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Necessary and sufficient conditions for robust global asymptotic stabilization of discrete-time systems†

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We give Lyapunov-like conditions for non-uniform in time output stabilization of discrete-time systems. Particularly, it is proved that for a discrete-time control system there exists a (continuous) output stabilizing feedback if and only if there exists a (strong) output control Lyapunov function (OCLF). Moreover, strategies for the construction of continuous robust feedback stabilizers are presented.

Keywords: Discrete-Time Systems; Lyapunov Functions; Closed-loop system; RGAOS

1. Introduction

In this paper, we study discrete-time systems of the form:

$$\begin{gathered} x(t+1) = F(t, x(t), d(t), u(t)), \quad Y(t) = H(t, x(t)) \\ x(t) \in \Re^n, \quad Y(t) \in \Re^k, \quad d(t) \in D, \quad u(t) \in \Re^m, \quad t \in Z^+ \end{gathered} \tag{1.1}$$

where $D \subseteq \Re^l$ is the set of disturbances or time-varying parameters and $F : Z^+ \times \Re^n \times D \times \Re^m \to \Re^n$, $H : Z^+ \times \Re^n \to \Re^k$ satisfy $F(t, 0, d, 0) = 0$, $H(t, 0) = 0$ for all $(t, d) \in Z^+ \times D$. Specifically, we present necessary and sufficient conditions for the existence of a (continuous) function $k : Z^+ \times \Re^n \to \Re^m$ with $k(t, 0) = 0$ for all $t \in Z^+$ such that the closed-loop system (1.1) with

$$u(t) = k(t, x(t)) \tag{1.2}$$

is non-uniformly in time robustly globally asymptotically output stable (RGAOS, see [8]). Notice that the case of state stabilization is also accounted in this framework, since it is equivalent to the output stabilization of system (1.1) with output map $H(t, x) := x$.

Time-varying discrete-time systems were recently studied in [6,8,17,21,22]. A preliminary version of the present paper studying time-varying discrete-time systems without disturbances and with the whole state vector as output map (i.e. state stabilization) was given in [10]. It should be emphasized that in the present paper, we study the discrete-time

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systems *per se*, and not necessarily as the sampled-data representation of a continuous-time system (see for instance [23] and the references therein).

1.1. Motivation

Time-varying control systems of the form (1.1) appear naturally when tracking control problems are studied. However, the reader should not think that the use of time-varying feedback is restricted to time-varying control systems. Time-varying feedback may be used for stabilization of autonomous control systems. Section 2 of the present paper is devoted to the presentation of an example: an autonomous planar system, which cannot be uniformly stabilized by continuous time-invariant state feedback or (discontinuous) time-invariant partial state feedback. On the other hand, there exists a continuous time-varying partial state feedback, which robustly globally asymptotically stabilizes the origin non-uniformly in time, even in the presence of measurement errors. Continuity of the dynamics of the closed-loop system is a desired property, since it guarantees certain robustness properties (see [13,14]). Thus, as the example presented in section 2 shows, time-varying feedback laws that induce non-uniform in time asymptotic stability for the resulting closed-loop system may be the only option for the solution of certain stabilization problems (and sometimes is not a "bad" option).

Necessary and sufficient Lyapunov-like conditions for the existence of stabilizing ordinary feedback have been given in the pioneering papers [3,5,25,27] for continuous-time systems, where the concept of the (Robust) control Lyapunov function (CLF) was introduced. Explicit formulae for the feedback law are also given in [19] for autonomous continuous-time control systems. Recently, in [7], necessary and sufficient Lyapunov-like conditions were given for the existence of a stabilizing ordinary time-varying feedback for continuous-time systems. CLFs are also used for the expression of Lyapunov-like conditions for asymptotic controllability of continuous-time systems (see for instance [1,16]).

For discrete-time systems Lyapunov functions were proposed in [28] for the construction of continuous state stabilizing feedback. In [29,30] a special class of Lyapunov functions (norms of $\Re^n$) was used to express necessary and sufficient conditions for local asymptotic stabilization by means of time-invariant feedback. Recently, it was shown in [12,15] that the existence of a smooth CLF for discrete-time systems is equivalent to asymptotic controllability. This important result allowed the authors in [12,15] to construct stabilizing (in general discontinuous) feedback for asymptotically controllable discrete-time systems. A similar result for the existence of (discontinuous) feedback stabilizer was given in [2]. However, in [12,15] the authors prove in addition that the closed-loop system is robust for perturbations of appropriate magnitude.

In the present paper, uncertain control systems of the form (1.1) are considered, where the magnitude of the perturbation (i.e. the "size" of the set D) is *a priori* given and the problem of the existence of an output stabilizing feedback law (1.2), which stabilizes the output of the system for all possible disturbances with $d(t) \in D$ for all $t \in Z^+$, is studied. We show that the existence of a output control Lyapunov function (OCLF) is a necessary and sufficient condition for the existence of a stabilizing feedback that stabilizes the output of the system (in general) non-uniformly in time (see [8,9] for the notion of non-uniform in time RGAOS for discrete-time systems). Moreover, we provide necessary and sufficient Lyapunov-like conditions for the continuity of the feedback stabilizer. Thus, the result contained in [2] is generalized in more than one direction. Since explicit formulae for the feedback law cannot be given for the general nonlinear time-varying discrete-time case, we show that the actual computation of the control action at each time can be achieved via a minimization procedure (Proposition 3.10).

In section 4, we focus on the problem of state stabilization (i.e. when the stabilized output is the state vector of the system), where the notion of the CLF is applicable. Exactly as in the continuous-time case, in some cases the construction of a CLF is simpler than the construction of a state stabilizing feedback. For the case of controllable triangular discrete-time single input systems (i.e. $u \in \Re$) of the form (1.1), where we suppose that there exist continuous functions $f_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, \ldots, n)$ with $f_i(t, 0) = 0$ for all $t \geq 0$, such that

$$\begin{aligned} F(t,x,0,u) &= \big(f_1(t,x_1,x_2), f_2(t,x_1,x_2,x_3), \ldots, f_n(t,x_1,\ldots,x_n,u)\big)', \\ &\forall\, (t,x,u) \in Z^+ \times \Re^n \times \Re \end{aligned} \tag{1.3}$$

and $D \subset \Re^l$ is a compact set with $0 \in D$, we show that a CLF can be constructed in parallel with the construction of stabilizing feedback. Under appropriate assumptions we are in a position to provide a robust feedback state stabilizer of this system, which guarantees exponential convergence of the solutions of the closed-loop system to the origin (Theorem 4.3). Moreover, sufficient conditions that guarantee the existence of a partial state feedback that stabilizes the state of the system are provided in Corollary 4.7. Our work is based on ideas given in [11,24], where the authors present backstepping strategies for the construction of feedback stabilizers that guarantee the so-called "dead beat" property for the closed-loop system, when the dynamics of (1.1) are autonomous and do not contain uncertainties. We notice that controllers, which guarantee the "dead beat" property for discrete-time systems, were also considered in [20].

Finally, it should be emphasized that all results presented in this paper automatically cover the corresponding results in the time-invariant case. In order to illustrate this point, we also present examples of autonomous systems (see Example 4.6 as well as the example of section 2).

1.2. Notations

Throughout this paper, we adopt the following notations:

* For $x \in \Re^n$, $|x|$ denotes its Euclidean norm and x' its transpose.
* Z^+ denotes the set of non-negative integers.
* K^+ denotes the class of positive C^0 functions $\phi : \Re^+ \to (0, +\infty)$. K_∞ denotes the class of continuous, strictly increasing functions $a : \Re^+ \to \Re^+$ with $a(0) = 0$ and $\lim_{s \to +\infty} a(s) = +\infty$. By KL, we denote the set of all continuous functions $\sigma = \sigma(s,t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\bullet, t)$ is continuous and increasing with $\sigma(0,t) = 0$; (ii) for each $s \geq 0$, the mapping $\sigma(s, \bullet)$ is non-increasing with $\lim_{t \to +\infty} \sigma(s,t) = 0$.
* M_D denotes the set of all sequences $\{d(t)\}_0^\infty$ with $d(t) \in D$ for all $t \in Z^+$, where $D \subset \Re^l$.
* B denotes the open unit sphere in $\Re^n$. Let $\varepsilon > 0$ and $S \subseteq \Re^n$. We define $S + \varepsilon B := \cup_{s \in S} \{x \in \Re^n; |x - s| < \varepsilon\}$.

Remark 1.1. We would like to point out that for every mapping $F : Z^{+} \times \Omega \to A$, where $\Omega \subseteq \Re^{n}$ and $A \subseteq \Re^{m}$ is a convex set, we may define the mapping $\tilde{F} : \Re \times \Omega \to A$, which satisfies $\tilde{F}(t,x) = F(t,x)$ for all $(t,x) \in Z^{+} \times \Omega$ and $\tilde{F}(t,x) := (1 - t + [t])F([t],x) + (t - [t])F([t]+1,x)$ for all $(t,x) \in (\Re^{+} \backslash Z^{+}) \times \Omega$, where $[t] := \max\left\{\tau \in Z^{+}; \tau \le t\right\}$ denotes the integer part of $t \in \Re^{+}$ and $\tilde{F}(t,x) = F(0,x)$ for all $(t,x) \in (-\infty, 0) \times \Omega$ (notice that $\tilde{F}$ is continuous with respect to $t \in \Re$).

2. Motivating Example

In this section, an example of an autonomous system is presented, which cannot be robustly globally uniformly asymptotically stabilized by continuous time-invariant state feedback or (discontinuous) partial state feedback but it can be non-uniformly in time robustly globally asymptotically stabilized by continuous time-varying partial state feedback even in the presence of measurement errors. This example shows that continuous time-varying feedback laws may be used for the robust stabilization of autonomous control systems as well. The notion of robust uniform global asymptotic stability is given in [6] for discrete-time systems with continuous dynamics but can be directly extended to the case of discontinuous dynamics.

Consider the planar autonomous control system:

$$\begin{aligned} & x(t+1) = x(t)(2 + x(t)u(t)) + d(t)y(t)u(t) \\ & y(t+1) = \exp(-2\mu)y(t) \\ & (x(t), y(t)) \in \Re^{2}, \quad u(t) \in \Re, \quad d(t) \in D := [-1, 1], \quad t \in Z^{+} \end{aligned} \tag{2.1}$$

We claim that for every $\mu > 0$ there is no continuous time-invariant state feedback, which robustly globally asymptotically stabilizes system (2.1). The proof of this claim is made by contradiction. Suppose that there exists $k \in C^{0}(\Re^{2}; \Re)$ so that zero is robustly globally uniformly asymptotically stable for the closed-loop system (2.1) with $u(t) = k(x(t), y(t))$. Let $R > 0$, such that $|k(x,0)| \le R$ for all $|x| \le 1$. By virtue of the uniform robust Lyapunov stability property for the closed-loop system (2.1) with $u(t) = k(x(t), y(t))$, we obtain the existence of $\delta > 0$ such that if $|x_0| < \delta$ then the solution of the closed-loop system (2.1) with $u(t) = k(x(t), y(t))$, initial condition $(x(t_0), y(t_0)) = (x_0, 0)$, corresponding to zero input, i.e. $d(t) \equiv 0$ for all $t \in Z^{+}$, satisfies $|x(t)| < 1/(2R+2)$ for all $t \ge t_0$ (notice that $y(t) = 0$ for all $t \ge t_0$). Let $x_0 \ne 0$ with $|x_0| < \delta$. It is clear that the solution of the closed-loop system (4.1) with $u(t) = k(x(t), y(t))$, initial condition $(x(t_0), y(t_0)) = (x_0, 0)$, corresponding to zero input, i.e. $d(t) \equiv 0$ for all $t \in Z^{+}$, satisfies $|x(t)k(x(t), y(t))| \le (1/2)$ for all $t \ge t_0$. It follows from system (2.1) that $|x(t+1)| \ge (3/2)|x(t)|$ for all $t \ge t_0$. Thus, we obtain $(3/2)^{t-t_0}|x_0| \le |x(t)| < 1/(2R+2)$ for all $t \ge t_0$, which is clearly a contradiction.

System (2.1) can be stabilized by discontinuous state feedback. For example, the discontinuous feedback:

$$k(x,y) := \begin{cases} 0 & \text{if} \quad |x| \le |y|^{\frac{1}{2}} \\ -2x^{-1} & \text{if} \quad |x| > |y|^{\frac{1}{2}} \end{cases} \tag{2.2}$$

robustly globally asymptotically stabilizes system (2.1). To prove this fact notice that the continuous, positive definite and radially unbounded function

$$V(x,y) := |x| + \frac{2\exp(\mu)}{\exp(\mu/2)-1}|y|^{\frac{1}{2}},$$

satisfies the following inequality:

$$V\big(x(2+xk(x,y)) + d\,yk(x,y), \exp(-2\mu)y\big) \le \exp\big(-\mu/2\big)V(x,y), \quad \forall\,(x,y,d) \in \Re^2 \times [-1,1]$$

and the above inequality implies uniform robust global asymptotic stability of zero for the closed-loop system (2.1) with $u(t) = k(x(t), y(t))$.

However, for every $\mu > 0$, there is no (discontinuous) time-invariant partial state feedback, which depends only on x that robustly globally asymptotically stabilizes system (2.1). Again, the proof of this claim is made by contradiction. Suppose that there exists $k : \Re \to \Re$ so that zero is robustly globally uniformly asymptotically stable for the closed-loop system (2.1) with $u(t) = k(x(t))$. Similar arguments previously show that for every $r > 0$, we must necessarily have $\sup\{|k(x)|; |x| \le r\} = +\infty$. This implies the existence of a sequence $\{x_i \in \Re\}$ with $x_i \to 0$ and $|k(x_i)| \to +\infty$. Let $\varepsilon > 0$ be arbitrary. By virtue of the uniform robust Lyapunov stability property for the closed-loop system (2.1) with $u(t) = k(x(t))$, we obtain the existence of $\delta \in (0, \varepsilon)$ such that if $|(x_0, y_0)| \le \delta$ then the solution of the closed-loop system (2.1) with $u(t) = k(x(t))$, initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$, corresponding to input $d \in M_D$, satisfies $|x(t)| \le \varepsilon$ for all $t \ge t_0$. Without loss of generality we may assume that the sequence $\{x_i \in \Re\}$ considered previously with $x_i \to 0$ and $|k(x_i)| \to +\infty$ satisfies $|x_i| < \delta/2$. Let $y_0 = \delta/2$ and $d = 1$. We obtain:

$$\big|x_i\big(2 + x_i k(x_i)\big) + dy_0 k(x_i)\big| \ge \big|x_i^2 + dy_0\big||k(x)| - 2|x_i| \ge \left(\frac{\delta}{2}|k(x_i)| - 2|x_i|\right) \to +\infty$$

Since $|(x_i, y_0)| \le \delta$ the solution of the closed-loop system (2.1) with $u(t) = k(x(t))$, initial condition $(x(t_0), y(t_0)) = (x_i, y_0)$, corresponding to input $d \in M_D$ with $d(t_0) = 1$, should satisfy $\varepsilon \ge \big|x_i\big(2 + x_i k(x_i)\big) + d(t_0)y_0 k(x_i)\big| \to +\infty$, which is clearly a contradiction.

