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Stabilization by means of time-varying hybrid feedback

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Abstract In this work characterizations of the notion of non-uniform in time robust global asymptotic output stability for hybrid systems with disturbances are given. Based on the provided characterizations, it is shown that every asymptotically output controllable time-varying control system can be stabilized (in general non-uniformly in time) by means of time-varying hybrid feedback.

Keywords Hybrid systems · Asymptotic controllability

1 Introduction

In this paper systems of the following form are considered:

$$\begin{aligned}
&\dot{x}(t) = f\left(t, d(t), d\left(\tau_i\right), x(t), x\left(\tau_i\right), u(t), u\left(\tau_i\right)\right), \quad \tau_i \leq t < \tau_{i+1},\\
&Y(t) = H(t, x(t)),\\
&x(t) \in \Re^n, \quad Y(t) \in \Re^p, \ \ u(t) \in \Re^m, \ \ t \geq 0, \ \ d(t) \in D,
\end{aligned} \tag{1}$$

where $\pi = \{\tau_i\}_{i=0}^{\infty}$ is a partition of $\Re^+$with diameter $r > 0$, i.e., an increasing sequence of times with $\tau_0 = 0$, $\sup\{\tau_{i+1} - \tau_i;\ i = 0, 1, 2, \ldots\} = r$ and $\tau_i \to +\infty$, $d(t)$ represents the disturbance vector or the vector of time-varying uncertainties taking values in the set $D \subset \Re^l$, $Y(t)$ represents the output of the system and $u(t)$ represents the input vector. A wide class of systems described by impulsive differential equations with impulses at fixed times (see [15]), as well as hybrid systems of the form:

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), u(t), w(i)), \quad \tau_i \leq t < \tau_{i+1}\\
&w(i) = g(i, x(\tau_i), u(\tau_i)),
\end{aligned} \tag{2}$$

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where $\pi = \{\tau_i\}_{i=0}^{\infty}$ is a partition of $\Re^+$of diameter $r > 0$, can be represented by the time-varying case (1). Fundamental properties of the solutions of systems of the form (2) are studied in [3,15].

The most important motive to study systems of the form (1) is that such systems arise naturally when sampled solutions are considered for finite-dimensional continuous-time control systems:

$$\begin{aligned} &\dot{x}(t) = f(t, d(t), x(t), u(t)) \\ &Y(t) = H(t, x(t)) \\ &x(t) \in \Re^n, \quad Y(t) \in \Re^p, \ d(t) \in D, \ u(t) \in \Re^m, \ t \geq 0 \end{aligned} \tag{3}$$

under a discontinuous feedback law:

$$u(t) = k\left(t, x(t), x\left(\tau_i\right)\right), \quad \tau_i \leq t < \tau_{i+1}. \tag{4}$$

It is clear that the closed-loop system (3) with (4) is a system of the form (1). This particular approach of feedback stabilization of the equilibrium point of a control system has been used in [13,19–21,37], where it was shown that this approach is successful even in systems that cannot be stabilized by continuous time-invariant feedback. Particularly, in [21] it was shown that this type of feedback law guarantees important robustness properties for the closed-loop system and in [19] it was shown that every autonomous asymptotically controllable system can be stabilized by hybrid feedback of this type. Feedback of the form (4) is usually characterized as hybrid strategy (see [13,34]) or as a controller with synchronous switching (see [27]).

Many researchers have studied the particular case where the feedback law in (4) is independent of t (time) and $x(t)$ (current state), i.e., $u(t) = k\left(x\left(\tau_i\right)\right)$, where $\pi = \{\tau_i\}_{i=0}^{\infty}$ is the sampling partition (or sampling schedule). More specifically, in the pioneering papers [4,17,34] the problem of the existence of such feedback laws that practically stabilize the equilibrium point of an autonomous control system for arbitrary sampling partitions is studied, in connection with the property of asymptotic controllability. In these works the concepts of sampled solutions or π-solutions of control systems as well as the fundamental concept of the Control Lyapunov Function (CLF, see [12,26,29,31]) are utilized. In [5, 6,18] sufficient conditions for the local asymptotic stability of hybrid systems that arise from sampled-data feedback laws are provided. Recently, attempts for extending results to the time-varying case are given in [1,14,39,40] and the problem of “adding an integrator” for this type of feedback is examined in [38]. It should be emphasized that in [16,17,34] the authors also study the robustness properties of the resulting closed-loop system and important results are provided in terms of the input-to-state stability (ISS, introduced by E.D. Sontag in [30]) property with respect to modeling, actuator and measurement errors. The work of C. Prieur in [22–25] is similiar to the previously mentioned works and provides links to the hybrid feedback approach as well as with the patchy vector field approach (see [2]). In [28] sampled-data output feedback was used for the stabilization of asymptotically controllable and observable control systems.

In the present paper, we intend to extend the results provided in [19] to the cases:

- Where only the output can be led to zero (output regulation problems);
- Of asymptotically controllable time-varying control systems;
- Of partitions with not necessarily positive lower diameter.

For these purposes, we use the concepts of non-uniform in time Robust Global Asymptotic Output Stability (RGAOS, see [8–11]), which is the "natural" extension of the notions of Uniform RGAOS for autonomous finite-dimensional continuous-time systems (see [7,35,36]). It should be emphasized that the construction of the feedback law (4) is based directly on the open-loop controls that lead the output of (3) to zero and does not rely on the knowledge of a CLF for (3). This is a major methodological difference between the approach proposed in the present paper and previous works. The proof of our main result (Proposition 3.2) is simpler than the proofs of analogous results in the literature and is based on the construction of a CLF, which is different from the notion of a CLF used in [4] and closer to the notion of a CLF given in [32].

Moreover, in the present work we intend to provide Lyapunov characterizations of the notion of non-uniform in time RGAOS for hybrid systems with minimal regularity hypotheses concerning the right-hand sides of the differential equations. It is expected that the stability characterizations provided in this work will be utilized in the future by researchers that work on hybrid systems.

The structure of this paper is as follows. In Sect. 2 the definitions of the stability notions used in this paper and results on the characterization of the stability notions for the case (1) are provided. In Sect. 3, the main results are presented and proved. Section 4 is devoted to the case of linear time-varying control systems under the assumption of complete controllability. It is shown that there exists a linear hybrid time-varying feedback that guarantees finite-time stability for the nominal closed-loop system and the non-uniform in time ISS property from the inputs that quantify the effect of modeling, actuator and state measurement errors for partitions of positive lower diameter. Finally, the conclusions of the paper are provided in Sect. 5.

Notations Throughout this paper we adopt the following notations:

- For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose;
- We denote by $[R]$ the integer part of the real number R, i.e., the greatest integer, which is less than or equal to R;
- By $C^j(A)$ ($C^j(A\ ;\ \Omega)$), where $j \geq 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of the order j on A;
- Z^+ denotes the set of positive integers;
- E denotes the class of non-negative C^0 functions $\mu : \Re^+ \to \Re^+$, for which it holds: $\int_0^{+\infty} \mu(t)\mathrm{d}t < +\infty$ and $lim_{t\to+\infty}\,\mu(t) = 0$;
- We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. By K we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho : \Re^+ \to \Re^+$ is of class K_∞ if $\underset{s\to+\infty}{lim}\,\rho(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s,t) : \Re^+ \times \Re^+ \to \Re^+$ with

the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is non-decreasing with $\sigma(0, t) = 0$; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim\limits_{t \to +\infty} \sigma(s, t) = 0$.

2 Definitions and preliminary results

Consider system (1) under the following assumptions.

(H1) $\pi = \{\tau_i\}_{i=0}^{\infty}$ is a partition of $\Re^+$with finite diameter $r > 0$, i.e., an increasing sequence of times with $\tau_0 = 0$, $\sup\{\tau_{i+1} - \tau_i;\ i = 0, 1, 2, \dots\} = r$ and $\tau_i \to +\infty$.
(H2) $H : \Re^+ \times \Re^n \to \Re^p$ is continuous with $H(t, 0) = 0$, for all $t \geq 0$.
(H3) $f(t, d, d_0, x, x_0, u, u_0)$ is measurable with respect to $t \geq 0$, continuous with respect to $(d, u) \in D \times \Re^m$ and such that for every compact $S \subset \Re^n \times \Re^n \times \Re^m \times \Re^m$ and for every compact $I \subset \Re^+$ there exists constant $L \geq 0$ such that

$$\begin{aligned}&(x - y)'\,(f(t, d, d_0, x, x_0, u, u_0) - f(t, d, d_0, y, x_0, u, u_0)) \leq L\ |x - y|^2 \\ &\quad \forall t \in I,\ \forall (d, d_0) \in D \times D,\ \forall (x, x_0, u, u_0) \in S,\ \forall (y, x_0, u, u_0) \in S. \end{aligned} \tag{5}$$

(H4) There exist functions $\gamma \in K^+, a \in K_\infty$ such that $|f(t, d, d_0, x, x_0, u, u_0)| \leq \gamma(t)\, a\,(|x_0| + |x| + |u| + |u_0|)$ for all $(t, d, d_0, x, x_0, u, u_0) \in \Re^+ \times D \times D \times \Re^n \times \Re^n \times \Re^m \times \Re^m$.

Using the method of steps on consecutive intervals, it is clear that system (1) under hypotheses (H1–4) defines a continuous-time control system with outputs as defined in [10], with state space $\mathsf{X} = \Re^n \times \Re^n$, output space $\mathsf{Y} = \Re^p$, set of structured uncertainties M_D being the set of mappings $t \in \Re^+ \to d(t) = \left\{ \tilde{d}(t + \theta)\ ;\ \theta \in [-r, 0] \right\}$ where $\tilde{d} \colon \Re \to D$ is any measurable and locally bounded function, input space $U = \mathsf{U}$ is the normed linear space of measurable and bounded functions on $[-r,\ 0]$ taking values in $\Re^m$ with the sup norm and set of external inputs M_U being the set of mappings $t \in \Re^+ \to u(t) = \{\tilde{u}(t + \theta)\ ;\ \theta \in [-r,\ 0]\} \in U$, where $\tilde{u} \colon \Re \to \Re^m$ is any measurable and locally bounded function. The reader may be surprised by the complicated definition of M_D and M_U, but it should be emphasized that this definition guarantees that the causality property of the control system (1) holds. However, in what follows we may identify elements $t \in \Re^+ \to u(t) = \{\tilde{u}(t + \theta)\ ;\ \theta \in [-r,\ 0]\} \in U$ and $t \in \Re^+ \to d(t) = \left\{ \tilde{d}(t + \theta)\ ;\ \theta \in [-r,\ 0] \right\}$ in M_U and M_D, respectively, with the measurable and locally bounded functions $\tilde{u} \colon \Re \to \Re^m$ and $\tilde{d} \colon \Re \to D$. Thus when we write $(d, u) \in M_D \times M_U$ we mean that $u \colon \Re \to \Re^m$ and $d \colon \Re \to D$ are measurable and locally bounded functions.

