SIAM J. CONTROL OPTIM.
Vol. 45, No. 1, pp. 320–342

FINITE-TIME GLOBAL STABILIZATION BY MEANS OF TIME-VARYING DISTRIBUTED DELAY FEEDBACK*

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Abstract. The paper contains certain results concerning the finite-time global stabilization for triangular control systems described by retarded functional differential equations by means of time-varying distributed delay feedback. These results enable us to present solutions to feedback stabilization problems for systems with delayed input. The results are obtained by using the backstepping technique.

Key words. nonlinear time delay systems, global feedback stabilization

AMS subject classifications. 93D15, 93C23, 93C10

DOI. 10.1137/040616383

1. Introduction. It is known that for finite-dimensional continuous-time control systems with locally Lipschitz dynamics (e.g., $\dot{x} = f(x, u)$, where f is locally Lipschitz with respect to $(x, u) \in \Re^n \times \Re^m$), finite-time global stabilization cannot be achieved by means of a locally Lipschitz feedback law. However, it has been shown that finite-time global stabilization is possible by means of continuous (see [2, 3, 4, 6, 8] as well as the reported results in [1]) or discontinuous feedback laws (see [14]).

Recently, the option of using feedback with delays has been considered for the stabilization of continuous-time systems in various problems. The closed-loop system may be considered as a system of time-varying retarded functional differential equations (RFDEs). For example, in [20] analytic driftless control systems of the following form are considered:

$$\begin{aligned}&\dot{x}(t) = f(x(t),\ u(t)) := \sum_{i=1}^{m} f_i(x(t))u_i(t)\\&x(t) \in \Re^n,\ u(t) := (u_1(t), \ldots, u_m(t))' \in \Re^m.\end{aligned}$$

The authors in [20] provide strategies for the construction of control laws of the form $u(t) = k(t, x(t), x(lT))$ for $t \in [lT, (l+1)T)$, where l is a nonnegative integer and $T > 0$ denotes the updating time-period of the control. Notice that this type of feedback is a time-varying feedback with delays of the form $u(t) = k(t, x(t), x([t/T]T))$, where $[t/T]$ denotes the integer part of t/T, which is time-varying even if k is independent of time, i.e., $k(t, x, \xi) = k(x, \xi)$. The same comments apply for the synchronous controller switching strategies proposed in [22]. The possibility of switching control laws using distributed delays was recently exploited in [17]. Observers that make use of past values of the state estimate and guarantee convergence in finite-time were considered in [5, 19]. The ability of output discrete delay feedback to stabilize minimum phase linear systems was studied in [9]. Recently, there has been an increasing interest in the feedback stabilization problems of systems with delayed input (see [15, 18]) as well as the application of the backstepping technique for the stabilization of nonlinear

*Received by the editors October 5, 2004; accepted for publication (in revised form) October 20, 2005; published electronically March 15, 2006.
http://www.siam.org/journals/sicon/45-1/61638.html
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time-delay systems (see [10, 16]). An account of the use of delays in linear feedback design is given in [21].

In the present work it is shown that finite-time global stabilization can be achieved by time-varying locally Lipschitz distributed delay feedback. It is known that systems described by time-varying RFDEs admit solutions that converge to the equilibrium point in finite time (e.g., Property 5.1 in Chapter 3 in [7]). Using the backstepping technique (see [13, 24]), the problem of finite-time global stabilization for nonlinear triangular systems is studied and solved. Moreover, the approach proposed in this paper is not limited to triangular finite-dimensional continuous-time control systems but can be directly applied to nonlinear triangular systems described by RFDEs. The case of delayed inputs is also considered. Among other cases, we address the finite-time global stabilization problems for the following cases:

- The case of triangular control systems with no delays,

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, x_1(t), \dots, x_i(t)) + x_{i+1}(t), \quad i = 1, \dots, n-1; \\ &\dot{x}_n(t) = f_n(t, x(t)) + u(t), \\ &x(t) := (x_1(t), \dots, x_n(t)) \in \Re^n, \; u(t) \in \Re, \; t \geq 0. \end{aligned} \tag{1.1}$$

- The case of a chain of delayed integrators with no limitation on the size of the delays,

$$\dot{x}_i(t) = x_{i+1}(t - \tau_i), i = 1, \dots, n-1; \quad \dot{x}_n(t) = u(t - \tau_n), \tag{1.2}$$

  where $x(t) = (x_1(t), \dots, x_n(t)) \in \Re^n, u(t) \in \Re$ and $\tau_i \geq 0$ $i = 1, \dots, n$ are the delays.

- The case of triangular control systems with delayed drift terms:

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, x_1(t - \tau_{i,1}), \dots, x_i(t - \tau_{i,i})) + x_{i+1}(t), \quad i = 1, \dots, n-1, \\ &\dot{x}_n(t) = f_n(t, x_1(t - \tau_{n,1}), \dots, x_n(t - \tau_{n,n})) + u(t), \\ &x(t) := (x_1(t), \dots, x_n(t)) \in \Re^n, u(t) \in \Re, t \geq 0, \end{aligned} \tag{1.3}$$

  where $\min_{i=1,\dots,n-1} \quad \min_{j=1,\dots,i} \tau_{i,j} > 0$.

The construction of the proposed distributed delay feedback control is based on a backstepping method, which is applicable to systems with delays. Roughly speaking, the main idea that lies behind the integrator backstepping lemma is described next (here $(T_r(t)x = \{x(t+\theta); \theta \in [-r, 0]\}$ denotes the *r-history* of the state):

> Suppose that the feedback $y(t) = k(t, T_r(t)x)$ leads the state of the system $\dot{x}(t) = f(t, x(t), y(t))$ to zero in finite time (say $T > 0$) and that the feedback $u(t) = \widetilde{k}(t, T_r(t)x, T_r(t)y)$ guarantees the equality $y(t) = k(t, T_r(t)x)$ in finite time (say $T' > 0$) for the augmented system $\dot{x}(t) = f(t, x(t), y(t)); \dot{y}(t) = u(t)$. Then the feedback $u(t) = \widetilde{k}(t, T_r(t)x, T_r(t)y)$ leads the state of the augmented system to zero in finite time.

The explanation of the conclusion is simple: the first T' time units are used in order to achieve $y(t) = k(t, T_r(t)x)$; the next T time units are used in order to achieve $x(t) = 0$ and finally the last r time units are used in order to achieve $x(t + \theta) = 0; \theta \in [-r, 0]$, which directly implies $y(t) = 0$. The idea above can be applied for the design of controllers using a step-by-step procedure (backstepping method).

The structure of the paper is as follows. In section 2, the reader can find definitions and technical results that will be used in later sections of this work. Section 2 is divided into two subsections. In the first subsection (section 2.1) we present differentiability notions for functionals. In the second subsection (section 2.2) we give the notion of finite-time stabilizability of control systems as well as some preliminary results for the scalar case. Section 3 contains the statement and the proofs of the main results of the paper. Examples are presented in section 4, where the main results of the paper can be directly applied. The conclusions of the paper are given in section 5.

Notation. Throughout this paper we adopt the following notation:

- For a vector $x \in \Re^n$ we denote by $|x|$ its Euclidean norm. For $x \in C^0([-r,0];\Re^n)$ we define $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$.
- By $C^j(A)(C^j(A;\Omega))$, where $j \geq 0$ is a nonnegative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
- We denote by K^+ the class of positive C^∞ functions defined on $\Re^+$. We say that an increasing and continuous function $\rho : \Re^+ \to \Re^+$ with $\rho(0) = 0$ is of class K_∞ if $\lim_{s\to+\infty} \rho(s) = +\infty$.
- Z^+ denotes the set of positive integers and $\Re^+$ the set of nonnegative real numbers. For every real number R, $[R]$ denotes its integer part, i.e., $[R] := \max\{x; x \leq R, x\, integer\}$.
- A continuous mapping $f : I \times C^0([-r,0];\Re^n) \times U \to \Re^k$, where $\Re^+ \subseteq I \subseteq \Re, U \subseteq \Re^m$, is said to be completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$ if for every bounded set $S \subset I \times C^0([-r,0];\Re^n) \times U$ there exists $L \geq 0$ such that $|f(t,x,u) - f(t,y,v)| \leq L\|x-y\|_r + L|u-v|$ for all $(t,x,u) \in S, (t,y,v) \in S$. Notice that a mapping $f : I \times C^0([-r,0];\Re^n) \times U \to \Re^k$, where $\Re^+ \subseteq I \subseteq \Re, U \subseteq \Re^m$, which is completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$, is also defined on $I \times C^0([-r-\sigma,0];\Re^n) \times U$, for every $\sigma \geq 0$, and is completely locally Lipschitz with respect to $(x,u) \in C^0([-r-\sigma,0];\Re^n) \times U$, for every $\sigma \geq 0$.
- Let $x : [a-r,b) \to \Re^n$ with $b > a \geq 0$ and $r \geq 0$. By $T_r(t)x$ we denote the "history" of x from $t-r$ to t, i.e., $T_r(t)x := \{x(t+\theta); \theta \in [-r,0]\}$, for $t \in [a,b)$.

2. Definitions and technical results. Consider the following control system described by retarded functional differential equations (RFDEs):

$$\dot{x}(t) = f(t, T_r(t)x, u(t-\tau(t))); \quad x(t) \in \Re^n, u(t) \in U, t \geq 0, \tag{2.1}$$

where $0 \in U \subseteq \Re^m$, $f : \Re^+ \times C^0([-r,0];\Re^n) \times U \to \Re^n$ is completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$ with $f(t,0,0) = 0$ for all $t \geq 0$ and $\tau : \Re^+ \to \Re^+$ is a bounded continuous function. We denote by $x(t) = x(t,t_0,x_0,u) \in \Re^n$ the solution of (2.1) initiated from $t_0 \geq 0$ with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ and corresponding to $u \in C^0(\Re;U)$. By virtue of Theorem 3.2 in [7], for every $(t_0,x_0,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0(\Re;U)$ there exists $t_{\max}(t_0,x_0,u) > t_0$ (called the maximal existence time) such that the solution $x(t) = x(t,t_0,x_0,u) \in \Re^n$ of (2.1) initiated from $t_0 \geq 0$ with initial condition $T_r(t_0)x = x_0$ and corresponding to $u \in C^0(\Re;U)$, is defined on $[t_0-r, t_{\max})$, is continuous on $[t_0-r, t_{\max})$ and continuously differentiable on $[t_0, t_{\max})$ and cannot be further continued, i.e., if $t_{\max} < +\infty$, then $\limsup_{t\to t_{\max}^-} |x(t)| = +\infty$. When $r = 0$ we identify the space $C^0([-r,0];\Re^n)$ with the finite-dimensional space $\Re^n$ and we obtain the familiar finite-dimensional continuous-time case. Consequently, all of the following definitions and results hold also for finite-dimensional continuous-time systems.

This section is divided into two subsections. In the first subsection (section 2.1) we present the differentiability notions for functionals used in the present paper. In the second subsection (section 2.2) we give the notion of finite-time stabilizability of control systems as well as some preliminary results for the scalar case.

2.1. Differentiability notions. In order to study the properties of control system (2.1), we must clarify the differentiability properties of functionals along the solutions of (2.1). The following definition provides the notions of ultimate differentiability and differentiability of functionals along the solutions of (2.1).