On the other hand, consider the time-varying continuous partial state feedback law:

$$k(t,x) := \begin{cases} -2x^{-1} & \text{if } \exp(\mu t)|x| \ge 2 \\ -2\big(\exp(\mu t)|x| - 1\big)x^{-1} & \text{if } 1 < \exp(\mu t)|x| < 2 \\ 0 & \text{if } \exp(\mu t)|x| \le 1 \end{cases} \tag{2.3}$$

It is proved next that the feedback defined above robustly globally stabilizes the origin for system (2.1) non-uniformly in time even in the presence of measurement errors. Thus, it should be emphasized that the (common) claim that time-varying feedback designs that involve terms, which are unbounded with respect to time, are highly sensitive to measurement errors is not generally true.

Consider the solution of the closed-loop system (2.1) with $u(t) = k(t, x(t) + e(t))$, where $e(t)$ denotes the measurement error. We assume 10% measurement error, i.e. $|e(t)| \le (1/10)|x(t)|$ for all $t \ge t_0$. The solution of the closed-loop system (2.1) with $u(t) = k(t, x(t) + e(t))$, initial condition $(x(t_0), y(t_0)) = (x_0, y_0) \in \Re^2$, corresponding to input $d \in M_D$, satisfies the following estimates:

$$|y(t)| \le \exp\big(-2\mu(t-t_0)\big)|y_0|, \quad \forall\, t \ge t_0 \tag{2.4a}$$

$$|x(t)| \leq \exp(-rt)\big(|x_0| + K_1 + K_2 \exp(2\mu t_0)|y_0|\big), \quad \forall\, t \geq t_0 + 1 \tag{2.4b}$$

for certain constants $r, K_1, K_2 > 0$. Indeed, equation (2.4a) is immediate ($y(t)$ is the solution of a linear difference equation). Inequality (2.4b) follows from the consideration of the following cases:

* $\exp(\mu t)|x(t) + e(t)| \geq 2$. In this case, using definitions (2.3) and (2.1) and the trivial inequality $|x(t) + e(t)| \geq (9/10)|x(t)|$, which is combined with the inequality $|e(t)| \leq (1/10)|x(t)|$ and gives

$$\frac{|e(t)|}{|x(t) + e(t)|} \leq \frac{1}{9},$$

we obtain:

$$|x(t+1)| \leq \frac{2|x(t)||e(t)|}{|x(t) + e(t)|} + \frac{2|y(t)|}{|x(t) + e(t)|} \leq \frac{2}{9}|x(t)| + \exp(\mu t)|y(t)|$$

* $1 < \exp(\mu t)|x(t) + e(t)| < 2$. In this case, using definition (2.3) we obtain $|u(t)| \leq 2\exp(\mu t)$. The trivial inequality $|x(t)| \leq |x(t) + e(t)| + |e(t)|$, combined with the inequality $|e(t)| \leq (1/10)|x(t)|$ gives $|x(t)| \leq (10/9)|x(t) + e(t)|$ and consequently $|x(t)| \leq (20/9)\exp(-\mu t)$. Thus, we obtain:

$$|x(t+1)| \leq 2|x(t)| + |x(t)|^2|u(t)| + |y(t)||u(t)| \leq 24\exp(-\mu t) + 2\exp(\mu t)|y(t)|$$

* $\exp(\mu t)|x(t) + e(t)| \leq 1$. In this case, using definitions (2.3) and (2.1) and the trivial inequality $|x(t)| \leq |x(t) + e(t)| + |e(t)|$, which is combined with the inequality $|e(t)| \leq (1/10)|x(t)|$ and gives $|x(t)| \leq (10/9)|x(t) + e(t)|$ we obtain: $|x(t+1)| \leq 2|x(t)| \leq 3\exp(-\mu t)$.

Thus in any case it holds that: $|x(t+1)| \leq (2/9)|x(t)| + 24\exp(-\mu t) + 2\exp(\mu t)|y(t)|$. The previous inequality in conjunction with equation (2.4a) gives: $|x(t+1)| \leq (2/9)|x(t)| + 24\exp(-\mu t) + 2\exp(-\mu t)\exp(2\mu t_0)|y_0|$, which implies inequality (2.4b) for appropriate $r, K_1, K_2 > 0$. By virtue of Lemma 3.3 in [8], estimates (2.4a,b) imply that zero is non-uniformly in time RGAS for the closed-loop system (2.1) with $u(t) = k(t, x(t) + e(t))$, $|e(t)| \leq (1/10)|x(t)|$.

It should be also emphasized that the time-varying feedback law defined by equation (2.3) has better properties than the discontinuous time-invariant feedback defined by equation (2.2): (1) it is continuous and (2) it depends only on x. Continuity of the dynamics of the closed-loop system is a desired property since it guarantees robustness to modeling errors (see [13,14]). Of course, both feedback laws become ill-conditioned for large times (notice that in both cases equation (2.4a) holds and consequently both feedback laws will have exactly the same implementation problems for large times), although the discontinuous feedback defined by equation (2.2) has additional problems when $|y|$ is small (it becomes unbounded even for small times).

3. Control Lyapunov functions for discrete-time systems

We study time-varying discrete-time systems of the form (1.1) under the following hypothesis:

(H1). *There exist functions* $a \in K_\infty, \gamma \in K^+$ *such that:* $|F(t,x,d,u)| \le a\big(\gamma(t)|(x,u)|\big)$ *for all* $(t,x,d,u) \in Z^+ \times \Re^n \times D \times \Re^m$ *and* $H : Z^+ \times \Re^n \to \Re^k$ *is continuous with* $H(t,0) = 0$ *for all* $t \in Z^+$.

Lemma 3.2 in [8] implies that hypothesis (H1) is fulfilled if the following (stronger) hypothesis holds:

(H2). *The mapping* $F : Z^+ \times \Re^n \times D \times \Re^m \to \Re^n$ *is continuous with* $F(t,0,d,0) = 0$ *for all* $(t,d) \in Z^+ \times D, D \subset \Re^l$ *is a compact set and* $H : Z^+ \times \Re^n \to \Re^k$ *is continuous with* $H(t,0) = 0$ *for all* $t \in Z^+$.

We next provide the notion of non-uniform in time stabilizability for general discrete-time systems.

Definition 3.1 *We say that equation* (1.1) *is non-uniformly in time* (*continuously*) *robustly globally output stabilizable, if there exist functions* $\sigma \in KL$, $a \in K_\infty$ *and* $\beta, \gamma, \mu \in K^+$, *a* (*continuous*) *mapping* $k : Z^+ \times \Re^n \to \Re^m$ (*called the feedback function*) *with* $|F(t,x,d,k(t,x))| \le a\big(\gamma(t)|x|\big)$ *for all* $(t,x,d) \in Z^+ \times \Re^n \times D$ *such that for every* $(t_0, x_0, \{d(t)\}_0^\infty) \in Z^+ \times \Re^n \times M_D$ *the unique solution* $x(t)$, $Y(t) = H(t,x(t))$ *of the closed-loop system* (1.1) *with* (1.2) *and initial condition* $x(t_0) = x_0$, *corresponding to input* $\{d(t)\}_0^\infty \in M_D$, *satisfies the following estimate:*

$$|Y(t)| + \mu(t)|x(t)| \le \sigma\big(\beta(t_0)|x_0|, t - t_0\big), \forall t \ge t_0 \tag{3.1}$$

Specifically, we say that the function $k : Z^+ \times \Re^n \to \Re^m$ is a non-uniform in time (continuous) robust global asymptotic output stabilizer for equation (1.1). Moreover, if equation (1.1) is non-uniformly in time (continuously) robustly globally output stabilizable with output $H(t,x) \equiv x$ then we say that equation (1.1) is non-uniformly in time (continuously) robustly globally state stabilizable and the function $k : Z^+ \times \Re^n \to \Re^m$ is a non-uniform in time (continuous) robust global asymptotic state stabilizer for equation (1.1).

Remark 3.2. (i) Notice that by virtue of Lemma 3.2 in [8], the existence of $a \in K_\infty, \gamma \in K^+$ with $\big|\tilde{F}(t,x,d)\big| \le a\big(\gamma(t)|x|\big)$ for all $(t,x,d) \in Z^+ \times \Re^n \times D$, where $\tilde{F}(t,x,d) := F(t,x,d,k(t,x))$, is equivalent to the following properties: (i) For every bounded set $S \subset Z^+ \times \Re^n$ the image set $\tilde{F}(S \times D)$ is bounded, (ii) $\tilde{F}(t,0,d) = 0$ for all $(t,d) \in Z^+ \times D$ and (iii) for every $\varepsilon > 0$, $t \in Z^+$ there exists $\delta(\varepsilon,t) > 0$ such that $\sup\big\{\big|\tilde{F}(t,x,d)\big|; |x| \le \delta(\varepsilon,t), d \in D\big\} < \varepsilon$. Clearly, these requirements are automatically fulfilled if hypothesis (H1) holds and $k : Z^+ \times \Re^n \to \Re^m$ is continuous with $k(t,0) = 0$ for all $t \in Z^+$.

(ii) The main result in [10] shows that if there exist functions $\sigma \in KL$, $a \in K_\infty, \beta, \gamma \in K^+$, a (continuous) mapping $k : Z^+ \times \Re^n \to \Re^m$ with $|F(t,x,d,k(t,x))| \le a\big(\gamma(t)|x|\big)$ for all $(t,x,d) \in Z^+ \times \Re^n \times D$ such that for every $\big(t_0, x_0, \{d(t)\}_0^\infty\big) \in Z^+ \times \Re^n \times M_D$ the unique solution $x(t), Y(t) = H(t,x(t))$ of the closed-loop system (1.1) with (1.2) and initial condition $x(t_0) = x_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$, satisfies the estimate $|Y(t)| \le \sigma\big(\beta(t_0)|x_0|, t - t_0\big)$, for all $t \ge t_0$, then equation (1.1) is non-uniformly in time (continuously) robustly globally output stabilizable. However, this fact is not going to be used in the present paper.

The following class of upper semi-continuous functions plays an important role in the exploitation of the properties of Lyapunov functions for discrete-time systems.

DEFINITION 3.3 *Let the upper semi-continuous function* $\Psi : Z^+ \times \Re^n \times U \to \Re$, *where* $U \subseteq \Re^m$ *is a convex set. The function* $\Psi(t,x,u)$ *is quasi-convex with respect to* $u \in U$, *if for each fixed* $(t,x) \in Z^+ \times \Re^n$ *it holds that*

$$\begin{aligned} &\Psi(t,x,\lambda u_1 + (1-\lambda)u_2) \leq \max\{\Psi(t,x,u_1), \Psi(t,x,u_2)\}, \\ &\textit{for all} \quad \lambda \in (0,1), (u_1,u_2) \in U \times U \end{aligned} \tag{3.2a}$$

The function $\Psi(t,x,u)$ *is strictly quasi-convex with respect to* $u \in U$, *if for each fixed* $(t,x) \in Z^+ \times \Re^n$ *it holds that*

$$\begin{aligned} \Psi(t,x,\lambda u_1 + (1-\lambda)u_2) < \max\{\Psi(t,x,u_1), \Psi(t,x,u_2)\}, \quad \textit{for all} \quad \lambda \in (0,1), \\ (u_1,u_2) \in U \times U, \quad \textit{with} \quad u_1 \neq u_2 \end{aligned} \tag{3.2b}$$

Clearly, if the mapping $u \in U \to \Psi(t,x,u)$ is (strictly) convex for each fixed $(t,x) \in Z^+ \times \Re^n$, then $\Psi(t,x,u)$ is (strictly) quasi-convex with respect to $u \in U$. For example, for all functions $a \in K_\infty$, $k \in C^0(Z^+ \times \Re^n; \Re^m)$ and $V \in C^0(Z^+ \times \Re^n; \Re)$, the function $\Psi(t,x,u) := V(t,x) + a\left(|u - k(t,x)|\right)$ is strictly quasi-convex with respect to $u \in \Re^m$. Notice that if $\Psi(t,x,u)$ is strictly quasi-convex with respect to $u \in U$, then for each fixed $(t,x) \in Z^+ \times \Re^n$ there exists at most one $u \in U$ such that $\Psi(t,x,u) = \inf_{v \in U} \Psi(t,x,v)$.

We next give the definition of the notion of (strong) OCLF for a discrete-time system used in this paper. The definition is in the same spirit with the definition of the notion of robust CLF given in [5] for continuous-time finite-dimensional control systems.

DEFINITION 3.4 *We say that equation* (1.1) *admits an OCLF if there exists a function* $V : Z^+ \times \Re^n \to \Re^+$, *which satisfies the following properties:*

(i) *There exists* $a_1, a_2 \in K_\infty$, $\beta, \mu \in K^+$ *such that:*

$$a_1\left(|(\mu(t)x, H(t,x))|\right) \leq V(t,x) \leq a_2\left(\beta(t)|x|\right), \forall (t,x) \in Z^+ \times \Re^n \tag{3.3a}$$

(ii) *There exist functions* $\rho \in K_\infty$ *with* $\rho(s) \leq s$ *for all* $s \geq 0$, $q \in C^0(Z^+; \Re^+)$ *with* $\lim_{t \to +\infty} q(t) = 0$ *such that:*

$$\begin{aligned} &\inf_{u \in \Re^m} \sup_{d \in D} V(t+1, F(t,x,d,u)) \leq V(t,x) - \rho(V(t,x)) + q(t), \\ &\forall\, (t,x) \in Z^+ \times \Re^n \end{aligned} \tag{3.3b}$$

We say that equation (1.1) *admits a strong-OCLF if there exists a continuous function* $V : Z^+ \times \Re^n \to \Re^+$, *which satisfies property* (*i*) *above as well as the following property:*

(iii) *There exist functions* $\rho \in K_\infty$ *with* $\rho(s) \leq s$ *for all* $s \geq 0$, $q \in C^0(Z^+; \Re^+)$ *with* $\lim_{t \to +\infty} q(t) = 0$ *and an upper semi-continuous function* $\Psi : Z^+ \times \Re^n \times \Re^m \to \Re$

with $\Psi(t,0,0)=0$ for all $t\in Z^{+}$, which is quasi-convex with respect to $u\in\Re^{m}$ such that the following inequalities hold:

$$\sup_{d\in D} V(t+1,F(t,x,d,u))\leq \Psi(t,x,u),\forall(t,x,u)\in Z^{+}\times\Re^{n}\times\Re^{m}, \tag{3.3c}$$

$$\inf_{u\in\Re^{m}} \Psi(t,x,u)\leq V(t,x)-\rho(V(t,x))+q(t),\forall(t,x)\in Z^{+}\times\Re^{n} \tag{3.3d}$$

For the case $H(t,x)\equiv x$, we simply call $V:Z^{+}\times\Re^{n}\to\Re^{+}$ a CLF.