Let $p(t) = \max\{\tau_i\ ;\ \tau_i \in \pi\ ,\ \tau_i \leq t\}$. For all $(t_0, x_0, x_1, d, u) \in \Re^+ \times \Re^n \times \Re^n \times M_D \times M_U$, we denote by $x(t) = \phi(t, t_0, x_0, x_1; d, u) \in \Re^n$ the solution of (1) at time $t \geq t_0$ with initial condition $x(t_0) = x_0$ and the additional condition $x(p(t_0)) = x_1$, which holds only for the case $t_0 \notin \pi$, corresponding to inputs $(d, u) \in M_D \times M_U$ [this solution is unique by virtue of property (H3)]. Notice that the actual state of system (1) at time $t \geq t_0$ is given by $\tilde{\phi}(t, t_0, x_0, x_1; d, u) = (\phi(t, t_0, x_0, x_1; d, u), \phi(p(t), t_0, x_0, x_1; d, u)) \in \Re^n \times \Re^n$.

Remark 2.1 It is clear that under hypotheses (H1–4) the following are true.

(i) System (1) satisfies the "Boundedness-Implies-Continuation" (BIC) property as defined in [10], i.e., for all $(t_0, x_0, x_1, d, u) \in \Re^+ \times \Re^n \times \Re^n \times M_D \times M_U$ there exists a maximal existence time $t_{\max} > t_0$ such that the solution $x(t) = \phi(t, t_0, x_0, x_1; d, u) \in \Re^n$ of (1) may be defined on $t \in [t_0, t_{\max})$, cannot be continued for $t > t_{\max}$ and if $t_{\max} < +\infty$ then we have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$.

(ii) If $t_0 \in \pi$, the solution $x(t) \in \Re^n$ of (1) for $t \geq t_0$ does not depend on $x_1 \in \Re^n$ but depends only on $x_0 \in \Re^n$. Thus in this case we may write $x(t) = \phi(t, t_0, x_0; d, u) \in \Re^n$ with $t \geq t_0$, $x(t_0) = x_0 \in \Re^n$.

(iii) $0 \in \Re^n \times \Re^n$ is an equilibrium point for system (1).

(iv) Let $\tau_i = p(t_0)$. The solution $x(t) \in \Re^n$ of (1) for $t > \tau_{i+1}$ does not depend (in general) continuously on the initial condition $x_0 \in \Re^n$ and particularly does not depend continuously on $x_1 \in \Re^n$. However, using (5) it can be shown that for the case $\tau_i < t_0 \leq t \leq \tau_{i+1}$ the solution $x(t) \in \Re^n$ of (1) depends continuously on $x_0 \in \Re^n$.

(v) The solution $x(t) \in \Re^n$ of (1) for $t \geq t_0$ actually depends on $\{d(\tau); p(t_0) \leq \tau \leq t\}$ and $\{ u(\tau) \; ; \; p(t_0) \leq \tau \leq t \}$, where $(d, u) \in M_D \times M_U$.

Systems of the form (1) arise when hybrid feedback is used for the stabilization of systems described by ordinary differential equations. The following example illustrates this point.

Example 2.2 Consider the scalar system:

$$\begin{aligned} \dot{x}(t) &= \exp(t)x(t) + v(t) \\ Y(t) &= x(t) \in \Re, \quad t \in \Re^+. \end{aligned} \tag{6}$$

Suppose that we apply the feedback law $v(t) = -\left[\exp(t)\,(1 - t + [t]\,) + 1\right] x\,(\,[t]\,) + u(t)$, where $u(t) \in \Re$ represents the possible modeling errors and control actuator errors and $[t]$ denotes the integer part of $t \in \Re^+$. The resulting closed-loop system is described by

$$\begin{aligned} \dot{x}(t) &= \exp(t)x(t) - \left[\exp(t)\,(1 - t + [t]\,) + 1\right] x\,(\,[t]\,) + u(t) \\ Y(t) &= x(t). \end{aligned} \tag{7}$$

It is easy to verify that system (7) is a system of the form (1) with $\tau_i = i$ and $H(t, x) = x$, which satisfies hypotheses (H1–4). Moreover the solution of (7) can be found analytically by the following formulae:

$$\begin{aligned} x(t) = {} & \phi\,(t, t_0)\,(\,x(t_0) - (1 + [t_0] - t_0)\,x\,(\,[t_0]\,)\,) + (1 + [t_0] - t)x\,(\,[t_0]\,) \\ & + \int_{t_0}^{t} \phi\,(\,t, \tau)\,u(\tau)\mathrm{d}\tau, \quad \text{for } t_0 \leq t \leq [t_0] + 1, \end{aligned}$$

$$x(t) = (1 - t + [t])\,\phi([t], t_0)\,(x(t_0) - (1 + [t_0] - t_0)\,x([t_0])) + (1 - t + [t]) \int_{t_0}^{[t]} \phi([t], \tau)\,u(\tau)\mathrm{d}\tau + \int_{[t]}^{t} \phi(t, \tau)\,u(\tau)\mathrm{d}\tau, \quad \text{for } [t_0] + 1 \le t < [t_0] + 2,$$

$$x(t) = (1 - t + [t]) \int_{[t]-1}^{[t]} \phi([t], \tau)\,u(\tau)\mathrm{d}\tau + \int_{[t]}^{t} \phi(t, \tau)\,u(\tau)\mathrm{d}\tau, \quad \text{for } t \ge [t_0] + 2,$$

where $\phi(t, \tau) := \exp(\exp(t) - \exp(\tau))$. This example will be further studied. $\triangleleft$

The following proposition guarantees that under hypotheses (H1–4), $0 \in \Re^n \times \Re^n$ is a robust equilibrium point for system (1), in the sense described in [10]. Its proof is provided in the Appendix.

Proposition 2.3 $0 \in \Re^n \times \Re^n$ *is a robust equilibrium point for system (1) under hypotheses (H1–4).*

We remind the reader that according to [10]:

(1) with $u(t) \equiv 0$ is robustly forward complete (RFC) if for every $s \ge 0$, $T \ge 0$, it holds that

$$\sup\{\,|\phi(t_0 + h, t_0, x_0, x_1; d, 0)|\ ;\ \ h \in [0, T]\,,\ \max\{\,|x_0|\ ;\ |x_1|\,\} \le s\ ,\ t_0 \in [0, T]\ ,\ d \in M_D\} < +\infty$$

(2) is RFC from the input u if for every $s \ge 0$, $T \ge 0$, it holds that

$$\sup\left\{ |\phi(t_0 + h, t_0, x_0, x_1; d, u)|\ ;\ u \in M_U,\ \sup_{\tau \ge 0} |u(\tau)| \le s, h \in [0, T]\,,\ \max\{\,|x_0|\ ;\ |x_1|\,\} \le s\ ,\ t_0 \in [0, T]\ ,\ d \in M_D \right\} < +\infty.$$

The following lemma provides the criteria for establishing robust forward completeness for the case (1). Its proof is provided in the Appendix. The result of Lemma 2.4 is going to be used in Sect. 3.

Lemma 2.4 *System (1) with $u(t) \equiv 0$ is RFC if and only if for every $s \ge 0$, $\tau_i \in \pi$, it holds that*

$$\sup\{\,|\phi(t, t_0, x_0, x_1; d, 0)|\ ;\ \ t \in [t_0, \tau_{i+1}]\,,\ \max\{\,|x_0|\ ;\ |x_1|\,\} \le s\ ,\ t_0 \in [\tau_i, \tau_{i+1})\ ,\ d \in M_D\} < +\infty. \tag{8}$$

Moreover, system (1) is RFC from the input u if and only if for every $s \geq 0$, $\tau_i \in \pi$, it holds that

$$\sup\left\{ |\phi(t,t_0,x_0,x_1;d,u)| \ ;\ u \in M_U \ ,\ \sup_{\tau \geq 0} |u(\tau)| \leq s \ ,\ t \in [t_0,\tau_{i+1}] \ , \right.$$
$$\left. \max\{|x_0| \ ;\ |x_1|\} \leq s \ ,\ t_0 \in [\tau_i,\tau_{i+1}) \ ,\ d \in M_D \right\} < +\infty. \qquad (9)$$

Next the definition of the notion of non-uniform in time RGAOS for (1) with $u(t) \equiv 0$ is provided.

Definition 2.5 *Consider system (1) under hypotheses (H1–4). We say that (1) with $u(t) \equiv 0$ is non-uniformly in time RGAOS if (1) with $u(t) \equiv 0$ is RFC and the following properties hold:*

P1 *(1) with $u(t) \equiv 0$ is Robustly Lagrange Output Stable, i.e., for every $\varepsilon > 0$, $T \geq 0$, it holds that*

$$\sup\{|H(t,\phi(t,t_0,x_0,x_1;d,0))| \ ;\ t \geq t_0 \ ,\ \max\{|x_0| \ ;\ |x_1|\} \leq \varepsilon \ ,$$
$$t_0 \in [0,T] \ ,\ d \in M_D\} < +\infty.$$

(Robust Lagrange Output Stability)

P2 *(1) with $u(t) \equiv 0$ is Robustly Lyapunov Output Stable, i.e., for every $\varepsilon > 0$ and $T \geq 0$, there exists a $\delta := \delta(\varepsilon,T) > 0$ such that:*

$$\max\{|x_0| \ ;\ |x_1|\} \leq \delta, \quad t_0 \in [0,T] \ \Rightarrow \ |H(t,\phi(t,t_0,x_0,x_1;d,0))| \leq \varepsilon, \quad \forall t \geq t_0, \ \forall d \in M_D.$$

(Robust Lyapunov Output Stability)

P3 *(1) with $u(t) \equiv 0$ satisfies the Robust Output Attractivity Property, i.e., for every $\varepsilon > 0$, $T \geq 0$ and $R \geq 0$, there exists a $\tau := \tau(\varepsilon,T,R) \geq 0$, such that:*

$$\max\{|x_0| \ ;\ |x_1|\} \leq R \ , \quad t_0 \in [0,T] \ \Rightarrow \ |H(t,\phi(t,t_0,x_0,x_1;d,0))| \leq \varepsilon, \quad \forall t \geq t_0 + \tau, \ \forall d \in M_D.$$

Moreover, if there exists $a \in K_\infty$ such that $a(|x|) \leq |H(t,x)|$ for all $(t,x) \in \Re^+ \times \Re^n$ then we say that (1) with $u(t) \equiv 0$ is non-uniformly in time robustly globally asymptotically stable (RGAS).