DEFINITION 2.1. *Let a functional $\varphi : I \times C^0([-r, 0]; \Re^n) \to \Re$, where $\Re^+ \subseteq I \subseteq \Re$, $\varphi(t, x)$ being completely locally Lipschitz with respect to $x \in C^0([-r, 0]; \Re^n)$, with $\varphi(t, 0) = 0$ for all $t \in I$. We say that*

- *φ is ultimately differentiable along the solutions of* (2.1) *with time constant $T \geq 0$, if there exists a constant $T \geq 0$ and a functional $D\varphi : I \times C^0([-r, 0]; \Re^n) \times U \to \Re$ (called the derivative of φ along the solutions of* (2.1) *with $D\varphi(t, x, u)$ being completely locally Lipschitz with respect to $(x, u) \in C^0([-r, 0]; \Re^n) \times U$ and $D\varphi(t, 0, 0) = 0$ for all $t \in I$, such that for every $(t_0, x_0, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times C^0(\Re; U)$ for which $t_0 + T < t_{\max}(t_0, x_0, u)$, the mapping $t \to \varphi(t, T_r(t)x)$ is of class C^1 on $[t_0 + T, t_{\max})$ and it holds that*

$$\frac{d}{dt}\varphi(t, T_r(t)x) = D\varphi(t, T_r(t)x, u(t - \tau(t))) \quad \forall t \in [t_0 + T, t_{\max}).$$

- *φ is differentiable along the solutions of* (2.1), *if φ is ultimately differentiable along the solutions of* (2.1) *with time constant $T = 0$.*

If $\varphi(t, x) = q(t, x(0))$, where $q \in C^1(\Re^+ \times \Re^n; \Re)$ is a function with locally Lipschitz derivatives and $q(t, 0) = 0$ for all $t \geq 0$, then φ is differentiable along the solutions of (2.1) with $D\varphi(t, x, u) := \frac{\partial q}{\partial t}(t, x(0)) + \frac{\partial q}{\partial x}(t, x(0))f(t, x, u)$ for all $(t, x, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times U$. This remark holds particularly for the case $r = 0$, where the functional $\varphi(t, x)$ is an ordinary function $\varphi : \Re^+ \times \Re^n \to \Re$ with $\varphi(t, 0) = 0$ for all $t \geq 0$. However, when delays are involved, the notion of ultimate differentiability of a functional is useful. The following example illustrates this point.

Example 2.2. Consider the following system:

$$\begin{aligned} &\dot{x}_1(t) = x_2(t); \qquad \dot{x}_2(t) = u(t) \\ &(x_1(t), x_2(t)) \in \Re^2, \quad u(t) \in \Re. \end{aligned} \tag{2.2}$$

Let $r > 0$ and consider the functional $\varphi(t, x) := x_1(-r)$ defined on $\Re^+ \times C^0([-r, 0]; \Re^2)$. It can be shown that φ is ultimately differentiable along the solutions of (2.2) with time constant $T = r > 0$ but φ is *not differentiable* along the solutions of (2.2). Moreover, we have $D\varphi(t, x, u) := x_2(-r)$, since $\dot{x}_1(t - r) = x_2(t - r)$ for all $t \geq t_0 + r$. A *more demanding* notion of differentiability of functionals along the solutions of (2.1) is given next.

DEFINITION 2.3. *Let $0 \leq \mu \leq r$. A functional p defined on $I \times C^0([-(r - \mu), 0]; \Re^n)$, where $\Re^+ \subseteq I \subseteq \Re$, $p(t, x)$ being completely locally Lipschitz with respect to $x \in C^0([-(r - \mu), 0]; \Re^n)$, with $p(t, 0) = 0$ for all $t \in I$, is called l-differentiable along the solutions of* (2.1) *with delay μ, if there exist functionals $D^i p$, $i = 1, 2, \ldots, l$, called the ith derivatives of p, defined on $I \times C^0([-(r - \mu), 0]; \Re^n)$, each $D^i p(t, x)$ being completely locally Lipschitz with respect to $x \in C^0([-(r - \mu), 0]; \Re^n)$ with $D^i p(t, 0) = 0$ for all $t \in I$, such that for every $(t_0, x_0, u) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times C^0(\Re; U)$, the* *mapping* $t \to p(t, T_{r-\mu}(t-\mu)x)$ *is of class* C^l *on* $[t_0, t_{\max}+\mu)$ *and it holds that*

$$\frac{d^i}{dt^i} p(t, T_{r-\mu}(t-\mu)x) = D^i p(t, T_{r-\mu}(t-\mu)x) \quad \forall t \in [t_0, t_{\max}+\mu), i = 1, 2, \dots, l.$$

Remark 2.4. (i) If $p : I \times C^0([-(r-\mu), 0]; \Re^n) \to \Re$, where $\Re^+ \subseteq I \subseteq \Re$, $0 \le \mu \le r$, is l-differentiable along the solutions of (2.1) with delay μ, then its derivatives $D^i p : I \times C^0([-(r-\mu), 0]; \Re^n) \to \Re$, $i = 1, 2, \dots, l-1$, are functionals which are $(l-i)$-differentiable along the solutions of (2.1) with delay μ.

(ii) If $p : I \times C^0([-(r-\mu), 0]; \Re^n) \to \Re$, where $\Re^+ \subseteq I \subseteq \Re$, $0 \le \mu \le r$, is l-differentiable along the solutions of (2.1) with delay μ, then for every $\tau \in [0, \mu]$ the functional $k(t,x) := p(t+\tau, x)$ is l-differentiable along the solutions of (2.1) with delay $\mu - \tau$ with derivatives $D^i k(t,x) = D^i p(t+\tau, x)$. Moreover, the functional $k(t,x) := p(t+\tau, T_{r'}(\tau-\mu)x)$, where $r' \ge r+\tau-\mu$ is differentiable along the solutions of (2.1) with derivative $Dk(t,x) = Dp(t+\tau, T_{r'}(\tau-\mu)x)$.

The following definition clarifies the notion of a periodic functional.

Definition 2.5. *Let a functional* $\varphi : I \times C^0([-r, 0]; \Re^n) \to \Re$, *where* $\Re^+ \subseteq I \subseteq \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-r, 0]; \Re^n)$, *with* $\varphi(t, 0) = 0$ *for all* $t \in I$. *We say that* φ *is* T*-periodic if there exists* $T > 0$ *such that* $\varphi(t+T, x) = \varphi(t,x)$ *for all* $(t,x) \in I \times C^0([-r, 0]; \Re^n)$.

The following lemma provides classes of functionals, which are l-differentiable with delay μ along the solutions of (2.1) and are going to be used extensively in the next section.

Lemma 2.6. *The following statements hold:*

(i) *The functional*

$$k(t,x) := \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) \varphi(t+s, T_R(s+\mu)x) ds \tag{2.3}$$

defined on $\Re \times C^0([-R-\mu, 0]; \Re^n)$, *where* $\mu > 0$, $R \ge 0$, $h \in C^l(\Re; \Re^+)$ *with* $h(s) = 0$ *for all* $s \notin (-2, -1)$ *and* $\varphi : \Re \times C^0([-R, 0]; \Re^n) \to \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-R, 0]; \Re^n)$, *with* $\varphi(t, 0) = 0$ *for all* $t \in \Re$, *is* l*-differentiable along the solutions of* (2.1) *with delay* μ, *with derivatives for* $i = 1, 2, \dots, l$:

$$D^i k(t,x) := \frac{(-1)^i}{\mu^{i+1}} \int_{-2\mu}^{-\mu} \frac{d^i h}{dt^i}\left(\frac{s}{\mu}\right) \varphi(t+s, T_R(s+\mu)x) dw \tag{2.4}$$

Moreover, if φ *is* T*-periodic (or linear), then* k *and its derivatives* $D^i k$ $(i = 1, \dots, l)$ *are* T*-periodic (or linear).*

(ii) *Let* $a \in C^l(\Re; \Re)$ *and* $p : \Re \times C^0([-R-\mu, 0]; \Re^n) \to \Re$ *a* $(l+i)$*-differentiable functional along the solutions of* (2.1) *with delay* μ. *The functional*

$$k(t,x) := D^i p(t,x) + a(t) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) \varphi(t+s, T_R(s+\mu)x) ds \tag{2.5}$$

defined on $\Re \times C^0([-R-\mu, 0]; \Re^n)$, *where* $\mu > 0$, $R \ge 0$, $h \in C^l(\Re; \Re^+)$ *with* $h(s) = 0$ *for all* $s \notin (-2, -1)$, $D^i p$ *is the ith derivative of* p *and* $\varphi : \Re \times C^0([-R, 0]; \Re^n) \to \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-R, 0]; \Re^n)$, *with* $\varphi(t, 0) = 0$ *for all* $t \in \Re$, *is* l*-differentiable along* *the solutions of* (2.1) *with delay* μ. *Moreover, if* $a \in C^l(\Re;\Re^+)$, φ, p, *and its derivatives* $D^j p$ $(j = 1,\dots,l+i)$ *are* T*-periodic, then it follows that* k *and its derivatives* $D^j k$ $(j = 1,\dots,l)$ *are* T*-periodic. Finally, if* φ, p, *and its derivatives* $D^j p$ $(j = 1,\dots,l+i)$ *are linear, then it follows that* k *and its derivatives* $D^j k$ $(j = 1,\dots,l)$ *are linear.*

The proof of statement (i) of Lemma 2.6 is an immediate consequence of the following equalities:

$$\begin{aligned} k(t, T_{R+\mu}(t-\mu)x) &= \int_{t-2\mu}^{t-\mu} \mu^{-1} h\left(\frac{w-t}{\mu}\right) \varphi(w, T_R(w)x)dw \\ \frac{d^i}{dt^i} k(t, T_{R+\mu}(t-\mu)x) &= D^i k(t, T_{R+\mu}(t-\mu)x) \\ &= \frac{(-1)^i}{\mu^{i+1}} \int_{t-2\mu}^{t-\mu} \frac{d^i h}{dt^i}\left(\frac{w-t}{\mu}\right) \varphi(w, T_R(w)x)dw. \end{aligned}$$

The proof of statement (ii) of Lemma 2.6 is an immediate consequence of statement (i) of Lemma 2.6 and is omitted.

2.2. Finite-time stabilizability. Having clarified the notions of differentiability of functionals along the solutions of a control system, we next proceed to the notion of finite-time stabilizability of a control system.

Definition 2.7. *Let* $b := \sup_{t \ge 0} \tau(t)$. *We say that system* (2.1) *is finite-time stabilizable if there exists constant* $T \ge 0$ *and a functional* $k : [-b,+\infty) \times C^0([-R,0];\Re^n) \to U$, $k(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-R,0];\Re^n)$ *with* $k(t,0) = 0$ *for all* $t \ge -b$ *such that*

(P1) *for every* $(t_0,x_0) \in \Re^+ \times C^0([-\widetilde{r},0];\Re^n)$, *where* $\widetilde{r} := \max(r, R+b)$, *the solution* $x(t) = x(t,t_0,x_0) \in \Re^n$ *of the closed-loop system* (2.1) *with* $u(t) = k(t, T_R(t)x)$ *initiated from* $t_0 \ge 0$ *with initial condition* $T_{\widetilde{r}}(t_0)x = x_0 \in C^0([-\widetilde{r},0];\Re^n)$ *exists for all* $t \ge t_0$ *and satisfies* $x(t) = 0$ *for all* $t \ge t_0 + T$

(P2) $\sup\{|x(t_0+h,t_0,x_0)|;\ h \in [-\widetilde{r},s], \|x_0\|_{\widetilde{r}} \le s, t_0 \in [0,s]\} < +\infty\ \forall s \ge 0.$

Particularly, if system (2.1) *is finite-time stabilizable, then we say that the closed-loop system* (2.1) *with* $u(t) = k(t,T_R(t)x)$ *satisfies the dead-beat property of order* T.

