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Non-uniform robust global asymptotic stability for discrete-time systems and applications to numerical analysis

IASSON KARAFYLLIS
Division of Mathematics, Department of Economics, University of Athens, 8 Pesmazoglou Street, 10559, Athens, Greece

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The notion of non-uniform Robust Global Asymptotic Stability (RGAS) presented in this paper generalizes the notion of non-uniform in time RGAS for finite- or infinite-dimensional discrete-time systems. Lyapunov characterizations for this stability notion are provided. The results are applied to finite-dimensional discrete-time systems obtained by time discretization of continuous-time systems by the explicit Euler method.

Keywords: discrete-time systems; time discretization; Lyapunov functions.

1. Introduction

In this paper we study discrete-time systems of the form:

$$\begin{aligned} x(t+1) &= f(w(t), x(t), d(t)), \\ w(t+1) &= g(w(t), x(t), d(t)), \\ x(t) \in \mathcal{X}, \quad w(t) &\in W \subseteq \mathcal{W}, \quad d(t) \in D, \quad t \in \mathbb{Z}^+, \end{aligned} \tag{1.1}$$

where $\mathcal{X}, \mathcal{W}$ is a pair of normed linear spaces, $\mathbb{Z}^+$ denotes the set of non-negative integers, $W \subseteq \mathcal{W}$ and D are sets with $0 \in W$, $f\colon W \times \mathcal{X} \times D \to \mathcal{X}$, $g\colon W \times \mathcal{X} \times D \to W$ are mappings with $f(w, 0, d) = 0 \in \mathcal{X}$, for all $(w, d) \in W \times D$. Notice that time-varying discrete-time systems

$$\begin{aligned} x(t+1) &= f(t, x(t), d(t)), \\ x(t) &\in \mathcal{X}, \quad d(t) \in D, \quad t \in \mathbb{Z}^+, \end{aligned} \tag{1.2}$$

where $f\colon \mathbb{Z}^+ \times \mathcal{X} \times D \to \mathcal{X}$ is a mapping with $f(t, 0, d) = 0 \in \mathcal{X}$, for all $(t, d) \in \mathbb{Z}^+ \times D$ can be described by the evolution equation (1.1), since such systems take the form

$$\begin{aligned} x(t+1) &= f(w(t), x(t), d(t)), \\ w(t+1) &= g(w(t)), \\ x(t) \in \mathcal{X}, \quad w(t) &\in \mathbb{Z}^+ \subseteq \mathbb{R}, \quad d(t) \in D, \quad t \in \mathbb{Z}^+, \end{aligned} \tag{1.3}$$

with $g(w) := w + 1$, for all $w \in W := \mathbb{Z}^+ \subseteq \mathbb{R} := \mathcal{W}$.

Specifically, in this paper we provide necessary and sufficient conditions and Lyapunov characterizations for non-uniform Robust Global Asymptotic Stability (RGAS). The notion of non-uniform RGAS is extension of non-uniform in time RGAS introduced in Karafyllis & Tsinias (2003a) for continuous-time systems and of non-uniform in time Robust Global Asymptotic Output Stability introduced in Karafyllis

(2004) for a wide class of systems. This notion has been proved to be fruitful for the solution of several problems in Mathematical Control Theory (see Karafyllis & Tsinias, 2003a,b). We also specialize the results to the important time-varying case (1.2).

Discrete-time systems have been studied extensively, since discrete-time systems arise by the process of sampling, which is used in practice for computer control (see Nijmeijer & van der Schaft, 1990; Sontag, 1998a). The dynamics of discrete-time systems are studied in Devaney (1989) and the topological dynamics of discrete-time control systems are studied in Kloeden & Schmalfuss (1998), Kotsios (1993), and Tsinias *et al.* (1989). In the series of papers Jiang & Wang (2001); Jiang & Wang (2002) and Jiang *et al.* (1999, 2000, 2004), the authors generalize the uniform Input-to-State Stability notion to the discrete-time systems, provide converse Lyapunov theorems for the notions of uniform Global Asymptotic Stability and Global Exponential Stability and establish Small-Gain theorems for discrete-time systems. A converse Lyapunov theorem for local asymptotic stability is also presented in Stuart & Humphries (1998). Converse Lyapunov theorems for more general types of asymptotic stability and results concerning the existence of control Lyapunov functions are provided in Kellett & Teel (2003a,b, 2004a,b). Sufficient conditions for practical stability of sampled-data systems are given in Nesic *et al.* (1999) by making use of the asymptotic stability of an approximate discrete-time system. Finally, in Loria & Nesic (2003), discrete-time systems in cascade form are studied.