Remark 3.5. Notice that property (iii) of Definition 3.4 implies property (ii). If the mapping $u\to V(t+1,F(t,x,d,u))$ is (quasi-) convex for each fixed $(t,x,d)\in Z^{+}\times\Re^{n}\times D$, then the mapping $u\to\sup_{d\in D} V(t+1,F(t,x,d,u))$ is (quasi-) convex for each fixed $(t,x)\in Z^{+}\times\Re^{n}$. It follows that property (iii) is satisfied with $\Psi(t,x,u):=\sup_{d\in D} V(t+1,F(t,x,d,u))$.

The following proposition shows that the existence of a (strong) OCLF is a necessary and sufficient condition for the existence of a (continuous) non-uniform in time robust global asymptotic output stabilizer for equation (1.1). Let the following statements:

(a) System (1.1) admits an OCLF.
(b) System (1.1) admits a strong OCLF.
(c) System (1.1) is non-uniformly in time robustly globally output stabilizable.
(d) System (1.1) is non-uniformly in time continuously robustly globally output stabilizable.

Proposition 3.6. *Consider system* (1.1) *under hypothesis* (H1). *Then the following implications hold:* (a) $\Leftrightarrow$ (c)*,* (b) $\Rightarrow$ (d)*. Moreover, if hypothesis* (H2) *holds for system* (1.1) *then the implication* (d) $\Rightarrow$ (b) *also holds.*

Remark 3.7. (i) The procedure of partition of unity used in the proof of implication (b) $\Rightarrow$ (d) of Proposition 3.6 guarantees that the constructed feedback is actually a function of class C^{∞}. Thus, under hypothesis (H2), the existence of a non-uniform in time continuous robust global stabilizer is equivalent to the existence of a non-uniform in time smooth robust global stabilizer.

(ii) The result of Proposition 3.6 extends the corresponding results in [2,13] in many directions. For example, we do not have to impose the "small control" property in order to guarantee that $\lim_{x\to 0} k(t,x)=k(t,0)=0$. Moreover, Proposition 3.6 covers the general discrete-time case with disturbances (not covered by the corresponding results in [2,13]) under minimal assumptions concerning the regularity of the dynamics of (1.1). On the other hand, Proposition 3.6 presents conditions that guarantee the regularity of the constructed feedback (this is not the case with the corresponding results in [2,13]).

The proof of Proposition 3.6 relies on the following lemma, which provides sufficient Lyapunov-like conditions for non-uniform in time RGAOS. For reasons of completeness we state it here and we provide its proof in the Appendix.

Lemma 3.8 *Consider the time-varying discrete-time system:*

$$\begin{aligned} &x(t+1)=F(t,x(t),d(t)), \quad Y(t)=H(t,x(t))\\ &x(t)\in\Re^{n}, Y(t)\in\Re^{k}, \quad d(t)\in D, t\in Z^{+}\end{aligned} \tag{3.4}$$

where $D\subseteq\Re^{l}$ and $F:Z^{+}\times\Re^{n}\times D\to\Re^{n}$ satisfies the following hypothesis:

(A1). *There exist functions $a\in K_{\infty},\gamma\in K^{+}$ such that $|F(t,x,d)|\leq a(\gamma(t)|x|)$, for all $(t,x,d)\in Z^{+}\times\Re^{n}\times D$ and $H:Z^{+}\times\Re^{n}\to\Re^{k}$ is continuous with $H(t,0)=0$ for all $t\in Z^{+}$.*

Suppose that there exist functions $V : Z^+ \times X \to \Re^+$, $a_1, a_2, a_3 \in K_\infty$ *with* $a_3(s) \le s$ *for all* $s \ge 0$, $\tilde{\beta}, q, \tilde{\mu} \in K^+$ *with* $\lim_{t\to+\infty} q(t) = 0$, *satisfying the following inequalities for all* $(t,x,d) \in Z^+ \times \Re^n \times D$:

$$a_1\left(|(\tilde{\mu}(t)x, H(t,x))|\right) \le V(t,x) \le a_2\left(\tilde{\beta}(t)|x|\right) \tag{3.5a}$$

$$V(t+1, F(t,x,d)) \le V(t,x) - a_3(V(t,x)) + q(t) \tag{3.5b}$$

Then there exist functions $\sigma \in KL, \beta, \mu \in K^+$, *such that for every* $\left(t_0, x_0, \{d(t)\}_0^\infty\right) \in Z^+ \times \Re^n \times M_D$ *the unique solution* $x(t)$, $Y(t) = H(t, x(t))$ *of system* (3.4) *with initial condition* $x(t_0) = x_0$, *corresponding to input* $\{d(t)\}_0^\infty \in M_D$, *satisfies estimate* (3.1).

The following example illustrates the use of the OCLF for discrete-time systems.

Example 3.9. Consider the following linear time-varying planar discrete-time system:

$$\begin{aligned}
&x_1(t+1) = tx_1(t) + x_2(t) + \frac{d(t)}{1+2(t+1)(t+3)}x_1(t)\\
&x_2(t+1) = x_3(t) + u(t)\\
&x_3(t+1) = \exp(1)x_3(t)\\
&Y(t) = (x_1(t), x_2(t)) \in \Re^2,\\
&x(t) := (x_1(t), x_2(t), x_3(t)) \in \Re^3, u(t) \in \Re,\\
&d(t) \in D := \left[-\frac{1}{2}, \frac{1}{2}\right], t \in Z^+
\end{aligned} \tag{3.6}$$

Define the continuous function:

$$V(t, x_1, x_2, x_3) := |x_1| + |x_2| + 2(t+2)|tx_1 + x_2| + \exp(-2t)|x_3| \tag{3.7}$$

Notice that the following inequality holds for all $(t,x) \in Z^+ \times \Re^3$:

$$\frac{1}{2}|(Y, \exp(-2t)x)| \le V(t,x) \le 3(t+1)(t+2)|x| \tag{3.8}$$

Furthermore, we obtain:

$$\begin{aligned}
&\inf_{u\in\Re}\sup_{d\in D} V\left(t+1, tx_1 + x_2 + \frac{dx_1}{1+2(t+1)(t+3)}, x_3 + u, \exp(1)x_3\right)\\
&\quad = \inf_{u\in\Re}\sup_{d\in D}\left(\left|tx_1 + x_2 + \frac{dx_1}{1+2(t+1)(t+3)}\right| + |x_3+u| + 2(t+3)\right.\\
&\quad\quad \left.\left|(t+1)tx_1 + (t+1)x_2 + \frac{(t+1)\,dx_1}{1+2(t+1)(t+3)} + x_3 + u\right| + \exp(-2t-1)|x_3|\right)\\
&\quad \le \inf_{u\in\Re}\sup_{d\in D}(|tx_1 + x_2| + |x_3+u| + 2(t+3)|(t+1)tx_1 + (t+1)x_2 + x_3 + u|\\
&\quad\quad + |d||x_1| + \exp(-2t-1)|x_3|)\left(|d| \le \frac{1}{2}\right)\\
&\quad\quad \le \inf_{u\in\Re}(|tx_1 + x_2| + |x_3+u| + 2(t+3)|(t+1)tx_1 + (t+1)x_2 + x_3 + u| + \frac{1}{2}|x_1|\\
&\quad\quad + \exp(-2t-1)|x_3|)(\text{set} \quad u = -x_3 - (t+1)(tx_1 + x_2))\\
&\quad\quad \le (t+2)|tx_1 + x_2| + \frac{1}{2}|x_1| + \exp(-2t-1)|x_3| \le \frac{1}{2}V(t, x_1, x_2, x_3)
\end{aligned}$$

Thus, we conclude that V as defined by equation (3.7) is a strong OCLF for system (3.6) and particularly satisfies properties (i), (iii) of Definition 3.4 with

$$\Psi(t,x,u) := |tx_1 + x_2| + |x_3 + u| + 2(t+3)|(t+1)tx_1 + (t+1)x_2 + x_3 + u| + \frac{1}{2}|x_1| + \exp(-2t-1)|x_3|$$

$$a_1(s) : = \frac{1}{2}s, a_2(s) := s, \beta(t) := 3(t+1)(t+2), \mu(t) := \exp(-2t), \rho(s) := \frac{1}{2}s \quad \text{and} \quad q(t) \equiv 0$$

Lemma 3.8 implies that the continuous feedback function $k(t,x) := -x_3 - (t+1)(tx_1 + x_2)$ is a non-uniform in time continuous robust global asymptotic output stabilizer.

Universal explicit formulae for the stabilizing feedback law (1.2) cannot be given for the general nonlinear time-varying discrete-time case (1.1). However, the following proposition provides means for the computation of the control action, via a minimization procedure for a class of discrete-time systems (1.1) that satisfies the following hypothesis:

(**H3**). There exist $a_3 \in K_\infty$ and continuous positive mappings $\gamma, \delta : Z^+ \times \Re^n \to (0, +\infty)$ such that the following inequality holds:

$$a_3(\gamma(t,x))\max\{|u| - \delta(t,x), 0\}) \le |F(t,x,d,u)|, \quad \forall (t,x,d,u) \in Z^+ \times \Re^n \times D \times \Re^m \tag{3.9}$$

Hypothesis (H3) guarantees that $\lim |F(t,x,d,u)| = +\infty$ as $|u| \to +\infty$, i.e. the dynamics of equation (1.1) are "radially unbounded" with respect to $u \in \Re^m$. The proof of the following proposition can be found in the Appendix.

Proposition 3.10 *Consider system* (1.1) *under hypotheses* (H1) *and* (H3) *and suppose that equation* (1.1) *admits a strong OCLF* $V : \Re^+ \times \Re^n \to \Re^+$*, which in addition to properties* (*i*) *and* (*iii*) *of Definition* 3.4 *satisfies the following property:*

(iv) The function $\Psi(t,x,u)$ *defined in property (iii) of Definition 3.4 is continuous and strictly quasi-convex with respect to* $u \in \Re^m$.

Define:

$$\tilde{V}(t,x) := \inf\{\Psi(t,x,u); u \in \Re^m\} \tag{3.10}$$

$$\mathbf{M}(t,x) := \{u \in \Re^m : \tilde{V}(t,x) = \Psi(t,x,u)\} \tag{3.11}$$

Then there exists a continuous mapping $k : Z^+ \times \Re^n \to \Re^m$ *with* $k(t,0) = 0$ *for all* $t \in Z^+$ *such that*

$$\mathbf{M}(t,x) := \{k(t,x)\}, \forall (t,x) \in Z^+ \times \Re^n \tag{3.12}$$

Moreover, the continuous mapping $k(t,x)$ *is a non-uniform in time robust global asymptotic stabilizer for equation* (1.1).

Example 3.11. Consider the state stabilization problem for the following three-dimensional discrete-time system:

$$\begin{aligned}
x_1(t+1) &= d_1(t)\exp(t)x_2(t) \\
x_2(t+1) &= x_2(t)x_3(t) + d_2(t)\exp(-t-1)x_1(t) \\
x_3(t+1) &= f(t,x(t)) + u(t) \\
Y(t) = x(t) &:= (x_1(t),x_2(t),x_3(t)) \in \Re^3,\ u(t)\in\Re, \\
d(t) &:= (d_1(t),d_2(t)) \in D \\
&:= [-1,1]\times\left[-\frac{1}{4},\frac{1}{4}\right],\ t\in Z^+
\end{aligned} \tag{3.13}$$

where $f:\Re^+\times\Re^3\to\Re$ is a continuous mapping with $f(t,0,0,0)=0$ for all $t\in Z^+$. Define the following smooth function:

$$V(t,x_1,x_2,x_3) := x_1^2 + (1+2\exp(2t))x_2^2 + x_3^2 + 8(1+\exp(2t+2))x_2^2x_3^2 \tag{3.14}$$

Notice that the following inequality holds for all $(t,x_1,x_2,x_3)\in Z^+\times\Re^3$:

$$|x|^2 \le V(t,x_1,x_2,x_3) \le 3\exp(2t)|x|^2 + 16\exp(2t+2)|x|^4 \tag{3.15}$$

and that by virtue of the trivial inequality $\left(x_2x_3 + d_2\exp(-t-1)x_1\right)^2 \le 2x_2^2x_3^2 + (1/8)\exp(-2t-2)x_1^2$ (which holds since $|d_2|\le(1/4)$), we obtain for all $(t,x,d)\in Z^+\times\Re^3\times D$

$$\begin{aligned}
&V\left(t+1, d_1\exp(t)x_2, x_2x_3 + d_2\exp(-t-1)x_1, f(t,x)+u\right) \\
&:= \left|d_1^2\right|\exp(2t)x_2^2 + (1+2\exp(2t+2))\left(x_2x_3 + d_2\exp(-t-1)x_1\right)^2 \\
&\quad + \left[1 + 8(1+\exp(2t+4))\left(x_2x_3 + d_2\exp(-t-1)x_1\right)^2\right](f(t,x)+u)^2 \\
&\le \exp(2t)x_2^2 + 2(1+2\exp(2t+2))x_2^2x_3^2 + \frac{3}{8}x_1^2 \\
&\quad + \left[1 + 16(1+\exp(2t+4))\left(x_2^2x_3^2 + \exp(-2t-2)x_1^2\right)\right](f(t,x)+u)^2
\end{aligned}$$

Notice that the continuous map

$$\begin{aligned}
\Psi(t,x,u) := {} & \exp(2t)x_2^2 + 2(1+2\exp(2t+2))x_2^2x_3^2 + \frac{3}{8}x_1^2 \\
& + \left[1 + 16(1+\exp(2t+4))\left(x_2^2x_3^2 + \exp(-2t-2)x_1^2\right)\right](f(t,x)+u)^2
\end{aligned}$$

is strictly convex with respect to $u\in\Re$ for each fixed $(t,x)\in Z^+\times\Re^3$. Furthermore, we obtain:

$$\sup_{d\in D} V\left(t+1, d\exp(t)x_2, x_2x_3, f(t,x)+u\right) \le \Psi(t,x,u);\ \inf_{u\in\Re}\Psi(t,x,u) \le \frac{1}{2}V(t,x_1,x_2,x_3)$$