Remark 2.8. Using Lemma 3.3 in [12] it can be shown that if the closed-loop system (2.1) with $u(t) = k(t,T_R(t)x)$ satisfies the dead-beat property of order T, then the equilibrium point $0 \in C^0([-\widetilde{r},0];\Re^n)$ is nonuniformly in time globally asymptotically stable for the closed-loop system (2.1) with $u(t) = k(t,T_R(t)x)$. The notion of nonuniform in time robust global asymptotic stability was introduced in [11] for finite-dimensional continuous-time systems and was extended to a wide class of systems in [12], including systems described by RFDEs.

As already remarked in the introduction, it is generally known that for finite-dimensional continuous-time control systems with locally Lipschitz dynamics, finite-time global stabilization cannot be achieved by means of locally Lipschitz feedback. The following lemma studies the scalar case and shows that this is no longer true if time-varying distributed delay feedback is used.

Lemma 2.9. *Consider the one-dimensional control system:*

$$\dot{x}(t) = u(t-\tau) + v(t); \quad x(t) \in \Re, u(t) \in \Re, v(t) \in \Re, t \ge 0, \tag{2.6}$$

where $\tau \ge 0$ *is a constant. Then for every* $\mu > \tau$, *the solutions of the closed-loop system* (2.6) *with*

$$u(t) = -p_\mu(t+\tau, T_\mu(t+\tau-\mu)x), \tag{2.7}$$

where $p_\mu : \Re \times C^0([-\mu, 0]; \Re) \to \Re$ *is the linear* 3μ*-periodic functional defined by*

$$p_\mu(t, x) := \mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(s+\mu) ds \tag{2.8}$$

for certain $a \in C^0(\Re; \Re^+)$ *being a periodic function with period* 3 *with* $a(t) = 0$ *for* $t \in [0, 2]$ *and* $\int_2^3 a(t)dt = 1$ *and* $h \in C^0(\Re; \Re^+)$ *with* $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1} h(\frac{s}{\mu})ds = 1$, *initiated from arbitrary* $t_0 \geq 0$ *with arbitrary initial condition* $T_{2\mu}(t_0)x = x_0 \in C^0([-2\mu, 0]; \Re)$ *and corresponding to arbitrary input* $v \in C^0(\Re^+; \Re)$ *satisfy*

$$|x(t)| \leq \exp(3L)\sigma\left(\frac{t - t_0}{\mu}\right) \|x_0\|_{2\mu} + 10\mu \exp(3L) \sup_{\max(t_0, t-6\mu) \leq s \leq t} |v(s)| \quad \forall t \geq t_0, \tag{2.9}$$

where $\sigma(t) := \left\{\begin{smallmatrix} 1 & if & t < 6 \\ 0 & if & t \geq 6 \end{smallmatrix}\right.$ *and* $L := \max_{t \in [0,3]} a(t)$.

Proof. The closed-loop system (2.6) with (2.7) is described by the differential equation

$$\dot{x}(t) = -\mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(t+s) ds + v(t). \tag{2.10}$$

As in the proof of Theorem 1.1, Chapter 6, page 168 in [7] it can be shown that for every initial condition $x_0 \in C^0([-2\mu, 0]; \Re)$ the solution of the closed-loop system (2.6) with (2.7) exists for all $t \geq t_0$ and satisfies

$$|x(t)| \leq \exp(L\mu^{-1}(t - t_0)) \left(\|x_0\|_{2\mu} + (t - t_0) \sup_{t_0 \leq \tau \leq t} |v(\tau)| \right) \quad \forall t \geq t_0. \tag{2.11}$$

Define the time sequence $t_i = 3\mu([\frac{t_0}{3\mu}] + i)$, $i \in Z^+$. We next prove the following claim.

Claim. For the solution of the closed-loop system (2.6) with (2.7), the following inequalities hold for all $i \in Z^+$:

$$\begin{aligned} &|x(t)| \leq |x(t_i)| + 5\mu \sup_{t_i \leq \tau \leq t} |v(\tau)| \quad \forall t \in [t_i, t_{i+1}] \quad \text{and} \\ &|x(t_{i+1})| \leq 5\mu \sup_{t_i \leq \tau \leq t_{i+1}} |v(\tau)|. \end{aligned} \tag{2.12}$$

Proof of Claim. Let arbitrary $i \in Z^+$. Indeed, since $a \in C^0(\Re; \Re^+)$ is a periodic function with period 3, with $a(t) = 0$ for $t \in [0, 2]$, we have $a(\frac{t}{\mu}) = 0$ for $t \in [t_i, t_i + 2\mu]$. Consequently, it follows that

$$x(t) = x(t_i) + \int_{t_i}^{t} v(\tau) d\tau \quad \forall t \in [t_i, t_i + 2\mu]. \tag{2.13}$$

For $t \in [t_i + 2\mu, t_{i+1}]$ we have

$$x(t) = x(t_i + 2\mu) + \int_{t_i + 2\mu}^{t} v(\tau) d\tau - \mu^{-1} \int_{t_i + 2\mu}^{t} a\left(\frac{\tau}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(\tau + s) ds\, d\tau.$$

The latter relation in conjunction with (2.13) implies that

$$
\begin{aligned}
x(t) = x(t_i)\left[1 - \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) ds \int_{t_i+2\mu}^{t} \mu^{-1} a\left(\frac{\tau}{\mu}\right) d\tau\right] + \int_{t_i}^{t} v(\tau) d\tau \\
- \mu^{-1} \int_{t_i+2\mu}^{t} a\left(\frac{\tau}{\mu}\right) \int_{-2\mu}^{-\mu} \left(\mu^{-1} h\left(\frac{s}{\mu}\right)\left(\int_{t_i}^{\tau+s} v(w) dw\right)\right) ds\, d\tau \\
\forall t \in [t_i + 2\mu, t_{i+1}].
\end{aligned} \tag{2.14}
$$

Since $h \in C^0(\Re;\Re^+)$ with $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1} h(\frac{s}{\mu})ds = 1$ and $a \in C^0(\Re;\Re^+)$ is a periodic function with period 3 with $\int_2^3 a(t)dt = 1$, it follows from (2.13), (2.14) that inequalities (2.12) hold. Thus the claim made above holds.

An immediate consequence of the previous claim is the following estimate for the solution of closed-loop system (2.6) with (2.7):

$$
|x(t)| \leq 10\mu \sup_{t_i \leq \tau \leq t} |v(\tau)| \quad \forall t \in [t_{i+1}, t_{i+2}]. \tag{2.15}
$$

Combining estimates (2.11) (for the interval $[t_0, t_1]$), (2.12) (for the interval $[t_1, t_2]$) and (2.15) (for the interval $[t_2, +\infty)$) we obtain the desired estimate (2.9). The proof is complete. □

Remark 2.10. (a) Although the input of system (2.6) is delayed, when the state of the closed-loop system (2.6) with (2.7) and $v(t) \equiv 0$ hits zero, then the control action is indeed zero. Notice that the control action is given by the following formulae:

$$
\begin{aligned}
u(t) &= -\mu^{-1} a\left(\frac{t+\tau}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(t+\tau+s) ds; \\
u(t-\tau) &= -\mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(t+s) ds.
\end{aligned}
$$

The reader can verify that under the hypotheses of Lemma 2.9, if $v(t) \equiv 0$ and t_0 is a multiple of 3μ, then we have

$$
\begin{aligned}
a\left(\frac{t}{\mu}\right) &= 0, u(t-\tau) = 0, x(t) = x(t_0) \quad \forall t \in [t_0, t_0 + 2\mu] \quad \text{and} \\
x(t) &= x(t_0)\left(1 - \mu^{-1} \int_{t_0+2\mu}^{t} a\left(\frac{\tau}{\mu}\right) d\tau\right), \\
u(t-\tau) &= -\mu^{-1} a\left(\frac{t}{\mu}\right) x(t_0) \quad \forall t \in [t_0 + 2\mu, t_0 + 3\mu].
\end{aligned}
$$

For $t = t_0 + 3\mu$ both state and input become zero (and thus the control cannot push the system away from zero). The same analysis repeated for the next interval $t \in [t_0 + 3\mu, t_0 + 6\mu]$ (with $x(t_0 + 3\mu) = 0$) shows that $x(t) = 0$, $u(t-\tau) = 0$, for all $t \in [t_0 + 3\mu, t_0 + 6\mu]$. Thus we have $x(t) = 0$, $u(t-\tau) = 0$ for all $t \geq t_0 + 3\mu$.

(b) The functions $a \in C^0(\Re;\Re^+)$ and $h \in C^0(\Re;\Re^+)$ are, in a generalized sense, *time-varying gains* of the linear feedback law $u(t) = -\mu^{-1} a(\frac{t+\tau}{\mu}) \int_{-2\mu}^{-\mu} \mu^{-1} h(\frac{s}{\mu}) x(t + \tau + s) ds$. Clearly, we have $|u(t)| \leq \frac{L}{\mu} \|T_{2\mu}(t)x\|_{2\mu}$ for all $t \geq 0$, where $L := \max_{t\in[0,3]} a(t)$.

3. Main results. In the present paper we consider triangular time-varying systems described by RFDEs:

$$
\begin{aligned}
&\dot{x}_i(t) = f_i(t, T_r(t)x_1, \dots, T_r(t)x_i) + x_{i+1}(t-\tau_i), \quad i = 1, \dots, n-1,\\
&\dot{x}_n(t) = f_n(t, T_r(t)x) + u(t-\tau_n),\\
&x(t) := (x_1(t), \dots, x_n(t)) \in \Re^n,\ u(t) \in \Re, t \geq 0,
\end{aligned} \tag{3.1}
$$

where $r \geq \tau_i \geq 0, i = 1, \dots, n$, the mappings $f_i : \Re^+ \times C^0([-r, 0]; \Re^i) \to \Re\, i = 1, \dots, n$ are completely locally Lipschitz with respect to $x \in C^0([-r, 0]; \Re^n)$ with $f_i(t, 0) = 0$ for all $t \geq 0$ and satisfy one of the following assumptions:

(A1) *There exist mappings $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re, i = 1, \dots, n$, which are differentiable along the solutions of* (3.1) *and satisfy the following identities for all $t \geq 0$ and $x \in C^0([-r, 0]; \Re^n)$:*

$$
\varphi_1(t-\tau_1, T_{r-\tau_1}(-\tau_1)x_1) := f_1(t, x_1) \tag{3.2a}
$$

(3.2b)

$$
\begin{aligned}
&\varphi_{i+1}(t-\tau_{i+1}, T_{r-\tau_{i+1}}(-\tau_{i+1})x_1, \dots, T_{r-\tau_{i+1}}(-\tau_{i+1})x_i, T_{r-\tau_{i+1}}(-\tau_{i+1})x_{i+1})\\
&:= D\varphi_i(t, x_1, \dots, x_i, x_{i+1}(-\tau_i)) + f_{i+1}(t, x_1, \dots, x_i, x_{i+1}) \quad i = 1, \dots, n-1.
\end{aligned}
$$

(A2) *There exist mappings $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re, i = 1, \dots, n$, which are ultimately differentiable along the solutions of* (3.1) *with time constant $T > 0$ and satisfy identities* (3.2). *Moreover, there exists a constant $R \in (0, r]$ and a continuous function $L : \Re^+ \times \Re^+ \to \Re^+$ such that*

$$
\begin{aligned}
&\sum_{i=1}^{n-1} x_i(0) f_i(t, x_1, \dots, x_i) - x_n(0) D\varphi_{n-1}(t, x)\\
&\leq L\left(t, \sup_{-r \leq \theta \leq -R} |x(\theta)|\right)(|x(0)|^2 + 1),\\
&\forall (t, x) \in \Re^+ \times C^0([-r, 0]; R^n).
\end{aligned} \tag{3.3}
$$

Our first main result states that system (3.1) is finite-time stabilizable under assumption (A1). Particularly, we have the following theorem.