Discrete-time systems of the form (1.1) arise naturally by time discretization of continuous-time systems. For example, for the continuous-time infinite-dimensional system described by the following system of retarded functional differential equations (see Hale & Lunel, 1993):

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t_{\mathrm{c}}}(t_{\mathrm{c}}) = \tilde{f}\,(t_{\mathrm{c}}, \{x(t_{\mathrm{c}}); \theta \in [-r, 0]\}, d(t_{\mathrm{c}}\,))\,, \\ x(t_{\mathrm{c}}) \in \mathbb{R}^n, \quad d(t_{\mathrm{c}}) \in D, \quad t_{\mathrm{c}} \geqslant 0,\end{gathered} \tag{1.4}$$

where t_{c} denotes the 'continuous' time variable (in order to differentiate with t, the 'discrete' time variable). By applying time discretization, we obtain the following infinite-dimensional discrete-time system:

$$\begin{gathered}x(t+1) = f(t_{\mathrm{c}}(t), x(t), d(t), h(t)), \\ t_{\mathrm{c}}(t+1) = t_{\mathrm{c}}(t) + h(t), \\ x(t) \in C^0([-r, 0]; \mathbb{R}^n), \quad t_{\mathrm{c}}(t) \in \mathbb{R}^+ \subset \mathbb{R}, \quad h(t) \in [0, 1], \quad d(t) \in C^0(\mathbb{R}^+; D), \quad t \in \mathbb{Z}^+,\end{gathered} \tag{1.5}$$

where

$$f(t_{\mathrm{c}}, x, d, h) := \begin{cases} x(\theta + h), \quad \text{for } \theta \in [-r, -h], \\ x(0) + \int_{t_{\mathrm{c}}}^{t_{\mathrm{c}}+h+\theta} \tilde{f}(s, \{x(\xi); \xi \in [-r, 0]\}, d(s))\,\mathrm{d}s, \\ \quad \text{for } \theta \in (-h, 0] \text{ and } h > 0. \end{cases} \tag{1.6}$$

The method of time discretization is crucial for the properties of the corresponding discrete-time system. In the above example, the discrete-time system (1.5) is obtained from the continuous-time system (1.4) by an immediate extension of the so-called explicit Euler method, which is the simplest method of time discretization of continuous-time finite-dimensional systems described by ordinary differential equations of the form:

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t_{\mathrm{c}}}(t_{\mathrm{c}}) = \tilde{f}(t_{\mathrm{c}}, x(t_{\mathrm{c}}), d'(t_{\mathrm{c}})), \\ x(t_{\mathrm{c}}) \in \mathbb{R}^n, \quad t_{\mathrm{c}} \geqslant 0, \quad d'(t_{\mathrm{c}}) \in \varOmega \subset \mathbb{R}^m.\end{gathered} \tag{1.7}$$

The explicit Euler method for system (1.7) yields a finite-dimensional discrete-time system:

$$\begin{aligned} x(t+1) &= F\left(t_c(t), x(t), d'(t), \frac{h(t)}{1+\theta(t)}\right), \\ t_c(t+1) &= t_c(t) + \frac{h(t)}{1+\theta(t)}, \\ (x(t), t_c(t)) &\in \mathbb{R}^n \times \mathbb{R}^+, \quad h(t) \in [0,1], \quad t \in \mathbb{Z}^+, \\ d(t) &:= (d'(t), \theta(t)) \in D := \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \Theta, \end{aligned} \tag{1.8}$$

where $\Theta \subseteq \mathbb{R}^+$ and

$$F(t_c, x, d', h) := x + \int_{t_c}^{t_c+h} \tilde{f}(\tau, x, d'(\tau))\,d\tau. \tag{1.9}$$

