ the unique solution $(x(k), z(k))$ of (14), with initial condition $(x(k_0), z(k_0)) = (x_0, z_0)$ and corresponding to $(d,v) \in M_{\tilde{D}}$, satisfies

$$\|h(k, x(k))\|_{\mathcal{Y}} \leq \sigma(\beta(k_0)(\|x_0\|_{\mathcal{X}} + |z_0|), k - k_0) \qquad \forall k \geq k_0. \tag{18}$$

Moreover, notice that that for every $x_0 \in \mathcal{X}, k_0 \in Z^+, k \in Z^+$ with $k \geq k_0, (u,d) \in M_U \times M_D$, the unique solution $x(k)$ of (8), with initial condition $x(k_0) = x_0$ and corresponding to $(u,d) \in M_U \times M_D$ coincides on the time set $\{k_0, \ldots, k\}$ with the component $x(k)$ of the solution of (14) with initial condition $(x(k_0), z(k_0)) = (x_0, \sup_{l \in \{k_0, \ldots, k\}} \|u(l)\|_{\mathcal{U}})$ and corresponding to $(d,v) \in M_{\tilde{D}}$, where

$$v(\tau) := \frac{u(\tau)}{\sup_{l \in \{k_0, \ldots, k\}} \|u(l)\|_{\mathcal{U}}} \in V, \quad \text{if } \sup_{l \in \{k_0, \ldots, k\}} \|u(l)\|_{\mathcal{U}} > 0 \text{ and } \tau \in \{k_0, \ldots, k\}$$

and

$$v(\tau) = 0 \in V \text{ if otherwise}$$

The previous observation in conjunction with (18) implies the desired estimate (17).

ii) $\Rightarrow$ i) This implication is an immediate consequence of the properties of the KL functions.

The proof is complete. ◁

Remark 6: Some comments regarding hypotheses **H1)–H3)**. The role of hypotheses **H1)–H3)** is to guarantee the estimates derived in this section. The estimates of the solutions of difference equations are used extensively in the following sections in order to prove the main results of the paper. The following remarks apply for each one of the hypotheses **H1)–H3)**.

a) Hypothesis **H1)** is automatically satisfied if f is of class CU and D is compact. Furthermore, there are certain classes of discrete-time systems with discontinuous right-hand sides which satisfy hypothesis **H1)**. Notice that hypothesis **H1)** implies continuity of f only at $(x,u) = (0,0)$ and local boundedness of f (f maps bounded sets into bounded sets).

b) Hypothesis **H2)** is a technical hypothesis, which usually holds in applications. If hypothesis **H2)** is not satisfied, then it may be satisfied after input redefinition, i.e., we may introduce a locally bounded mapping $q : V \to U$ with $q(0) = 0$ and $q(V) = U$, where $V \subseteq \mathcal{U}$ is a positive cone with $0 \in V$ and define $u = q(v)$. Notice that in the case of input redefinition, the mapping $q : V \to U$ is not required to be continuous [only continuity at $0 \in U$ is required in order to regain hypothesis **H1)**]. For example, when $U \subseteq \Re^k$ is a closed convex set with $0 \in U$, we may define $q : \Re^k \to U$ as the (continuous) projection on $U \subseteq \Re^k$.

c) Hypothesis **H3)** is a standard continuity hypothesis for the output map in the finite-dimensional case. However, in the infinite-dimensional case it is not necessary that a continuous map h maps bounded sets into bounded sets and consequently, hypothesis **H3)** guarantees exactly this additional property.

III. Observers and Observability for Discrete-Time Systems With Inputs

The following definitions introduce notions of estimators and observers for discrete-time systems.

Definition 7: Let $\theta : Z^+ \times \mathcal{X} \times U \to \Theta$, where Θ is a normed linear space with $\theta(\cdot, 0, 0) = 0$ and $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$. Let $\mathcal{Z}$ be a normed linear space and consider the system

$$\begin{aligned} z(k+1) &= g(k, z(k), h(k, x(k)), u(k)) \\ \bar{\theta}(k) &= \Psi(k, z(k), h(k, x(k)), u(k)) \end{aligned} \tag{19}$$

where $g : Z^+ \times \mathcal{Z} \times \mathcal{Y} \times U \to Z, \Psi : Z^+ \times \mathcal{Z} \times \mathcal{Y} \times U \to \Theta$ with $g(k, 0, 0, 0) = 0$ and $\Psi(k, 0, 0, 0) = 0$ for all $k \in Z^+$. Suppose that there exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}, k_0 \in Z^+, (u, d) \in M_U \times M_D$, the unique solution of (8), (9), and (19) with initial condition $(x(k_0), z(k_0)) = (x_0, z_0)$ corresponding to inputs $(u, d) \in M_U \times M_D$, satisfies for all $k \geq k_0$; see (20), as shown at the bottom of the page. Then system (19) is called a robust ***q*-estimator** for θ with respect to (8), (9). System (19) is called a robust ***q*-estimator** for system (8), (9) if θ is the identity map $\theta(k, x, u) := x$. If $q(k) \equiv 1$, then (19) is simply called a robust ***estimator*** for θ with respect to (8), (9). In any case, the map $\Psi : Z^+ \times \mathcal{Z} \times \mathcal{Y} \times U \to \Theta$ is called the reconstruction map of the robust (q)-estimator for θ with respect to (8), (9).

Definition 8: Let $\theta : Z^+ \times \mathcal{X} \times U \to \Theta$, where Θ is a normed linear space with $\theta(\cdot, 0, 0) = 0$ and $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$. Let $\mathcal{Z}$ be a normed linear space and consider system (19) where $g : Z^+ \times \mathcal{Z} \times \mathcal{Y} \times U \to Z, \Psi : Z^+ \times \mathcal{Z} \times \mathcal{Y} \times U \to \Theta$ with $g(k, 0, 0, 0) = 0$ and $\Psi(k, 0, 0, 0) = 0$ for all $k \in Z^+$. We say that (19) satisfies the ***consistent initialization Property*** for θ if for every $(k_0, x_0) \in Z^+ \times \mathcal{X}$ there exists $z_0 \in Z$ such that the unique solution of (8), (9), and (19) with initial condition $(x(k_0), z(k_0)) = (x_0, z_0)$ and corresponding to arbitrary $(d, u) \in M_D \times M_U$, satisfies

$$\begin{aligned} \theta(k, x(k), u(k)) &= \Psi(k, z(k), y(k), u(k)) \\ &\forall k \geq k_0 \qquad \forall (d, u) \in M_D \times M_U. \end{aligned} \tag{21}$$

Definition 9: Let $\theta : Z^+ \times \mathcal{X} \times U \to \Theta$, where Θ is a normed linear space with $\theta(\cdot, 0, 0) = 0$ and $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$. Suppose that (19) is a q-estimator for θ with respect to (8), (9), which satisfies the consistent initialization property for θ. Then, we say that system (19) is a robust global q-observer for θ with respect to (8), (9), or that the robust global q-observer problem for θ with respect to (8), (9) is solvable. If $q(k) \equiv 1$ then we say that system (19) is a robust global observer for θ with respect to (8), (9), or that the robust global observer problem for θ with respect to (8), (9) is solvable. If θ is the identity map $\theta(k, x, u) := x$, then we say that system (19) is a robust global q-observer for (8), (9). Finally, if $\mathcal{Z} = \mathcal{X}$ and $\Psi(k, z, h(k, x), u) \equiv z$ then we say that (19) is an identity observer.