THEOREM 3.1. *Consider system* (3.1) *and suppose that assumption* (A1) *holds. Let $b_{i,m} := \sum_{k=i}^{m} \tau_k$. Then for every $\mu > b_{1,n}$ and $l \in Z^+$, there exist functions $\gamma \in K^+$, $\rho \in K_\infty$, functionals $p_i : \Re \times C^0([-r_n + \mu, 0]; \Re^i) \to \Re, i = 1, \dots, n$, where $r_n := r + 2n\mu$, which are l-differentiable along the solutions of* (3.1) *with delay $\mu > 0$ and a constant $T > 0$, such that*

(i) *the closed-loop system* (3.1) *with $u(t) = k(t, T_{r_n - \mu}(t)x)$ satisfies the dead-beat property of order T, where $k : \Re \times C^0([-r_n + \mu, 0]; \Re^n) \to \Re$ is completely locally Lipschitz with respect to $x \in C^0([-r_n + \mu, 0]; \Re^n)$ with $k(\cdot, 0) = 0$ and is defined by*

(3.4)

$$
k(t, x) := -\varphi_n(t, x) - \sum_{i=1}^{n} p_i(t + b_{i,n}, T_{r_n - \mu}(b_{i,n} - \mu)x_1, \dots, T_{r_n - \mu}(b_{i,n} - \mu)x_i),
$$

(ii) *for every* $(t_0, x_0, v) \in \Re^+ \times C^0([-r_n, 0]; \Re^n) \times C^0(\Re; \Re)$ *the solution of the closed-loop system* (3.1) *with* $u(t) = k(t, T_{r_n-\mu}(t)x) + v(t)$ *satisfies the estimate*

$$|x(t)| \leq \gamma(t)\rho\left(\|x_0\|_{r_n} + \sup_{t_0-\tau_n \leq s \leq t} |v(s)|\right) \quad \forall t \geq t_0. \tag{3.5}$$

Moreover, if the mappings $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \dots, n$, *are independent of* t, *then the functionals* p_i $i = 1, \dots, n$ *and* k *as defined by* (3.4) *can be chosen to be* 3μ*-periodic. Finally, if the mappings* $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re$, $i = 1, \dots, n$, *are linear, then the functionals* p_i $i = 1, \dots, n$ *and* k *as defined by* (3.4) *can be chosen to be linear.*

The proof of Theorem 3.1 relies on the following technical lemma.

LEMMA 3.2 (adding a delayed integrator). *Consider the system*

$$\dot{x}(t) = f(t, T_r(t)x, y(t-\tau)), \tag{3.6a}$$

$$\begin{aligned} &\dot{y}(t) = g(t, T_r(t)x, T_r(t)y) + u(t-\tau'), \\ &x(t) \in \Re^n, y(t) \in \Re, u(t) \in \Re, t \geq 0, \end{aligned} \tag{3.6b}$$

where $r \geq \tau \geq 0$, $\tau' \geq 0$ *are constants and the mappings* $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times \Re \to \Re^n$, $g : \Re^+ \times C^0([-r, 0]; \Re^n) \times C^0([-r, 0]; \Re) \to \Re$ *are completely locally Lipschitz with respect to* $(x, y) \in C^0([-r, 0]; \Re^n) \times C^0([-r, 0]; \Re)$ *with* $f(t, 0, 0) = 0, g(t, 0, 0) = 0$ *for all* $t \geq 0$. *Suppose there exists a functional* $k : \Re \times C^0([-r + \tau, 0]; \Re^n) \to \Re$, *with* $k(\cdot, 0) = 0$, *which is differentiable along the solutions of* (3.6a) *such that*

(H1) *The closed-loop system* (3.6a) *with* $y(t) = k(t, T_{r-\tau}(t)x)$ *satisfies the dead-beat property of order* $T > 0$.

(H2) *There exist functions* $\gamma \in K^+$ *and* $\rho \in K_\infty$ *such that for every* $(t_0, x_0, z) \in \Re^+ \times C^0([-r, 0]; \Re^n) \times C^0(\Re; \Re)$ *the solution of the closed-loop system* (3.6a) *with* $y(t) = k(t, T_{r-\tau}(t)x) + z(t)$, *initiated from* $t_0 \geq 0$ *with initial condition* $T_r(t_0)x = x_0 \in C^0([-r, 0]; \Re^n)$ *and corresponding to* $z \in C^0(\Re; \Re)$ *satisfies the estimate*

$$|x(t)| \leq \gamma(t)\rho\left(\|x_0\|_r + \sup_{t_0-\tau \leq s \leq t} |z(s)|\right) \quad \forall t \geq t_0. \tag{3.7}$$

(H3) *There exists a functional* $\varphi : \Re \times C^0([-r + \tau', 0]; \Re^{n+1}) \to \Re$, $\varphi(t, x, y)$ *being completely locally Lipschitz with respect to* $(x, y) \in C^0([-r + \tau', 0]; \Re^n) \times C^0([-r - \tau', 0]; \Re)$ *with* $\varphi(t, 0, 0) = 0$ *for all* $t \in \Re$ *and such that the following identity holds for all* $t \geq 0$ *and* $(x, y) \in C^0([-r, 0]; \Re^n) \times C^0([-r, 0]; \Re)$*:*

$$\varphi(t - \tau', T_{r-\tau'}(-\tau')x, T_{r-\tau'}(-\tau')y) := -g(t, x, y) + Dk(t, x, y(-\tau)), \tag{3.8}$$

where $Dk : \Re \times C^0([-r, 0]; \Re^n) \times \Re \to \Re$ *denotes the derivative of* k *along the solutions of* (3.6a).

Then for every $\mu > \tau'$, *there exist functions* $\widetilde{\gamma} \in K^+$, $\widetilde{\rho} \in K_\infty$ *and a functional* $\widetilde{k} : \Re \times C^0([-r - 2\mu + \tau', 0]; \Re^{n+1}) \to \Re$, *which is completely locally Lipschitz with respect to* $(x, y) \in C^0([-r - 2\mu + \tau', 0]; \Re^{n+1})$ *with* $\widetilde{k}(t, 0, 0) = 0$ *for all* $t \in \Re$ *and is defined by*

$$\begin{aligned} \widetilde{k}(t, x, y) := \varphi(t, x, y) - \mu^{-1} a\left(\frac{t + \tau'}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) (y(\tau' + s) \\ - k(t + \tau' + s, T_{r-\tau}(\tau' + s)x)) ds, \end{aligned} \tag{3.9}$$

where $a \in C^0(\Re;\Re^+)$ is a periodic function with period 3, with $a(t) = 0$ for $t \in [0,2]$ and $\int_2^3 a(t)dt = 1$, $h \in C^0(\Re;\Re^+)$ with $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1}h(\frac{s}{\mu})ds = 1$, such that the closed-loop system (3.6) *with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y)$ satisfies the dead-beat property of order $T + r + 6\mu$ and for every $(t_0, x_0, y_0, v) \in \Re^+ \times C^0([-r-2\mu, 0];\Re^{n+1}) \times C^0(\Re;\Re)$ the solution of the closed-loop system* (3.6) *with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y) + v(t)$ satisfies the estimate*

$$|(x(t), y(t))| \leq \widetilde{\gamma}(t)\widetilde{\rho}\left(\|(x_0, y_0)\|_{r+2\mu} + \sup_{t_0-\tau'\leq s\leq t} |v(s)|\right) \quad \forall t \geq t_0. \tag{3.10}$$

Remark 3.3. Notice that the stabilizing feedback is given by

$$u(t) = \varphi(t, T_r(t)x, T_r(t)y) - \mu^{-1}a\left(\frac{t+\tau'}{\mu}\right)\int_{-2\mu}^{-\mu} \mu^{-1}h\left(\frac{s}{\mu}\right)z(t+\tau'+s)ds,$$

where $z(t) := y(t) - k(t, T_{r-\tau}(t)x)$.

Proof of Lemma 3.2. Let arbitrary $(t_0, x_0, y_0, v) \in \Re^+ \times C^0([-r-2\mu, 0];\Re^{n+1}) \times C^0(\Re;\Re)$ and define

$$z(t) := y(t) - k(t, T_{r-\tau}(t)x). \tag{3.11}$$

By virtue of definitions (3.9), (3.11), and identity (3.8), we guarantee that as long as the solution of the closed-loop system (3.6) with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y) + v(t)$ and $T_{r+2\mu}(t_0)x = x_0$, $T_{r+2\mu}(t_0)y = y_0$ exists, it coincides with the solution of the following system:

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, k(t-\tau, T_{r-\tau}(t-\tau)x) + z(t-\tau)) \\ &\dot{z}(t) = -\mu^{-1}a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu} \mu^{-1}h\left(\frac{s}{\mu}\right)z(t+s)ds + v(t-\tau') \\ &x(t) \in \Re^n,\ z(t) \in \Re,\ v(t) \in \Re,\ t \geq 0 \end{aligned} \tag{3.12}$$

corresponding to the same input $v \in C^0(\Re;\Re)$ with $y(t) = z(t) + k(t, T_{r-\tau}(t)x)$ and $T_{r+2\mu}(t_0)x = x_0$, $z(t_0+\theta) = y_0(\theta) - k(t_0+\theta, T_{r-\tau}(t_0+\theta)x); \theta \in [-\widetilde{\tau}, 0]$, where $\widetilde{\tau} := \max(\tau, 2\mu)$. The solution of (3.12) exists for all $t \geq t_0$, since by virtue of Lemma 2.9, we obtain

$$\begin{aligned} |z(t)| \leq{}& \exp(3L)\sigma\left(\frac{t-t_0}{\mu}\right)\|T_{\widetilde{\tau}}(t_0)z\|_{\widetilde{\tau}} \\ &+ 10\mu\exp(3L)\sup_{\max(t_0, t-6\mu)\leq s\leq t} |v(s-\tau')| \\ &\forall t \geq t_0 - \widetilde{\tau}, \end{aligned} \tag{3.13}$$

where $\sigma(t) := \left\{\begin{smallmatrix} 1 & if & t<6 \\ 0 & if & t \geq 6 \end{smallmatrix}\right.$ and $L := \max_{t\in[0,3]} a(t)$ and inequality (3.13) in conjunction with estimate (3.7) implies the following estimate:

$$|x(t)| \leq \gamma(t)\bar{a}\left(\|T_r(t_0)x\|_r + \|T_{\widetilde{\tau}}(t_0)z\|_{\widetilde{\tau}} + \sup_{t_0-\tau'\leq s\leq t} |v(s)|\right) \quad \forall t \geq t_0, \tag{3.14}$$

where $\bar{a}(s) := \rho(\exp(3L)(1+10\mu)s) \in K_\infty$. We next prove the following claim.

Claim. There exist functions $\widetilde{\gamma} \in K^+$, $\widetilde{\rho} \in K_\infty$ such that for every $(t_0, x_0, y_0, v) \in \Re^+ \times C^0([-r-2\mu, 0]; \Re^{n+1}) \times C^0(\Re; \Re)$ the solution of (3.12) corresponding to $v \in C^0(\Re; \Re)$ with $y(t) = z(t) + k(t, T_{r-\tau}(t)x)$, $T_{r+2\mu}(t_0)x = x_0$, $z(t_0+\theta) = y_0(\theta) - k(t_0 + \theta, T_{r-\tau}(t_0+\theta)x); \theta \in [-\widetilde{\tau}, 0]$ satisfies estimate (3.10).