Thus, we conclude that V as defined by equation (3.14) is a strong CLF for system (3.13) and satisfies properties (i), (iii) of Definition 3.4, $a_1(s):=s^2/2$, $a_2(s):=3s^2+16s^4$, $\mu(t)=1$, $\beta(t):=\exp(t+1)$, $\rho(s):=s/2$ and $q(t)\equiv 0$. Notice that hypotheses (H2) and (H3) are satisfied and particularly inequality (3.9) holds since we have: $\max\{0,|u|-|f(t,x)|\}\le|u+f(t,x)|\le|F(t,x,d,u)|$, for all $(t,x,d,u)\in Z^+\times\Re^n\times D\times\Re$. Moreover, the set-valued map $\mathbf{M}(t,x)$ as defined by equation (3.11) satisfies: $\mathbf{M}(t,x):=\{-f(t,x)\}$, for all $(t,x)\in Z^+\times\Re^3$. We conclude that the continuous feedback function $k(t,x):=-f(t,x)$, robustly globally asymptotically stabilizes the origin for equation (3.13). □

4. Sufficient conditions for robust feedback state stabilization

In this section, we focus on the continuous state stabilization problem for discrete-time systems. Exactly as in the continuous-time case, in some cases the construction of a CLF is performed simultaneously with the construction of a state stabilizing feedback. In this section, we present certain classes of discrete-time systems of the form (1.1) for which a CLF can be constructed in parallel with the construction of robust feedback state stabilizers that guarantee exponential convergence of the solutions of the closed-loop system to the origin. First, the reader is introduced to the notion of non-uniform robust global K-exponential stability (RGK-ES) for discrete-time systems, which is a generalization of the corresponding notion introduced in [18] for continuous-time systems. Consider a finite-dimensional discrete-time system:

$$\begin{aligned} &x(t+1) = F(t, x(t), d(t)) \\ &x(t) \in \Re^n, d(t) \in D \subset \Re^l, t \in Z^+ \end{aligned} \tag{4.1}$$

where $D \subset \Re^l$ and $F : Z^+ \times \Re^n \times D \to \Re^n$ with $F(t,0,d) = 0$ for all $(t,d) \in Z^+ \times D$.

Definition 4.1 *We say that zero is non-uniformly RGK-ES for equation* (4.1) *with constant* $c > 0$ *if there exist functions* $a \in K_\infty$, $\beta \in K^+$ *such that for every* $(t_0, x_0, \{d(t)\}_0^\infty) \in Z^+ \times \Re^n \times M_D$ *the solution* $x(t)$ *of equation* (4.1) *with initial condition* $x(t_0) = x_0$ *and corresponding to* $\{d(t)\}_0^\infty \in M_D$ *satisfies the estimate:*

$$|x(t)| \le \exp\left(-c(t-t_0)\right) a\left(\beta(t_0)|x_0|\right), \forall t \ge t_0 \tag{4.2}$$

The following lemma provides sufficient conditions for non-uniform in time RGK-ES and its proof can be found in the Appendix.

Lemma 4.2 *Suppose that there exist functions* $V : Z^+ \times \Re^n \to \Re^+$, $a_2 \in K_\infty$, $\beta \in K^+$ *and constants* $\lambda \in (0,1)$, $p, K > 0$ *such that the following inequalities are satisfied for all* $(t,x,d) \in Z^+ \times \Re^n \times D$:

$$K|x|^p \le V(t,x) \le a_2\left(\beta(t)|x|\right) \tag{4.3}$$

$$V(t+1, F(t,x,d)) \le \lambda\, V(t,x) \tag{4.4}$$

Then zero is non-uniformly in time RGK-ES for equation (4.1) *with constant* $c = -\log(\lambda)/p$.

The following theorem is the main result of this section and provides sufficient conditions for the robust stabilization of system (1.1).

Theorem 4.3 *Consider system* (1.1) *under hypothesis* (*H*2) *with* $u \in \Re, D \subset \Re^l$ *being compact with* $0 \in D$ *and suppose that there exist continuous functions* $f_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, ..., n)$ *with* $f_i(t,0) = 0$ *for all* $t \in Z^+$, *such that equation* (1.3) *holds. Furthermore, suppose that there exist continuous functions* $k_i : Z^+ \times \Re^i \to \Re$ $(i = 1, \ldots, n)$ *with* $k_i(t,0) = 0$ *for all* $t \in Z^+$ *such that the following identities hold for all* $(t,x) \in Z^+ \times \Re^n$:

$$f_1(t, x_1, k_1(t, x_1)) = 0 \tag{4.5}$$

$$\begin{aligned} &f_i(t, x_1, \ldots, x_i, k_i(t, x_1, \ldots, x_i)) = k_{i-1}(t+1, f_1(t, x_1, x_2), \ldots, f_{i-1}(t, x_1, \ldots, x_i)), \\ &\quad \textit{for} \quad i = 2, \ldots, n \end{aligned} \tag{4.6}$$

Consider the following vector fields defined on $\Re^+ \times \Re^n$*:*

$$\tilde{F}(t,x) := F(t,x,0,k_n(t,x)) \tag{4.7}$$

$$F^{(0)}(t,x) := x \tag{4.8a}$$

$$F^{(i)}(t,x) := \tilde{F}(t+i-1, F^{(i-1)}(t,x)), \quad \text{for} \quad i \geq 1 \tag{4.8b}$$

Let $p \in C^0(\Re^n;\Re^+)$ *a positive definite function with* $p(0)=0$ *that satisfies* $p(x) \geq K|x|$ *for all* $x \in \Re^n$ *for certain constant* $K>0$ *and let a pair of constants* $\gamma > 1$, $\lambda \in (0,1)$. *Let* $D(\gamma,\lambda) \subseteq D$ *the set of all* $d \in D$ *that satisfies the following property:*

$$\sum_{i=0}^{n-1} \gamma^i p\big(F^{(i)}(t+1, F(t,x,d,k_n(t,x)))\big) \leq \lambda \sum_{i=0}^{n-1} \gamma^i p\big(F^{(i)}(t,x)\big), \quad \forall\, (t,x) \in Z^+ \times \Re^n \tag{4.9}$$

Then the following statements hold:

(i) *For every pair* $\gamma > 1$, $\lambda \in (0,1)$ *with* $\lambda\gamma \geq 1$, *the set* $D(\gamma,\lambda) \subseteq D$ *is a non-empty compact set with* $0 \in D(\gamma,\lambda)$.
(ii) *For every pair* $\gamma > 1$, $\lambda \in (0,1)$ *with* $\lambda\gamma \geq 1$, *zero is non-uniformly in time RGK-ES with constant* $c = -\log(\lambda)$ *for the closed-loop system* (1.1) *with* $u(t) = k_n(t,x(t))$ *and* $d(t) \in D(\gamma,\lambda)$.

The main idea that lies behind the proof of Theorem 4.3 is to construct a continuous feedback that guarantees the so-called "dead-beat property of order n" (see [24]) for the nominal system (1.1) with $d=0$. Then by making use of a CLF for the nominal closed-loop system, we establish RGK-ES for disturbances that belong to an appropriate set, namely, the set $D(\gamma,\lambda) \subseteq D$. The proof of Theorem 4.3 is based on the following lemma, which is similar to Theorem 3.2 in [24].

LEMMA 4.4 (FINITE-TIME STABILIZATION AND EXPLICIT CONSTRUCTION OF CLFS FOR TRIANGULAR SINGLE INPUT SYSTEMS). *Consider the single input discrete-time system*

$$\begin{gathered} x(t+1) = F(t,x(t),u(t)) \\ u(t) \in \Re, x(t) := (x_1(t),\ldots,x_n(t)) \in \Re^n, t \in Z^+ \end{gathered} \tag{4.10a}$$

where

$$F(t,x,u) = \big(f_1(t,x_1,x_2), f_2(t,x_1,x_2,x_3), \ldots, f_n(t,x_1,\ldots,x_n,u)\big)' \tag{4.10b}$$

for certain continuous mappings $f_i : Z^+ \times \Re^{i+1} \to \Re$ $(i=1,...,n)$ *with* $f_i(t,0)=0$ *for all* $t \in Z^+$. *Suppose that there exist continuous functions* $k_i : Z^+ \times \Re^i \to \Re$ $(i=1,...,n)$ *with* $k_i(t,0)=0$ *for all* $t \in Z^+$ *such that the identities* (4.5), (4.6) *hold for all* $(t,x) \in Z^+ \times \Re^n$. *Then zero is non-uniformly in time RGK-ES for the closed-loop system* (4.10*a*) *with* $u(t) = k_n(t,x(t))$, *namely, the system*

$$x(t+1) = \tilde{F}(t,x(t)) \tag{4.11}$$

where

$$\tilde{F}(t,x) := F(t,x,k_n(t,x)) \tag{4.12}$$

Moreover, the closed-loop system (4.10a) *with* $u(t) = k_n(t,x(t))$ *(system (4.11)), has the dead-beat property of order n, i.e. for every* $(t_0,x_0) \in Z^+ \times \Re^n$, *the unique solution* $x(t)$ *of the closed-loop system (4.10a) with* $u(t) = k_n(t,x(t))$ *and initial condition* $x(t_0) = x_0$ *satisfies:*

$$x(t) = 0, \quad \text{for all} \quad t \geq t_0 + n \tag{4.13}$$

Furthermore, let $p \in C^0(\Re^n; \Re^+)$ *a positive definite function with* $p(0) = 0$ *that satisfies* $p(x) \geq K|x|$ *for all* $x \in \Re^n$ *for certain constant* $K > 0$ *and the vector fields* $(t, x) \in Z^+ \times \Re^n \to F^{(i)}(t, x) \in \Re^n$ *defined by* (4.8a *and* b) *with* $\tilde{F}$ *defined by equation* (4.12). *Then for every* $\gamma > 1$ *the continuous function* $V_\gamma : Z^+ \times \Re^n \to \Re^+$ *defined by:*

$$V_\gamma(t, x) := \sum_{i=0}^{n-1} \gamma^i p\big(F^{(i)}(t, x)\big) \tag{4.14}$$

is a CLF for equation (4.10a). *Particularly, for every* $\gamma > 1$ *there exist functions* $a_2 \in K_\infty$ *and* $\beta \in K^+$ *such that*

$$K|x| \leq V_\gamma(t, x) \leq a_2\big(\beta(t)|x|\big), \quad \forall\, (t, x) \in Z^+ \times \Re^n \tag{4.15a}$$

$$V_\gamma(t + 1, F(t, x, u)) \leq \Psi_\gamma(t, x, u), \quad \forall\, (t, x, u) \in Z^+ \times \Re^n \times \Re \tag{4.15b}$$

$$\inf_{u \in \Re^m} \Psi_\gamma(t, x, u) \leq \Psi_\gamma(t, x, k_n(t, x)) \leq \frac{1}{\gamma} V_\gamma(t, x), \quad \forall\, (t, x) \in Z^+ \times \Re^n \tag{4.15c}$$

where the function $\Psi_\gamma : \Re^+ \times \Re^n \times \Re \to \Re^+$ *is quasi-convex with respect to* $u \in \Re$ *with* $\Psi_\gamma(t, 0, 0) = 0$ *for all* $t \in Z^+$ *and is defined by:*

$$\Psi_\gamma(t, x, u) := \sup\big\{V_\gamma(t + 1, F(t, x, k_n(t, x) + v); |v| \leq |u - k_n(t, x)|\big\} \tag{4.16}$$

Proof. By definition of the vector fields $F^{(i)}(t, x)$, we notice that for every $(t_0, x_0) \in Z^+ \times \Re^n$ the solution of equation (4.11) with initial condition $x(t_0) = x_0$ satisfies $x(t_0 + i) = F^{(i)}(t_0, x_0)$ for all $i \geq t$. In order to prove that system (4.11) satisfies the dead-beat property of order n, it suffices to show that $F^{(n)}(t, x) \equiv 0$ for all $(t, x) \in Z^+ \times \Re^n$. Using induction arguments, it is established that

$$F^{(i)}(t + 1, \tilde{F}(t, x)) = F^{(i+1)}(t, x) \quad \text{and} \quad F^{(i)}(t, 0) := 0, \quad \text{for} \quad \text{all} \quad i \geq 0 \tag{4.17}$$

The proof of the above relations is easy and is left to the reader. Define the sets $S^{(i)}(t) \subseteq \Re^n$ for $t \in Z^+$ by the following formulae:

$$S^{(0)}(t) := \Re^n \tag{4.18a}$$

$$S^{(i)}(t) := \big\{x \in S^{(i-1)}(t) : x_{n-i+1} = k_{n-i}(t, x_1, ..., x_{n-i})\big\}, \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{4.18b}$$

$$S^{(i)}(t) := \{0\}, \quad \text{for} \quad i \geq n \tag{4.18c}$$

where $k_i : \Re^+ \times \Re^i \to \Re (i = 1, \ldots, n)$ are the functions involved in equations (4.5) and (4.6). Notice that the definitions of the sets $S^{(i)}(t)$ above imply that for $1 \leq i \leq n - 1$:

$$S^{(i)}(t) := \{x \in \Re^n : x_n = k_{n-1}(t, x_1, \ldots, x_{n-1}), \ldots, x_{n-i+1} = k_{n-i}(t, x_1, \ldots, x_{n-i})\} \tag{4.19}$$

Next, we make the following claim.

Claim: $F^{(i)}(t, x) \in S^{(i)}(t + i)$, for all $i \geq 0$.

Clearly, definitions (4.8a) and (4.18a) implies that the above claim is true for $i = 0$. In order to prove the above claim, by virtue of definition (4.8b), it suffices to prove the following implication:

" If $x \in S^{(i)}(t + i)$ then $\tilde{F}(t + i, x) \in S^{(i+1)}(t + i + 1)$"

Since $\tilde{F}(t,0)=0$ for all $t\geq 0$, it follows that the above implication is true for $i\geq n$. For the case $0\leq i\leq n-1$, the above implication is an immediate consequence of equation (4.17) and properties (4.5) and (4.6).

Notice that the previous claim and definition (4.18c) implies that:

$$F^{(n)}(t,x)\equiv 0, \quad \text{for} \quad \text{all} \quad (t,x)\in Z^+\times\Re^n \tag{4.20}$$

Thus, system (4.11) satisfies the dead-beat property of order n.