Remark 10: We next discuss the consequences of Definition 9.

i) Notice that according to Definition 9, an observer for (8), (9) guarantees convergence of the estimates of the states only for bounded inputs $u \in M_U$. The phenomenon, that the state estimates of an observer converge to the states for bounded inputs and not necessarily for all possible inputs, is a purely nonlinear one. For linear finite-dimensional systems with linear identity observers this phenomenon cannot happen. Moreover, it should be emphasized that there exist more demanding versions of the notion of an observer (see, for example, [22]), where convergence for all inputs is required. The following example illustrates this point.

ii) The notion of state observer given by Definition 9 is weaker than the one of [22, Def. 7.1.3], even when we consider autonomous systems without inputs and full order autonomous observers because Definition 9 does not guarantee the "Lyapunov stability" property for the error $e = \Psi(k, z, y, u) - x$, i.e., if the initial value for the error $e_0 = \bar{x}_0 - x_0$ is "sufficiently small," we cannot guarantee that all future values of the error will be "small." This stronger property would be satisfied if instead of (20) with $\theta(k, x, u) \equiv x$ and $q(k) \equiv 1$ the following estimate were satisfied for all $k \geq k_0$ for the solution $(x(\cdot), z(\cdot))$ of system (8), (9) with (19)

$$\begin{aligned} &|\Psi(k, z(k), y(k), u(k)) - x(k)| \\ &\qquad = |e(k)| \leq \sigma(\beta(k_0)|e(k_0)|, k - k_0). \end{aligned} \tag{22}$$

Instead, our asymptotic property (20) guarantees that if the initial condition (z_0, x_0) and the input u are "sufficiently small," then all future values of the error will be "small." For example, if zero is a non-uniformly in time globally asymptotically stable equilibrium point for the system without inputs $x(k+1) = f(k, x(k))$, then according to Definition 9 the system $z(k+1) = f(k, z(k)), \bar{x} = z$ is a global identity observer. On the other hand, if the definition of the observer were based on (22) instead of (20), then the system $z(k+1) = f(k, z(k)), \bar{x} = z$ would not be an observer for the system $x(k+1) = f(k, x(k))$, unless the system $x(k+1) = f(k, x(k))$ had special structure (e.g., linear systems).

$$q(k)\|\theta(k, x(k), u(k)) - \Psi(k, z(k), y(k), u(k))\|_\Theta \leq \sigma\left(\beta(k_0)\left(\|x_0\|_\mathcal{X} + \|z_0\|_\mathcal{Z} + \sup_{l\in\{k_0,\ldots,k\}} \|u(l)\|_\mathcal{U}\right), k - k_0\right) \tag{20}$$

Example 11: Consider the following autonomous system with input:

$$x(k+1) = \frac{|u(k)|}{1+|u(k)|}x(k) + u(k)$$
$$x(k) \in \Re, u(k) \in \Re \tag{23}$$

and zero output map, i.e., $h(k,x) \equiv 0$. We will show that the following system:

$$z(k+1) = \frac{|u(k)|}{1+|u(k)|}z(k) + u(k), z(k) \in \Re$$
$$\bar{x}(k) = z(k) \tag{24}$$

is an observer for (23) in the sense of Definition 9, but is not an observer in the sense described in [22]. Define $e(k) = x(k) - \bar{x}(k)$ and notice that

$$|e(k+1)| = \frac{|u(k)|}{1+|u(k)|}|e(k)|. \tag{25}$$

It can be proved inductively using (25) that the following inequality holds:

$$|e(k)| \leq \left(\frac{\sup_{l\in\{k_0,\ldots,k\}}|u(l)|}{1+\sup_{l\in\{k_0,\ldots,k\}}|u(l)|}\right)^{(k-k_0)} |e(k_0)| \quad \forall k \geq k_0. \tag{26}$$

Inequality $(1+t)((r)/(1+r))^t \leq R+a(r)$, which holds for all $t, r \geq 0$ for $R := (1)/(\log(2))$ and $a \in K_\infty$ defined by $a(0) = 0, a(r) := (1)/(\log(1+r^{-1}))$ for $r > 0$, in conjunction with (25) implies the estimate

$$|x(k) - \bar{x}(k)| \leq \sigma\left(|x_0| + |z_0| + \sup_{l\in\{k_0,\ldots,k\}}|u(l)|, k-k_0\right) \quad \forall k \geq k_0$$

where $\sigma(s,t) := (\kappa(s))/(1+t)$ and $\kappa(s) := 2Rs + s^2 + (a(s))^2$. Thus, system (24) is an identity observer for (23) according to Definition 9. On the other hand, since by virtue of (25) we have $|e(k)| = \exp(\sum_{l=k_0}^{l=k-1}\log((|u(l)|)/(1+|u(l)|)))|e(k_0)|$, it is clear that the input $u(l) = (\exp((1)/(1+l^2))-1)^{-1}$ does not allow the error to converge to zero. Thus, system (24) is not an observer in the sense described in [22]. ◃

Next, the notion of robust (strong) complete observability is introduced. The definition of the notion of robust (strong) complete observability for discrete-time systems is completely analogous to the corresponding notions for continuous-time autonomous systems given in [24].

Definition 12: Consider system (8), (9) under hypotheses **H1)–H3)** and let $(d_i, u_i) \in D \times U, i = 0, 1, \ldots$ and define recursively the family of mappings shown in the equation at the bottom of the page, where $d^{(i)} := (d_0, \ldots, d_{i-1}), u^{(i)} := (u_0, \ldots, u_{i-1})$ for $i \geq 1$. Let an integer $p \geq 1$ and define the following mapping for all $(k, x, d^{(p)}, u^{(p)}) \in Z^+ \times \mathcal{X} \times D^p \times U^p$:

$$y^{(p+1)}\left(k, x, d^{(p)}, u^{(p)}\right) : = \left(y_0(k,x), \ldots, y_{p-1}\left(k, x, d^{(p-1)}, u^{(p-1)}\right), y_p\left(k, x, d^{(p)}, u^{(p)}\right)\right).$$

We say that a continuous function $\theta \in CU(Z^+ \times \mathcal{X}; \Theta)$ where Θ is a normed linear space, is ***robustly completely observable from the output*** $y = h(k,x)$ ***with respect to (8), (9)*** if there exists an integer $p \in Z^+$ and a continuous function (called the reconstruction map) $\Psi \in CU(Z^+ \times \mathcal{U}^p \times \mathcal{Y}^p \times \mathcal{Y}; \Theta)$, such that for all $(k, x, d^{(p)}, u^{(p)}) \in Z^+ \times \mathcal{X} \times D^p \times U^p$ it holds that

$$\theta\left(k+p, F_p\left(k, x, d^{(p)}, u^{(p)}\right)\right) = \Psi\left(k+p, u^{(p)}, y^{(p+1)}\left(k, x, d^{(p)}, u^{(p)}\right)\right). \tag{27}$$

Furthermore, we say that $\theta \in CU(Z^+ \times \mathcal{X}; \Theta)$ is ***robustly strongly completely observable from the output*** $y = h(k,x)$ ***with respect to (8), (9)*** if θ is robustly completely observable from the output $y = h(k,x)$ with respect to (8), (9) and for every $k \in Z^+, x \in \mathcal{X}$ there exists $w = (w_1, \ldots, w_p, w_{p+1}, \ldots, w_{2p}) \in U^p \times \mathcal{Y}^p$ such that for all $(d^{(p-1)}, u^{(p-1)}) \in D^{p-1} \times U^{p-1}$ it holds that

$$\theta\left(k+i, F_i\left(k, x, d^{(i)}, u^{(i)}\right)\right) = \Psi\left(k+i, w_1, \ldots, w_{p-i}, u^{(i)}, w_p, \ldots, w_{2p-i}, y^{(i+1)}\left(k, x, d^{(i)}, u^{(i)}\right)\right) \quad \text{for all } i = 1, \ldots, p-1 \tag{28}$$

$$\theta(k,x) = \Psi(k, w_1, \ldots w_p, w_{p+1}, \ldots, w_{2p}, h(k,x)) \tag{29}$$

We say that system (8), (9) is ***robustly (strongly) completely observable from the output*** $y = h(k,x)$ if the identity function $\theta(k,x) = x$ is robustly (strongly) completely observable from the output $y = h(k,x)$ with respect to (8), (9).