The work in [10] in conjunction with Remark 2.1 and Proposition 2.3 gives the following results.

Lemma 2.6 *Suppose that system (1) under hypotheses (H1–4) with $u(t) \equiv 0$ is RFC and satisfies the robust output attractivity property (property P3 of Definition 2.5). Then (1) with $u(t) \equiv 0$ is non-uniformly in time RGAOS.*

Theorem 2.7 *Consider system (1) under hypotheses (H1–4). The following statements are equivalent.*

(i) *(1) with $u(t) \equiv 0$ is RGAOS.*

(ii) *There exist functions $\mu, \beta \in K^+, \sigma \in KL$ such that for every $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$, we have:*

$$|H(t, \phi(t, t_0, x_0, x_1; d, 0))| + \mu(t)\,|\phi(t, t_0, x_0, x_1; d, 0)| \\ \leq \sigma\ (\beta(t_0)\, max\{|x_0|\ ;\ |x_1|\}\ ,\ t - t_0),\quad \forall t \geq t_0. \tag{10}$$

(iii) *There exist functions $\mu, \beta \in K^+$, $a \in K_\infty$, $\sigma \in KL$ and a constant $R \geq 0$ such that for every $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$, we have:*

$$|H(t, \phi(t, t_0, x_0, x_1; d, 0))| \\ \leq \sigma\ (\beta(t_0)\ (max\{|x_0|\ ;\ |x_1|\} + R)\ ,\ t - t_0),\quad \forall t \geq t_0. \tag{11}$$

$$|\phi(t, t_0, x_0, x_1; d, 0)| \leq \mu(t) a\,(max\{|x_0|\,;|x_1|\} + R),\quad \forall t \geq t_0. \tag{12}$$

Example 2.8 Consider the closed-loop system (7) with $u(t) \equiv 0$. It is clear from the analytical solution given in Example 2.2 that inequality (10) is satisfied for $\sigma(s, t) = \begin{cases} 3(3-t)s & \text{if } \ t \in [0, 3] \\ 0 & \text{if } \ t > 3 \end{cases}$, $\mu(t) \equiv 1$ and $\beta(t) := \exp((e-1)\exp(t))$. Thus we conclude that (7) with $u(t) \equiv 0$ is non-uniformly in time RGAS. It should be noted that the phenomenon of finite-time stability occurs for the closed-loop system (7) (i.e., the solution approaches the equilibrium point in finite time). ◁

The following proposition provides a Lyapunov characterization of non-uniform in time RGAOS for (1) under hypotheses (H1–4). It should be emphasized that the constructed Lyapunov function has no regularity properties and this feature is expected since the property of continuous dependence on the initial conditions is not satisfied for the solutions of (1) (see Remark 2.1). The following proposition is going to be used in Sect. 3 and its proof is given in the Appendix.

Proposition 2.9 *Suppose that system (1) with $u(t) \equiv 0$ is RFC. System (1) with $u(t) \equiv 0$ under hypotheses (H1–4) is non-uniformly in time RGAOS if and only if there exist functions $V\colon \Re^+ \times \Re^n \to \Re^+$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)\mathrm{d}t = +\infty$, $\varphi \in E$, $a_1, a_2 \in K_\infty$ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$, such that for every $(\tau_i, x, d) \in \pi \times \Re^n \times M_D$, the unique solution $\phi(t, \tau_i, x; d, 0) \in \Re^n$ of system (1) with $u(t) \equiv 0$, starting from $x(\tau_i) = x \in \Re^n$ and corresponding to input $d \in M_D$, satisfies:*

$$a_1\,(|H(t, \phi(t, \tau_i, x; d, 0))|) \leq V(\tau_i, x) \leq a_2(\beta(\tau_i)\,|x|),\ \forall t \in [\tau_i, \tau_{i+1}], \tag{13}$$

$$V(\tau_{i+1}, \phi(\tau_{i+1}, \tau_i, x; d, 0)) \leq \eta(\tau_{i+1}, \tau_i, V(\tau_i, x)), \tag{14}$$

where $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial value problem:

$$\dot{\eta} = -\gamma(t)\rho(\eta) + \gamma(t)\varphi\left(\int_0^t \gamma(s)ds\right);\quad \eta(t_0) = \eta_0 \geq 0. \tag{15}$$

Particularly, if system (1) with $u(t) \equiv 0$ *under hypotheses (H1–4) is non-uniformly in time RGAOS then there exist functions* $V : \Re^+ \times \Re^n \to \Re^+$, $\mu, \beta \in K^+$, $a_1, a_2 \in K_\infty$ *such that (13, 14, 15) are satisfied with* $\eta(t, t_0, s) := \exp(-(t - t_0))\, s$, $\rho(s) := s$, $\gamma(t) \equiv 1$ *and* $\varphi(t) \equiv 0$.

The notion of uniform robust global asymptotic output stability (URGAOS) for finite-dimensional continuous-time systems was recently given in [35,36] (where the name robust output stability was used). New characterizations for versions of this property for finite-dimensional continuous-time systems were given in [7]. The following definition extends the notion of URGAOS to hybrid systems of the form (1).

Definition 2.10 *Consider system (1) under hypotheses (H1–4). We say that (1) with* $u(t) \equiv 0$ *is URGAOS if (1) with* $u(t) \equiv 0$ *is RFC and there exists a function* $\sigma \in KL$ *such that for every* $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$*, we have:*

$$|H(t, \phi(t, t_0, x_0, x_1; d, 0))| \leq \sigma\ (max\{\,|x_0|\,;\,|x_1|\,\}, t - t_0\,)\,, \quad \forall t \geq t_0. \quad (16)$$

The following corollary shows that for periodic systems non-uniform in time RGAOS is equivalent to URGAOS. Thus it becomes clear that the non-uniform in time RGAOS is a natural extension to time-varying hybrid systems of the notion of URGAOS. The proof of Corollory 2.11 is given in the Appendix. We say that (1) is *T-periodic* if there exists $T > 0$ such that $\pi = \{\, i\,T\,\}_{i=0}^{\infty}$ (uniform partition), $f(t+T, d, d_0, x, x_0, u, u_0) = f(t, d, d_0, x, x_0, u, u_0)$ and $H(t+T, x) = H(t, x)$ for all $(t, x, x_0, d, d_0, u, u_0) \in \Re^+ \times \Re^n \times \Re^n \times D \times D \times \Re^m \times \Re^m$.

Corollary 2.11 *Suppose that (1) with* $u(t) \equiv 0$ *is non-uniformly in time RGAOS and that (1) is T-periodic. Then (1) with* $u(t) \equiv 0$ *is URGAOS.*

The stability properties of (1) subject to the external input u can be characterized by means of the non-uniform in time and uniform input-to-output stability (IOS) property, as described next as extensions of the corresponding notions for the finite-dimensional continuous-time case (see [10,30,35,36]).

Definition 2.12 *Consider a control system (1) under hypotheses (H1–4). We say that (1) satisfies the* non-uniform in time *IOS property from the input u if (1) is RFC from the input u and there exist functions* $\sigma \in KL$, $\beta, \gamma \in K^+$, $\rho \in K_\infty$ *such that the following estimate holds for all measurable and locally bounded u*: $\Re^+ \to \Re^m$, $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ *and* $t \geq t_0$*:*

$$\begin{aligned} &|H(t, \phi(t, t_0, x_0, x_1; d, u))| \\ &\leq max\left\{ \sigma(\beta(t_0)\, max\{\,|x_0|\;;\;|x_1|\,\}\;,\; t - t_0\,)\;, \right. \\ &\qquad \left. \sup_{s \in [p(t_0), t]} \sigma(\beta(s)\, \rho\,(\gamma(s)\;|u(s)|)\;,\; t - s\,) \right\}, \end{aligned} \quad (17)$$

where $p(t) = max\{\,\tau_i\;;\;\tau_i \in \pi\;,\;\tau_i \leq t\,\}$. *Moreover, if there exists* $a \in K_\infty$ *such that* $a(\,|x|) \leq |H(t, x)|$ *for all* $(t, x) \in \Re^+ \times \Re^n$ *then we say that (1) satisfies the non-uniform in time (ISS) property from the input u. We say that (1) satisfies the uniform IOS (ISS) property from the input u if (1) satisfies the non-uniform in time IOS (ISS) property from the input u with bounded functions* $\beta, \gamma \in K^+$.

Example 2.13 Consider again the closed-loop system (7). It is clear from the analytical solution given in Example 2.2 that estimate (17) is satisfied for

$$\sigma(s,t) = \begin{cases} 3(3-t)s & \text{if } t \in [0,3] \\ 0 & \text{if } t > 3 \end{cases},$$

$\rho(s) := 2s/3$, $\gamma(t) \equiv 1$ and $\beta(t) := \exp\left((e^2-1)\exp(t)\right)$. Thus we conclude that the closed-loop system (7) satisfies the non-uniform in time ISS property from the input u. ◁

3 Asymptotic controllability implies hybrid feedback stabilizability

Consider the system:

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u(t)), \\ &Y(t) = H(t, x(t)), \\ &x(t) \in \Re^n, \quad u(t) \in \Re^m, \;\; Y(t) \in \Re^p, \;\; t \geq 0, \end{aligned} \tag{18}$$

where $H : \Re^+ \times \Re^n \to \Re^p$ is continuous, $f(t,x,u)$ is measurable with respect to $t \geq 0$ and locally Lipschitz with respect to (x,u) in the sense that for every compact $S \subset \Re^n \times \Re^m$ and for every compact $I \subset \Re^+$ there exists a constant $L \geq 0$ such that

$$\begin{aligned} &|f(t,x,u) - f(t,y,v)| \leq L\,(|x-y| + |u-v|) \quad \forall t \in I, \\ &\quad \forall (x,u) \in S\,, \; \forall (y,v) \in S. \end{aligned} \tag{19}$$

Moreover, we assume that $f(t,0,0) = 0$ and $H(t,0) = 0$, for all $t \geq 0$.

We denote by $x(t) = \phi(t, t_0, x_0; u)$ with $t \geq t_0 \geq 0$ the solution of (18) which corresponds to some measurable and locally essentially bounded input $u : \Re^+ \to \Re^m$, initiated from $x(t_0) = x_0 \in \Re^n$.