Let $\gamma>1$ and consider the function $V_\gamma:\Re^+\times\Re^n\to\Re^+$ defined by equation (4.14). Since all vector fields $F^{(i)}(t,x)(i\geq 0)$ defined by equation (4.8a and b) and p are continuous on $\Re^+\times\Re^n$ with $F^{(i)}(t,0)=0$, $p(0)=0$ for all $t\geq 0$, by virtue of Lemma 3.2 in [8], there exist $a_2\in K_\infty$ and $\beta\in K^+$ such that the right-hand side inequality (4.15a) holds. The left-hand side inequality (4.15a) is immediate consequence of definitions (4.8a) and (4.14) and the fact that $p(x)\geq K|x|$ for all $x\in\Re^n$. Inequality (4.15b) is immediate consequence of definition (4.16). We next prove inequality (4.15c). Notice that by virtue of definition (4.14) and property (4.17) we have for all $(t,x)\in Z^+\times\Re^n$:

$$\begin{aligned} V_\gamma(t+1,\tilde{F}(t,x)) &= \sum_{i=0}^{n-1}\gamma^i p\big(F^{(i)}(t+1,\tilde{F}(t,x))\big) = \sum_{i=0}^{n-1}\gamma^i p\big(F^{(i+1)}(t,x)\big) \\ &= \sum_{i=1}^{n}\gamma^{i-1}p\big(F^{(i)}(t,x)\big) \end{aligned}$$

Consequently, it follows from equation (4.20), definitions (4.12) and (4.16) and the above equality:

$$\begin{aligned} \Psi_\gamma(t,x,k_n(t,x)) &= V_\gamma(t+1,\tilde{F}(t,x)) = \frac{1}{\gamma}\sum_{i=1}^{n-1}\gamma^i p\big(F^{(i)}(t,x)\big) \leq \frac{1}{\gamma}\sum_{i=0}^{n-1}\gamma^i p\big(F^{(i)}(t,x)\big) \\ &= \frac{1}{\gamma}V_\gamma(t,x) \end{aligned}$$

We conclude that inequality (4.15c) holds. The proof of the fact that the function $\Psi_\gamma(t,x,u)$ is quasi-convex with respect to $u\in\Re$ is identical to the proof of implication (d) $\Rightarrow$ (b) of Proposition 3.6 and is omitted. It follows from Lemma 4.2 that zero is non-uniformly in time RGK-ES with constant $c=\log(\gamma)>0$ for the closed-loop system (4.10a) with $u(t)=k_n(t,x(t))$. The proof is complete. □

Proof of Theorem 4.3. Notice that for the case $d=0$, it follows from equation (1.3) that system (1.1) has the triangular structure (4.10). Thus, Lemma 4.4 holds and equation (4.9) for $d=0$ and $1/\gamma\leq\lambda<1$, is a consequence of inequality (4.15c). This proves that $D(\gamma,\lambda)\subseteq D$ is non-empty set with $0\in D(\gamma,\lambda)$. Compactness of statement $D(\gamma,\lambda)\subseteq D$ follows from compactness of D and continuity of all mappings involved in equation (4.9) with respect to d. We next prove statement (ii). Let the Lyapunov function $V_\gamma:Z^+\times\Re^n\to\Re^+$ defined by equation (4.14) and notice that inequality (4.9) in conjunction with definition (4.14) implies:

$$V_\gamma(t+1,F(t,x,d,k_n(t,x)))\leq\lambda V_\gamma(t,x), \quad \forall\,(t,x,d)\in Z^+\times\Re^n\times D(\gamma,\lambda) \tag{4.21}$$

It follows from equations (4.15) and (4.21) in conjunction with Lemma 4.2, that zero is non-uniformly in time RGK-ES with constant $c=-\log(\lambda)$ for the closed-loop system (1.1) with $u(t)=k_n(t,x(t))$ and $d(t)\in D(\gamma,\lambda)$. The proof is complete. □

We illustrate the result of Theorem 4.3 by presenting two examples. Both examples illustrate a trade-off between the convergence rate (c) and the size of the disturbance set $D(\gamma,\lambda)\subseteq D$. The first example is an application of Theorem 4.3 to a linear two-dimensional time-varying discrete-time system.

Example 4.5. Consider the linear planar time-varying system:

$$\begin{aligned} x_1(t+1) &= tx_1(t)+\left(1+\frac{d(t)}{(t+1)(t+3)}\right)x_2(t)\\ x_2(t+1) &= t^2x_2(t)+u(t)\\ &(x_1(t),x_2(t))\in\Re^2, u(t)\in\Re, d(t)\in D:=[-1,1], t\in Z^+ \end{aligned}\tag{4.22}$$

Notice that the dynamics of system (4.22) satisfy equation (1.3) with $f_1(t,x_1,x_2):=tx_1+x_2$ and $f_2(t,x_1,x_2,u):=t^2x_2+u$. Moreover, equations (4.5) and (4.6) are satisfied for $k_1(t,x_1):=-tx_1$ and $k_2(t,x_1,x_2):=-t^2x_2-t(t+1)x_1-(t+1)x_2$. Consequently, the vector fields $F^{(i)}(t,x)$ are defined by equation (4.8a and b):

$$F^{(0)}(t,x):=\left(x_1,x_2\right)',\ F^{(1)}(t,x):=\left(tx_1+x_2,-t(t+1)x-1(t+1)x_2\right)'$$

We select $p(x_1,x_2):=|x_1|+|x_2|$. Thus, inequality (4.9) is equivalent to the following inequality:

$$\begin{aligned}\left|tx_1+\left(1+\frac{d}{(t+1)(t+3)}\right)x_2\right|+(t+1)|tx_1+x_2|+\gamma\,|\mathrm{d}x_2|\le\lambda|x_1|+\lambda|x_2|\\ +\gamma\lambda(t+2)|tx_1+x_2|\end{aligned}\tag{4.23}$$

Clearly, inequality (4.23) is satisfied for all $(t,x_1,x_2)\in\Re^+\times\Re^2$ if $\lambda\gamma\ge 1$ and $|d|\le(3\lambda)/(1+3\gamma)$. Thus, we conclude that zero is non-uniformly RGK-ES with constant $c>0$, for the closed-loop system (4.22) with $u(t)=-t^2x_2(t)-t(t+1)x_1(t)-(t+1)x_2(t)$ and $d(t)\in D(\gamma,\lambda)$ for $\gamma=\exp(c)=1/\lambda$, i.e. for $d(t)\in\{d\in[-1,1]:|d|\le(3/(\exp(c)+3\exp(2c))\}$. Notice that larger values for the convergence constant $c>0$ give smaller values for the radius of the disturbance set $D(\gamma,\lambda)$.

Our second example is the application of Theorem 4.3 to an autonomous non-linear discrete-time system.

Example 4.6. Consider the non-linear planar autonomous system:

$$\begin{aligned} x_1(t+1) &= (1+d(t))|x_1(t)|-x_2^2(t)\\ x_2(t+1) &= x_2(t)+u(t)\\ &(x_1(t),x_2(t))\in\Re^2, u(t)\in\Re, d(t)\in D:=[-1,1], t\in Z^+ \end{aligned}\tag{4.24}$$

Notice that the dynamics of system (4.24) satisfies equation (1.3) with $f_1(t,x_1,x_2):=|x_1|-x_2^2$ and $f_2(t,x_1,x_2,u):=x_2+u$. Moreover, equations (4.5) and (4.6) are satisfied for $k_1(t,x_1):=|x_1|^{(1/2)}$ and $k_2(t,x_1,x_2):=-x_2+\left||x_1|-x_2^2\right|^{(1/2)}$. Consequently, the vector fields $F^{(i)}(t,x)$ are defined by equation (4.8a and b):

$$F^{(0)}(t,x):=\left(x_1,x_2\right)',$$

$$F^{(1)}(t,x) := \left(|x_1| - x_2^2, \left||x_1| - x_2^2\right|^{\frac{1}{2}} \right)'$$

We select $p(x_1,x_2) := |x_1| + |x_2|$. Thus, inequality (4.9) is equivalent to the following inequality:

$$\begin{aligned} &\left|(1+d)|x_1| - x_2^2\right| + \left||x_1| - x_2^2\right|^{\frac{1}{2}} + \gamma\left|\left|(1+d)|x_1| - x_2^2\right| - \left||x_1| - x_2^2\right|\right| \\ &\quad + \gamma\left|\left|(1+d)|x_1| - x_2^2\right| - \left||x_1| - x_2^2\right|\right|^{\frac{1}{2}} \\ &\quad \leq \lambda|x_1| + \lambda|x_2| + \gamma\lambda\left||x_1| - x_2^2\right| + \gamma\lambda\left||x_1| - x_2^2\right|^{\frac{1}{2}} \end{aligned} \tag{4.25}$$

Clearly, inequality (4.25) is satisfied for all $(x_1,x_2) \in \Re^2$ if $\lambda\gamma > 1$ and $|d| \leq \min\left\{\frac{\lambda}{1+\gamma}, \frac{\lambda^2}{\gamma^2}, \frac{(\gamma\lambda-1)^2}{\gamma^2}\right\}$. Thus, we conclude that zero is non-uniformly RGK-ES with constant $c > 0$, for the closed-loop system (4.24) with $u(t) = -x_2(t) + \left||x_1(t)| - x_2^2(t)\right|^{(1/2)}$ and $d(t) \in D(\gamma,\lambda)$ for $\gamma = \exp(c) + 1$ and $\lambda = \exp(-c)$, i.e. for $d(t) \in \{d \in [-1,1] : |d| \leq (1/(\exp(2c)(\exp(c)+1)^2))\}$. Notice again that larger values for the convergence constant $c > 0$ give smaller values for the radius of the disturbance set $D(\gamma,\lambda)$.

The following corollary provides sufficient conditions for robust partial state feedback stabilization of time-varying discrete-time systems with guaranteed exponential rate of convergence of the solutions of the closed-loop system to the equilibrium.

Corollary 4.7 (Partial State Feedback Stabilization). *Consider the single input discrete-time system:*

$$\begin{aligned} &w(t+1) = G(t, w(t), d(t), x(t), u(t)) \\ &x_i(t+1) = f_i(t, x_1(t), \ldots, x_i(t), x_{i+1}(t)) \quad i = 1, \ldots, n-1 \\ &x_n(t) \qquad = f_n(t, x(t), u(t)) \\ &x(t) : \qquad = (x_1(t), \ldots, x_n(t)) \in \Re^n, w(t) \in \Re^l, u(t) \in \Re, d(t) \in D, t \in Z^+ \end{aligned} \tag{4.26}$$

where $D \subset \Re^m$ is a compact set, $f_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, \ldots, n)$ and $G : Z^+ \times \Re^l \times D \times \Re^n \times \Re \to \Re^l$ are continuous mappings with $f_i(t,0) = 0$, $G(t,0,d,0,0) = 0$ for all $(t,d) \in Z^+ \times D$. Suppose that there exist continuous functions $k_i : Z^+ \times \Re^i \to \Re$ $(i = 1, \ldots, n)$ with $k_i(t,0) = 0$ for all $t \in Z^+$ such that the identities (4.5) *and* (4.6) *hold for all $(t,x) \in Z^+ \times \Re^n$. Moreover, suppose that $0 \in \Re^l$ is non-uniformly in time RGK-ES with constant $c > 0$ for the system:*

$$\begin{aligned} &w(t+1) = G(t, w(t), d(t), 0, 0) \\ &w(t) \in \Re^l, d(t) \in D, t \in Z^+ \end{aligned} \tag{4.27}$$

Then $0 \in \Re^l \times \Re^n$ is non-uniformly in time RGK-ES with constant $c > 0$ for the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$.

Proof. Since zero is non-uniformly in time RGK-ES with constant $c > 0$ for equation (4.27), there exist functions $a \in K_\infty$, $\beta \in K^+$ such that for every $\left(t_0, w_0, \{d(t)\}_0^\infty\right) \in Z^+ \times \Re^l \times M_D$ the solution $w(t)$ of equation (4.27) with initial condition $w(t_0) = w_0$ and corresponding to $\{d(t)\}_0^\infty \in M_D$ satisfies the estimate:

$$|w(t)| \leq \exp\left(-c(t-t_0)\right) a\left(\beta(t_0)|w_0|\right), \quad \forall\, t \geq t_0 \tag{4.28}$$

Moreover, by virtue of Lemma 4.3, the component $x(t)$ of the solution of the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$ satisfies equation (4.13). Thus, we obtain that for every $\left(t_0, x_0, w_0, \{d(t)\}_0^\infty\right) \in Z^+ \times \Re^n \times \Re^l \times M_D$, the solution $w(t)$ of the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$, initial condition $(x(t_0), w(t_0)) = (x_0, w_0)$ and corresponding to $\{d(t)\}_0^\infty \in M_D$ satisfies the estimate:

$$|w(t)| \leq \exp\left(-c(t - t_0 - n)\right) a\left(\beta\left(t_0 + n\right) |w(t_0 + n)|\right), \quad \forall\, t \geq t_0 + n \tag{4.29}$$

By virtue of continuity of the mappings $f_i : Z^+ \times \Re^{i+1} \to \Re$ $(i = 1, \ldots, n)$, $k_n : Z^+ \times \Re^n \to \Re$, $G : Z^+ \times \Re^l \times D \times \Re^n \times \Re \to \Re^l$ and the facts that $D \subset \Re^m$ is a compact set, $f_i(t, 0) = 0$, $G(t, 0, d, 0, 0) = 0$, $k_n(t, 0) = 0$ for all $(t, d) \in Z^+ \times D$, we may prove that the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$ is robustly forward complete (RFC, see [8]) and that $0 \in \Re^l \times \Re^n$ is a robust equilibrium point for the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$. The proof of this observation is made by using Remark 3.2(i) and following the same methodology with the proof of Lemma 3.8 (see Appendix). Details are left to the reader. Using Lemma 3.5 in [8], we guarantee the existence of $a' \in K_\infty$, $\mu \in K^+$ such that for every $(t_0, x_0, w_0, \{d(t)\}_0^\infty) \in Z^+ \times \Re^n \times \Re^l \times M_D$ the solution $(x(t), w(t))$ of the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$, initial condition $(x(t_0), w(t_0)) = (x_0, w_0)$ and corresponding to $\{d(t)\}_0^\infty \in M_D$ satisfies the estimate:

$$|(x(t), w(t))| \leq \mu(t)\, a'(|(x_0, w_0)|), \quad \forall t \geq t_0 \tag{4.30}$$