If $\tilde{f}(t_c, 0, d') = 0$, for all $(t_c, d') \in \mathbb{R}^+ \times \Omega$, then it can be verified that (1.8) has the form of system (1.1). System (1.8) is called the Euler discrete-time approximation of (1.7). The reason for introducing the uncertainty $\theta(t) \in \Theta$ is to allow some flexibility on the chosen time step (one usually wants to be free to apply smaller step sizes and this can be quantified appropriately by selecting large values for the sequence $\theta(t)$). For linear time-invariant finite-dimensional continuous-time systems of the form $\dot{x} = Ax$, $x \in \mathbb{R}^n$, it is known that if the matrix A is Hurwitz, i.e. zero is a globally asymptotically stable equilibrium point for the continuous-time system, then for every $r \geqslant 0$, zero will be robustly globally asymptotically stable for its Euler discrete-time approximation with $\Theta := [0, r]$ and constant step size (i.e. $h = h(0) = h(1) = h(2) = \cdots$), namely, for every $r \geqslant 0$, the discrete-time finite-dimensional system $x(t+1) = \left(I + \frac{h}{1+\theta(t)}A\right)x(t)$, $t_c(t+1) = t_c(t) + \frac{h}{1+\theta(t)}$, where $x(t) \in \mathbb{R}^n$, $h > 0$, $\theta(t) \in [0, r]$ and I is the identity matrix, will be robustly globally asymptotically stable, if and only if $0 < h\,|\lambda_i|^2 < -2\,\mathrm{Re}(\lambda_i)$, for all $i = 1, \ldots, n$, where $\lambda_i, i = 1, \ldots, n$, are the eigenvalues of the matrix A. For a general non-linear continuous-time system the assumption of robust global asymptotic stability of zero does not necessarily imply that there exists $h > 0$ sufficiently small such that its Euler discrete-time approximation with constant step size is globally asymptotically stable (see the discussion in Stuart & Humphries (1998)). For example, zero is globally asymptotically stable for the continuous-time finite-dimensional system $\dot{x} = -x^3$, $x \in \mathbb{R}$, but for every $h > 0$, its Euler discrete-time approximation with constant step size $x(t+1) = (1 - hx^2(t))x(t)$, $t_c(t) = ht$, $x(t) \in \mathbb{R}$, is not globally asymptotically stable.

The qualitative behaviour of the solutions of discrete-time systems, which are obtained via time discretization from continuous-time finite-dimensional systems, was the subject of intensive research during the last few years. The existence of discretization methods that conserve invariants of the corresponding continuous-time system is studied in Hairer *et al.* (2002). The questions concerning the relation between the attracting sets of the continuous-time (original) system and its numerical approximation are answered in Grune (2002) and Stuart & Humphries (1998). Both monographs present results that apply to discretization methods with fixed time step (or integration step size). Adaptive discretization schemes or discretization schemes with step-size control are also used in the literature (see Schwarz, 1989).

In this paper we consider the explicit Euler method with variable time step. The time step h is used as a control variable to stabilize the numerical approximation and to this end we use the results obtained for non-uniform robust global asymptotic stability for discrete-time systems of the form (1.1). As far as we know, this idea is novel even for uniform asymptotic stability. We show that if zero is non-uniformly in time robustly globally asymptotically stable, then there exists a continuous function

ϕ: $\mathbb{R}^+ \times \mathbb{R}^n \to (0, 1]$, such that for every $r \geqslant 0$, system (1.8) with $0 < h(t) = \phi(t_c(t), x(t))$ is non-uniformly robustly globally asymptotically stable. Moreover, we explicitly construct the continuous function ϕ: $\mathbb{R}^+ \times \mathbb{R}^n \to (0, 1]$ based on the knowledge of a Lyapunov function for the continuous-time system (1.7). The obtained result implies that the global discretization error between the solution of explicit Euler discrete-time approximation of (1.7) with $0 < h(t) = \phi(t_c(t), x(t))$ and the exact solution of (1.7) is bounded on the positive semi-axis. This implication is important for numerical analysis.