Definition 3.1 *We say that (18) is non-uniformly in time globally asymptotically output controllable (GAOC), if there exist functions $\mu, \beta, \gamma \in K^+$, $\sigma \in KL$ and $a \in K_\infty$, such that for each $(t_0, x_0) \in \Re^+ \times \Re^n$ there exists a measurable and locally bounded input $u(\,\cdot\,, t_0, x_0) : \Re^+ \to \Re^m$ with the following properties:*

- **(S1)** *the solution $\phi(t, t_0, x_0; u(\,\cdot\,, t_0, x_0))$ of (18) exists for all $t \geq t_0$ and satisfies $|H(t, \phi(t, t_0, x_0; u(\,\cdot\,, t_0, x_0)))| + \mu(t)\,|\phi(t, t_0, x_0; u(\,\cdot\,, t_0, x_0))| \leq \sigma\,(\,\beta(t_0)\,|x_0|\;,\; t - t_0\,)$ for all $t \geq t_0$;*
- **(S2)** $|f(t, x, u(t, t_0, x_0))| \leq \gamma(t)\, a\,(|x_0| + |x|)$*, for all $t \geq t_0$;*
- **(S3)** *for every $\xi \in \Re^n$ and $t_1 \geq t_0$ the solution $\phi(t, t_1, \xi; u(\,\cdot\,, t_0, x_0))$ of (18) exists for all $t \geq t_1$ and satisfies $|\phi(t, t_1, \xi; u(\,\cdot\,, t_0, x_0))| \leq \gamma(t)\, a\,(|x_0| + |\xi|)$ for all $t \geq t_1$.*

The first two requirements of Definition 3.1 (S1–2) are in the same spirit with the definition of non-uniform in time global asymptotic controllability given in [40]. However, in this work it is necessary to impose the additional property (S3) for certain purposes that are explained next.

The main result of the present section states that non-uniform in time GAOC for the control system (18) implies the existence of a hybrid feedback such that the closed-loop system is a non-uniform in time RGAOS hybrid system for appropriate partitions.

Proposition 3.2 *Suppose that (18) is non-uniformly in time GAOC. Then for every bounded function* $\delta \in K^+$ *there exists a function* $k_\delta : \Re^+ \times \Re^+ \times \Re^n \to \Re^m$ *with* $k_\delta(\cdot, \tau, x)$ *being measurable and locally bounded for each fixed* $(\tau, x) \in \Re^+ \times \Re^n$ *such that for every partition* $\pi = \{\tau_i\}_{i=0}^{\infty}$ *of* $\Re^+$ *with finite diameter and* $\inf\{\tau_{i+1} - \tau_i - \delta(\tau_i);\ i = 0, 1, \dots\} \geq 0$*, the closed-loop system (18) with* $u(t) = k_\delta(t, \tau_i, x(\tau_i))$ *for* $t \in [\tau_i, \tau_{i+1})$ *satisfies hypotheses (H1–4) and is non-uniformly in time RGAOS.*

Proof Let an arbitrary bounded function $\delta \in K^+$. The proof is divided into two parts: in the first part we construct the feedback function and a function $V : \Re^+ \times \Re^n \to \Re^+$, which is going to be used as a Lyapunov function. As remarked already in the introduction $V : \Re^+ \times \Re^n \to \Re^+$ has certain properties such that this function may be characterized as a CLF for (18) in the sense of [32]. In the second part we assume a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with finite diameter, $\inf\{\tau_{i+1} - \tau_i - \delta(\tau_i);\ i = 0, 1, \dots\} \geq 0$ and it is shown that the Lyapunov function $V : \Re^+ \times \Re^n \to \Re^+$ satisfies the requirements of Proposition 2.9 for the closed-loop system. Thus we may conclude that the closed-loop system satisfies hypotheses (H1–4) and is non-uniformly in time RGAOS.

Step 1: Construction of $k_\delta : \Re^+ \times \Re^+ \times \Re^n \to \Re^m$ and $V : \Re^+ \times \Re^n \to \Re^+$.

Suppose that (18) is non-uniformly in time GAOC. Then there exist functions $\mu, \beta, \gamma \in K^+$, $\sigma \in KL$ and $a \in K_\infty$, such that for each $(t_0, x_0) \in \Re^+ \times \Re^n$ there exists a measurable input $u(\cdot, t_0, x_0) : \Re^+ \to \Re^m$ satisfying properties (S1–3). We define $\tilde{\phi}(t, t_0, x_0) := \phi(t, t_0, x_0; u(\cdot, t_0, x_0))$. Without loss of generality we may assume that $\beta(t) \geq 1$ for all $t \geq 0$ and that $\beta \in K^+$ is non-decreasing. Moreover, by virtue of Proposition 7 in [33] there exist functions a_1, a_2 of class K_∞, such that the KL function $\sigma(s, t)$ is dominated by $a_1^{-1}(\exp(-2t)a_2(s))$. Thus we obtain:

$$a_1\left(\left|H(t, \tilde{\phi}(t, t_0, x_0))\right| + \mu(t)\left|\tilde{\phi}(t, t_0, x_0)\right|\right) \\ \leq \exp(-2(t - t_0))\, a_2(\beta(t_0)\,|x_0|) \quad \text{for all } t \geq t_0. \tag{20}$$

For each $(\tau, x_0) \in \Re^+ \times \Re^n$ and each $0 \leq t_1 \leq \tau$ define the (possibly empty) set:

$$P(t_1, \tau, x_0) := \left\{\xi \in \Re^n\ ;\ \tilde{\phi}(\tau, t_1, \xi) = x_0\right\}. \tag{21}$$

Clearly, $P(\tau, \tau, x_0) := \{x_0\}$ for all $(\tau, x_0) \in \Re^+ \times \Re^n$. It follows from (20) that

$$\xi \in P(t_1, \tau, x_0) \quad \text{implies that } a_1(|H(\tau, x_0)| + \mu(\tau)\,|x_0|) \\ \leq \exp(-2(\tau - t_1))\, a_2(\beta(t_1)\,|\xi|). \tag{22}$$

Next define the function:

$$V(\tau, x_0) := inf\,\{\exp(-2(\tau - t_1))a_2(\beta(t_1)\,|\xi|)\,; \\ \xi \in P(t_1, \tau, x_0),\ 0 \leq t_1 \leq \tau\}. \tag{23}$$

Implication (22) and definition (23) imply

$$a_1\left(|H(\tau,x_0)|+\mu(\tau)\,|x_0|\right)\le V(\tau,x_0)$$
$$\le a_2(\beta(\tau)\,|x_0|),\quad \forall(\tau,x_0)\in\Re^+\times\Re^n. \tag{24}$$

Clearly, since for each $(\tau,x_0)\in\Re^+\times(\Re^n\backslash\{0\})$ it follows from (24) that $V(\tau,x_0)>0$, we may conclude that for each $(\tau,x_0)\in\Re^+\times(\Re^n\backslash\{0\})$ there exists $(t_1(\tau,x_0),\xi(\tau,x_0))\in[0,\tau]\times\Re^n$ with $\xi(\tau,x_0)\in P(t_1(\tau,x_0),\tau,x_0)$ such that $\exp(-2(\tau-t_1(\tau,x_0)))a_2\left(\beta(t_1(\tau,x_0))\,|\xi(\tau,x_0)|\right)\le\exp(\delta(\tau))V(\tau,x_0)$. Notice that, since $\beta(t)\ge 1$ for all $t\ge 0$, we obtain from (24) and the previous inequality

$$a_2\left(|\xi(\tau,x_0)|\right)\le\exp(2\tau+R)V(\tau,x_0)$$
$$\le\exp(2\tau+R)a_2(\beta(\tau)\,|x_0|),$$
$$\forall(\tau,x_0)\in\Re^+\times(\Re^n\backslash\{0\}), \tag{25}$$

where $R:=\sup\limits_{t\ge 0}\delta(t)$. Define for each $(t,\tau,x_0)\in\Re^+\times\Re^+\times\Re^n$:

$$k_\delta(t,\tau,x_0):=u(t,t_1(\tau,x_0),\xi(\tau,x_0))\quad\text{for } t\ge\tau \text{ and } x_0\ne 0. \tag{26}$$

$$k_\delta(t,\tau,0):=u(t,\tau,0)\quad\text{for all } t\ge\tau. \tag{27}$$

$$k_\delta(t,\tau,x_0):=k_\delta(\tau,\tau,x_0)\quad\text{for } t<\tau. \tag{28}$$

Clearly, since $u(\cdot,t_0,x_0):\Re^+\to\Re^m$ is measurable and locally bounded, it follows that $k_\delta(\cdot,\tau,x)$ is measurable and locally bounded for each fixed $(\tau,x)\in\Re^+\times\Re^n$.

Step 2: Stability of the closed-loop system.

Let the arbitrary partition $\pi=\{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$ with finite diameter and $\inf\{\tau_{i+1}-\tau_i-\delta(\tau_i);\ i=0,1,\dots\}\ge 0$. The dynamics of the closed-loop system (18) with $u(t)=k_\delta(t,\tau_i,x(\tau_i))$ for $t\in[\tau_i,\tau_{i+1})$ are described by the equation $\dot{x}(t)=\tilde{f}(t,x(t),x(\tau_i))$ for $t\in[\tau_i,\tau_{i+1})$, where $\tilde{f}(t,x,x_0):=f(t,x,k_\delta(t,\tau_i,x_0))$, for $t\in[\tau_i,\tau_{i+1})$. Notice that, property (S2) of Definition 3.1 in conjunction with definition (26, 27, 28), inequalities (24) and (25) and the fact that $\beta\in K^+$ is non-decreasing, imply that

$$\left|\tilde{f}(t,x,x_0)\right|\le\gamma(t)\,a\left(a_2^{-1}\left(\exp(2t+R)a_2(\beta(t)\,|x_0|)\right)+|x|\right)$$

for all $(t,x,x_0)\in\Re^+\times\Re^n\times\Re^n$. The above inequalities combined with (the repeated use of) Corollary 10 and Remark 11 in [33] imply that property (H4) is satisfied for appropriate functions $\gamma\in K^+$ and $a\in K_\infty$ for the closed-loop system (18) with $u(t)=k_\delta(t,\tau_i,x(\tau_i))$ for $t\in[\tau_i,\tau_{i+1})$, with the feedback function k as defined by (26, 27, 28). Property (H3) for the closed-loop system (18) with $u(t)=k_\delta(t,\tau_i,x(\tau_i))$ for $t\in[\tau_i,\tau_{i+1})$ and feedback function k as defined by (26, 27, 28) is an immediate consequence of inequality (19). Particularly, the fact that the vector field $\tilde{f}(t,x,x_0):=f(t,x,k_\delta(t,\tau_i,x_0))$ for $t\in[\tau_i,\tau_{i+1})$, is measurable and locally bounded with respect to $t\ge 0$ is an immediate consequence of the above inequality and the following facts: (i) the vector field $f(t,x,u)$ is measurable and locally bounded with respect to $t \geq 0$ and continuous with respect to $u \in \Re^m$ and (ii) $k_\delta(t, \tau, x_0)$ is measurable with respect to $t \geq 0$.