Remark 13:

a) Notice that if (27) holds, then for every input $(d,u) \in M_D \times M_U$ and for every $(k_0, x_0) \in Z^+ \times \mathcal{X}$, the unique solution $x(k)$ of (8), (9) corresponding to (d,u) and initiated from x_0 at time k_0, satisfies the following relation for all $k \geq k_0 + p$:

$$\theta(k, x(k)) = \Psi(k, u(k-p), \ldots, u(k-1) \; y(k-p), y(k-p+1), \ldots, y(k-1), y(k)).$$

$$F_0(k,x) = x \quad F_1\left(k, x, d^{(1)}, u^{(1)}\right) = f(k, d_0, x, u_0)$$
$$F_i\left(k, x, d^{(i)}, u^{(i)}\right) := f\left(k+i-1, d_{i-1}, F_{i-1}\left(k, x, d^{(i-1)}, u^{(i-1)}\right), u_{i-1}\right), \qquad \text{for } i \geq 2$$
$$y_0(k,x) = h(k,x) \quad y_i\left(k, x, d^{(i)}, u^{(i)}\right) := h\left(k+i, F_i\left(k, x, d^{(i)}, u^{(i)}\right)\right), \qquad i \geq 1$$

Following the terminology in [22], if system (8), (9) is robustly completely observable from the output $y = h(k, x)$ then every control $(d, u) \in M_D \times M_U$ final-state distinguishes between any two events in time $p \in Z^+$.

b) Notice that every continuous function of the measured output $\theta(k, x) = \varphi(k, h(k, x))$, where $\varphi \in CU(Z^+ \times \mathcal{Y}; \Theta)$ is robustly strongly completely observable from the measured output with $\Psi(k, y) := \varphi(k, y)$.

There are classes of uncertain finite-dimensional discrete-time systems, for which it can be verified that they satisfy the requirements for robust complete observability, as described in Definition 12. The first class generalizes example (3) presented in the Introduction.

Proposition 14: Let $D \subset \Re^l$ a non-empty compact set, $G : Z^+ \times \Re^s \times \Re^n \times \Re^m \times D \to \Re^s, \varphi : Z^+ \times \Re^s \times \Re^n \to \Re^q, f_i : Z^+ \times \Re^s \times \Re^n \times \Re^m \times D \to \Re (i = 1, \ldots, n)$ continuous mappings with $G(k, 0, 0, 0, d) = 0, \varphi(k, 0, 0) = 0, f_i(k, 0, 0, 0, d) = 0 (i = 1, \ldots, n)$ for all $(k, d) \in Z^+ \times D$. Consider the following finite-dimensional discrete-time system:

$$
\begin{aligned}
&z(k+1) = G(k, z(k), x(k), u(k), d(k)) \\
&x_i(k+1) = f_i(k, z(k), x(k), u(k), d(k)), i = 1, \ldots, n \\
&y(k) = (\varphi(k, z(k), x(k)), x_1(k)) \\
&z \in \Re^s, x = (x_1, \ldots, x_n)' \in \Re^n, u \in \Re^m, d \in D \subseteq \Re^l, k \in Z^+
\end{aligned}
\tag{30}
$$

and suppose that there exist continuous mappings $g_i : Z^+ \times \Re^q \times \Re^i \times \Re^i \times \Re^m \to \Re (i = 1, \ldots, n-1)$ with $g_i(k, 0, 0, 0, 0) = 0 (i = 1, \ldots, n-1)$ for all $k \in Z^+$ such that the following identities hold for all $(k, z, x, u, d) \in Z^+ \times \Re^s \times \Re^n \times \Re^m \times D$ and $i = 1, \ldots, n-1$:

$$
\begin{aligned}
f_{i+1}(k, z, x, u, d) = g_i(k, \varphi(k, z, x), x_1, \ldots \\
x_i, f_1(k, z, x, u, d), \ldots, f_i(k, z, x, u, d), u).
\end{aligned}
\tag{31}
$$

Then every continuous function $\theta(k, \varphi(k, z, x), x)$ with $\theta(k, 0, 0) = 0$ for all $k \in Z^+$ is robustly completely observable from the output $y = (\varphi(k, z, x), x_1)$ with respect to (30).

Sketch of Proof: Robust complete observability for the function $\theta(x) = x$ from the output $y = (\varphi(k, z, x), x_1)$ with respect to system (30) follows directly from the following claim.

Claim: For each $i = 2, \ldots, n$, there exists a continuous function $\psi_i : Z^+ \times (\Re^q)^{(i-1)} \times \Re^i \times \Re^{m(i-1)} \to \Re$ with $\psi_i(k, 0, 0, 0, 0) = 0$ for all $k \in Z^+$ such that for every $(d, u) \in M_D \times M_{\Re^m}, (k_0, z_0, x_0) \in Z^+ \times \Re^s \times \Re^n$ and $k \geq k_0 + (i-1)$ it holds that

$$
\begin{aligned}
&x_i(k) = \psi_i(k, \varphi(k-i+1), \ldots, \varphi(k-1) \\
&\quad x_1(k-i+1), \ldots, x_1(k), u(k-i+1), \ldots, u(k-1))
\end{aligned}
\tag{32}
$$

where $(z(k), x(k))$ denotes the solution of (30) corresponding to (d, u) with initial condition $(z(k_0), x(k_0)) = (z_0, x_0)$ and $\varphi(k) = \varphi(k, z(k), x(k))$.

The proof of the claim is straightforward and is left to the reader. ◁

The second class of discrete-time systems generalizes what is known in the literature as "triangular" systems (see [21]).

Proposition 15: Let $D \subset \Re^l$ a non-empty compact set, $G : Z^+ \times \Re^s \times \Re^n \times \Re^m \times D \to \Re^s, \varphi : Z^+ \times \Re^s \times \Re^n \to \Re^q, f_i : Z^+ \times \Re^q \times \Re^{i+1} \times \Re^m \to \Re (i = 1, \ldots, n-1), f_n : Z^+ \times \Re^q \times \Re^n \times \Re^m \to \Re$ continuous mappings with $G(k, 0, 0, 0, d) = 0, \varphi(k, 0, 0) = 0, f_i(k, 0, 0, 0) = 0 (i = 1, \ldots, n)$ for all $(k, d) \in Z^+ \times D$. Consider the following finite-dimensional discrete-time system:

$$
\begin{aligned}
&z(k+1) = G(k, z(k), x(k), u(k), d(k)) \\
&x_i(k+1) = f_i(k, \varphi(k, z(k), x(k)), x_1(k), \ldots \\
&\quad x_{i+1}(k), u(k)), \qquad i = 1, \ldots, n-1 \\
&x_n(k+1) = f_n(k, \varphi(k, z(k), x(k)), x(k), u(k)) \\
&y(k) = (\varphi(k, z(k), x(k)), x_1(k)) \\
&z \in \Re^s, \qquad x = (x_1, \ldots, x_n)' \in \Re^n \\
&\quad u \in \Re^m, \qquad d \in D, \quad k \in Z^+.
\end{aligned}
\tag{33}
$$

Suppose that there exist continuous mappings $g_i : Z^+ \times \Re^q \times \Re^i \times \Re \times \Re^m \to \Re (i = 1, \ldots, n-1)$ with $g_i(k, 0, 0, 0, 0) = 0 (i = 1, \ldots, n-1)$ for all $k \in Z^+$ such that the following identities hold for all $(k, \varphi, x, u) \in Z^+ \times \Re^q \times \Re^n \times \Re^m$ and $i = 1, \ldots, n-1$:

$$
g_i(k, \varphi, x_1, \ldots, x_i, f_i(k, \varphi, x_1, \ldots, x_i x_{i+1}, u), u) = x_{i+1}.
\tag{34}
$$

Then, every continuous function $\theta(k, \varphi(k, z, x), x)$ with $\theta(k, 0, 0) = 0$ for all $k \in Z^+$ is robustly completely observable from the output $y = (\varphi(k, z, x), x_1)$ with respect to system (33).

Sketch of Proof: Robust complete observability for the function $\theta(x) = x$ from the output $y = (\varphi(k, z, x), x_1)$ with respect to system (33) follows directly from the following claim.