Proof of Claim. Since $k : [-2\mu, +\infty) \times C^0([-r+\tau, 0]; \Re^n) \to \Re$ is completely locally Lipschitz with respect to $x \in C^0([-r+\tau, 0]; \Re^n)$ with $k(t, 0) = 0$ for all $t \geq -2\mu$, it follows from Lemma 3.2 in [12] that there exist functions $\delta \in K^+$, being nondecreasing and $q \in K_\infty$ such that

$$|k(t,x)| \leq q(\delta(t)\|x\|_{r-\tau}) \quad \forall (t,x) \in [-2\mu, +\infty) \times C^0([-r+\tau, 0]; \Re^n). \tag{3.15}$$

Notice that definition (3.11) in conjunction with (3.15) implies

$$\|T_{\widetilde{\tau}}(t_0)z\|_{\widetilde{\tau}} \leq \|y_0\|_{\widetilde{\tau}} + q(\delta(t)\|x_0\|_{r+2\mu}) \quad \forall t \geq t_0. \tag{3.16}$$

Using Corollary 10 and Remark 11 in [23] we may find a function $\kappa \in K_\infty$ such that

$$q(rs) + \overline{a}(rs) + \exp(3L)(1+10\mu)rs \leq \kappa(r)\kappa(s) \quad \forall r, s \geq 0. \tag{3.17}$$

Without loss of generality we may assume that the function $\gamma \in K^+$ involved in (3.14) is nondecreasing. Thus, using estimate (3.14), we obtain the following estimate for the solution of (3.12):

$$\|T_r(t)x\|_r \leq \gamma(t)\overline{a}\left(\|x_0\|_r + \|T_{\widetilde{\tau}}(t_0)z\|_{\widetilde{\tau}} + \sup_{t_0-\tau' \leq s \leq t} |v(s)|\right) \quad \forall t \geq t_0. \tag{3.18}$$

Combining (3.15), (3.16), (3.17), and (3.18) we obtain for the solution of (3.12),

$$|k(t, T_{r-\tau}(t)x)| \leq p(t)\zeta\left(\|(x_0, y_0)\|_{r+2\mu} + \sup_{t_0-\tau' \leq s \leq t} |v(s)|\right) \quad \forall t \geq t_0, \tag{3.19}$$

where $p(t) := \kappa(\delta(t)\gamma(t)\kappa(1+\kappa(\delta(t))))$ and $\zeta(s) := \kappa(\kappa(2s+\kappa(s)))$. Estimates (3.13), (3.14), (3.19) in conjunction with inequalities (3.16), (3.17) and the trivial inequality $|y(t)| \leq |z(t)| + |k(t, T_{r-\tau}(t)x)|$, show that the solutions of (3.12) satisfy estimate (3.10) for $\widetilde{\gamma}(t) := (1+\gamma(t))\kappa(1+\kappa(\delta(t))) + p(t)$ and $\widetilde{a}(s) := \kappa(2s+\kappa(s)) + \zeta(s)$. Thus the claim is proved.

Since for every $(t_0, x_0, y_0, v) \in \Re^+ \times C^0([-r-2\mu, 0]; \Re^{n+1}) \times C^0(\Re; \Re)$ the solution of the closed-loop system (3.6) with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y) + v(t)$ and $T_{r+2\mu}(t_0)x = x_0$, $T_{r+2\mu}(t_0)y = y_0$ coincides with the solution of (3.12) corresponding to the same input $v \in C^0(\Re; \Re)$ with $y(t) = z(t) + k(t, T_{r-\tau}(t)x)$ and $T_{r+2\mu}(t_0)x = x_0$, $z(t_0+\theta) = y_0(\theta) - k(t_0+\theta, T_{r-\tau}(t_0+\theta)x); \theta \in [-\widetilde{\tau}, 0]$, where $\widetilde{\tau} := \max(\tau, 2\mu)$, we conclude that for every $(t_0, x_0, y_0, v) \in \Re^+ \times C^0([-r-2\mu, 0]; \Re^{n+1}) \times C^0(\Re; \Re)$ the solution of the closed-loop system (3.6) with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y) + v(t)$ and $T_{r+2\mu}(t_0)x = x_0$, $T_{r+2\mu}(t_0)y = y_0$ exists for all $t \geq t_0$ and satisfies estimate (3.10). Finally, the fact that the closed-loop system (3.6) with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y)$ satisfies the dead-beat property of order $T + r + 6\mu$ follows from the observation that $z(t) = 0$ for all $t \geq t_0 + 6\mu$ (recall (3.13) with $v(t) \equiv 0$), which implies $y(t) = k(t, T_{r-\tau}(t)x)$ for all $t \geq t_0 + 6\mu$ (recall definition (3.11)). Thus for all $t \geq t_0 + 6\mu$ the solution of the closed-loop system (3.6) with $u(t) = \widetilde{k}(t, T_{r+2\mu-\tau'}(t)x, T_{r+2\mu-\tau'}(t)y)$ coincides with the solution of (3.6a) with $y(t) = k(t, T_{r-\tau}(t)x)$ initiated from $T_r(t_0+6\mu)x$. It follows that $x(t) = 0$ for all

$t \geq t_0 + T + 6\mu$, which implies $y(t) = k(t, T_{r-\tau}(t)x) = 0$ for all $t \geq t_0 + T + r + 6\mu$. The proof is complete. $\square$

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof is made by induction.

Step 1: We show that the statement of Theorem 3.1 holds for $n = 1$. Let $l \in Z^+$ and $\mu > b_{1,n} = \tau_1$. The statement of Theorem 3.1 for $n = 1$ is an immediate consequence of Lemma 2.9, identity (3.2a), (3.4), and the definition

$$p_1(t, x_1) := -\mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x_1(s+\mu) ds, \tag{3.20}$$

where $a \in C^l(\Re; \Re^+)$ is a periodic function with period 3 with $a(t) = 0$ for $t \in [0, 2]$ and $\int_2^3 a(t)dt = 1$ and $h \in C^l(\Re; \Re^+)$ with $h(s) = 0$ for all $s \notin (-2, -1)$, $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1} h(\frac{s}{\mu})ds = 1$. Notice that the fact that $p_1(t, x_1)$, as defined by (3.11), is l-differentiable with delay μ follows directly from Lemma 2.6. Moreover, if $\varphi_1 : \Re \times C^0([-r + \tau_1, 0]; \Re) \to \Re$ is independent of t, then the functionals p_1 and k (defined by (3.4)) are 3μ-periodic. Finally, if $\varphi_1 : \Re \times C^0([-r + \tau_1, 0]; \Re) \to \Re$ is linear, then the functionals p_1 and k (defined by (3.4)) are linear.

Induction step: Suppose that the statement of Theorem 3.1 holds for $n - 1$. We show that the statement of Theorem 3.1 holds for n. By assumption for every $\mu > b_{1,n-1}, l \in Z^+$, there exist functions $\gamma \in K^+, \rho \in K_\infty$, functionals $p_i : \Re \times C^0([-r_{n-1} + \mu, 0]; \Re^i) \to \Re$, $i = 1, \dots, n-1$, where $r_{n-1} := r + 2(n-1)\mu$, which are $(l+1)$-differentiable along the solutions of the following system:

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, T_r(t)x_1, \dots, T_r(t)x_i) + x_{i+1}(t - \tau_i), \quad i = 1, \dots, n-1, \\ &\xi(t) := (x_1(t), \dots, x_{n-1}(t)) \in \Re^{n-1}, x_n(t) \in \Re, t \geq 0 \end{aligned} \tag{3.21}$$

with delay $\mu > 0$ and a constant $T > 0$, such that

(a) the closed-loop system (3.21) with $x_n(t) = k(t, T_{r_{n-1}-\mu}(t)\xi)$ satisfies the dead-beat property of order T, where

$$\begin{aligned} k(t, \xi) := &- \varphi_{n-1}(t, \xi) \\ &- \sum_{i=1}^{n-1} p_i(t + b_{i,n-1}, T_{r_{n-1}-\mu}(b_{i,n-1} - \mu)x_1, \dots, T_{r_{n-1}-\mu}(b_{i,n-1} - \mu)x_i), \end{aligned} \tag{3.22}$$

(ii) for every $(t_0, \xi_0, z) \in \Re^+ \times C^0([-r_{n-1}, 0]; \Re^n) \times C^0(\Re; \Re)$ the solution of the closed-loop system (3.21) with $x_n(t) = k(t, T_{r_{n-1}-\mu}(t)\xi) + z(t)$ satisfies the estimate

$$|\xi(t)| \leq \gamma(t)\rho\left(\|\xi_0\|_{r_{n-1}} + \sup_{t_0 - \tau_{n-1} \leq s \leq t} |z(s)|\right) \quad \forall t \geq t_0. \tag{3.23}$$

Remark 2.4(ii) and hypothesis (A1) show that k as defined by (3.22) is differentiable along the solutions of (3.21) with derivative

$$\begin{aligned} Dk(t, \xi, x_n(-\tau_{n-1})) := &- D\varphi_{n-1}(t, \xi, x_n(-\tau_{n-1})) \\ &- \sum_{i=1}^{n-1} Dp_i(t + b_{i,n-1}, T_{r_{n-1}-\mu}(b_{i,n-1} - \mu)x_1, \dots, \\ &\qquad T_{r_{n-1}-\mu}(b_{i,n-1} - \mu)x_i). \end{aligned}$$

Clearly, hypotheses (H1) and (H2) of Lemma 3.2 hold for system (3.1). Next we show that hypothesis (H3) of Lemma 3.2 also holds for system (3.1). Notice that

$$\begin{aligned}&-f_n(t,x)+Dk(t,\xi,x_n(-\tau_{n-1})),\\&=-f_n(t,x)-D\varphi_{n-1}(t,\xi,x_n(-\tau_{n-1}))\\&\quad-\sum_{i=1}^{n-1}Dp_i(t+b_{i,n-1},T_{r_{n-1}-\mu}(b_{i,n-1}-\mu)x_1,\dots,T_{r_{n-1}-\mu}(b_{i,n-1}-\mu)x_i).\end{aligned}$$

Hypothesis (A1) implies that there exists a mapping $\varphi_n : \Re \times C^0([-r+\tau_n,0];\Re^n) \to \Re$, which is differentiable along the solutions of (3.1) and satisfies $\varphi_n(t-\tau_n, T_{r-\tau_n}(-\tau_n)x) := D\varphi_{n-1}(t,\xi,x_n(-\tau_{n-1})) + f_n(t,x)$ for all $t \geq 0$ and $x \in C^0([-r,0];\Re^n)$. Thus we obtain

$$-f_n(t,x)+Dk(t,\xi,x_n(-\tau_{n-1}))=\varphi(t-\tau_n,T_{r_{n-1}-\tau_n}(-\tau_n)x), \tag{3.24}$$

where $\varphi(t,x) := -\varphi_n(t,x) - \sum_{i=1}^{n-1} Dp_i(t+\tau_n+b_{i,n-1}, T_{r_{n-1}-\mu}(b_{i,n-1}+\tau_n-\mu)x_1,\dots, T_{r_{n-1}-\mu}(b_{i,n-1}+\tau_n-\mu)x_i)$. Since $b_{i,n-1}+\tau_n = b_{i,n}$ we have

$$\varphi(t,x):=-\varphi_n(t,x)-\sum_{i=1}^{n-1}Dp_i(t+b_{i,n},T_{r_{n-1}-\mu}(b_{i,n}-\mu)x_1,\dots,T_{r_{n-1}-\mu}(b_{i,n}-\mu)x_i). \tag{3.25}$$

Notice that the mapping $\varphi : \Re\times C^0([-r_{n-1}+\tau_n,0];\Re^n) \to \Re$ defined by (3.25) is well defined for all $\mu > b_{1,n}$, with $\varphi(t,x)$ being completely locally Lipschitz with respect to $x \in C^0([-r+\tau_n,0];\Re^n)$ and $\varphi(t,0) = 0$ for all $t \in \Re$.