Next we show that the closed-loop system (18) with $u(t) = k_\delta\,(t, \tau_i, x\,(\tau_i))$ for $t \in [\tau_i, \tau_{i+1})$ and feedback function k as defined by (26, 27, 28) is RFC. By virtue of Lemma 2.4 it suffices to show that (8) holds for every $s \geq 0$, $\tau_i \in \pi$. Consider $x(t) \in \Re^n$ as the solution of the closed-loop system (18) with $u(t) = k_\delta\,(t, \tau_i, x\,(\tau_i))$ for $t \in [\tau_i, \tau_{i+1})$ at time $t \geq t_0 \in [\tau_i, \tau_{i+1})$ with the initial condition $x(t_0) = x_0$ and the additional condition $x(\tau_i) = x_1$, which holds only for the case $t_0 \neq \tau_i$. Assume first that $x_1 \neq 0$. Then for all $t \in [t_0, \tau_{i+1})$ definition (26,27,28) implies $u(t) = k(t, x_1) = u(t, t_1(\tau_i, x_1), \xi(\tau_i, x_1))$ and property (S3) in conjunction with (25) guarantees the estimate: $|x(t)| \leq \gamma(t)\,a\left(a_2^{-1}\,(\exp(2t + R)a_2(\beta(t)\,|x_1|)) + |x_0|\right)$ for all $t \in [t_0, \tau_{i+1}]$. Assume next that $x_1 = 0$. Then by virtue of property (S2) and definition (26, 27, 28) for all $t \in [t_0, \tau_{i+1})$ we have $u(t) = k_\delta(t, \tau_i, 0) = u(t, \tau_i, 0)$. In this case property (S3) guarantees the estimate $|x(t)| \leq \gamma(t)\,a\,(|x_0|)$ for all $t \in [t_0, \tau_{i+1}]$. Thus in any case the estimate $|x(t)| \leq \gamma(t)\,a\left(a_2^{-1}\,(\exp(2t + R)a_2(\beta(t)\,|x_1|)) + |x_0|\right)$ holds for all $t \in [t_0, \tau_{i+1}]$ and consequently (8) holds.

Now let $i \in Z^+$, $x_0 \in \Re^n \backslash \{0\}$ and consider the solution $x(t) \in \Re^n$ of the closed-loop system (18) with $u(t) = k_\delta\,(t, \tau_i, x\,(\tau_i))$ for $t \in [\tau_i, \tau_{i+1})$ at time $t \geq \tau_i$ with the initial condition $x(\tau_i) = x_0$. It is clear that for all $\tau_i \leq t < \tau_{i+1}$ we have $u(t) = k_\delta\,(t, \tau_i, x(\tau_i)) = k_\delta(t, \tau_i, x_0) = u(t, t_1(\tau_i, x_0), \xi(\tau_i, x_0))$ and since $\xi(\tau_i, x_0) \in P(t_1(\tau_i, x_0), \tau_i, x_0)$ we have by virtue of definition (21)

$$x(t) = \tilde{\phi}(t, t_1(\tau_i, x_0), \xi(\tau_i, x_0)), \quad \text{for all } \tau_i \leq t \leq \tau_{i+1}, \tag{29}$$

where the reader should be reminded that $\tilde{\phi}(t, t_0, x_0) := \phi(t, t_0, x_0; u(\,\cdot\,, t_0, x_0))$. Definition (21) implies that $\xi(\tau_i, x_0) \in P(t_1(\tau_i, x_0), \tau_{i+1}, x(\tau_{i+1}))$and definition (23) gives

$$\begin{aligned} V(\tau_{i+1}, x(\tau_{i+1})) \leq \exp(-2(\tau_{i+1} - \tau_i)) \exp(-2(\tau_i - t_1(\tau_i, x_0))) \\ \times a_2\,(\beta(t_1(\tau_i, x_0))\,|\xi(\tau_i, x_0)|)\,. \end{aligned}$$

The above inequality in conjunction with the fact that $\inf\{\tau_{i+1} - \tau_i - \delta(\tau_i);\ i = 0,\ 1,\ \ldots\} \geq 0$ and the inequality $\exp(-2(\tau_i - t_1(\tau_i, x_0)))a_2\,(\beta(t_1(\tau_i, x_0))\,|\xi(\tau_i, x_0)|) \leq \exp(\delta(\tau_i))V(\tau_i, x_0)$ implies

$$V(\tau_{i+1}, x(\tau_{i+1})) \leq \exp(-(\tau_{i+1} - \tau_i))V(\tau_i, x(\tau_i)). \tag{30}$$

Moreover, inequality (20) in conjunction with (29) and inequality $\exp(-2(\tau_i - t_1(\tau_i, x_0)))a_2\left(\beta(t_1(\tau_i, x_0))\left|\xi(\tau_i, x_0)\right|\right) \leq \exp(\delta(\tau_i))V(\tau_i, x_0)$ gives

$$\exp(-R)a_1\,(|H(t, x(t))|) \leq V(\tau_i, x(\tau_i)) \quad \text{for all } \tau_i \leq t \leq \tau_{i+1}, \tag{31}$$

where $R := \sup_{t \geq 0}, \delta(t)$. For the case $x_0 = 0$, property (S3) in conjunction with definition (27) implies $x(t) = 0$ for all $\tau_i \leq t \leq \tau_{i+1}$ and estimates (30), (31) hold for this case too. Consequently, by virtue of inequalities (24), (30) and (31) all requirements of Proposition 2.9 are fulfilled [specifically, (14,15) hold with $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$ and $\rho(s) := s$]. Thus the closed-loop system (18) with $u(t) = k_\delta\,(t, \tau_i, x\,(\tau_i))$ for $t \in [\tau_i, \tau_{i+1})$ satisfies hypotheses (H1–4) and is non-uniformly in time RGAOS. The proof is complete. ◃

Remark 3.3

(i) It is clear that properties (S1–2) in Definition 3.1 are direct extensions of the analogous notions used for the autonomous case of state regulation. Property (S3) in the definition of GAOC is introduced because we cannot guarantee that $t_0 \in \pi$ (if the partition contains the initial time, i.e., $t_0 \in \pi$ then property (S3) can be omitted). In the literature, where autonomous systems are studied it is always assumed that $t_0 = 0 \in \pi$ and thus property (S3) is omitted.

(ii) Proposition 3.2 guarantees robustness of the closed-loop system with respect to the sampling times. This feature is important for implementation purposes.

(iii) Compared to the main result in [19], Proposition 3.2 has certain advantages: **(a)** it is more general (covers the case of time-varying systems with not necessarily uniform in time convergence to the equilibrium point), **(b)** applies to the case where the open-loop controls are not necessarily bounded, **(c)** applies to the case where only the output – and not necessarily the whole state vector – can be led to zero and finally **(d)** the constructed feedback guarantees stabilization even in the case of partitions with zero lower diameter (notice that the only requirement for $\delta \in K^+$ is to be bounded).

(iv) Following the proof of Proposition 3.2, a special case should be noted where the feedback function can be explicitly given: if there exists constant $T > 0$ such that the solution $\phi(t, t_0, x_0; u(\cdot, t_0, x_0))$ of (18), corresponding to the open-loop control $u(\cdot, t_0, x_0)$ provided by the definition of GAOC, satisfies $\phi(t, t_0, x_0; u(\cdot, t_0, x_0)) = 0$ for all $t \geq t_0 + T$ (case of null complete controllability) then the feedback function $k_T : \Re^+ \times \Re^+ \times \Re^n \to \Re^m$ that corresponds to a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ with finite diameter and $\inf\{\tau_{i+1} - \tau_i;\ i = 0,\ 1,\ \ldots\} \geq T$ coincides with the open-loop control, i.e., $k_T(t, \tau, x) := u(t, \tau, x)$.

(v) If system (18) satisfies properties (S1–2) of Definition 3.1, then a preliminary ordinary feedback may be used in order to satisfy all the properties (S1–3). Particularly, it may be possible to construct a C^0 function $k : \Re^+ \times \Re^n \to \Re^m$ with $k(t, 0) = 0$ for all $t \geq 0$ being locally Lipschitz in x, in such a way that system (18) with $u(t) = k(t, x(t)) + v(t)$ satisfies properties (S1–3) of Definition 3.1 from the input v. In this case it should be noted that property (S1) of Definition 3.1 for system (18) with $u(t) = k(t, x(t)) + v(t)$ is indeed satisfied by the family of inputs $v(t, t_0, x_0) := -k(t, \phi(t, t_0, x_0; u(\cdot, t_0, x_0))) + u(t, t_0, x_0)$, where $u(\cdot, t_0, x_0) : \Re^+ \to \Re^m$ is the "original" family of inputs for system (18). The following example illustrates this point.

Example 3.4 Consider Artstein's system:

$$\begin{aligned} \dot{x} &= u(x^2 - y^2) \\ \dot{y} &= 2uxy \\ (x, y) &\in \Re^2, \quad u \in \Re \end{aligned} \tag{32}$$

or in complex form $\dot{z}(t) = u(t)z^2(t)$ with $z(t) = x(t) + iy(t)$ (i here denotes the imaginary unit). Consider the family of inputs $u(t, t_0, x_0) \equiv -1$ if $x_0 \geq 0$ and $u(t, t_0, x_0) \equiv 1$ if $x_0 < 0$ [i.e., $u(t, t_0, x_0) \equiv -\mathrm{sgn}(x_0)$]. It is clear that properties (S1–2) of Definition 3.1 are satisfied with $H(t, x) := x$, $\beta(t) \equiv 1$, $\gamma(t) \equiv 1$, $\mu(t) \equiv 1$, $a(s) := 4s^2$ and $\sigma(s, t) := 2s/\sqrt{1 + s^2 t^2}$. However, property (S3) of

Definition 3.1 is not satisfied for this family of inputs [take for example $\xi = (2, 0)'$ and $(x_0, y_0) = (-1, 0)'$]. In order to satisfy all properties (S1–3) of Definition 3.1 we consider the system (32) with $u(t) = k(t, x(t)) + v(t) = -x(t) + v(t)$. Moreover, we consider the family of inputs

$$\begin{aligned} v(t, t_0, x_0, y_0) &:= -k(t, \phi(t, t_0, (x_0, y_0); u(\cdot, t_0, x_0))) + u(t, t_0, x_0) \\ &= \frac{x_0\,(1 + |x_0|\,(t - t_0)) + \operatorname{sgn}(x_0) y_0^2 (t - t_0)}{(1 + |x_0|\,(t - t_0))^2 + y_0^2 (t - t_0)^2} + u(t, t_0, x_0). \end{aligned}$$