Claim: For each $i = 2, \ldots, n$, there exists a continuous function $\psi_i : Z^+ \times (\Re^q)^{(i-1)} \times \Re^i \times \Re^{m(i-1)} \to \Re$ with $\psi_i(k, 0, 0, 0, 0) = 0$ for all $k \in Z^+$ such that for every $(d, u) \in M_D \times M_{\Re^m}, (k_0, z_0, x_0) \in Z^+ \times \Re^s \times \Re^n$ and $k \geq k_0 + (i-1)$ it holds that

$$
\begin{aligned}
&x_i(k-i+1) = \psi_i(k, \varphi(k-i+1), \ldots, \varphi(k-1) \\
&\quad x_1(k-i+1), \ldots, x_1(k), u(k-i+1), \ldots, u(k-1))
\end{aligned}
\tag{35}
$$

where $(z(k), x(k))$ denotes the solution of (33) corresponding to (d, u) with initial condition $(z(k_0), x(k_0)) = (z_0, x_0)$ and $\varphi(k) = \varphi(k, z(k), x(k))$.

The proof of the claim is straightforward and is left to the reader. The knowledge of $x(k-n+1), (\varphi(k-n+1), \ldots, \varphi(k-1))$ and $(u(k-n+1), \ldots, u(k-1))$ allows the calculation of the vector $x(k)$ (using (33) recursively). ◁

IV. Main Results

The following proposition is our first main result of the present paper and provides sufficient conditions for the solvability of the robust q-estimator problem for (8), (9).

Proposition 16: Let $\theta \in CU(Z^+ \times \mathcal{X};\Theta)$ where Θ is a normed linear space. Suppose that $\theta \in CU(Z^+ \times \mathcal{X};\Theta)$ is robustly completely observable from the output $y = h(k,x)$ with respect to (8), (9). Then, under hypotheses **H1)–H3)**, for every $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$, the robust global q-estimator problem for θ with respect to (8), (9) is solvable.

Proof: Since $\theta \in CU(Z^+ \times \mathcal{X};\Theta)$ is robustly completely observable from the output $y = h(k,x)$ with respect to (8), (9), there exists an integer $p \geq 1$ and a reconstruction map $\Psi \in CU(Z^+ \times \mathcal{U}^p \times \mathcal{Y}^p \times \mathcal{Y};\Theta)$ such that for all $(k,x,d^{(p)},u^{(p)}) \in Z^+ \times \mathcal{X} \times D^p \times U^p$ identity (27) holds. Consider the following system:

$$\begin{aligned}
&w_1(k+1) = y(k)\\
&w_2(k+1) = w_1(k)\\
&\vdots\\
&w_p(k+1) = w_{p-1}(k)\\
&w_{p+1}(k+1) = u(k)\\
&w_{p+2}(k+1) = w_{p+1}(k)\\
&\vdots\\
&w_{2p}(k+1) = w_{2p-1}(k)\\
&\tilde{\theta}(k) = \Psi(k, w_{2p}(k), \ldots, w_{p+1}(k), w_p(k), \ldots, w_1(k), y(k))\\
&w(k) := (w_1(k), \ldots, w_{2p}(k)) \in W := \mathcal{Y}^p \times \mathcal{U}^p, \qquad k \in Z^+
\end{aligned} \tag{36}$$

or

$$\begin{aligned}
&w(k+1) = g(w(k), y(k), u(k))\\
&\tilde{\theta}(k) = \Psi(k, w_{2p}(k), \ldots, w_{p+1}(k), w_p(k), \ldots, w_1(k), y(k))\\
&w(k) \in W, \qquad k \in Z^+
\end{aligned} \tag{37}$$

where

$$g(w,y,u) := (y, w_1, \ldots, w_{p-1}, u, w_{p+1}, \ldots, w_{2p-1}). \tag{38}$$

Clearly, for every $(k_0, x_0, w_0, d, u) \in Z^+ \times \mathcal{X} \times W \times M_D \times M_U$ the solution of (8), (9) with (36) (or equivalently (37)) with initial condition $(x(k_0), w(k_0)) = (x_0, w_0)$ and corresponding to input $(d,u) \in M_D \times M_U$ satisfies for all $k \geq k_0 + p$

$$w_i(k) = y(k-i), \qquad i = 1, \ldots, p \tag{39}$$

$$w_{p+i}(k) = u(k-i), \qquad i = 1, \ldots, p. \tag{40}$$

It follows from (39) and (40) and Remark 13(a) that the following equality holds for all $k \geq k_0 + p$:

$$\theta(k, x(k)) = \tilde{\theta}(k) = \Psi(k, w_{2p}(k), \ldots \; w_{p+1}(k), w_p(k), \ldots, w_1(k), y(k)). \tag{41}$$

Let $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$ arbitrary. It follows from (41) that system (8), (9) with (36) and output $\tilde{h}(k,x,w) = q(k)[\theta(k,x) - \Psi(k, w_{2p}, \ldots, w_1, h(k,x))]$ satisfies the robust output attractivity property as stated in Lemma 5. Next we show that hypotheses **H1)–H3)** are satisfied for the composite system (8), (9) with (36). Clearly, hypothesis **H2)** is automatically satisfied. Moreover, notice that the output map $\tilde{h}(k,x,w) = q(k)[\theta(k,x) - \Psi(k, w_{2p}, \ldots, w_1, h(k,x))]$ is of class CU (this follows from the facts that $\theta \in CU(Z^+ \times \mathcal{X};\Theta), \Psi \in CU(Z^+ \times \mathcal{U}^p \times \mathcal{Y}^p \times \mathcal{Y};\Theta)$ and $h \in CU(Z^+ \times \mathcal{X};\mathcal{Y})$) and consequently, hypothesis **H3)** is satisfied for the composite system (8), (9) with (36). Finally, we show that hypothesis **H1)** holds for system (8), (9) with (36). It is clear that by virtue of definition (38), the following inequality holds:

$$\|g(w,y,u)\|_W \leq \|w\|_W + \|y\|_{\mathcal{Y}} + \|u\|_{\mathcal{U}} \quad \forall (w,y,u) \in W \times Y \times U \tag{42}$$

Moreover, [10, Lemma 3.2] and the fact that $h \in CU(Z^+ \times \mathcal{X};\mathcal{Y})$ imply that there exist functions $a_1 \in K_\infty, \gamma_1 \in K^+$ such that $\|h(k,x)\|_{\mathcal{Y}} \leq a_1(\gamma_1(k)\|x\|_{\mathcal{X}})$, for all $(k,x) \in Z^+ \times \mathcal{X}$. This fact, in conjunction with inequalities (42) and hypothesis **H1)** for system (8) implies that there exist functions $a \in K_\infty, \gamma \in K^+$ such that

$$\begin{aligned}
&\|(f(k,d,x,u), g(w, h(k,x), u))\|_{\mathcal{X}\times W}\\
&\quad \leq a(\gamma(k)\|x\|_{\mathcal{X}}) + a(\gamma(k)\|u\|_{\mathcal{U}})\\
&\qquad + a_1(\gamma_1(k)\|x\|_{\mathcal{X}}) + \|w\|_W + \|u\|_{\mathcal{U}}\\
&\quad \forall (k,x,d,u,w) \in Z^+ \times \mathcal{X} \times D \times U \times W.
\end{aligned} \tag{43}$$

Inequality (43) guarantees that hypothesis **H1)** also holds for system (8), (9) with (36). Consequently, by virtue of Lemma 5 it follows that the system (36) is a robust q-estimator for θ with respect to (8), (9). The proof is complete. ◁

The following corollary shows that robust strong complete observability for a discrete-time system implies solvability of the observer problem and is a direct consequence of equalities (3.10a,b).

Corollary 17: Consider system (8), (9) under hypotheses (H1-2) and suppose that $\theta \in CU(Z^+ \times \mathcal{X};\Theta)$ is robustly strongly completely observable from the output $y = h(k,x)$ with respect to (8), (9). Then, under hypotheses **H1)–H3)**, for every $q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$, the robust global q-Observer problem for θ with respect to (8), (9) is solvable.

We are now in a position to state our second main result, which establishes the transitive observer property.