Notation

* By $\|\cdot\|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. By $|\cdot|$ we denote the Euclidean norm of $\mathbb{R}^n$.
* For definitions of classes K, K_∞ and KL, see Nesic *et al.* (1999).
  By K^+, we denote the set of all continuous positive functions defined on $\mathbb{R}^+ := [0, +\infty)$.
* By $C^j(A)$ ($C^j(A; \Omega)$), where $j \geqslant 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
* By M_D, we denote the set of all sequences $d = (d(0), d(1), d(2), \ldots)$ with values in D, i.e. $d(t) \in D$, for all $t \in \mathbb{Z}^+$.
* $\mathcal{L}^\infty_{\mathrm{loc}}(A)$ denotes the set of all measurable functions u: $A \to \mathbb{R}^m$ that are essentially bounded on every non-empty compact subset of A, and $\mathcal{L}^\infty(A)$ denotes the set of all measurable functions u: $A \to \mathbb{R}^m$ that are essentially bounded on A.
* By $(x(t), w(t)) = (x(t, x_0, w_0; d), w(t, x_0, w_0; d)) \in \mathcal{X} \times \mathcal{W}$, we denote the solution of (1.1) at time $t \in \mathbb{Z}^+$ with initial condition $(x_0, w_0) \in \mathcal{X} \times W$ and corresponding to some sequence $d \in M_D$.
* By $[x]$, we denote the integer part of a real number $x \in \mathbb{R}$, i.e. $[x] := \max\{k \in \mathbb{Z}; k \leqslant x\}$. It holds that $x - 1 < [x] \leqslant x$, for all $x \in \mathbb{R}$.
* The function $\mathrm{sgn}(x)$, where $x \in \mathbb{R}$, is defined by $\mathrm{sgn}(x) := \begin{cases} 0, & \text{if } x = 0, \\ \frac{x}{|x|}, & \text{if } x \neq 0. \end{cases}$

2. Definitions and main results

In this section the reader is introduced to the notions used in this paper and the main results of the paper are presented without proofs. The notion of Robust Forward Completeness introduced in Karafyllis (2004) is given first.

DEFINITION 2.1 We say that (1.1) is *robustly forward complete (RFC)* if for every $r \geqslant 0$, $T \in \mathbb{Z}^+$, it holds that

$$\begin{aligned} &\sup\{\|x(t, x_0, w_0; d)\|_{\mathcal{X}} + \|w(t, x_0, w_0; d)\|_{\mathcal{W}}; 0 \leqslant t \leqslant T, \\ &\|x_0\|_{\mathcal{X}} \leqslant r, \|w_0\|_{\mathcal{W}} \leqslant r, w_0 \in W, d \in M_D\} < +\infty. \end{aligned} \tag{2.1}$$

The following proposition provides characterizations of RFC for discrete-time systems.

PROPOSITION 2.2 The following statements are equivalent:

(i) The discrete-time system (1.1) is RFC.
(ii) For every bounded subset $S \subset W \times \mathcal{X}$ the image sets $f(S \times D) \subseteq \mathcal{X}$ and $g(S \times D) \subseteq W \subseteq \mathcal{W}$ are bounded.

(iii) There exist $\mu \in K^+$, $a \in K_\infty$ and $R \geqslant 0$ such that for all $(x_0, w_0) \in \mathcal{X} \times W$ and $d \in M_D$ the solution of (1.1) satisfies:

$$\|x(t, x_0, w_0; d)\|_{\mathcal{X}} + \|w(t, x_0, w_0; d)\|_{\mathcal{W}} \leqslant \mu(t)\, a(R + \|x_0\|_{\mathcal{X}} + \|w_0\|_{\mathcal{W}}), \quad \forall\, t \in \mathbb{Z}^+. \tag{2.2}$$