Lemma 3.2 shows that there exist functions $\widetilde{\gamma} \in K^+$, $\widetilde{\rho} \in K_\infty$ and a functional $\widetilde{k} : \Re\times C^0([-r_{n-1}-2\mu,0];\Re^n) \to \Re$, which is completely locally Lipschitz with respect to $x \in C^0([-r_{n-1}-2\mu,0];\Re^n)$ with $\widetilde{k}(t,0) = 0$ for all $t \in \Re$ and is defined by

$$\begin{aligned}&\widetilde{k}(t,x):=\varphi(t,x)-p_n(t+\tau_n,T_{r_n-\mu}(\tau_n-\mu)x),\\&p_n(t,x):=\mu^{-1}a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)(x_n(s+\mu)-k(t+s,T_{r_{n-1}}(s+\mu)\xi))ds,\end{aligned} \tag{3.26}$$

where $a \in C^l(\Re;\Re^+)$ is a periodic function with period 3, with $a(t) = 0$ for $t \in [0,2]$ and $\int_2^3 a(t)dt = 1$, $h \in C^l(\Re;\Re^+)$ with $h(s) = 0$ for all $s \notin (-2,-1)$, $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1}h(\frac{s}{\mu})ds = 1$, such that the closed-loop system (3.1) with $u(t) = \widetilde{k}(t, T_{r_n-\mu}(t)x)$ satisfies the dead-beat property of order $T + r_{n-1} + 6\mu$ and for every $(t_0, x_0, v) \in \Re^+ \times C^0([-r_n,0];\Re^n) \times C^0(\Re;\Re)$ the solution of the closed-loop system (3.1) with $u(t) = \widetilde{k}(t, T_{r_n-\mu}(t)x) + v(t)$ satisfies estimate (3.5) with $\widetilde{\gamma} \in K^+$ and $\widetilde{\rho} \in K_\infty$ in place of $\gamma \in K^+$ and $\rho \in K_\infty$, respectively. Lemma 2.6 implies that the functional p_n is l-differentiable along the solutions of (3.1) with delay $\mu > 0$. Moreover, if φ_n is independent of t and k is 3μ-periodic then the functionals p_n and $\widetilde{k}$ are 3μ-periodic. Finally, if φ_n and k are linear functionals, then the functionals p_n and $\widetilde{k}$ are linear. ∎

Our second main result states that system (3.1) is finite-time stabilizable under assumption (A2).

THEOREM 3.4. *Consider system* (3.1) *and suppose that assumption* (A2) *holds. Then for every* $\mu \geq R + \sum_{i=1}^{n} \tau_i$*, where* $R > 0$ *is the constant involved in assumption* (A2)*, there exists a functional* $k : \Re \times C^0([-\widetilde{r}, 0]; \Re^n) \to \Re$*, where* $\widetilde{r} := 2r + 2n\mu$*, which is completely locally Lipschitz with respect to* $x \in C^0([-\widetilde{r}, 0]; \Re^n)$ *and a constant* $T' > T$ *(where* $T \geq 0$ *is the time constant involved in assumption* (A2)*, such that the closed-loop system* (3.1) *with* $u(t) = k(t, T_{\widetilde{r}}(t)x)$ *satisfies the dead-beat property of order* T'*. Moreover, if the mappings* $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re$*,* $i = 1, \ldots, n$ *are independent of* t*, then the functional* k *can be chosen to be* 3μ*-periodic. Finally, if the mappings* $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re$*,* $i = 1, \ldots, n$ *are linear, then the functional* k *can be chosen to be linear.*

Proof. Consider the system

$$\begin{aligned} &\dot{z}_i(t) = z_{i+1}(t - \tau_i), i = 1, \ldots, n-1; \ \dot{z}_n(t) = v(t - \tau_n) \\ &z(t) = (z_1(t), \ldots, z_n(t)) \in \Re^n, v(t) = \Re, \end{aligned} \tag{3.27}$$

where $\tau_i \geq 0$ are exactly the delays appearing in (3.1). Since assumption (A1) holds for system (3.27) with $\varphi_i \equiv 0$ for $i = 1, \ldots, n$, by virtue of Theorem 3.1 we conclude that for every $\mu \geq \sum_{i=1}^{n} \tau_i + R$ there exists a linear 3μ-periodic time-varying distributed delay feedback $v(t) = \widetilde{k}(t, T_{\widetilde{r}-r-R}(t - R)z)$ and a constant $T' > 0$ such that the closed-loop system (3.27) with $v(t) = \widetilde{k}(t, T_{\widetilde{r}-r-R}(t - R)z)$ satisfies the dead-beat property of order T'.

Consider the mapping $(t, x) \in \Re \times C^0([-\widetilde{r}, 0]; \Re^n) \to P(t, x) = y \in C^0([-(\widetilde{r} - r), 0]; \Re^n)$ defined by

$$\begin{aligned} &y_1(\theta) := x_1(\theta), \quad \theta \in [-(\widetilde{r} - r), 0], \\ &y_i(\theta) := x_i(\theta) + \varphi_{i-1}(t + \theta, T_r(\theta)x_1, \ldots, T_r(\theta)x_{i-1}), \quad i = 2, \ldots, n, \theta \in [-(\widetilde{r} - r), 0]. \end{aligned} \tag{3.28}$$

Notice that since the mappings $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$, are all completely locally Lipschitz with respect to $x \in C^0([-r, 0]; \Re^n)$, it follows that for every bounded set $S \subset \Re \times C^0([-\widetilde{r}, 0]; \Re^n)$ there exists $L \geq 0$ such that $\|P(t, x) - P(t, y)\|_{\widetilde{r}-r} \leq L\|x - y\|_{\widetilde{r}}$ for all $(t, x) \in S$, $(t, y) \in S$. Moreover, notice that $P(t, 0) = 0$ for all $t \in \Re$. We next prove that the time-varying distributed delay feedback that induces the dead-beat property for the corresponding closed-loop system is defined by

$$k(t, x) := -\varphi_n(t, T_r(0)x) + \widetilde{k}(t, P(t - R, T_{\widetilde{r}}(-R)x)). \tag{3.29}$$

Notice that if the mappings $\varphi_i : \Re \times C^0([-r + \tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$ are independent of t (linear), then the functional k as defined by (3.29) is 3μ-periodic (linear). Let the solution $x(t)$ of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}+R}(t)x)$ with arbitrary initial condition $T_{\widetilde{r}+R+\tau_n}(t_0)x = x_0$.

We make the following claim.

CLAIM. *The solution* $x(t)$ *of the closed-loop system* (3.1) *with* $u(t) = k(t, T_{\widetilde{r}+R}(t)x)$ *exists for all* $t \geq t_0$ *and satisfies Property* (P2) *of Definition* 2.7.

Proof of Claim. Since

- $P(t, 0) = 0$ for all $t \in \Re$ and for every bounded set $S \subset \Re \times C^0([-\widetilde{r}, 0]; \Re^n)$ there exists $L \geq 0$ such that $\|P(t, x) - P(t, y)\|_{\widetilde{r}-r} \leq L\|x - y\|_{\widetilde{r}}$ for all $(t, x) \in S$, $(t, y) \in S$.
- $\widetilde{k}(t, z)$ is a linear 3μ-periodic time-varying functional

it follows from Lemma 3.2 in [12] that there exist functions $\delta \in K^+$, being nondecreasing and $q \in K_\infty$ such that

$$|\widetilde{k}(t-\tau_n, P(t-\tau_n-R, T_{\widetilde{r}}(-\tau_n-R)x))| \leq q(\delta(t)\|T_{\widetilde{r}}(-\tau_n-R)x\|_{\widetilde{r}}) \\ \forall (t,x) \in \Re^+ \times C^0([-\widetilde{R},0];\Re^n) \tag{3.30}$$

with $\widetilde{R} := \widetilde{r} + \tau_n + R$. In order to prove the claim it suffices to show that if there exists an integer N such that

$$\sup\big\{|x(t_0+h,t_0,x_0)|;\ h \in [-\widetilde{R}, NR], \|x_0\|_{\widetilde{R}} \leq s, t_0 \in [0,s]\big\} < +\infty \quad \forall s \geq 0, \tag{3.31}$$

then we have

$$\sup\big\{|x(t_0+h,t_0,x_0)|;\ h \in [-\widetilde{R}, (N+1)R], \|x_0\|_{\widetilde{R}} \leq s, t_0 \in [0,s]\big\} < +\infty \\ \forall s \geq 0, \tag{3.32}$$

where $x(t,t_0,x_0)$ denotes the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ with initial condition $T_{\widetilde{R}}(t_0)x = x_0$. Indeed, since (3.31) holds for $N = 0$, it will follow by induction that (3.31) holds for all nonnegative integers N.

Suppose that (3.31) holds for some nonnegative integer N. Let arbitrary $s \geq 0$ and define $V(t) := \frac{1}{2}|x(t)|^2$, where $x(t)$ denotes the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ with arbitrary initial condition $T_{\widetilde{R}}(t_0)x = x_0$ such that $\|x_0\|_{\widetilde{R}} \leq s$, $t_0 \in [0,s]$. Using identity (3.2b) for $i = n-1$, inequality (3.3) in conjunction with (3.30) imply that as long as the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ exists, it holds that

$$\dot{V}(t) \leq L\left(t, \sup_{-r\leq\theta\leq -R}|x(t+\theta)|\right)(2V(t)+1) + V(t) \\ + q^2\left(\delta(t)\sup_{-r\leq\theta\leq -R}|x(t+\theta)|\right). \tag{3.33}$$

Let $a := \sup\big\{|x(t_0+h,t_0,x_0)|;\ h \in [-\widetilde{R}, NR], \|x_0\|_{\widetilde{R}} \leq s, t_0 \in [0,s]\big\} < +\infty$. Clearly, for all $t \in [NR, (N+1)R]$ for which the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ exists, we have

$$\dot{V}(t) \leq \left(1 + q^2(\delta((N+1)R)a) + \max_{\substack{0\leq t\leq (N+1)R \\ 0\leq s\leq a}} 2L(t,s)\right)(V(t)+1).$$

The previous differential inequality implies that the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ exists for all $t \in [NR, (N+1)R]$ and satisfies

$$|x(t)|^2 \leq 2V(t) \leq \exp(M(N,R,a))(a^2+2) \quad \forall t \in [NR, (N+1)R],$$

where $M(N,R,a) := R(1 + q^2(\delta((N+1)R)a) + \max_{\substack{0\leq t\leq (N+1)R \\ 0\leq s\leq a}} 2L(t,a))$. The previous property shows that inequality (3.32) holds. Thus the claim is proved.