Combining estimate (4.29) with (4.30) we obtain:

$$\begin{aligned} |w(t)| \leq \exp\left(-c(t - t_0)\right) &\left[\exp(nc) a\left(M\left(t_0\right) a'\left(|(x_0, w_0)|\right)\right) + M(t_0) a'\left(|(x_0, w_0)|\right)\right] \\ &\forall\, t \geq t_0, \end{aligned} \tag{4.31}$$

where $M(t_0) := \exp\left(c(t_0 + n)\right) \max_{t_0 \leq t \leq t_0 + n} (1 + \mu(t))(1 + \beta(t))$. Thus, using Lemma 2.3 in [7], we conclude that there exist functions $\tilde{a} \in K_\infty$, $\tilde{\beta} \in K^+$ such that for every $(t_0, x_0, w_0, \{d(t)\}_0^\infty) \in Z^+ \times \Re^n \times \Re^l \times M_D$, the solution $(x(t), w(t))$ of the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$, initial condition $(x(t_0), w(t_0)) = (x_0, w_0)$ and corresponding to $\{d(t)\}_0^\infty \in M_D$ satisfies the estimate:

$$|w(t)| \leq \exp\left(-c(t - t_0)\right) \tilde{a}\left(\tilde{\beta}(t_0) |(x_0, w_0)|\right), \quad \forall\, t \geq t_0 \tag{4.32}$$

By virtue of Lemma 4.3, the component $x(t)$ of the solution of the closed-loop system (4.26) with $u(t) = k_n(t, x(t))$ satisfies a similar estimate of the form (4.32). The proof is complete. □

Example 4.8. Consider the following planar system:

$$\begin{aligned} w(t+1) &= \exp(-c) w(t) + d(t)\, g(t, x(t), u(t)) \\ x(t+1) &= f(t, x(t)) + u(t) \\ (w(t), x(t)) &\in \Re \times \Re, \quad d(t) \in [-1, 1], \quad u(t) \in \Re, \quad t \in Z^+ \end{aligned} \tag{4.33}$$

where $c > 0$ is a constant, $f : Z^+ \times \Re \to \Re$ and $g : Z^+ \times \Re^2 \to \Re$ are continuous mappings with $f(t, 0) = g(t, 0, 0) = 0$ for all $t \in Z^+$. The above system has the form (4.26) and furthermore zero is RGK-ES with constant $c > 0$ for the subsystem obtained for $x = u = 0$: $w(t+1) = \exp(-c) w(t)$, $w(t) \in \Re, t \in Z^+$. Thus, Corollary 4.7 guarantees that zero will be non-uniformly in time RGK-ES with constant $c > 0$ for the closed-loop system (4.33) with the partial state feedback law $u(t) = -f(t, x(t))$.

5. Conclusions

Necessary and sufficient Lyapunov-like conditions for non-uniform in time robust output stabilization of discrete-time systems are given. Particularly, it is proved that for a finite-dimensional discrete-time control system there exists a (continuous) output stabilizing feedback if and only if there exists a (strong) OCLF. Moreover, methodologies for the construction of continuous robust feedback stabilizers are presented.

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References

[1] Albertini, F. and Sontag, ED., 1999, Continuous control-Lyapuonv functions for asymptotically controllable time-varying systems. *International Journal of Control*, **72**, 1163–1173.

[2] Amicucci, G.L., Monaco, S. and Normand-Cyrot, D., 1997, Control Lyapunov stabilization of affine discrete-time systems. In *Proceedings of the 36th Conference on Decision and Control,* San Diego, California, USA, pp. 923–924.

[3] Artstein, Z., 1983, *Stabilization with Relaxed Controls.* TMA 7, *Nonlinear Analysis*, 1163–1173.

[4] Aubin, J.P. and Frankowska, H., 1990, *Set-Valued Analysis* (Boston: Birkhauser).

[5] Freeman, R.A. and Kokotovic, PV., 1996, Inverse optimality in robust stabilization. *SIAM Journal on Control and Optimization*, **34**(4), 1365–1391.

[6] Jiang, Z.P. and Wang, Y., 2002, A converse Lyapunov theorem for discrete-time systems with disturbances. *Systems and Control Letters*, **45**(1), 49–58.

[7] Karafyllis, I. and Tsinias, J., 2003, A converse Lyapunov theorem for non-uniform in time global asymptotic stability and its application to feedback stabilization. *SIAM Journal on Control and Optimization*, **42**(3), 936–965.

[8] Karafyllis, I., 2004, The non-uniform in time small-gain theorem for a wide class of control systems with outputs. *European Journal of Control*, **10**, 307–323.

[9] Karafyllis, I., 2006, Non-uniform in time robust global asymptotic output stability for discrete-time systems. *International Journal of Robust and Nonlinear Control*, **16**, 191–214.

[10] Karafyllis, I. and Kotsios, S., 2004, Conditions for global asymptotic stabilization of discrete-time systems. In *Proceedings of NOLCOS*, Stuttgart, Germany.

[11] Kazakos, D. and Tsinias, J., 1994, The input to state stability condition and global stabilization of discrete-time systems. *IEEE Transactions on Automatic Control*, **39**(10), 2111–2113.

[12] Kellett, C.M. and Teel, AR., 2004, Discrete-time asymptotic controllability implies smooth control-Lyapunov function. *Systems and Control Letters*, **52**(5), 349–359.

[13] Kellett, C.M. and Teel, AR., 2004, Smooth Lyapunov functions and robustness of stability for difference inclusions. *Systems and Control Letters*, **52**(5), 395–405.

[14] Kellett, C.M. and Teel, A.R., 2003, Results on converse Lyapunov theorems for difference inclusions. In *Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii, USA, pp. 3627–3632.

[15] Kellett, C.M. and Teel, AR., 2003, Results on Discrete-time Control-Lyapunov Functions In *Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii, USA, pp. 5961–5966.

[16] Kellett, C.M. and Teel, AR., 2004, Weak converse Lyapunov theorems and control-Lyapunov functions. *SIAM Journal on Control and Optimization*, **42**(6), 1934–1959.

[17] Kloeden, P.E. and Schmalfuss, B., 1998, Asymptotic behaviour of nonautonomous difference inclusions. *Systems and Control Letters*, **33**, 275–280.

[18] Lefeber, E., Robertsson, A. and Nijmeijer, H., 1999, Linear controllers for tracking chained-form systems. In: D. Aeyels, F. Lamnabhi-Lagarrigue and A. van der Schaft (Eds) *Stability and Stabilization of Nonlinear Systems*, Lecture Notes in Control and Information Sciences 246 (London: Springer-Verlag), pp. 183–199.

[19] Lin, Y. and Sontag, ED., 1991, A universal formula for stabilization with bounded controls. *Systems and Control Letters*, **16**, 393–397.

[20] Nesic, D. and Bastin, G., 1999, Stabilizability and dead-beat controllers for two classes of Wiener-Hammerstein models. *IEEE Transactions on Automatic Control*, **44**(11), 2068–2072.
[21] Nesic, D. and Loria, A., 2003, On uniform boundedness of parameterized discrete-time cascades with decaying inputs: applications to cascades. *Systems and Control Letters*, **49**(3), 163–174.
[22] Nesic, D. and Loria, A., 2004, On uniform asymptotic stability of time-varying parameterized discrete-time cascades. *IEEE Transactions on Automatic Control*, **49**(6), 875–887.
[23] Nesic, D. and Teel, A., 2004, A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control*, **49**(7), 1103–1122.
[24] Simoes, C., Nijmeijer, H. and Tsinias, J., 1996, Nonsmooth stabilizability and feedback linearization of discrete-time nonlinear systems. *International Journal of Robust and Nonlinear Control*, **6**(3), 171–188.
[25] Sontag, E.D., 1989, A "Universal" construction of Artstein's theorem on nonlinear stabilization. *Systems and Control Letters*, **13**, 117–123.
[26] Sontag, E.D., 1998, Comments on integral variants of ISS. *Systems and Control Letters*, **34**, 93–100.
[27] Tsinias, J., 1989, Sufficient Lyapunov like conditions for stabilization. *Mathematics of Control, Signals, and Systems*, **2**, 343–357.
[28] Tsinias, J., Kotsios, S. and Kalouptsidis, N., 1989, Topological dynamics of discrete-time systems. In *Proc. of MTNS*, II (Basle: Birkhauser), pp. 457–463.
[29] Tsinias, J., 1989, Stabilizability of discrete-time nonlinear systems. *IMA Journal of Mathematical Control and Information*, 135–150.
[30] Tsinias, J. and Kalouptsidis, N., 1990, Output feedback stabilization of discrete-time control systems. *IMA Journal of Mathematical Control and Information*, 257–268.

Appendix

Proof of Lemma 3.8. It suffices to show that system (3.4) is non-uniformly in time robustly globally asymptotically output stable (RGAOS, see [8]) and $0 \in \Re^n$ is a robust equilibrium point for equation (3.4) (see [8] for the precise definition of a robust equilibrium point). Then by virtue of Theorem 3.6 in [8], there exist functions $\sigma \in KL$, $\beta, \mu \in K^+$, such that for every $\left(t_0, x_0, \{d(t)\}_0^\infty\right) \in Z^+ \times \Re^n \times M_D$, the unique solution $x(t)$, $Y(t) = H(t, x(t))$ of system (3.4) with initial condition $x(t_0) = x_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$, satisfies estimate (3.1). In order to show that system (3.4) is non-uniformly in time RGAOS, it suffices to show that:

(1) System (3.4) is RFC (see [8])
(2) $0 \in \Re^n$ is a robust equilibrium point for (3.4)
(3) The output attractivity property for system (3.4) is satisfied, i.e. for every $\varepsilon > 0$, $T \in Z^+$ and $R \geq 0$, there exists a $\tau := \tau(\varepsilon, T, R) \in Z^+$, such that:

$$|x_0| \leq R, t_0 \in [0, T] \Rightarrow |H(t, x(t, t_0, x_0; d))| \leq \varepsilon, \quad \forall\, t \in [t_0 + \tau, +\infty), \quad \forall\, d \in M_D$$

where $x(t, t_0, x_0; d)$ denotes the unique solution of system (3.4) with initial condition $x(t_0) = x_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$.

Then by virtue of Lemma 3.3 in [8], it follows that system (3.4) is non-uniformly in time RGAOS. Concerning the proof of (1), we notice that by virtue of Definition 2.2 in [8] it suffices to show that for every $r \geq 0$, $T \geq 0$, it holds that $\sup\{|x(t, t_0, x_0; d)|;\ s \in [0, T], |x_0| \leq r, t_0 \in [0, T], d \in M_D\} < +\infty$. Particularly, this follows by considering arbitrary $r \geq 0$, $T \in Z^+$, then defining recursively the sequence of sets in $\Re^n$ by $A(k) := f([0, 2T] \times A(k-1) \times D)$ for $k = 1, ..., T$ with $A(0) := \{x \in X; |x| \leq r\}$, which are bounded by virtue of hypothesis (A1) and finally noticing that $\{x(t_0 + k, t_0, x_0; d); |x_0| \leq r, t_0 \leq T, k \leq T, d \in M_D\} \subseteq A(k)$ for all $k = 0, ..., T$, where $x(t, t_0, x_0; d)$ denotes the unique solution of equation (3.4) initiated from $x_0 \in \Re^n$ at time $t_0 \geq 0$ and corresponding to input $d \in M_D$.

Concerning the proof of (2), we notice that by virtue of Definition 2.3 in [8] it suffices to show that for every $\varepsilon > 0$, $N \in Z^+$ and $T \geq 0$ there exists $\bar{\delta} := \bar{\delta}(\varepsilon, N, T) \in (0, \varepsilon]$ such that:

$$|x_0| \leq \bar{\delta}, t_0 \in [0, T] \Rightarrow \sup\{|x(t, t_0, x_0; d)|; t_0 \leq t \leq t_0 + N, d \in M_D\} \leq \varepsilon$$

We prove this fact by induction on $N \in Z^+$. First notice that the fact holds for $N = 0$ (by selecting $\bar{\delta}(\varepsilon, 0, T) = \varepsilon$). We next assume that the fact holds for some $N \in Z^+$ and we prove it for the next integer $N + 1$. In order to have $|x(t_0 + N + 1, t_0, x_0; d)| \leq \varepsilon$, by virtue of hypothesis (A1) it suffices to have:

$$|x(t_0 + N, t_0, x_0; d)| \leq R(\varepsilon, N, T) := \frac{a^{-1}(\varepsilon)}{\max\{\gamma(t); 0 \leq t \leq T + N\}}$$

It follows that the selection $\bar{\delta}(\varepsilon, N + 1, T) := \min\left\{\bar{\delta}(\varepsilon, N, T), \bar{\delta}(R(\varepsilon, N, T), N, T)\right\} > 0$ guarantees that $\sup\{|x(t, t_0, x_0; d)|; t_0 \leq t \leq t_0 + N, d \in M_D\} \leq \min\{\varepsilon, R(\varepsilon, N, T)\}$ and $\sup\{|x(t_0 + N + 1, t_0, x_0; d)|; d \in M_D\} \leq \varepsilon$, for all $|x_0| \leq \bar{\delta}, t_0 \in [0, T]$.

Concerning the proof of (3), we notice that by virtue of inequality equation (3.5a), it suffices to show that for every $\varepsilon > 0$, $R \geq 0$ there exists $\tau := \tau(\varepsilon, R) \in Z^+$ such that the following implication holds:

$$V(t_0) \leq R \Rightarrow V(t) \leq \varepsilon, \forall t \geq t_0 + \tau, \quad \forall\, d \in M_D \tag{A1}$$

where $V(t) := V(t, x(t, t_0, x_0; d))$.

First we prove inductively the following fact:

$$V(t) \leq V(t_0) + a^{-1}(M) + M, \quad \forall\, t \geq t_0 \in Z^+, \quad \forall\, d \in M_D \tag{A2}$$

where $M := \sup_{t \geq 0} q(t)$. Indeed, notice that (A2) holds for $t = t_0$. Suppose that (A2) holds for some $t \in Z^+$ with $t \geq t_0$. Consider the cases:

* if $a(V(t)) \geq M$ then equation (3.5b) implies $V(t + 1) \leq V(t)$ and consequently (A2) holds for $t + 1$.
* if $a(V(t)) < M$ or equivalently if $V(t) < a^{-1}(M)$ then equation (3.5b) implies $V(t + 1) \leq V(t) + M < a^{-1}(M) + M$ and consequently (A2) holds for $t + 1$.