Theorem 18 (Transitive Observer Property): Let $\theta \in CU(Z^+ \times \mathcal{X};\Theta), q \in K^+$ with $\inf_{k\geq 0} q(k) > 0$. Suppose that:

A1) θ is robustly strongly completely observable from the output $y = h(k,x)$ with respect to (8), (9)

A2) The robust global q-observer problem for (8) with output $y = \theta(k,x)$ is solvable. Particularly, there exists $\varphi : Z^+ \times \mathcal{Z} \times \Theta \times U \to Z, \tilde{\Psi} \in CU(Z^+ \times \mathcal{Z} \times \Theta;\mathcal{X})$, functions $p \in K_\infty, q \in K^+$ such that $\|\varphi(k,z,\theta,u)\|_{\mathcal{Z}} \leq p(q(k)\|z\|_{\mathcal{Z}}) + p(q(k)\|\theta\|_\Theta) + p(q(k)\|u\|_{\mathcal{U}})$ for all $(k,z,\theta,u) \in Z^+ \times \mathcal{Z} \times \Theta \times U$ in such a way that the following system

$$\begin{aligned}
&z(k+1) = \varphi(k, z(k), \theta(k, x(k)), u(k))\\
&\bar{x}(k) = \tilde{\Psi}(k, z(k), \theta(k, x(k)))\\
&z(k) \in \mathcal{Z}, \bar{x}(k) \in \mathcal{X}, k \in Z^+
\end{aligned} \tag{44}$$

is a robust global q-observer for (8) with output $y = \theta(k,x)$.

Then, under hypotheses **H1)–H3)**, the robust global q-observer problem for (8), (9) with output $y = h(k,x)$ is solvable.

Proof: We consider the system

$$\begin{aligned}
&w(k+1) = g(w(k), y(k), u(k)) \\
&\tilde{\theta}(k) = \Psi(k, w_{2p}(k), \ldots, w_1(k), y(k)) \\
&w(k) \in W, \qquad \tilde{\theta}(k) \in \Theta, \quad k \in Z^+ \\
&z(k+1) = \varphi(k, z(k), \tilde{\theta}(k), u(k)) \\
&\bar{x}(k) = \tilde{\Psi}(k, z(k), \tilde{\theta}(k)) \\
&z(k) \in \mathcal{Z}, \quad \bar{x}(k) \in \mathcal{X}, \quad k \in Z^+.
\end{aligned} \tag{45}$$

where g is defined by (38). Since (44) is a robust global q-observer for (8) with output $y = \theta(k, x)$, there exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}, k_0 \in Z^+, (u, d) \in M_U \times M_D$, the unique solution $(x(k), z(k))$ of (8) and (44) with initial condition $(x(k_0), z(k_0)) = (x_0, z_0)$ corresponding to inputs $(u, d) \in M_U \times M_D$, satisfies for all $k \geq k_0$

$$\begin{aligned}
&q(k)\|x(k) - \tilde{\Psi}(k, z(k), \theta(k, x(k)))\|_{\mathcal{X}} \\
&\quad \leq \sigma\left(\beta(k_0)\left(\|x_0\|_{\mathcal{X}} + \|z_0\|_{\mathcal{Z}} + \sup\nolimits_{l\in\{k_0,\ldots,k\}}\|u(l)\|_{\mathcal{U}}\right),\right. \\
&\left. k - k_0\right).
\end{aligned} \tag{46}$$

By virtue of (41), the component z of the solution of (45) coincides for $k \geq k_0 + p$ with the solution of (44). Thus, for every $(x_0, w_0, z_0) \in \mathcal{X} \times W \times \mathcal{Z}, k_0 \in Z^+, (u, d) \in M_U \times M_D$, the unique solution $(x(k), w(k), z(k))$ of (8), (9), and (45) with initial condition $(x(k_0), w(k_0), z(k_0)) = (x_0, w_0, z_0)$ corresponding to inputs $(u, d) \in M_U \times M_D$, satisfies for all $k \geq k_0 + p$:

$$\begin{aligned}
&q(k)\|x(k) - \tilde{\Psi}(k, z(k), \tilde{\theta}(k))\|_{\mathcal{X}} \\
&\quad \leq \sigma\left(\beta(k_0+p)\left(\|x(k_0+p)\|_{\mathcal{X}} + \|z(k_0+p)\|_{\mathcal{Z}}\right.\right. \\
&\qquad \left.\left. + \sup_{l\in\{k_0,\ldots,k\}}\|u(l)\|_{\mathcal{U}}\right), k - k_0 - p\right).
\end{aligned} \tag{47}$$

Next, we prove the following fact.

Fact: System (8), (9) with (45) and output $\tilde{h}(k, x, w, z) = q(k)[x - \tilde{\Psi}(k, z, \Psi(k, w_{2p}, \ldots, w_1, h(k, x)))]$ satisfies hypotheses **H1)–H3)**.

Proof of the Fact: Hypothesis **H2)** is automatically satisfied. Notice that the output map $\tilde{h}(k, x, w, z) = q(k)[x - \tilde{\Psi}(k, z, \Psi(k, w_{2p}, \ldots, w_1, h(k, x)))]$ is of class CU, since $\tilde{\Psi} \in CU(Z^+ \times \mathcal{Z} \times \Theta; \mathcal{X}), h \in CU(Z^+ \times \mathcal{X}; \mathcal{Y})$ and $\Psi \in CU(Z^+ \times \mathcal{U}^p \times \mathcal{Y}^p \times \mathcal{Y}; \Theta)$. Next, we show hypothesis **H1)**. Lemma 3.2 in [10], in conjunction with the facts that $h \in CU(Z^+ \times \mathcal{X}; Y), \Psi \in CU(Z^+ \times \mathcal{U}^p \times \mathcal{Y}^p \times \mathcal{Y}; \Theta)$, implies that there exist functions $a_1 \in K_\infty, \gamma_1 \in K^+$ such that

$$\begin{aligned}
&\|h(k,x)\|_Y + \|\Psi(k, w_{2p}, \ldots, w_1, h(k,x))\|_\Theta \\
&\qquad \leq a_1(\gamma_1(k)\|x\|_{\mathcal{X}}) + a_1(\gamma_1(k)\|w\|_W)
\end{aligned}$$

for all $(k, x, w) \in Z^+ \times \mathcal{X} \times W$. Making use of hypothesis **H1)** for system (8), assumption **A2)** and the previous inequality, we obtain

$$\begin{aligned}
&\|(f(k,d,x,u), g(w,y,u), \varphi(k, z, \Psi(k, w_{2p}, \ldots, w_1, y), u)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|_{\mathcal{X}\times W\times\mathcal{Z}} \\
&\quad \leq a(\gamma(k)\|x\|_{\mathcal{X}}) + a(\gamma(k)\|u\|_{\mathcal{U}}) + a_1(\gamma_1(k)\|x\|_{\mathcal{X}}) + \|w\|_W \\
&\qquad + p(q(k)\|z\|_{\mathcal{Z}}) + p(2q(k)a_1(\gamma_1(k)\|x\|_{\mathcal{X}})) + \|u\|_{\mathcal{U}} \\
&\qquad + p(2q(k)a_1(\gamma_1(k)\|w\|_W)) + p(q(k)\|u\|_{\mathcal{U}}) \\
&\qquad \forall (k, x, d, u, w, z) \in Z^+ \times \mathcal{X} \times D \times U \times W \times \mathcal{Z}.
\end{aligned} \tag{48}$$

The previous inequality in conjunction shows that hypothesis **H1)** is satisfied for system (8), (9) with (45). The proof of the fact is complete.