Since $|v(t, t_0, x_0, y_0)| \le 1 + |x_0| + |y_0|$ for all $t \ge t_0$, we find by evaluating the derivative of the function $V(x, y) = x^2 + y^2$ along the trajectories of system (32) with $u(t) = k(t, x(t)) + v(t) = -x(t) + v(t)$:

$$\dot{V} \le |v(t)|^2\, V(x, y).$$

It follows that for every $\xi \in \Re^2$ and $t_1 \ge t_0$ the solution of system (32) with $u(t) = -x(t) + v(t, t_0, x_0, y_0)$ exists for all $t \ge t_1$ and satisfies $|(x(t), y(t))| \le \exp\left((t - t_1)\,(1 + |x_0| + |y_0|)^2\right) |\xi| \le \exp\left(t^2 + (1 + |x_0| + |y_0|)^2\right) |\xi|$ for all $t \ge t_1$. Thus the previous inequality implies that property (S3) of Definition 3.1 is satisfied for $\gamma(t) := \exp(t^2)$ and $a(s) := e\ s + \left(\exp\left((1 + 2s)^2\right) - e\right)^2$. Properties (S1–2) of Definition 3.1 are satisfied as well for the family of inputs $v(t, t_0, x_0, y_0)$ for system (32) with $u(t) = -x(t) + v(t, t_0, x_0, y_0)$. Following the proof of Proposition 3.2 and for every partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$with finite diameter we obtain the hybrid feedback:

$$\begin{aligned} u(t, \tau_i, x(t), x(\tau_i), y(\tau_i)) = &-x(t) \\ &+ \frac{x(\tau_i)\,(1 + |x(\tau_i)|\,(t - \tau_i)) + \operatorname{sgn}(x(\tau_i)) y^2(\tau_i)(t - \tau_i)}{(1 + |x(\tau_i)|\,(t - \tau_i))^2 + y^2(\tau_i)(t - \tau_i)^2} \\ &- \operatorname{sgn}(x(\tau_i)), \quad \text{for } t \in [\tau_i, \tau_{i+1}). \end{aligned}$$

The hybrid feedback given by the formula above guarantees that the closed-loop system (32) is non-uniformly in time RGAS for every partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\Re^+$with finite diameter. $\triangleleft$

4 Applications to linear controllable systems

In this section we consider linear controllable time-varying systems of the form:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ x(t) &\in \Re^n, \quad u(t) \in \Re^m, \end{aligned} \tag{33}$$

where $A(t)$, $B(t)$ are matrices of appropriate dimensions with continuous components. It is shown that under the assumption of complete controllability there exists a linear time-varying hybrid feedback that guarantees finite-time stability for the nominal closed-loop system and induces the non-uniform in time ISS property from the inputs that quantify the modeling, actuator and state measurement errors for partitions with positive lower diameter. It should be noted that systems of the form (33) arise as parts of autonomous non-linear systems of the form:

$$\dot{\xi} = f(\xi),$$
$$\dot{x} = A(\xi)x + B(\xi)u,$$

where the solution $\xi(t, \xi_0)$ with the initial condition $\xi(0) = \xi_0$ may be substituted into the matrices A, B to produce the linear time-varying system (33) with $A(t) = A(\xi(t, \xi_0))$, $B(t) = B(\xi(t, \xi_0))$.

Let $\Phi(t, t_0)$ be the fundamental solution matrix corresponding to (33) with $u(t) \equiv 0$ and $\Phi(t_0, t_0) = I$ (I denotes the identity matrix). We will assume that system (33) is completely controllable in time T in the sense described in [32], where $T > 0$ is a constant. Particularly, we will assume that there exists a function $\gamma \in K^+$, such that for each $t_0 \in \Re^+$ there exists a matrix $R(\cdot, t_0) : \Re^+ \to \Re^{m \times n}$ with measurable and locally bounded components satisfying the following properties.

(S4) *The solution* $x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)R(s, t_0)\mathrm{d}s\ x_0$ *of (33) with initial condition* $x_0 \in \Re^n$ *corresponding to input* $u(t) = R(t, t_0)x_0$ *satisfies* $x(t) = 0$ *for all* $t \geq t_0 + T$.

(S5) $|R(t, t_0)| \leq \gamma(t)$, *for all* $t \geq t_0$.

We consider the hybrid feedback stabilization problem for sampling schedules $\pi = \{\tau_i\}_{i=0}^{\infty}$, with $\inf\{\tau_{i+1} - \tau_i;\ i = 0,\ 1,\ \ldots\} \geq T$ (i.e., the lower diameter of the partition must be greater than or equal to T). According to Remark 3.3(iv), the feedback function that solves this problem is given by

$$k(t, \tau, x) := R(t, \tau)x \quad \text{for } t \geq \tau. \tag{34}$$

It is clear that hypotheses (S4–5) imply that for all $(t_0, x_0, x_1) \in \Re^+ \times \Re^n \times \Re^n$ the solution of the closed-loop system (33) with $u(t) = R(t, \tau_i)x(\tau_i)$ for $t \in [\tau_i, \tau_{i+1})$, initial condition $x(t_0) = x_0$ and the additional condition $x(p(t_0)) = x_1$, which holds only for the case $t_0 \notin \pi$ [recall that $p(t) = \max\{\tau_i\ ;\ \tau_i \in \pi\ ,\ \tau_i \leq t\}$], satisfies:

$$|x(t)| \leq \beta(t_0)\max\{|x_0|\ ;\ |x_1|\}, \quad \forall t \geq t_0, \tag{35}$$

$$x(t) = 0,\ \ \forall t \geq \tau_{j+2},\ \ p(t_0) = \tau_j, \tag{36}$$

where $\beta(t) := g(t)(1 + g(t + r))$, $g \in K^+$ is a non-decreasing function that satisfies

$$g(t) \geq \max_{t \leq s \leq t+r} |\Phi(s, t)| + \max_{t \leq s \leq t+r} \left| \int_t^s \Phi(s, l)B(l)R(l, p(t))\mathrm{d}l \right|$$

for all $t \geq 0$ and $r := \sup\{\tau_{i+1} - \tau_i;\ i = 0,\ 1,\ 2,\ \ldots\}$ is the diameter of the partition. Moreover, it follows from estimates (35, 36) that the following estimate holds:

$$|x(t)| \leq \sigma\ (t - t_0)\ \beta(t_0) \max\{|x_0|\,;\,|x_1|\}, \quad \forall t \geq t_0, \tag{37}$$

where

$$\sigma(s,t)=\begin{cases}\left(3-t\,r^{-1}\right) & \text{if} \quad t\in[0,\ 3]\\ 0 & \text{if} \quad t>3\end{cases}.$$

The phenomenon of finite-time stability occurs for the closed-loop system (33) with $u(t)=R(t,\tau_i)x(\tau_i)$ for $t\in[\tau_i,\tau_{i+1})$. Next we consider the perturbed version of system (33):

$$\begin{aligned}&\dot{x}(t)=A(t)x(t)+B(t)u(t)+w(t)\\ &x(t)\in\Re^n,\quad u(t)\in\Re^m,\quad w(t)\in\Re^n\end{aligned}\tag{38}$$

under the feedback law $u(t)=R(t,\tau_i)(x(\tau_i)+e(\tau_i))+v(t)$ for $t\in[\tau_i,\tau_{i+1})$, where the input $v(t)\in\Re^m$ represents the control actuator error, the input $w(t)\in\Re^n$ represents modeling errors and the input $e(t)\in\Re^n$ represents the state measurement error. Let $\gamma_i\in K^+\quad i=1,\ 2,\ 3$ functions that satisfy the following inequalities for all $t\geq 0$

$$\begin{aligned}&\gamma_1(t)\geq r\max_{0\leq s\leq r}|\Phi(t+s,t)|\,,\quad \gamma_2(t)\geq r\max_{0\leq s\leq r}|\Phi(t+s,t)B(t)|\,,\\ &\gamma_3(t)\geq r\max_{0\leq s\leq r}|\Phi(t+s,t)B(t)R(t,p(t))|\,.\end{aligned}\tag{39}$$

The following estimate is an immediate consequence of definition (39) and linearity of system (38) with $u(t)=R(t,\tau_i)(x(\tau_i)+e(\tau_i))+v(t)$ for $t\in[\tau_i,\tau_{i+1})$:

$$\begin{aligned}|x(t)|\leq g(p(t))\,|x(p(t))|+&\sup_{s\in[p(t),t]}\left(\gamma_1(s)\ |w(s)|+\gamma_2(s)\ |v(s)|\right.\\ &\left.+\gamma_3(s)\ |e(p(s))|\right),\quad \forall t\geq\tau_{j+1},\ \ p(t_0)=\tau_j\end{aligned}\tag{40}$$

$$\begin{aligned}|x(\tau_i)|\leq\sup_{s\in[\tau_{i-1},\tau_i]}&\left(\gamma_1(s)\ |w(s)|+\gamma_2(s)\ |v(s)|\right.\\ &\left.+\gamma_3(s)\ |e(p(s))|\right),\quad \forall i\geq j+2,\ \ p(t_0)=\tau_j.\end{aligned}\tag{41}$$

It follows from estimates (40, 41) that the following estimate holds for all $(t_0,x_0,x_1)\in\Re^+\times\Re^n\times\Re^n$ and $t\geq t_0$ for the solution of the closed-loop system (38) with $u(t)=R(t,\tau_i)(x(\tau_i)+e(\tau_i))+v(t)$ for $t\in[\tau_i,\tau_{i+1})$, initial condition $x(t_0)=x_0$ and the additional condition $x(p(t_0))=x_1$, which holds only for the case $t_0\notin\pi$:

$$\begin{aligned}|x(t)|\leq\ &\sigma\,(\,t-t_0\,)\,\beta(t_0)\max\{\,|x_0|\ ;\ |x_1|\,\}\\ &+\sup_{s\in[t_0,t]}\sigma\,(\,t-s\,)\,\beta(s)\,\gamma_1(s)\ |w(s)|\\ &+\sup_{s\in[t_0,t]}\sigma\,(\,t-s\,)\,\beta(s)\,\gamma_2(s)\ |v(s)|\\ &+\sup_{s\in[p(t_0),t]}\sigma\,(\,t-s\,)\,\beta(s)\,\gamma_3(s)\ |e(p(s))|\,.\end{aligned}\tag{42}$$

Thus we may conclude from estimate (42) that the closed-loop system (38) with $u(t)=R(t,\tau_i)(x(\tau_i)+e(\tau_i))+v(t)$ for $t\in[\tau_i,\tau_{i+1})$ satisfies the non-uniform in time ISS property from the inputs w, v, e. Moreover, if the matrices $A(t)$, $B(t)$ and $R(\,\cdot\,,t_0)\colon\Re^+\to\Re^{m\times n}$ are bounded with respect to time then it is clear from definitions (39) and $\beta(t):=g(t)(1+g(t+r))$ that the closed-loop system (38) with $u(t)=R(t,\tau_i)(x(\tau_i)+e(\tau_i))+v(t)$ for $t\in[\tau_i,\tau_{i+1})$ satisfies the uniform ISS property from the inputs w, v, e.