Let arbitrary $\varepsilon > 0$ and let $s(\varepsilon) > 0$ the unique solution of the equation $a^{-1}(2s) + s = \varepsilon$. Let $t_1(\varepsilon) \in Z^+$ such that $q(t) \leq s(\varepsilon)$ for all $t \geq t_1(\varepsilon)$. Clearly, by virtue of equation (3.5b), we have

$$V(t + 1) \leq V(t) - a(V(t)) + s(\varepsilon), \forall t \geq t_0 + t_1(\varepsilon) \tag{A3}$$

Next we prove the following claim: if $V(t) \leq \varepsilon$ for some $t = T \in Z^+$ with $T \geq t_0 + t_1(\varepsilon)$ then $V(t) \leq \varepsilon$ holds for all $t \geq T$. Consider the cases:

* if $a(V(t)) \geq s(\varepsilon)$ then (A3) implies $V(t + 1) \leq V(t)$ and consequently $V(t) \leq \varepsilon$ holds for $t + 1$.
* if $a(V(t)) < s(\varepsilon)$ or equivalently if $V(t) < a^{-1}(s(\varepsilon))$ then (A3) implies $V(t + 1) \leq V(t) + s(\varepsilon) < a^{-1}(s(\varepsilon)) + s(\varepsilon) \leq a^{-1}(2s(\varepsilon)) + s(\varepsilon) = \varepsilon$ and consequently $V(t) \leq \varepsilon$ holds for $t + 1$.

We are now in a position to show that implication (A1) holds for $\tau(\varepsilon, R) := 1 + t_1(\varepsilon) + [(R + a^{-1}(M) + M)/(s(\varepsilon))]$, where $M := \sup_{t \geq 0} q(t)$. The proof of implication (A1) is made by contradiction. Suppose that there exists $\varepsilon > 0$, $R \geq 0$, $(t_0, x_0, d) \in Z^+ \times \Re^n \times M_D$ with $V(t_0) \leq R$ and there exists $T \in Z^+$ with $T \geq t_0 + \tau(\varepsilon, R)$ such that $V(T) > \varepsilon$. By virtue of the previous claim, this implies that $V(t) > \varepsilon = a^{-1}(2s(\varepsilon)) + s(\varepsilon)$ for all $t = t_0 + t_1(\varepsilon), \ldots, T$. Consequently, we have $-a(V(t)) + s(\varepsilon) \leq -s(\varepsilon)$ for all $t = t_0 + t_1(\varepsilon), \ldots, T$. Thus, we obtain from (A3):

$$V(t+1) \le V(t) - s(\varepsilon), \quad \text{for all} \quad t = t_0 + t_1(\varepsilon), \ldots, T. \tag{A4}$$

Clearly, inequality (A4) implies that $V(T) \le V(t_0 + t_1(\varepsilon)) - s(\varepsilon)(T - t_0 - t_1(\varepsilon))$ and this estimate in conjunction with (A2) (which implies $V(t_0 + t_1(\varepsilon)) \le R + a^{-1}(M) + M$) and our assumption $T \ge t_0 + t_1(\varepsilon) + (R + a^{-1}(M) + M)/(s(\varepsilon))$ gives $V(T) \le 0$. Clearly, this implication is in contradiction with the assumption $V(T) > \varepsilon$. The proof is complete. □

Proof of Proposition 3.6. (b) ⇒ (d) Suppose that equation (1.1) admits a strong OCLF. Without loss of generality, we may assume that the function $q \in C^0(Z^+; \Re^+)$ with $\lim_{t \to +\infty} q(t) = 0$ involved in equation (3.3d) is positive for all $t \in Z^+$. We proceed by noticing some facts.

Fact I: For all $(t, x_0) \in Z^+ \times \Re^n$, there exists $u_0 \in \Re^m$ and a neighbourhood $\mathbf{N}(t, x_0) \subset \Re^n$, such that

$$x \in \mathbf{N}(t, x_0) \Rightarrow \Psi(t, x, u_0) \le V(t, x) - \rho(V(t, x)) + 4q(t) \tag{A5}$$

Moreover, if $x_0 = 0$, then we may select $u_0 = 0$.

Proof of Fact I: By virtue of equation (3.3d) and since $q(t) > 0$ for all $t \in Z^+$ it follows that for all $(t, x_0) \in Z^+ \times \Re^n$, there exists $u_0 \in \Re^m$ such that

$$\Psi(t, x_0, u_0) \le V(t, x_0) - \rho(V(t, x_0)) + 2q(t) \tag{A6}$$

If $x_0 = 0$ (and since $\Psi(t, 0, 0) = 0$ for all $t \in Z^+$) then we may select $u_0 = 0$. Since the mapping $x \to \Psi(t, x, u)$ is upper semi-continuous and the mapping $x \in \Re^n \to V(t, x) - \rho(V(t, x))$ is continuous, there exists a neighbourhood $\mathbf{N}(t, x_0) \subset \Re^n$ around x_0 such that for all $x \in \mathbf{N}(t, x_0)$:

$$\begin{aligned} &\Psi(t, x, u_0) \le \Psi(t, x_0, u_0) + q(t) \\ &V(t, x_0) - \rho(V(t, x_0)) \le V(t, x) - \rho(V(t, x)) + q(t) \end{aligned} \tag{A7}$$

Inequalities (A6) and (A7) imply property (A5).

Fact II: For each fixed $t \in Z^+$ there exists a family of open sets $(\Omega_j^{(t)})_{j \in J(t)}$ with $\Omega_j^{(t)} \subset \Re^n \backslash \{0\}$ for all $j \in J(t)$, which consists a locally finite open covering of $\Re^n \backslash \{0\}$ and a family of points $(u_j^{(t)})_{j \in J(t)}$ with $u_j^{(t)} \in \Re^m$ for all $j \in J(t)$, such that

$$x \in \Omega_j^{(t)} \Rightarrow \Psi\left(t, x, u_j^{(t)}\right) \le V(t, x) - \rho(V(t, x)) + 4q(t) \tag{A8}$$

This fact is an immediate consequence of Fact I.

By virtue of Fact II and standard partition of unity arguments, it follows that for each fixed $t \in Z^+$ there exists a family of smooth functions $\theta_0^{(t)} : \Re^n \to [0, 1]$, $\theta_j^{(t)} : \Re^n \to [0, 1]$, with $\theta_j^{(t)}(x) = 0$ if $x \notin \Omega_j^{(t)} \subset \Re^n \backslash \{0\}$ and $\theta_0^{(t)}(x) = 0$ if $x \notin \mathbf{N}(t, 0)$, where $\mathbf{N}(t, 0) \subset \Re^n$ is the neighbourhood provided by Fact I for $x_0 = 0$ and $u_0 = 0$, $\theta_0^{(t)}(x) + \sum_{j \in J(t)} \theta_j^{(t)}(x)$ being locally finite and $\theta_0^{(t)}(x) + \sum_j \theta_j^{(t)}(x) = 1$ for all $x \in \Re^n$. We define for each fixed $t \in Z^+$:

$$k(t, x) := \sum_{j \in J(t)} \theta_j^{(t)}(x) u_j^{(t)} \tag{A9}$$

Clearly, k as defined by (A9) is a smooth function. Notice that $0 \notin \Omega_j^{(t)}$ for all $j \in J$ and consequently by definition (A9) we have $k(t, 0) = 0$ for all $t \in Z^+$. It also follows from the fact that Ψ is quasi-convex with respect to $u \in \Re^m$ and definition (A9) that:

$$\Psi(t,x,k(t,x)) = \Psi\left(t,x,\sum_{j\in J'(t,x)} \theta_j^{(t)}(x)u_j^{(t)}\right) \leq \max_{j\in J'(t,x)} \left\{\Psi\left(t,x,u_j^{(t)}\right)\right\} \tag{A10}$$

where $J'(t,x) = \left\{j \in J(t) \cup \{0\};\ \theta_j^{(t)}(x) \neq 0\right\}$ is a finite set. For each $j \in J'(t,x)$ we obtain that $x \in \Omega_j^{(t)}$ or $x \in \mathbf{N}(t,0)$. Consequently, by virtue of (A8) or (A5) we have that $\Psi\left(t,x,u_j^{(t)}\right) \leq V(t,x) - \rho(V(t,x)) + 4q(t)$, for all $j \in J'(t,x)$. Combining the previous inequality with inequality (3.3c), we conclude that the following property holds:

$$\begin{gathered} V(t+1,F(t,x,d,k(t,x))) \leq \Psi(t,x,k(t,x)) \leq V(t,x) - \rho(V(t,x)) + 4q(t), \\ \forall\,(t,x,d) \in Z^+ \times \Re^n \times D \end{gathered} \tag{A11}$$

It follows from (A11), Remark 3.2(i) and Lemma 3.8 that system (1.1) is non-uniformly in time continuously robustly globally output stabilizable and k defined by equation (A9) is a non-uniform in time continuous robust global asymptotic output stabilizer.

(a) $\Rightarrow$ (c) Suppose that equation (1.1) admits an OCLF. Without loss of generality, we may assume that the function $q \in C^0(Z^+;\Re^+)$ with $\lim_{t\to+\infty} q(t) = 0$ involved in equation (3.3b) is positive for all $t \in Z^+$. By property (ii) of Definition 3.4 it follows that for every $(t,x) \in Z^+ \times \Re^n$, there exists $u(t,x) \in \Re^m$ such that

$$V(t+1,F(t,x,d,u(t,x))) \leq V(t,x) - \rho(V(t,x)) + 2q(t), \quad \forall\,(t,x,d) \in Z^+ \times \Re^n \times D \tag{A12}$$

Moreover, inequality (A12) in conjunction with inequality (3.3a) implies that:

$$a_1\left(\mu(t)|F(t,x,d,u(t,x))|\right) \leq a_2\left(\beta(t)|x|\right) + 2q(t), \quad \forall\,(t,x,d) \in Z^+ \times \Re^n \times D \tag{A13}$$

By virtue of hypothesis (H1) there exist functions $a \in K_\infty$, $\gamma \in K^+$ such that: $|F(t,x,d,u)| \leq a\left(\gamma(t)|(x,u)|\right)$ for all $(t,x,d,u) \in Z^+ \times \Re^n \times D \times \Re^m$. This fact in conjunction with equation (3.3a) implies that:

$$\begin{gathered} V(t+1,F(t,x,d,0)) \leq V(t,x) - \rho(V(t,x)) + 2q(t), \\ \forall\,(t,d) \in Z^+ \times D, \quad \forall\, x \in S(t) \end{gathered} \tag{A14}$$

where $S(t) := \{x \in \Re^n;\ |x| \leq (1/\gamma(t))a^{-1}((1/(\beta(t+1))))a_2^{-1}(2q(t))\}$. We define for all $(t,x) \in Z^+ \times \Re^n$:

$$k(t,x) := \begin{cases} 0 & if \quad x \in S(t) \\ u(t,x) & if \quad x \notin S(t) \end{cases} \tag{A15}$$

By virtue of hypothesis (H1), we obtain $|F(t,x,d,k(t,x))| \leq a\left(\gamma(t)|x|\right)$ for all $(t,d) \in Z^+ \times D$ and $x \in S(t)$. Clearly, by virtue of the previous inequality, inequality (A13) , definition (A15) and Remark 3.2(i), it follows that there exist functions $a' \in K_\infty$, $\gamma' \in K^+$ such that: $|F(t,x,d,k(t,x))| \leq a'\left(\gamma'(t)|x|\right)$ for all $(t,x,d) \in Z^+ \times \Re^n \times D$. Moreover, by virtue of equations (A12) and (A14) and definition (A15) we have:

$$V(t+1,F(t,x,d,k(t,x))) \leq V(t,x) - \rho(V(t,x)) + 2q(t), \quad \forall\,(t,x,d) \in Z^+ \times \Re^n \times D$$

It follows from Lemma 3.8 that system (1.1) is non-uniformly in time robustly globally output stabilizable and k defined by equation (A15) is a non-uniform in time robust global asymptotic output stabilizer.

(c)⇒(a) Suppose that there exist $\sigma \in KL$, $a \in K_\infty, \beta, \gamma, \mu \in K^+$ and a mapping $k : Z^+ \times \Re^n \to \Re^m$ with $|F(t,x,d,k(t,x))| \le a\big(\gamma(t)|x|\big)$ for all $(t,x,d) \in \Re^+ \times \Re^n \times D$, such that for every $(t_0, x_0, \{d(t)\}_0^\infty) \in Z^+ \times \Re^n \times M_D$ the unique solution $x(t) = x(t,t_0,x_0;d)$, $Y(t) = H(t, x(t,t_0,x_0;d))$ of the closed-loop system (1.1) with (1.2) and initial condition $x(t_0) = x_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$, satisfies estimate (3.1). By virtue of Proposition 7 in [26], there exist functions $a_1, a_2 \in K_\infty$ such that: $\sigma(s,t) \le a_1^{-1}\big(\exp(-2t)a_2(s)\big)$, for all $s, t \ge 0$. Thus estimate (3.1) in conjunction with the above inequality implies:

$$\begin{aligned} &a_1\Big(\Big|\big(\mu(t)x(t,t_0,x_0;d), H(t,x(t,t_0,x_0;d))\big)\Big|\Big) \le \exp\big(-2(t-t_0)\big)a_2\big(\beta(t_0)|x_0|\big) \\ &\quad \forall\ (t_0,x_0,d) \in Z^+ \times \Re^n \times M_D \quad \text{and} \quad t \ge t_0, \end{aligned} \tag{A16}$$

Define:

$$V(t_0,x_0) := \sup\Big\{\exp\big(t-t_0\big)a_1\Big(\Big|\big(\mu(t)x(t,t_0,x_0;d), H(t,x(t,t_0,x_0;d))\big)\Big|\Big);\ t \ge t_0,\ d \in M_D\Big\} \tag{A17}$$

Inequalities (3.3a) are immediate consequences of definition (A17) and estimate (A16). Moreover, notice that definition (A17) implies that:

$$\begin{aligned} &V(t_0+1, F(t_0,x_0,d(t_0),k(t_0,x_0))) = V(t_0+1, x(t_0+1,t_0,x_0;d)) \le \exp(-1)V(t_0,x_0), \\ &\quad \forall\ (t_0,x_0,d) \in Z^+ \times \Re^n \times M_D \end{aligned} \tag{A18}$$

Clearly, property (ii) of Definition 3.4 is an immediate consequence of inequality (A18). Particularly, property (ii) of Definition 3.4 holds with $\rho(s) := (1-\exp(-1))s$ and $q(t) \equiv 0$.