It follows from Lemma 4 that there exist functions $\mu \in K^+, a \in K_\infty$ such that for every $(k_0, x_0, w_0, z_0, d, u) \in Z^+ \times \mathcal{X} \times W \times \mathcal{Z} \times M_D \times M_U$, the corresponding solution $(x(k), w(k), z(k))$ of (8), (9) with (45) and initial condition $(x(k_0), w(k_0), z(k_0)) = (x_0, w_0, z_0)$ satisfies for all $k \geq k_0$

$$\begin{aligned}
&\|(x(k), w(k), z(k))\|_{\mathcal{X}\times W\times\mathcal{Z}} \\
&\quad \leq \mu(k) a\left(\|x_0\|_{\mathcal{X}} + \|w_0\|_W + \|z_0\|_{\mathcal{Z}}\right. \\
&\qquad \left. + \sup\nolimits_{l\in\{k_0,\ldots,k\}}\|u(l)\|_{\mathcal{U}}\right).
\end{aligned} \tag{49}$$

Estimate (47) combined with estimate (49) shows that system (8), (9) with (45) and output $\tilde{h}(k, x, w, z) = q(k)[x - \tilde{\Psi}(k, z, \Psi(k, w_{2p}, \ldots, w_1, h(k, x)))]$ satisfies the robust output attractivity property as stated in Lemma 5. By virtue of Lemma 5 and the fact proved previously, it follows that system (45) is a robust q-estimator for the identity mapping with respect to (8), (9). The fact that system (45) satisfies the consistent initialization property for θ is an immediate consequence of (28), (29) and the fact that (45) also satisfies the consistent initialization property for θ. The proof is complete.◁

We conclude this section with two illustrative examples. The first is a finite-dimensional example with "triangular" structure, whereas the second is an infinite-dimensional example.

Example 19: Consider the following four-dimensional discrete-time system:

$$\begin{aligned}
x_1(k+1) &= d(k) f_1(x(k), u(k)) \\
x_2(k+1) &= f_2(x_1(k), u(k)) \\
x_3(k+1) &= f_3(x_1(k), x_2(k), u(k))(x_3(k) + x_4(k)) \\
&\quad + b_3(x_1(k), x_2(k), u(k)) \\
x_4(k+1) &= f_4(x_1(k), x_2(k), u(k))(x_3(k) + x_4(k)) \\
&\quad + b_4(x_1(k), x_2(k), u(k)) \\
y(k) &= x_1(k) \in \Re \\
x(k) &= (x_1(k), x_2(k), x_3(k), x_4(k))' \in \Re^4 \\
&\quad u(k) \in \Re, d(k) \in D
\end{aligned} \tag{50}$$

where $D \subset \Re$ is a compact set, the mappings f_i and b_i are continuous and vanish at $(x, u) = (0, 0)$ with

$$|f_i(x_1, x_2, u)| \leq \frac{1}{4} \qquad \forall (x_1, x_2, u) \in \Re^3 \quad \text{for } i = 3, 4 \tag{51}$$

Notice that the following system is an observer for system (50) with output map $\theta(x) := (x_1, x_2)$:

$$\begin{aligned}
z_1(k+1) &= f_3(\theta(x(k)), u(k))(z_1(k) + z_2(k)) \\
&\quad + b_3(\theta(x(k)), u(k)) \\
z_2(k+1) &= f_4(\theta(x(k)), u(k))(z_1(k) + z_2(k)) \\
&\quad + b_4(\theta(x(k)), u(k)) \\
\bar{x}(k) &= (\theta(x(k)), z_1(k), z_2(k))' \\
z(k) &= (z_1(k), z_2(k))' \in \Re^2.
\end{aligned} \tag{52}$$

In order to prove this, we define the error vector $e(k) = (e_1(k), e_2(k)) := (x_3(k) - z_1(k), x_4(k) - z_2(k))'$ and the error Lyapunov function $V(e) := e_1^2 + e_2^2$, which satisfies the following difference inequality:

$$V(e(k+1)) \leq \frac{1}{4} V(e(k)) \qquad \forall k \geq k_0. \tag{53}$$

Thus, it can be proved using induction arguments and (53) that the solution of system (50) with (52) corresponding to arbitrary inputs $(d,u) \in M_D \times M_{\Re}$ satisfies

$$|e(k)| \leq \left(\frac{1}{2}\right)^{k-k_0} 2(|x(k_0)| + |z(k_0)|) \qquad \forall k \geq k_0.$$

On the other hand the mapping $\theta(x) := (x_1, x_2)$ is robustly completely observable from the output $y = x_1$ with respect to (36) since we have $\theta(x(k)) = (y(k), f_2(y(k-1), u(k-1)))$, for all $k \geq k_0 + 1$. Moreover, if $f_2(\Re \times \Re) = \Re$, then the mapping $\theta(x) := (x_1, x_2)$ is robustly strongly completely observable from the output $y = x_1$ with respect to (50) since we can always select appropriate $(w_1, w_2) \in \Re^2$ so that $f_2(w_1, w_2) = x_2(k_0)$. Thus, using the methodology of the proof of Theorem 18, we conclude that if $f_2(\Re \times \Re) = \Re$, then the following system:

$$\begin{aligned}
w_1(k+1) &= y(k)\\
w_2(k+1) &= u(k)\\
z_1(k+1) &= f_3(y(k), f_2(w_1(k), w_2(k)), u(k))(z_1(k) + z_2(k))\\
&\quad + b_3(y(k), f_2(w_1(k), w_2(k)), u(k))\\
z_2(k+1) &= f_4(y(k), f_2(w_1(k), w_2(k)), u(k))(z_1(k) + z_2(k))\\
&\quad + b_4(y(k), f_2(w_1(k), w_2(k)), u(k))\\
\bar{x}(k) &= (y(k), f_2(w_1(k), w_2(k)), z_1(k), z_2(k))' \in \Re^4
\end{aligned}$$

is a state observer for (50). It should be emphasized that even if we assumed that the mappings f_i and b_i are continuously differentiable and $d(k) = d$ is a known constant, the linearization of system (50) is not necessarily observable and this makes many methodologies for observer design appearing in the literature not directly applicable.

System (50) is a special case of the following class of systems:

$$\begin{aligned}
x_i(k+1) &= f_i(k, x(k), u(k), d(k)), i = 1, \ldots, n\\
z(k+1) &= A(k, x(k), u(k))z(k) + b(k, x(k), u(k))\\
y(k) &= x_1(k)\\
z &\in \Re^s, \quad x = (x_1, \ldots, x_n)' \in \Re^n\\
u &\in \Re^m, \quad d \in D \subseteq \Re^l, \quad k \in Z^+
\end{aligned}$$

where $Z^+ \times \Re^n \times \Re^m \ni (k,x,u) \to A(k,x,u) \in \Re^{s\times s}$ and $Z^+ \times \Re^n \times \Re^m \ni (k,x,u) \to b(k,x,u) \in \Re^s$ are continuous mappings, $f_i : Z^+ \times \Re^s \times \Re^n \times \Re^m \times D \to \Re (i = 1, \ldots, n)$ are continuous mappings with $f_i(k,0,0,d) = 0 (i = 1, \ldots, n)$ for all $(k,d) \in Z^+ \times D$, for which there exist continuous mappings $g_i : Z^+ \times \Re^i \times \Re^i \times \Re^m \to \Re (i = 1, \ldots, n-1)$ with $g_i(k,0,0,0) = 0 (i = 1, \ldots, n-1)$ for all $k \in Z^+$, such that the following identities hold for all: $(k,x,u,d) \in Z^+ \times \Re^n \times \Re^m \times D$ and $i = 1, \ldots, n-1$:

$$f_{i+1}(k,x,u,d) = g_i(k, x_1, \ldots, x_i \quad f_1(k,x,u,d), \ldots, f_i(k,x,u,d), u)$$

and furthermore there exists a positive–definite symmetric matrix P and a constant $\lambda \in [0,1)$ such that

$$A'(k,x,u)PA(k,x,u) \leq \lambda P \qquad \forall (k,x,u) \in Z^+ \times \Re^n \times \Re^m.$$

By virtue of Proposition 14, the vector x is robustly completely observable from the output $y = x_1$. Moreover, the vector z may be estimated by the observer $\xi(k+1) = A(k, x(k), u(k))\xi(k) + b(k, x(k), u(k))$ with linear dynamics for the estimation error $e(k) = \xi(k) - z(k)$. ◁

Finally, we present an example of an infinite-dimensional discrete-time system.