We continue the proof of Theorem 3.4 by considering the following variables defined by

$$z_1(t) := x_1(t); \quad z_i(t) := x_i(t) + \varphi_{i-1}(t, T_r(t)x_1, \dots, T_r(t)x_{i-1}), \quad i = 2, \dots, n. \tag{3.34}$$

Clearly, by virtue of definitions (3.28) and (3.34) we obtain $T_{\widetilde{r-r}}(t-R)z = P(t-R, T_{\widetilde{r}}(t-R)x)$ for all $t \geq t_0$. Moreover, by virtue of (3.34), identities (3.2), and the definition of the notion of ultimate differentiability along the solutions of (3.1) with time constant $T \geq 0$ (Definition 2.1), $z(t)$ as defined by (3.34) for the solution of the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r+R}}(t)x)$, coincides for all $t \geq t_0 + T$ with the solution of the closed-loop system (3.27) with $v(t) = \widetilde{k}(t, T_{\widetilde{r-r-R}}(t-R)z)$ with initial condition $T_{\widetilde{r-r}}(t_0+T)z = P(t_0+T, T_{\widetilde{r}}(t_0+T)x)$. Thus we obtain $z(t) = 0$ for all $t \geq t_0 + T + T'$. It follows from definition (3.34) that $x(t) = 0$ for all $t \geq t_0 + T + T' + (n-1)r$. Thus the closed-loop system (3.1) with $u(t) = k(t, T_{\widetilde{r}}(t)x)$ satisfies the dead-beat property of order $\widetilde{T} := T + T' + (n-1)r$. The proof is complete. ∎

4. Examples. The following examples present systems, which satisfy assumption (A1) and consequently, by virtue of Theorem 3.1, can be finite-time stabilized by means of time-varying distributed delay feedback. Our first example is concerned with finite-dimensional control systems.

Example 4.1. Consider the two-dimensional control system

$$\begin{aligned} &\dot{x}_1(t) = f_1(t, x_1(t)) + x_2(t); \quad \dot{x}_2(t) = f_2(t, x_1(t), x_2(t)) + u(t), \\ &(x_1(t), x_2(t)) \in \Re^2, u(t) \in \Re, t \geq 0, \end{aligned} \tag{4.1}$$

where $f_1 \in C^2(\Re \times \Re; \Re)$ and $f_2 \in C^1(\Re \times \Re^2; \Re)$ with $f_1(t, 0) = f_2(t, 0, 0) = 0$ for all $t \in \Re$. For system (4.1) finite-time global stabilization cannot be achieved by means of a locally Lipschitz feedback law $u(t) = k(t, x_1(t), x_2(t))$. On the other hand Theorem 3.1 guarantees that there exists a time-varying distributed delay feedback law that achieves finite-time stabilization for the closed-loop system. Indeed, system (4.1) satisfies hypothesis (A1) and thus Theorem 3.1 can be applied. Following the proof of Theorem 3.1, it can be shown that for every $\mu > 0$, $l \in Z^+ \cup \{+\infty\}$, there exist functions $\gamma \in K^+$ and $\rho \in K_\infty$ such that the feedback law

$$\begin{aligned} u(t) = {} & v(t) - f_2(t, x_1(t), x_2(t)) - \frac{\partial f_1}{\partial t}(t, x_1(t)) - \frac{\partial f_1}{\partial x_1}(t, x_1(t)) f_1(t, x_1(t)) \\ & - \frac{\partial f_1}{\partial x_1}(t, x_1(t)) x_2(t) - \mu^{-2} \dot{a}\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x_1(t+s) ds \\ & + \mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-2} h'\left(\frac{s}{\mu}\right) x_1(t+s) ds \\ & - \mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) \Bigg[x_2(t+s) + f_1(t+s, x_1(t+s)) \\ & + \mu^{-1} a\left(\frac{t+s}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{w}{\mu}\right) x_1(t+s+w) dw \Bigg] ds, \end{aligned} \tag{4.2}$$

where $a \in C^l(\mathfrak{R}; \mathfrak{R}^+)$ is a periodic function with period 3, with $a(t) = 0$ for $t \in [0, 2]$ and $\int_2^3 a(t) dt = 1$, $h \in C^l(\mathfrak{R}; \mathfrak{R}^+)$ with $h(s) = 0$ for all $s \notin (-2, -1)$, $\int_{-2}^{-1} h(s) ds = \int_{-2\mu}^{-\mu} \mu^{-1} h(\frac{s}{\mu}) ds = 1$

(i) achieves the dead-beat property of order $T = 14\mu$ for the corresponding closed-loop system with $v(t) \equiv 0$ and

(ii) for every $(t_0, x_0, v) \in \mathfrak{R}^+ \times C^0([-4\mu, 0]; \mathfrak{R}^2) \times C^0(\mathfrak{R}; \mathfrak{R})$ the solution of the closed-loop system (4.1) with (4.2) satisfies estimate (3.5).

In order to estimate the functions $\gamma \in K^+$ and $\rho \in K_\infty$ we consider the coordinate transformation

$$z_1(t) = x_1(t); z_2(t) = f_1(t, x_1(t)) + \mu^{-1} a\left(\frac{t}{\mu}\right)\int_{t-2\mu}^{t-\mu} \mu^{-1} h\left(\frac{w-t}{\mu}\right) x_1(w)dw + x_2(t). \tag{4.3}$$

The closed-loop system (4.1) with (4.2) is described in z-coordinates by the following set of retarded functional differential equations:

$$\dot{z}_1(t) = -\mu^{-1} a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) z_1(t+s)dw + z_2(t);$$
$$\dot{z}_2(t) = -\mu^{-1} a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) z_2(t+s)dw + v(t).$$

Repeated application of Lemma 2.9 for the above system provides the following estimate:

$$|z(t)| \leq 2(1+5\mu)\exp(9L)\sigma\left(\frac{t-t_0}{\mu} - 12\right)\|T_{2\mu}(t_0)z\|_{2\mu} + 100\mu(1+2\mu)\exp(9L)\sup_{t_0\leq s\leq t}|v(s)| \quad \forall t \geq t_0, \tag{4.4}$$

where $z := (z_1, z_2)', \sigma(t) := \left\{\begin{smallmatrix} 1 & if & t<0 \\ 0 & if & t\geq 0\end{smallmatrix}\right.$ and $L := \max_{t\in[0,3]} a(t)$. The functions $\gamma \in K^+$ and $\rho \in K_\infty$ can be determined directly from estimate (4.4). For example, consider the case that there exists $b \in K_\infty$ such that $|f_1(t, x_1)| \leq b(|x_1|)$ for all $(t, x_1) \in \Re \times \Re$ (this requirement is automatically satisfied if f_1 is independent of time). In this case transformation (4.3) implies the inequalities $|x(t)| \leq \beta(\|T_{2\mu}(t)z\|_{2\mu})$ and $|z(t)| \leq \beta(\|T_{2\mu}(t)x\|_{2\mu})$, where $\beta(s) := (2+\mu^{-1}L)s + b(s)$. Using estimate (4.4) and previous inequalities $\|T_{2\mu}(t_0)z\|_{2\mu} \leq \beta(\|T_{4\mu}(t_0)x\|_{4\mu})$ (which directly imply the inequality $\|T_{2\mu}(t_0)z\|_{2\mu} \leq \beta(\|T_{4\mu}(t_0)x\|_{4\mu})$), we obtain

$$|x(t)| \leq \beta\Bigg(2(1+5\mu)\exp(9L)\sigma\left(\frac{t-t_0}{\mu} - 14\right)\beta(\|T_{4\mu}(t_0)x\|_{4\mu}) + 100\mu(1+2\mu)\exp(9L)\sup_{t_0\leq s\leq t}|v(s)|\Bigg) \quad \forall t \geq t_0.$$

We conclude that estimate (3.5) holds for the closed-loop system (4.1), (4.2) with $\gamma(t) \equiv 1$ and $\rho(s) := \beta(M(s+\beta(s)))$, where $M := 100(1+5\mu)(1+\mu)\exp(9L)$.

Similarly, we can address the finite-time stabilization problem for the general triangular case (1.1) with no delays, where $f_i \in C^{n-i+1}(\Re\times\Re^i; \Re)$ with $f_i(t, 0, \ldots, 0) = 0$ for all $t \in \Re, i = 1, \ldots, n$. Proceeding exactly as in the 2-dimensional case, it can be shown that if (1.1) is autonomous then the proposed distributed delay feedback guarantees that estimate (3.5) holds for the corresponding closed-loop system with $\gamma(t) \equiv 1$.

The following example is a triangular autonomous control system with discrete delays.

Example 4.2. Consider the control system:

$$\begin{aligned} &\dot{x}_1(t) = x_1^2(t-\tau_1) + x_2(t-\tau_1);\\ &\dot{x}_2(t) = -2x_1(t)x_1^2(t-\tau_1) - 2x_1(t)x_2(t-\tau_1) + x_2(t-2\tau_2) + u(t-\tau_2),\\ &(x_1(t), x_2(t)) \in \Re^2, u(t) \in \Re. \end{aligned} \tag{4.5}$$

It can be verified that assumption (A1) holds for (4.5) with $\varphi_1(t,x_1) := x_1^2(0)$ and $\varphi_2(t,x_1,x_2) := x_2(-\tau_2)$. Thus Theorem 3.1 guarantees that for every $\mu > \tau_1+\tau_2$, there exists a 3μ-periodic time-varying distributed delay feedback such that the corresponding closed-loop system satisfies the dead-beat property. The following feedback law:

$$\begin{aligned} u(t) = &-x_2(t-\tau_2) - \mu^{-2}\dot{a}\left(\frac{t+\tau_1+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)x_1(t+\tau_1+\tau_2+s)ds \\ &+\mu^{-1}a\left(\frac{t+\tau_1+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-2}\dot{h}\left(\frac{s}{\mu}\right)x_1(t+\tau_1+\tau_2+s)ds \\ &-\mu^{-1}a\left(\frac{t+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)\Bigg[x_2(t+\tau_2+s)+x_1^2(t+\tau_2+s) \\ &+\mu^{-1}a\left(\frac{t+\tau_1+\tau_2+s}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{w}{\mu}\right)x_1(t+\tau_1+\tau_2+s+w)dw\Bigg]ds, \end{aligned}$$

where $a \in C^l(\Re;\Re^+)$ is a periodic function with period 3, with $a(t)=0$ for $t\in[0,2]$ and $\int_2^3 a(t)dt = 1$, $h\in C^l(\Re;\Re^+)$ with $h(s)=0$ for all $s\notin(-2,-1)$, $\int_{-2}^{-1}h(s)ds = \int_{-2\mu}^{-\mu}\mu^{-1}h(\frac{s}{\mu})ds = 1$, achieves the dead-beat property of order $T=14\mu$ for the corresponding closed-loop system.