(d)⇒(b) under hypothesis (**H2**). This implication is an immediate consequence of the main result in [9]. However, we are going to use instead the main result in [14] in order to prove the existence of a strong OCLF for (1.1). The methodology for the proof of this implication was suggested by Prof. Teel and shows the close connection between the non-uniform in time notions of stability and the notion of stability with respect to two measures for finite-dimensional discrete-time systems.

Suppose that there exist functions $\sigma \in KL$, $\beta, \mu \in K^+$, a continuous mapping $k : Z^+ \times \Re^n \to \Re^m$, such that for every $(t_0, x_0, d) \in Z^+ \times \Re^n \times M_D$ the unique solution $x(t) = x(t,t_0,x_0;d)$, $Y(t) = H(t,x(t))$ of the closed-loop system (1.1) with (1.2) and initial condition $x(t_0) = x_0$, corresponding to input $d \in M_D$, satisfies estimate (3.1). We next consider the following autonomous system:

$$\begin{aligned} &y(t+1) = \frac{1}{b(w(t)+1)}\bar{F}\big(w(t), b(w(t))y(t), d(t)\big), w(t+1) = w(t)+1 \\ &\bar{Y}(t) = \bar{H}(w(t), b(w(t))y(t)), (y(t),w(t)) \in \Re^n \times I, \bar{Y}(t) \in \Re^k, d(t) \in D \end{aligned} \tag{A19}$$

where $I := \cup_{k \in Z^+}\big(k-\frac{1}{4}, k+\frac{1}{4}\big)$, $I \subset \Re$, $\bar{F}(w,y,d) = F(p(w),y,d,k(p(w),y))$, $\bar{H}(w,y) := H(p(w),y)$, $b(w) := 1/(\mu(p(w)))$ and $p : I \to Z^+$ is the mapping that maps each $w \in I$ to its closest integer. Notice that this mapping is well defined on I and is continuous on I. Moreover, by virtue of hypothesis (H2) it follows that the mappings $\bar{F}, \bar{H}$ are continuous on their domains and that for each $(w_0, y_0, \{d(t)\}_0^\infty) \in I \times \Re^n \times M_D$ the unique solution of (A19) with initial condition $y(0) = y_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$, satisfies for all $t \in Z^+$:

$$\begin{aligned} y(t) &= \mu\big(p(w_0)+t\big)x(p(w_0)+t, p(w_0), b(w_0)y_0; \bar{d}) \\ \bar{Y}(t) &= H(p(w_0)+t, x(p(w_0)+t, p(w_0), b(w_0)y_0; \bar{d})) \end{aligned} \tag{A20}$$

where $\bar{d} \in M_D$ satisfies $\bar{d}(p(w_0)+t) = d(t)$ for all $t \in Z^+$. It follows from estimate (3.1) that for each $(w_0, y_0, \{d(t)\}_0^\infty) \in I \times \Re^n \times M_D$ the unique solution of equation (A19) with initial condition $y(0) = y_0$, corresponding to input $\{d(t)\}_0^\infty \in M_D$, satisfies for all $t \in Z^+$:

$$\left|\bar{Y}(t)\right| + |y(t)| \leq \sigma\big(\beta(p(w_0))b(w_0)|y_0|, t\big) \tag{A21}$$

Clearly, the solutions of equation (A19) are solutions of the difference inclusion:

$$\begin{aligned} &(y(t+1), w(t+1)) \in \tilde{F}(w(t), y(t)) \\ &\tilde{F}(w,y) := \left\{ \left(\frac{1}{b(w+1)} \bar{F}(w, b(w)y, d), w+1 \right); d \in D \right\} \subseteq \Re^n \times I \end{aligned} \tag{A22}$$

and each solution of the difference inclusion (A22) is a solution of equation (A19) for some $\{d(t)\}_0^\infty \in M_D$. Estimate (A21) implies that the difference inclusion (A22) is strongly *KL*-stable with respect to the measures (ω_1, ω_2) in the sense described in [14], where $\omega_1(y,w) := |y| + \left|\bar{H}(w, b(w)y)\right|$ and $\omega_2(y,w) := \beta(p(w))b(w)|y|$. Moreover, by virtue of hypothesis (**H2**), it is immediate to verify that the set-valued map $\tilde{F}$ as defined by equation (A22) satisfies the basic conditions and is continuous on the open set $\Re^n \times I \subset \Re^{n+1}$ in the sense of Definitions 5, 6 in [14], respectively. Thus by virtue of Theorem 2 in [14], it follows that the difference inclusion (A22) is robustly strongly *KL*-stable with respect to the measures (ω_1, ω_2) in the sense described in [14]. By virtue of Theorem 1 in [14], there exist functions $a_1, a_2 \in K_\infty$ and a continuous function $U : I \times \Re^n \to \Re^+$ such that

$$a_1\left(\left|(y, \bar{H}(w, b(w)y))\right|\right) \leq U(w,y) \leq a_2\big(\beta(p(w))b(w)|y|\big), \forall (w,y) \in I \times \Re^n \tag{A23}$$

$$U\left(w+1, \frac{1}{b(w+1)} \bar{F}(w, b(w)y, d)\right) \leq \exp(-1)U(w,y), \quad \forall\, (w,y,d) \in I \times \Re^n \times D \tag{A24}$$

Finally, we define:

$$V(t,x) := U(t, \mu(t)x), \quad \forall\, (t,x) \in Z^+ \times \Re^n \tag{A25}$$

We next prove that V is a strong OCLF for (1.1). Obviously property (i) of Definition 3.4 is a consequence of inequality (A23). Define

$$\Psi(t,x,u) := \sup\{V(t+1, F(t,x,d,k(t,x)+v);\ d \in D,\ |v| \leq |u - k(t,x)|\} \tag{A26}$$

Inequalities (3.3c,d) with $\rho(s) := (1 - \exp(-1))s$ and $q(t) \equiv 0$ are immediate consequences of inequality (A24) and definitions (A25), (A26). Finally, we prove that the function Ψ as defined by equation (A26) is quasi-convex with respect to $u \in \Re^m$. Notice that the continuous maps $F : Z^+ \times \Re^n \times D \times \Re^m \to \Re^n$, $k : Z^+ \times \Re^n \to \Re^m$ and $V : Z^+ \times \Re^n \to \Re^+$ can be continuously extended to $F : \Re \times \Re^n \times D \times \Re^m \to \Re^n$, $k : \Re \times \Re^n \to \Re^m$ and $V : \Re \times \Re^n \to \Re^+$, respectively (see Remark 1.1). Under hypothesis (H2), it follows from compactness of $D \subset \Re^l$, continuity of $F : \Re^+ \times \Re^n \times D \times \Re^m \to \Re^n$, $k : \Re^+ \times \Re^n \to \Re^m$, $V : \Re^+ \times \Re^n \to \Re^+$ and Theorem 1.4.16 in [4] that the function Ψ as defined by (A26) is continuous. Clearly, definition (A26) implies $\Psi(t,0,0) = 0$ for all $t \in Z^+$. Let $(t,x) \in Z^+ \times \Re^n$, $u_1, u_2 \in \Re^m$ and $\lambda \in [0,1]$. Definition (A26) implies:

$$\Psi(t,x,\lambda u_1 + (1-\lambda)u_2) \leq$$
$$\sup\{V(t+1, F(t,x,d,k(t,x)+v); d \in D, |v| \leq \lambda|u_1 - k(t,x)| + (1-\lambda)|u_2 - k(t,x)|\}$$
$$\leq \sup\{V(t+1, F(t,x,d,k(t,x)+v); d \in D, |v| \leq \max\{|u_1 - k(t,x)|, |u_2 - k(t,x)|\}\}$$

If $\max\{|u_1 - k(t,x)|, |u_2 - k(t,x)|\} = |u_1 - k(t,x)|$ then the above inequality implies $\Psi(t,x,\lambda u_1 + (1-\lambda)u_2) \leq \Psi(t,x,u_1)$. Similarly, if $\max\{|u_1 - k(t,x)|, |u_2 - k(t,x)|\} = |u_2 - k(t,x)|$ then we obtain $\Psi(t,x,\lambda u_1 + (1-\lambda)u_2) \leq \Psi(t,x,u_2)$. Thus in any case it holds that $\Psi(t,x,\lambda u_1 + (1-\lambda)u_2) \leq \max\{\Psi(t,x,u_1), \Psi(t,x,u_2)\}$. The proof is complete. □

Proof of Proposition 3.10. Define:

$$p(t,x) := \delta(t,x) + \frac{1}{\gamma(t,x)} a_3^{-1}\left(\frac{1}{\mu(t)} a_1^{-1}(\Psi(t,x,0)+1)\right) \tag{A27}$$

and notice that the mapping $x \to p(t,x)$ is a continuous, positive function for each fixed $t \in Z^+$. Definitions (3.10), (A27) and inequalities (3.3a,c) and (3.9) imply that for each fixed $t \in Z^+$ we have:

$$\begin{aligned}\tilde{V}(t,x) &= \min\big(\inf\{\Psi(t,x,u); |u| \leq p(t,x)\}, \inf\{\Psi(t,x,u); |u| > p(t,x)\}\big)\\ &\geq \min\big(\inf\{\Psi(t,x,u); |u| \leq p(t,x)\}, \inf\{a_1(\mu(t)|F(t,x,d,u)|); d \in D, |u| > p(t,x)\}\big)\\ &\geq \min\big(\inf\{\Psi(t,x,u); |u| \leq p(t,x)\},\ \Psi(t,x,0)+1\big)\end{aligned}$$

Clearly, since $\tilde{V}(t,x) \leq \Psi(t,x,0)$, the latter inequality implies that the case $\min\big(\inf\{\Psi(t,x,u);\ |u| \leq p(t,x)\},\ \ \Psi(t,x,0)+1\big) = \Psi(t,x,0)+1$ cannot happen. Thus, we conclude that:

$$\tilde{V}(t,x) = \inf\{\Psi(t,x,u) : |u| \leq p(t,x)\} = -\sup\{-\Psi(t,x,u) : |u| \leq p(t,x)\} \tag{A28}$$

Moreover, it follows that the set-valued map $\mathbf{M}(t,x) \subseteq \Re^m$, as defined by (3.11), is non-empty and bounded for each fixed $t \in Z^+$. Notice that the continuous maps $\Psi : Z^+ \times \Re^n \times \Re^m \to \Re^+$ and $p : Z^+ \times \Re^n \to \Re^+$ can be continuously extended to $\Psi : \Re \times \Re^n \times \Re^m \to \Re^+$ and $p : \Re \times \Re^n \to \Re^+$, respectively (see Remark 1.1). Continuity of the mapping $x \to \tilde{V}(t,x)$ follows immediately by equalities (A28) in conjunction with Theorem 1.4.16 in [4] and the lower and upper semi-continuity of the set-valued map $x \to S(t,x) := \{u \in \Re^m; |u| \leq p(t,x)\}$. Moreover, by continuity of the mapping $x \to \tilde{V}(t,x)$ it follows that for each fixed $t \in Z^+$ the set $\mathbf{M}(t,x) \subseteq \Re^m$ is compact. We finish the proof by establishing that the set-valued map $x \to M(t,x)$, as defined by (3.11), is upper semi-continuous for each fixed $t \in Z^+$. It suffices to prove that for every $(t,x) \in Z^+ \times \Re^n$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x' - x| < \delta \Rightarrow \mathbf{M}(t,x') \subset \mathbf{M}(t,x) + \varepsilon B \tag{A29}$$

The proof will be made by contradiction. Suppose the contrary: there exists $(t,x) \in Z^+ \times \Re^n$ and $\varepsilon > 0$, such that for all $\delta > 0$, there exists $x' \in \{x\} + \delta B$ and $u' \in \mathbf{M}(t,x')$ with $|u' - u| \geq \varepsilon$, for all $u \in \mathbf{M}(t,x)$. Clearly, this implies the existence of a sequence $\{(x'_j, u'_j)\}_{j=1}^{\infty}$ with $x'_j \to x$, $u'_j \in \mathbf{M}(t,x'_j)$ and $|u'_j - u| \geq \varepsilon$, for all $u \in \mathbf{M}(t,x)$ and $j = 1, 2, \ldots$. On the other hand, since u'_j is bounded, it contains a convergent subsequence $u'_i \to \bar{u} \notin \mathbf{M}(t,x)$. By continuity of the mappings $x \to \tilde{V}(t,x)$ and $(x,u) \in \Re^n \times \Re^m \to \Psi(t,x,u)$, we have: $\tilde{V}(t,x'_i) \to \tilde{V}(t,x)$ and $\tilde{V}(t,x'_i) = \Psi(t,x'_i,u'_i) \to \Psi(t,x,\bar{u})$. Consequently, we must have: $\tilde{V}(t,x) = \Psi(t,x,\bar{u})$, which, by virtue of definition (3.11) implies that $\bar{u} \in \mathbf{M}(t,x)$, a contradiction. Notice that the fact that the mapping $u \rightarrow \Psi(t,x,u)$ is strictly quasi-convex guarantees that the set-valued map $\mathbf{M}(t,x)$ defined by (3.11) is a singleton for all $(t,x) \in Z^+ \times \Re^n$. Thus there exists a function $k : Z^+ \times \Re^n \rightarrow \Re^m$ with $k(t,0)=0$ for all $t \in Z^+$ with the mapping $x \rightarrow k(t,x)$ being continuous for each fixed $t \in Z^+$ and in such a way that equation (3.12) is satisfied. Consequently, inequalities (3.3c,d) imply that the following property holds:

$$V(t+1, F(t,x,d,k(t,x))) \leq V(t,x) - \rho(V(t,x)) + 4q(t) \quad \forall\, (t,x,d) \in Z^+ \times \Re^n \times D,$$

It follows from Remark 3.2(i), hypothesis (H1) and Lemma 3.8 that system (1.1) is non-uniformly in time continuously robustly globally output stabilizable and k defined by (3.12) is a non-uniform in time robust global asymptotic output stabilizer for (1.1). $\square$

Proof of Lemma 4.2. Consider the solution $x(t)$ of equation (4.1) with arbitrary initial condition $x(t_0)=x_0$, corresponding to arbitrary input $\{d(t)\}_0^\infty \in M_D$. Using equation (4.4) and (trivial) induction arguments, it can be shown that for all $t \geq t_0$ it holds that: $V(t,x(t)) \leq \lambda^{t-t_0} V(t_0,x_0)$. The previous estimate in conjunction with inequality (4.3) implies (4.2) with $c := -\log(\lambda)/p$, $a(s) := \left(K^{-1} a_2(s)\right)^{\frac{1}{p}}$. It follows that zero is non-uniformly in time RGK-ES for (4.1) with constant $c = -\log(\lambda)/p$. $\square$