Example 20: Consider the observer design problem for the system of continuous-time difference equations

$$\begin{aligned}
x_1(t) &= d(t)x_1(t - r_1)\\
x_2(t) &= x_1^2(t - r_2) + u(t)\\
x_3(t) &= \frac{x_3(t - r_3)}{2 + x_2^2(t - r_1)} + u^2(t) + x_2(t - r_2)\\
&\quad (x_1(t), x_2(t), x_3(t))' \in \Re^3, \quad u(t) \in \Re\\
&\quad d(t) \in [-1,1]
\end{aligned} \tag{54}$$

with output

$$y(t) = x_1(t) \tag{55}$$

where $t \in \Re^+$ denotes the continuous time variable and $0 < r_1 \leq r_2 \leq r_3$ are constants. Stability issues for such systems are considered in [20]. In order to design an observer for this system we follow a three-step procedure: i) we convert the continuous-time system (54), (55) into a discrete-time system of the form (8), (9) satisfying hypotheses **H1)–H3)**, ii) we design an observer for the discrete-time system with output an appropriate map θ that satisfies hypothesis **A2)** of Theorem 18, and iii) we show that θ is robustly strongly completely observable from the output h of the discrete-time system. For simplicity reasons we also assume that $r_3 \leq 2r_1$, although the general case can be treated in exactly the same way.

First Step: Let $\mathcal{X} = \mathcal{F}([-r_3, 0); \Re^3)$ denote the normed linear space of bounded functions on $[-r_3, 0)$ with values in $\Re^3$ and norm $\|x\|_{\mathcal{X}} := \sup\{|x(s)|; s \in [-r_3, 0)\}$. Similarly, we may define the normed linear space $\mathcal{U} = \mathcal{F}([-r_1, 0); \Re)$ of bounded functions on $[-r_1, 0)$ with values in $\Re$ with norm $\|u\|_{\mathcal{U}} := \sup\{|u(s)|; s \in [-r_1, 0)\}$. Finally, let $U := \mathcal{U}$ and define the disturbance set $D = \mathcal{F}([-r_1, 0); [-1,1])$, which is the set of functions on $[-r_1, 0)$ with values in $[-1,1]$. The following nonlinear operator $f : D \times \mathcal{X} \times U \to \mathcal{X}$ is defined for all $(d,x,u) \in D \times \mathcal{X} \times U$ with $x = \{(x_1(s), x_2(s), x_3(s))'; s \in [-r_3, 0)\}, d = \{d(s); s \in [-r_1, 0)\}, u = \{u(s); s \in [-r_1, 0)\}$; see (56) as shown at the bottom of the next page. Consider the solution of (54) with arbitrary initial condition $x(0) = x_0 \in \mathcal{X}$, corresponding to locally bounded inputs $\tilde{u} : \Re^+ \to \Re$ and $\tilde{d} : \Re^+ \to [-1,1]$. Let $x(k) := \{(x_1(kr_1 + s), x_2(kr_1 + s), x_3(kr_1 + s))'; s \in [-r_3, 0)\} \in \mathcal{X}, d(k) = \{\tilde{d}((k+1)r_1 + s); s \in [-r_1, 0)\} \in D, u(k) = \{\tilde{u}((k+1)r_1 + s); s \in [-r_1, 0)\} \in U = \mathcal{U}$ for $k \in Z^+$.

Then, using the evolution equations (54) and definition (56), we obtain

$$x(k+1) = f(d(k), x(k), u(k)) \quad \text{for all } k \in Z^+ \text{ with } x(0) = x_0 \in \mathcal{X}. \tag{57}$$

Clearly, we have converted the continuous-time system (54) into the infinite-dimensional discrete-time system (57). Since the output (55) expresses the fact that the value of the state x_1 is measured for all times, it is reasonable to define the following output map $h : \mathcal{X} \to Y$ for the discrete-time system (57) with $x = \{(x_1(s), x_2(s), x_3(s))'; s \in [-r_3, 0)\} \in \mathcal{X}$:

$$y = h(x) = \{x_1(s); s \in [-r_3, 0)\} \in \mathcal{Y} := \mathcal{F}([-r_3, 0); \Re) \tag{58}$$

where $\mathcal{Y} = \mathcal{F}([-r_3, 0); \Re)$ denotes the space of bounded functions on $[-r_3, 0)$ with values in $\Re$ and norm $\|y\|_{\mathcal{Y}} := \sup\{|y(s)|; s \in [-r_3, 0)\}$. It can be easily verified that the above definitions guarantee that hypotheses **H1)–H3)** are satisfied for the discrete-time system (57), (58).

Second Step: Next, consider the map $\theta \in CU(\mathcal{X}; \Theta)$ defined for all $x = \{(x_1(s), x_2(s), x_3(s))'; s \in [-r_3, 0)\} \in \mathcal{X}$ by

$$\theta(x) = \{(x_1(s), x_2(s))'; s \in [-r_3, 0)\} \in \Theta := \mathcal{F}([-r_3, 0); \Re^2) \tag{59}$$

and the observer for system (57) with output $y(k) = \theta(x(k))$

$$\begin{aligned} z(k+1) &= \varphi(z(k), \theta(x(k)), u(k)) \\ \bar{x}(k) &= \tilde{\Psi}(z(k), \theta(x(k))) \\ z(k) &\in \mathcal{Z} \quad \bar{x}(k) \in \mathcal{X} \end{aligned} \tag{60}$$

where φ and $\tilde{\Psi}$ are defined by (61), as shown at the bottom of the page, holds. In order to show that (60) with definitions (59), (61) is an observer for system (57) with output $y(k) = \theta(x(k))$, we notice that if we define the error variable $e := x - \tilde{\Psi}(z, \theta)$ and the functional $V : \mathcal{X} \to \Re^+$ by

$$V(e) := \sup\{|e(s)| \exp(\mu s); s \in [-r_3, 0)\} \tag{62}$$

where $\mu := (\log(2))/(2r_3) > 0$. It follows that the solution of (57) and (60) satisfies the recursive relation:

$$V(e(k+1)) \leq \exp(-\mu r_1) V(e(k)) \qquad \text{for all } k \geq k_0.$$

Using induction, the previous inequality, and definition (62) we obtain

$$\|e(k)\|_{\mathcal{X}} \leq \exp(-\mu r_1 k)\sqrt{2}\|e(0)\|_{\mathcal{X}} \qquad \text{for all } k \geq k_0.$$

The previous inequality shows that system (60) with definitions (61) is an observer for system (57) with output $y(k) = \theta(x(k))$. Moreover, it can be immediately verified that $\tilde{\Psi} \in CU(\mathcal{Z} \times \Theta; \mathcal{X})$ and that there exist a function $p \in K_\infty$ such that $\|\varphi(z, \theta, u)\|_{\mathcal{Z}} \leq p(\|z\|_{\mathcal{Z}}) + p(\|\theta\|_{\Theta}) + p(\|u\|_{\mathcal{U}})$ for all $(z, \theta, u) \in \mathcal{Z} \times \Theta \times U$.

Third Step: Notice that $\theta \in CU(\mathcal{X}; \Theta)$ as defined by (59) is robustly strongly completely observable from the output $y = h(x)$ defined by (58). Particularly, since $r_3 \leq 2r_1$, it follows that for every input $(d, u) \in M_D \times M_U$ and for every $(k_0, x_0) \in Z^+ \times \mathcal{X}$, the unique solution $x(k)$ of (57) corresponding to (d, u) and initiated from x_0 at time k_0, satisfies the following relation:

$$\theta(x(k)) = \Psi(u(k-2), u(k-1), y(k-2), y(k-1), y(k)) \quad \forall k \geq k_0 + 2$$

where $\Psi \in CU(\mathcal{U}^2 \times \mathcal{Y}^2 \times \mathcal{Y}; \Theta)$ is the map shown in the last equation at the bottom of the page. Moreover, identities (28), (29) are also satisfied for appropriate choices of $w = (w_1, w_2, w_3, w_4) \in \mathcal{U}^2 \times \mathcal{Y}^2$.