5 Conclusions

In the present work characterizations of the notion of non-uniform in time RGAOS for hybrid systems with disturbances are given. Based on the provided characterizations, it is shown that every asymptotically output controllable time-varying control system can be stabilized (in general non-uniformly in time) by means of time-varying hybrid feedback. The obtained results cover the cases of time-varying control systems where only the output can be led to zero (output regulation problems) as well as the case of partitions with not necessarily positive lower diameter. The constructed feedback guarantees robustness of the closed-loop system with respect to the sampling times. For the linear time-varying case under the hypothesis of complete controllability it is shown that there exists a linear time-varying hybrid feedback that guarantees robustness with respect to modeling, actuator and measurement errors for partitions of positive lower diameter.

Appendix

Proof of Proposition 2.3 It suffices to show that for every $\varepsilon > 0$, $T, h \geq 0$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that if $\max\{ |x_0| ; |x_1| \} < \delta$, $t_0 \in [0, T]$ then the unique solution $x(t) = \phi(t, t_0, x_0, x_1; d, 0) \in \Re^n$ of (1) with $u(t) \equiv 0$ at time $t \geq t_0$ with initial condition $x(t_0) = x_0$ and the additional condition $x(p(t_0)) = x_1$, which holds only for the case $t_0 \notin \pi$, corresponding to the input $d \in M_D$ exists for all $(t, d) \in [t_0, t_0 + h] \times M_D$ and satisfies $\sup\{ |\phi(t, t_0, x_0, x_1; d, 0)| ;\ t \in [t_0, t_0 + h],\ \ d \in M_D \} < \varepsilon$. We first show the following claim.

Claim 1 *For every $\varepsilon > 0$, $i \in Z^+$ there exists $\delta_i > 0$, such that if $\max\{ |x(\tau_i)| ; |x(t_0)| \} < \delta_i$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(t_0)$ at time $t_0 \in [\tau_i, \tau_{i+1})$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [t_0, \tau_{i+1}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_{i+1}]$.*

Proof of Claim 1 Let $L > 0$ be the constant that satisfies (5) for the compact sets $S := B[0, \varepsilon] \times B[0, \varepsilon] \times \{0\} \times \{0\}$ and $I := [\tau_i, \tau_{i+1}]$, where $B[0, \varepsilon]$ denotes the closed sphere in $\Re^n$ of radius $\varepsilon > 0$ centered at zero. It follows from (5) and hypothesis (H4) that the following inequality holds for all $x, x_i \in B[0, \varepsilon]$, $t \in I$ and $d, d_i \in D$:

$$x' f(t, d, d_i, x, x_i, 0, 0) \leq (L + M) \left(|x|^2 + a^2(|x_i|) \right), \tag{43}$$

where

$$M := \frac{1 + \max\left\{ \gamma^2(t) \ ;\ t \in I \right\}}{2}$$

and $\gamma \in K^+$, $a \in K_\infty$ are the functions involved in hypothesis (H4). Let $\rho > 0$ be the unique solution of the equation

$$\varepsilon^2 = 4 \exp\left(2(L + M)\, r\right) \left(\rho^2 + 2a^2(\rho) \right) \tag{44}$$

and define

$$\delta_i = min\left\{ \frac{\varepsilon}{2} \ ;\ \rho \right\}. \tag{45}$$

Consider the arbitrary initial conditions $x(\tau_i)$, $x(t_0) \in \Re^n$ with $\max\{|x(\tau_i)|\,;\ |x(t_0)|\} < \delta_i$ and arbitrary input $d \in M_D$ and consider the unique solution $x(t) \in \Re^n$ of (1) with $u(t) \equiv 0$ starting from $x(t_0)$ and corresponding to input $d \in M_D$. Since (45) implies $\delta_i < \varepsilon$, it follows that $|x(t_0)| < \varepsilon$. Let t_ε be the maximal time in the interval $[t_0, \tau_{i+1}]$ such that $|x(t)| < \varepsilon$ for all $t \in [t_0, t_\varepsilon)$. By virtue of continuity of the solution with respect to time, the maximal time t_ε is well defined. We consider two cases: $t_\varepsilon < \tau_{i+1}$ and $t_\varepsilon = \tau_{i+1}$. The first case $t_\varepsilon < \tau_{i+1}$ cannot hold since by continuity of the solution with respect to time we must have $|x(t_\varepsilon)| = \varepsilon$. On the other hand, inequality (43) implies $\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le 2(L+M)\left(|x(t)|^2 + a^2(\rho)\right)$ for almost all $t \in [t_0, t_\varepsilon]$. The previous differential inequality, in conjunction with (44), $t_\varepsilon < \tau_{i+1} \le \tau_i + r$ and inequality $|x(t_0)| < \rho$ directly implies that $|x(t_\varepsilon)| \le \frac{\varepsilon}{2} < \varepsilon$ which contradicts $|x(t_\varepsilon)| = \varepsilon$. Thus we must have $t_\varepsilon = \tau_{i+1}$ and repeating the previous analysis we find $\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le 2(L+M)\left(|x(t)|^2 + a^2(\rho)\right)$ for almost all $t \in [t_0, t_{i+1}]$, which again implies $|x(\tau_{i+1})| \le \frac{\varepsilon}{2} < \varepsilon$. We conclude that $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_{i+1}]$. Consequently, Claim 1 is proved.

Claim 2 *For every $\varepsilon > 0$, $N, i \in Z^+$ there exists $\delta > 0$, such that if $|x(\tau_i)| < \delta$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(\tau_i)$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [\tau_i, \tau_{i+N}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [\tau_i, \tau_{i+N}]$.*

Proof of Claim 2 We prove Claim 2 by induction on $N \in Z^+$. Clearly, by virtue of Claim 1, Claim 2 holds for $N = 1$. Suppose that Claim 2 holds for certain $N \in Z^+$. We show next that there exists $\tilde{\delta} > 0$, such that if $|x(\tau_i)| < \tilde{\delta}$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(\tau_i)$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [\tau_i, \tau_{i+N+1}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [\tau_i, \tau_{i+N+1}]$.

By virtue of Claim 1, for every $\varepsilon > 0$, $N, i \in Z^+$ there exists $\delta_{i+N} > 0$, such that if $|x(\tau_{i+N})| < \delta_{i+N}$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(\tau_{i+N})$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [\tau_{i+N}, \tau_{i+N+1}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [\tau_{i+N}, \tau_{i+N+1}]$. Since Claim 2 holds for $N \in Z^+$, for every $\varepsilon > 0$, $i \in Z^+$ there exists $\delta > 0$, such that if $|x(\tau_i)| < \delta$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(\tau_i)$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [\tau_i, \tau_{i+N}] \times M_D$ and satisfies $|x(t)| < min\{\varepsilon;\ \delta_{i+N}\}$ for all $t \in [\tau_i, \tau_{i+N}]$. The two previous statements imply the desired statement for $\tilde{\delta} := min\{\varepsilon\,;\ \delta_{i+N}\}$. Consequently, Claim 2 is proved.

We complete the proof of the proposition by combining the two claims above. Let arbitrary $\varepsilon > 0$, $T, h \ge 0$. There exist $N, i \in Z^+$ such that $T \in [\tau_i, \tau_{i+1})$ and $T + h \le \tau_{i+N}$. By virtue of Claim 2, for each $j \in \{0, 1, \ldots, i\}$ there exists $\tilde{\delta}_j > 0$, such that if $\left|x(\tau_{j+1})\right| < \tilde{\delta}_j$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(\tau_{j+1})$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [\tau_{j+1}, \tau_{i+N}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [\tau_{j+1}, \tau_{i+N}]$. By virtue of Claim 1, for each $j \in \{0, 1, \ldots, i\}$ there exists $\delta_j > 0$, such that if $\max\left\{\left|x(\tau_j)\right|\ ;\ |x(t_0)|\right\} < \delta_j$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(t_0)$ at time $t_0 \in [\tau_j, \tau_{j+1})$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [t_0, \tau_{j+1}] \times M_D$ and satisfies $|x(t)| < \tilde{\delta}_j$ for all $t \in [t_0, \tau_{j+1}]$. Combining the two previous statements we find the following.

Property *For each $j \in \{0, 1, \ldots, i\}$ there exists $\delta_j > 0$, such that if $\max\left\{ \left|x(\tau_j)\right|; |x(t_0)| \right\} < \delta_j$ then the unique solution of (1) with $u(t) \equiv 0$ starting from $x(t_0)$ at time $t_0 \in [\tau_j, \tau_{j+1})$ and corresponding to input $d \in M_D$ exists for all $(t, d) \in [t_0, \tau_{i+N}] \times M_D$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, \tau_{i+N}]$*

We set $\delta(\varepsilon, T, h) := min\left\{ \delta_j; j = 0, \ldots, i \right\}$. Let arbitrary $d \in M_D$, $t_0 \in [0, T]$ and $(x_0, x_1) \in \Re^n \times \Re^n$ with $\max\{ |x_0|; |x_1| \} < \delta$. Since $T \in [\tau_i, \tau_{i+1})$, there exists $j \in \{0, 1, \ldots, i\}$ such that $t_0 \in [\tau_j, \tau_{j+1})$. Moreover, since $T + h \leq \tau_{i+N}$ (which implies $t_0 + h \leq \tau_{i+N}$) and $\max\{ |x_0|; |x_1| \} < \delta$ (which implies $\max\left\{ \left|x(\tau_j)\right|; |x(t_0)| \right\} < \delta_j$), the property stated above implies that the unique solution of (1) with $u(t) \equiv 0$ with the initial condition $x(t_0) = x_0$ and the additional condition $x(p(t_0)) = x_1$, which holds only for the case $t_0 \notin \pi$, corresponding to input $d \in M_D$ exists for all $t \in [t_0, t_0 + h]$ and satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, t_0 + h]$. The proof is complete. $\lhd$

Proof of Lemma 2.4 Notice that if system (1) is RFC from the input u then condition (9) is automatically satisfied and thus (9) is a necessary condition for RFC of system (1). We will show next that (9) is a sufficient condition for RFC of system (1). The following claim is established next.