Example 4.3. The chain of delayed integrators (1.2), where $\tau_i \geq 0$, has been considered in the literature for the particular case of $\tau_1=\cdots=\tau_{n-1}=0$ in [15] with the additional requirement that the stabilizing feedback must be bounded. Here we consider the general case and demand finite-time stabilization of the corresponding closed-loop system. Since assumption (A1) holds with $\varphi_i\equiv 0$ for $i=1,\ldots,n$, by virtue of Theorem 3.1 we conclude that for every $\mu > \sum_{i=1}^n \tau_i$ there exists a linear 3μ-periodic time-varying distributed delay feedback such that the closed-loop system satisfies the dead-beat property. Notice that there is no limitation on the size of the delays. Specifically, for the case $n=2$ we obtain the following feedback, which guarantees the dead-beat property of order $T=14\mu$:

$$\begin{aligned} u(t) = &-\mu^{-2}\dot{a}\left(\frac{t+\tau_1+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)x_1(t+\tau_1+\tau_2+s)ds \\ &+\mu^{-1}a\left(\frac{t+\tau_1+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-2}h'\left(\frac{s}{\mu}\right)x_1(t+\tau_1+\tau_2+s)ds \\ &-\mu^{-1}a\left(\frac{t+\tau_2}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)\Bigg[x_2(t+\tau_2+s) \\ &+\mu^{-1}a\left(\frac{t+\tau_1+\tau_2+s}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{w}{\mu}\right)x_1(t+\tau_1+\tau_2+s+w)dw\Bigg]ds, \end{aligned} \tag{4.6}$$

where $a \in C^l(\Re;\Re^+)$ is a periodic function with period 3, with $a(t)=0$ for $t\in[0,2]$ and $\int_2^3 a(t)dt = 1$, $h\in C^l(\Re;\Re^+)$ with $h(s)=0$ for all $s\notin(-2,-1)$, $\int_{-2}^{-1}h(s)ds = \int_{-2\mu}^{-\mu}\mu^{-1}h(\frac{s}{\mu})ds = 1$. Figures 1–3 show the evolution of the state and the input for the C^2 selection:

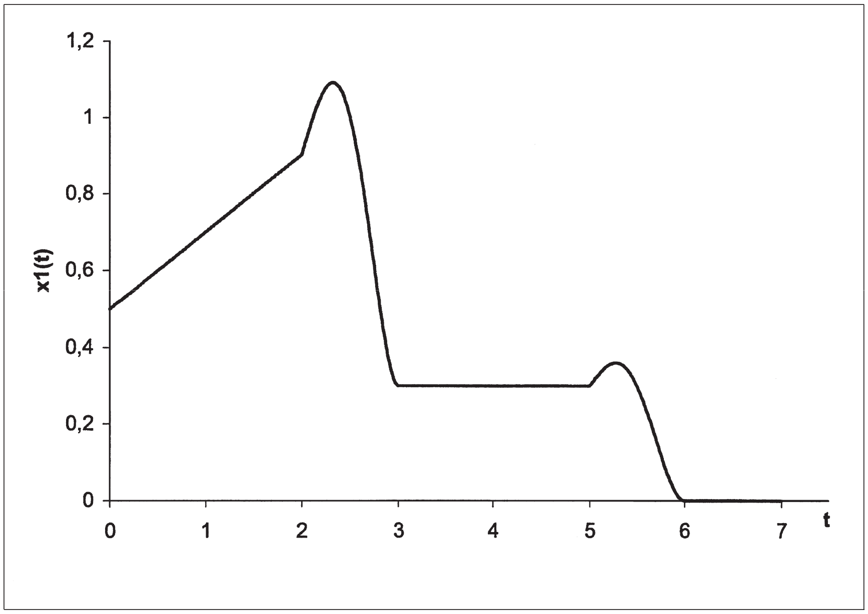

FIG. 1. *Evolution of $x_1(t)$ for the case* (1.2) *with $n = 2$ and $\tau_1 = \tau_2 = 0$ under the feedback law* (4.6), (4.7).

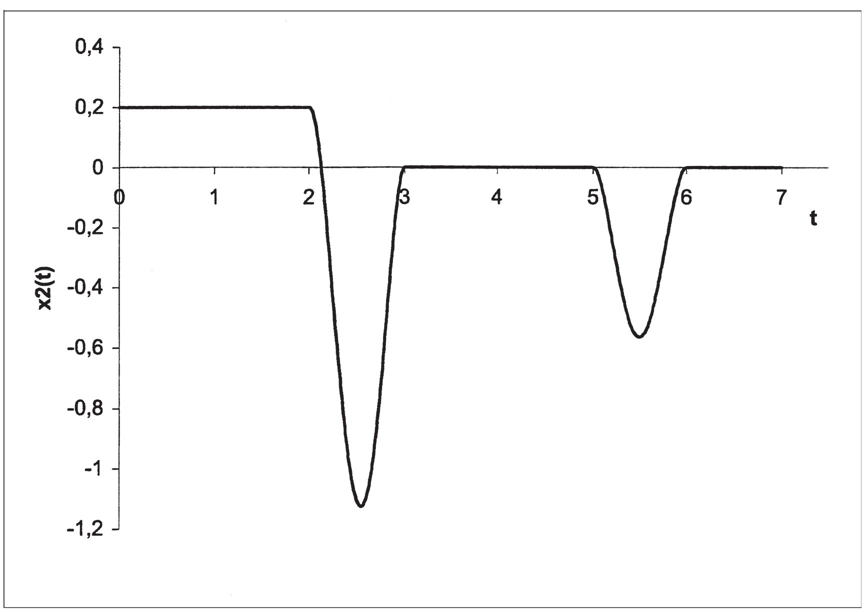

FIG. 2. *Evolution of $x_2(t)$ for the case* (1.2) *with $n = 2$ and $\tau_1 = \tau_2 = 0$ under the feedback law* (4.6), (4.7).

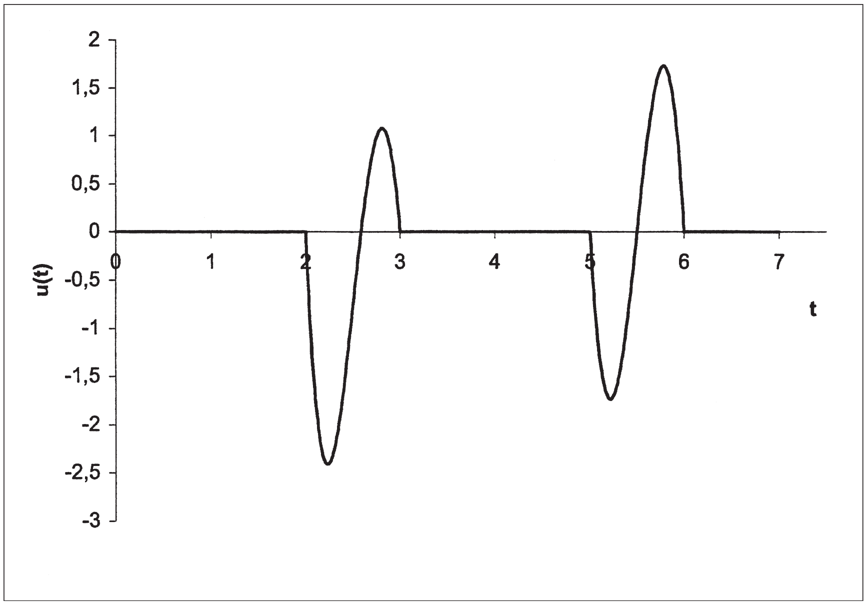

FIG. 3. *Evolution of $u(t)$ for the case* (1.2) *with $n = 2$ and $\tau_1 = \tau_2 = 0$ under the feedback law* (4.6)*,* (4.7)*.*

$$
\tag{4.7}
\begin{aligned}
&a(t) := p\left(t - 3\left[\frac{t}{3}\right]\right) \quad \text{and} \quad h(s) := a(s+4), \text{ where} \\
&p(x) := \begin{cases} 30(x-2)^2(3-x)^2 & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases}
\end{aligned}
$$

for the case (1.2) with $n = 2$ and $\tau_1 = \tau_2 = 0$ with initial condition $x_1(0) = 0.5$ and $x_2(0) = 0.2$.

The case of triangular control systems with delayed drift terms is considered in the following example.

Example 4.4. Consider the planar system

$$
\tag{4.8}
\begin{aligned}
\dot{x}_1(t) &= f_1(t, x_1(t - \tau_{1,1})) + x_2(t); \\
\dot{x}_2(t) &= f_2(t, x_1(t - \tau_{2,1}), x_2(t - \tau_{2,2})) + u(t), \\
&(x_1(t), x_2(t)) \in \Re^2, u(t) \in \Re, t \geq 0,
\end{aligned}
$$

where $\tau_{1,1} > 0, \tau_{2,1} \geq 0, \tau_{2,2} \geq 0$, and $f_1 \in C^2(\Re^+ \times \Re; \Re)$ and $f_2 \in C^1(\Re^+ \times \Re^2; \Re)$ with $f_1(t, 0) = f_2(t, 0, 0) = 0$ for all $t \geq 0$. Clearly, assumption (A2) holds for this system with $R := \tau_{1,1}$, since we have

$$
\begin{aligned}
&x_1(0) f_1(t, x_1(-\tau_{1,1})) - x_2(0) \left(\frac{\partial f_1}{\partial t}(t, x_1(-\tau_{1,1})) \right. \\
&\quad \left. + \frac{\partial f_1}{\partial x_1}(t, x_1(-\tau_{1,1})) f_1(t - \tau_{1,1}, x_1(-2\tau_{1,1})) + \frac{\partial f_1}{\partial x_1}(t, x_1(-\tau_{1,1})) x_2(-\tau_{1,1}) \right) \\
&\quad \leq \frac{1}{2} x_1^2(0) + \frac{1}{2} x_2^2(0) + L\left(t, \sup_{-2\tau_{1,1} \leq \theta \leq -\tau_{1,1}} |x(\theta)|\right),
\end{aligned}
$$

where $L(t,s) := \frac{1}{2}\max_{|\xi|\leq s}[(f_1(t,\xi_1))^2 + (\frac{\partial f_1}{\partial t}(t,\xi_1) + \frac{\partial f_1}{\partial x_1}(t,\xi_1)f_1(t-\tau_{1,1},\xi_2) + \frac{\partial f_1}{\partial x_1}(t,\xi_1)\xi_3)^2]$. Following the proof of Theorem 3.4, it can be shown that for every $\mu > \tau_{1,1}$ the feedback law

$$\begin{aligned}
u(t) = &-\frac{\partial f_1}{\partial t}(t, x_1(t-\tau_{1,1})) - \frac{\partial f_1}{\partial x_1}(t, x_1(t-\tau_{1,1}))f_1(t-\tau_{1,1}, x_1(t-2\tau_{1,1})) \\
&- \frac{\partial f_1}{\partial x_1}(t, x_1(t-\tau_{1,1}))x_2(t-\tau_{1,1}) - f_2(t, x_1(t-\tau_{2,1}), x_2(t-\tau_{2,2})) \\
&- \mu^{-2}\dot{a}\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)x_1(t+s)ds \\
&+ \mu^{-1}a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-2}h'\left(\frac{s}{\mu}\right)x_1(t+s)ds \\
&- \mu^{-1}a\left(\frac{t}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{s}{\mu}\right)\left[x_2(t+s) + f_1(t+s, x_1(t-\tau_{1,1}+s))\right. \\
&\left.+ \mu^{-1}a\left(\frac{t+s}{\mu}\right)\int_{-2\mu}^{-\mu}\mu^{-1}h\left(\frac{w}{\mu}\right)x_1(t+s+w)dw\right]ds
\end{aligned}$$

guarantees the dead-beat property of order $14\mu + 2\tau_{1,1}$. Similarly, we can address the finite-time stabilization problem for the general triangular case with delayed drift terms (1.3), where $\min_{i=1,\dots,n-1}\ \min_{j=1,\dots,i}\tau_{i,j} > 0,\ f_i \in C^{n-i+1}(\Re^+ \times \Re^i;\Re)$ with $f_i(t,0,\dots,0) = 0$ for all $t \geq 0,\ i = 1,\dots,n$.

5. Conclusions. In this paper it is shown that finite-time stabilization by means of time-varying distributed delay feedback is possible for specific classes of systems described by RFDEs. Based on our main results (Theorems 3.1 and 3.4) we have been able to construct locally Lipschitz feedback laws that guarantee the dead-beat property for the corresponding closed-loop systems for important cases such as the general triangular case with no delays, the case of a chain of delayed integrators with no limitation on the size of the delays and the general triangular case with delayed drift terms.

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