$$\begin{aligned} f(d,x,u) &:= (g_1(x,d,u), g_2(x,u), g_3(x,u)) \\ g_1(x,d,u) &:= \begin{cases} d(s)x_1(s) + u(s) & -r_1 \leq s < 0 \\ x_1(s+r_1) & -r_3 \leq s < -r_1 \end{cases} \\ g_2(x,u) &:= \begin{cases} x_1^2(s+r_1-r_2) + u(s) & -r_1 \leq s < 0 \\ x_2(s+r_1) & -r_3 \leq s < -r_1 \end{cases} \\ g_3(x,u) &:= \begin{cases} \frac{x_3(s+r_1-r_3)}{2+x_2^2(s)} + u^2(s) + x_2(s+r_1-r_2) & -r_1 \leq s < 0 \\ x_3(s+r_1) & -r_3 \leq s < -r_1. \end{cases} \end{aligned} \tag{56}$$

$$\begin{aligned} \varphi(z,\theta,u) &:= \begin{cases} \frac{z(s+r_1-r_3)}{2+\theta_2^2(s)} + u^2(s) + \theta_2(s+r_1-r_2) & -r_1 \leq s < 0 \\ z(s+r_1) & -r_3 \leq s < -r_1 \end{cases} \\ \tilde{\Psi}(z,\theta) &:= \{(\theta_1(s), \theta_2(s), z(s))'; s \in [-r_3, 0)\} \\ \mathcal{Z} &:= \mathcal{F}([-r_3, 0); \Re). \end{aligned} \tag{61}$$

$$\begin{aligned} \Psi(u_2, u_1, y_2, y_1, y_0) &:= \{(y_0(s), \omega(s))'; s \in [-r_3, 0)\} \in \Theta \\ \omega(s) &:= \begin{cases} y_1^2(s+r_1-r_2) + u_1(s), & \text{if } -r_1 \leq s < 0 \\ y_2^2(s+2r_1-r_2) + u_2(s+r_1), & \text{if } -r_3 \leq s < -r_1 \end{cases} \end{aligned}$$

V. Concluding Remarks

This work studied the construction of observers for nonlinear time-varying (possibly infinite-dimensional) discrete-time control systems with uncertainties. Appropriate notions of robust complete observability as well as appropriate notions of estimators and observers were proposed, which are suitable for nonlinear systems and generalize the familiar linear case. Under these notions, a constructive proof of existence of an observer was developed. Moreover, a "transitive observer property" was proven, with which a state observer can be generated as the series connection of two observers. This result can be used in conjunction with recently proposed methodologies for the observer design in discrete-time systems, since the replacement of the original output map by other maps that "carry" more information about the states can facilitate the observer design. It should be emphasized that the proposed observer notions play a critical role in establishing the transitive observer property, which would not hold under more stringent specifications for the observer.

One meaningful direction for future research is to try to establish a "transitive observer property" for continuous-time systems, in analogy to the results of the present work. This would, of course, require appropriate modifications to the notions of observer and complete observability, analogous to the modifications proposed in the present work. Additional technical issues will arise since there is no direct analog of deadbeat observers in continuous-time systems, unless observers with delays (infinite-dimensional systems) are used in a way that guarantees finite-time convergence, or analogies are sought for in the context of sliding-mode observers.

Appendix
Nonuniform in time Robust Global Asymptotic Output Stability for Discrete-Time Systems

The definition of the notion of non-uniform in time robust global asymptotic output stability is given next for the discrete-time case (8), (9) under hypotheses **H1), H3)**. Notice that the following definition is equivalent to [10, Def. 3.1] for the discrete-time case (8), (9) under hypotheses **H1), H3)**. In what follows u_0 denotes the identity zero input.

Definition 21: Consider system (8), (9) under hypotheses **H1), H3)** and suppose that $0 \in \mathcal{X}$ is a robust equilibrium point for (8), (9). We say that (8), (9) is ***non-uniformly in time RGAOS*** if following properties hold.

P1) System (8), (9) is **robustly Lagrange output stable**, i.e., *for every* $\varepsilon > 0, T \in Z^+$, *it holds that* the equation at the bottom of the page holds.

P2) System (8), (9) is **robustly Lyapunov output stable**, i.e., *for every* $\varepsilon > 0$ *and* $T \in Z^+$ *there exists a* $\delta := \delta(\varepsilon, T) > 0$ *such that*

$$\begin{aligned}&\|x_0\|_{\mathcal{X}} \le \delta, k_0 \in \{0,\ldots,T\}\\ &\quad\Rightarrow \|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}}\\ &\quad\le \varepsilon \qquad \forall k \ge k_0 \qquad \forall d \in M_D\end{aligned}$$

$$(\textbf{Robust Lyapunov Output Stability}).$$

P3) (8), (9) satisfies the **robust output attractivity property**, i.e., *for every* $\varepsilon > 0, T \in Z^+$ *and* $R \ge 0$, *there exists a* $\tau := \tau(\varepsilon, T, R) \in Z^+$, *such that*

$$\begin{aligned}&\|x_0\|_{\mathcal{X}} \le R, k_0 \in \{0,\ldots,T\}\\ &\quad\Rightarrow \|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}}\\ &\quad\le \varepsilon \qquad \forall k \ge k_0+\tau \qquad \forall d \in M_D\end{aligned}$$

Moreover, if there exists a function $a \in K_\infty$ such that $a(\|x\|_{\mathcal{X}}) \le \|h(k,x)\|_{\mathcal{Y}}$ for all $(k,x) \in Z^+ \times \mathcal{X}$, then we say that (8), (9) is ***nonuniformly in time RGAS***.

The following technical results provide essential characterizations of non-uniform in time RGAOS for discrete-time systems.

Lemma 22 [10, Lemma 3.3]: Suppose that system (8), (9) under hypotheses **H1), H3)** satisfies the robust output attractivity property (property **P3)** of Definition 21) and that $0 \in \mathcal{X}$ is a robust equilibrium point for (8), (9). Then, (8), (9) is non-uniformly in time RGAOS.

Lemma 23 [10, Lemma 3.4]: Suppose that system (8), (9) under hypotheses **H1), H3)** is nonuniformly in time RGAOS. Then there exist functions $\sigma \in KL, \beta \in K^+$ such that the following estimate holds for all $(k_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$ and $k \ge k_0$:

$$\|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}} \le \sigma(\beta(k_0)\|x_0\|_{\mathcal{X}}, k-k_0). \tag{63}$$

Theorem 24 [10, Th. 3.6]: Consider system (8), (9) under hypotheses **H1), H3)** and suppose that $0 \in \mathcal{X}$ is a robust equilibrium point for (8), (9). Then, the following statements are equivalent.

i) (8), (9) is nonuniformly in time RGAOS.

ii) There exist functions $\mu, \beta \in K^+, \sigma \in KL$ such that for every $(k_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$, we have for all $k \ge k_0$:

$$\begin{aligned}\|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}} + \mu(k)\|x(k,k_0,x_0,u_0,d)\|_{\mathcal{X}}\\ \le \sigma(\beta(k_0)\|x_0\|_{\mathcal{X}}, k-k_0).\end{aligned} \tag{64}$$

iii) There exist functions $\mu, \beta \in K^+, a \in K_\infty, \sigma \in KL$ and a constant $r \ge 0$ such that for every $(k_0, x_0, d) \in Z^+ \times \mathcal{X} \times M_D$, we have for all $k \ge k_0$:

$$\|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}} \le \sigma(\beta(k_0)(\|x_0\|_{\mathcal{X}} + r), k-k_0) \tag{65}$$

$$\|x(k,k_0,x_0,u_0,d)\|_{\mathcal{X}} \le \mu(k)a(\|x_0\|_{\mathcal{X}} + r). \tag{66}$$

$$\sup\left\{\|h(k, x(k,k_0,x_0,u_0,d))\|_{\mathcal{Y}};\quad \begin{matrix} k \in \{k_0, k_0+1,\ldots\}, \|x_0\|_{\mathcal{X}} \le \varepsilon \\ k_0 \in \{0,\ldots,T\}, d \in M_D\end{matrix}\right\} < +\infty$$

$$(\textbf{Robust Lagrange Output Stability}).$$

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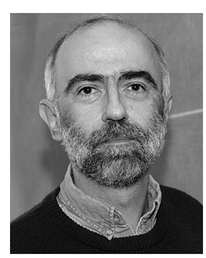

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