Claim 3 *For every $s \geq 0$, $N, i \in Z^+$ it holds that*

$$\delta_{i,N}(s) := \sup\left\{ |\phi(t, \tau_i, x_0;\ d, u)|;\ u \in M_U, \sup_{\tau \geq 0} |u(\tau)| \leq s,\ t \in [\tau_i, \tau_{i+N}], |x_0| \leq s,\ d \in M_D \right\} < +\infty.$$

Proof of Claim 3 We prove Claim 3 by induction on $N \in Z^+$. Clearly, by virtue of condition (9), Claim 3 holds for $N = 1$. Suppose that Claim 3 holds for certain $N \in Z^+$. Notice that $\delta_{i,(N+1)}(s) \leq \max\left\{ \delta_{i,N}(s);\ \delta_{N,(N+1)}\left(\delta_{i,N}(s)\right) \right\}$ and since $\delta_{i,N}(s) < +\infty$, $\delta_{N,(N+1)}\left(\delta_{i,N}(s)\right) < +\infty$ [an immediate consequence of the statement of (9)], we obtain $\delta_{i,(N+1)}(s) < +\infty$. Consequently, Claim 3 is proved.

Let arbitrary $s \geq 0$, $T \geq 0$. There exist $N, i \in Z^+$ such that $T \in [\tau_i, \tau_{i+1})$ and $2T \leq \tau_{i+N}$. Notice that we have

$$\begin{aligned} &\sup\left\{ |\phi(t_0 + h, t_0, x_0, x_1; d, u)|; u \in M_U, \sup_{\tau \geq 0} |u(\tau)| \leq s,\ h \in [0, T], \right. \\ &\quad \left. \max\{ |x_0|; |x_1| \} \leq s,\ t_0 \in [0, T],\ d \in M_D \right\} \\ &\leq \max\left\{ \delta_{j+1,N+i-j-1}(a_j(s));\ j = 0,\ 1, \ldots, i \right\}, \end{aligned}$$

where $a_j(s) := \sup\left\{ |\phi(t, t_0, x_0, x_1; d, u)|;\ u \in M_U, \sup_{\tau \geq 0} |u(\tau)| \leq s,\ t \in [t_0,\ \tau_{j+1}],\ \max\{ |x_0|;\ |x_1| \} \leq s,\ t_0 \in [\tau_j, \tau_{j+1}),\ d \in M_D \right\}$. Condition (9) implies that $a_j(s) < +\infty$ for $j = 0, \ldots, i$ and by virtue of Claim 3 it follows that $\max\left\{ \delta_{j+1,N+i-j-1}(a_j(s));\ j = 0, 1, \ldots, i\right\} < +\infty$. Thus we have established RFC for system (1). In order to establish the equivalence of (8) with RFC the reader may repeat all previous arguments with $u(t) \equiv 0$.

The proof is complete. $\lhd$

Proof of Proposition 2.9 Suppose first that system (1) with $u(t) \equiv 0$ is non-uniformly in time RGAOS. Then by virtue of statement (ii) of Theorem 2.7 there exist functions $\mu, \beta \in K^+$, $\sigma \in KL$ such that for every $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$, estimate (10) holds. Moreover, by recalling Proposition 7 in [33] there exist functions a_1, a_2 of class K_∞, such that the KL function $\sigma(s,t)$ is dominated by a_1^{-1} $(\exp(-2t)a_2(s))$. Furthermore, by virtue of Remark 2.1(ii), we know that if $t_0 \in \pi$, the solution $x(t) \in \Re^n$ of (1) for $t \geq t_0$ does not depend on $x_1 \in \Re^n$ but depends only on $x_0 \in \Re^n$ [and in this case we may write $x(t) = \phi(t, t_0, x_0; d, u) \in \Re^n$ with $t \geq t_0, x(t_0) = x_0 \in \Re^n$]. Combining the two previous observations with estimate (10) we obtain the following estimate that holds for all $(t_0, x_0, d) \in \pi \times \Re^n \times M_D$:

$$\begin{aligned} &a_1 \left(|H(t, \phi(t, t_0, x_0; d, 0))| + \mu(t)\,|\phi(t, t_0, x_0; d, 0)|\right) \\ &\quad \leq \exp\left(-2(t - t_0)\right) a_2 \ \left(\beta(t_0)\ |x_0|\,\right), \quad \forall t \geq t_0. \end{aligned} \tag{46}$$

We define for all $(t_0, x_0) \in \pi \times \Re^n$

$$\begin{aligned} V(t_0, x_0) := \sup\{&\exp\left(t - t_0\right) a_1 \left(|H(t, \phi(t, t_0, x_0; d, 0))|\right. \\ &\left. +\ \mu(t)\,|\phi(t, t_0, x_0; d, 0)|\right)\ ;\ t \geq t_0\,,\ d \in M_D\}\,. \end{aligned} \tag{47}$$

It can be verified that definition (47) in conjunction with estimate (46) guarantees that inequality (13) holds. Moreover, definition (47) guarantees that inequality (14) holds with $\eta(t, t_0, s) := \exp\left(-(t - t_0)\right) s$ [and consequently, (15) holds for $\rho(s) := s$, $\gamma(t) \equiv 1$ and $\varphi(t) \equiv 0$].

Conversely, suppose that there exist functions $V : \Re^+ \times \Re^n \to \Re^+$, $\mu, \beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t)\mathrm{d}t = +\infty$, $\varphi \in E$, $a_1, a_2, a \in K_\infty$ and a locally Lipschitz positive definite function $\rho : \Re^+ \to \Re^+$ with $\rho(s) < s$ for all $s > 0$, such that inequalities (13,14) hold with $\eta(t, t_0, \eta_0)$ being the unique solution of the initial value problem (15). By virtue of Lemma 2.6 and since system (1) with $u(t) \equiv 0$ is RFC, it suffices to show that the robust output attractivity property (property P3 of Definition 2.5) holds. Lemma 5.2 in [8] implies that there exists a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$0 \leq \eta(t, t_0, \eta_0) \leq \sigma\left(\eta_0 + M,\ q(t, t_0)\right), \quad \forall t \geq t_0,\ \ \forall \eta_0 \geq 0, \tag{48}$$

where $q(t, t_0) := \int_{t_0}^{t} \gamma(s)\mathrm{d}s$.

Let arbitrary $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ and consider $x(t) = \phi(t, t_0, x_0, x_1; d, 0) \in \Re^n$, the solution of (1) with $u(t) \equiv 0$, evaluated at time $t \geq t_0$ with initial condition $x(t_0) = x_0$ and the additional condition $x(p(t_0)) = x_1$, which holds only for the case $t_0 \notin \pi$, corresponding to input $d \in M_D$. Consider also the smallest integer i with $\tau_i > t_0$ and $\tau_i \in \pi$. The semigroup property for $\eta(t, t_0, \eta_0)$ in conjunction with inequality (14) and trivial induction arguments imply that

$$V(\tau_{i+N}, x(\tau_{i+N})) \leq \eta(\tau_{i+N}, \tau_i, V(\tau_i, x(\tau_i))) \tag{49}$$

for all non-negative integers N. Moreover, by virtue of Lemma 3.5 in [10] and Proposition 2.2, there exist functions $\mu \in K^+$ and $a \in K_\infty$ such that the following estimate holds for all $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$:

$$|x(t)| \leq \mu(t)a\left(\max\{\,|x_0|\,;\,|x_1|\,\}\right), \quad \forall t \geq t_0 \tag{50}$$

Combining estimates (48), (49), (50) and (13) we have

$$V(\tau_{i+N}, x(\tau_{i+N})) \le \sigma\ (a_2\,(\beta(\tau_i)\mu(\tau_i)a\,(\max\{\,|x_0|\,;\,|x_1|\,\})) \\ +M\ ,\ q(\tau_{i+N}, \tau_i)) \tag{51}$$

for all non-negative integers N. Estimate (51) in conjunction with (13) implies that

$$|H(t, x(t))| \le a_1^{-1}\,(\sigma\ (a_2\,(\beta(\tau_i)\mu(\tau_i)a\,(\max\{\,|x_0|\,;\,|x_1|\,\})) \\ +M\ ,\ q(\tau_{i+N}, \tau_i)))\,, \quad \text{for all } t \ge \tau_{i+N}. \tag{52}$$

The robust output attractivity property (property P3 of Definition 2.5) is an immediate consequence of estimate (52). The proof is complete. ◁

Proof of Corollary 2.11 The proof is based on the following observation: if (1) is *T-periodic* then for all $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ it holds that

$$\phi(t, t_0, x_0, x_1; d, 0) = \phi\left(t - \left[\frac{t_0}{T}\right]T, t_0 - \left[\frac{t_0}{T}\right]T, x_0, x_1; P(t_0)d, 0\right),$$

and

$$H(t, \phi(t, t_0, x_0, x_1; d, 0)) \\ = H\left(t - [t_0/T]\,T, \phi\left(t - \left[\frac{t_0}{T}\right]T, t_0 - \left[\frac{t_0}{T}\right]T, x_0, x_1; P(t_0)d, 0\right)\right),$$

where $[t_0/T]$ denotes the integer part of t_0/T and $P(t_0)d \in M_D$ is defined by

$$(P(t_0)d)\,(t) := d\left(t + \left[\frac{t_0}{T}\right]T\right), \quad \forall t \ge 0.$$

Since (1) with $u(t) \equiv 0$ is non-uniformly in time RGAOS, there exist functions $\sigma \in KL, \beta \in K^+$ such that (10) holds for all $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ and $t \in [t_0, +\infty)$. Consequently, it follows that the following estimate holds for all $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ and $t \in [t_0, +\infty)$:

$$|H(t, \phi(t, t_0, x_0, x_1; d, 0))| \\ \le \sigma\ \left(\beta\left(t_0 - \left[\frac{t_0}{T}\right]T\right)\max\{\,|x_0|\,;\,|x_1|\,\},\ t - t_0\right).$$

Since $0 \le t_0 - [t_0/T]\,T < T$, for all $t_0 \ge 0$, it follows that the following estimate holds for all $(t_0, x_0, x_1, d) \in \Re^+ \times \Re^n \times \Re^n \times M_D$ and $t \in [t_0, +\infty)$:

$$|H(t, \phi(t, t_0, x_0, x_1; d, 0))| \le \tilde{\sigma}\ \ (\max\{\,|x_0|\,;\,|x_1|\,\}\ ,\ t - t_0\,),$$

where $\tilde{\sigma}(s, t) := \sigma(R\,s, t)$ and $R := \max\{\,\beta(t);\ 0 \le t \le T\,\}$. The previous estimate in conjunction with Definition 2.10 implies that (1) with $u(t) \equiv 0$ is URGAOS. The proof is complete. ◁

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