Concerning the proof of Proposition 2.2, we note that the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are obvious. Particularly, the proof of implication (ii) $\Rightarrow$ (i) follows by considering arbitrary $r \geqslant 0$, $T \in \mathbb{Z}^+$, and then defining recursively the sequence of bounded sets in $W \times \mathcal{X}$ by $A(t) := g(A(t-1) \times D) \times f(A(t-1) \times D)$ for $t = 1, \ldots, T$ with $A(0) := \{w \in W; \|w\|_{\mathcal{W}} \leqslant r\} \times \{x \in \mathcal{X}; \|x\|_{\mathcal{X}} \leqslant r\}$ and finally noticing that

$$\{(w(t, x_0, w_0; d), x(t, x_0, w_0; d)); \|x_0\|_{\mathcal{X}} \leqslant r, \|w_0\|_{\mathcal{W}} \leqslant r, w_0 \in W, d \in M_D\} \subseteq A(t), \quad \text{for all } t = 0, \ldots, T.$$

The proof of the implication (i) $\Rightarrow$ (iii) is almost identical to the proof of Lemma 3.5 in Karafyllis (2004) and is omitted.

DEFINITION 2.3 We say that (1.1) is non-uniformly robustly globally asymptotically stable if (1.1) is RFC and the following properties hold:

(P1) (1.1) is *robustly Lagrange stable*, i.e. for every $\varepsilon \geqslant 0$, it holds that

$$\sup\{\|x(t, x_0, w_0; d)\|_{\mathcal{X}}; t \in \mathbb{Z}^+, \|x_0\|_{\mathcal{X}} \leqslant \varepsilon, \|w_0\|_{\mathcal{W}} \leqslant \varepsilon, w_0 \in W, d \in M_D\} < +\infty.$$

(Robust Lagrange Stability)

(P2) (1.1) is *robustly Lyapunov stable*, i.e. for every $\varepsilon > 0$ and $r \geqslant 0$, there exists a $\delta := \delta(\varepsilon, r) > 0$, such that:

$$\|x_0\|_{\mathcal{X}} \leqslant \delta, \|w_0\|_{\mathcal{W}} \leqslant r, w_0 \in W \Rightarrow \|x(t, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \varepsilon, \quad \forall\, t \in \mathbb{Z}^+, \ \forall\, d \in M_D.$$

(Robust Lyapunov Stability)

(P3) (1.1) satisfies the *Property of Robust Uniform Attractivity on bounded sets of initial data*, i.e. for every $\varepsilon > 0$ and $R \geqslant 0$, there exists a $\tau := \tau(\varepsilon, R) \in \mathbb{Z}^+$, such that:

$$\|x_0\|_{\mathcal{X}} \leqslant R, \|w_0\|_{\mathcal{W}} \leqslant R, w_0 \in W \Rightarrow \|x(t, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \varepsilon, \quad \forall\, t \geqslant \tau, \ \forall\, d \in M_D.$$

We say that (1.1) is non-uniformly Robustly Globally Exponentially Attracting and Stable (RGEAS) with constant $\mu > 0$ if (1.1) is RGAS and the following property holds:

(P4) (1.1) satisfies the Robust Global Exponential Attractivity Property with constant $\mu > 0$, i.e. for every $R \geqslant 0$, it holds that:

$$\sup\{\exp(\mu t)\|x(t, x_0, w_0; d)\|_{\mathcal{X}}; t \in \mathbb{Z}^+, \|x_0\|_{\mathcal{X}} \leqslant R, \|w_0\|_{\mathcal{W}} \leqslant R, w_0 \in W, d \in M_D\} < +\infty.$$

It is clear that the notion of non-uniform RGAS generalizes the notion of uniform RGAS (e.g. Jiang & Wang, 2002) as well as the notion of non-uniform in time RGAS as given in Karafyllis (2004) for discrete-time systems. The following lemma is an essential tool for establishing RGAS for (1.1). Its proof is similar to the proof of Lemma 3.3 in Karafyllis (2004) and is given in the Appendix.

LEMMA 2.4 Suppose that (1.1) is RFC and satisfies the Property of Robust Uniform Attractivity on bounded sets of initial data (property P3 of Definition 2.3). Moreover, suppose that the following hypothesis is also satisfied:

(H1) There exist functions $a \in K_\infty, \beta \in K^+$, such that $\| f(w, x, d)\|_{\mathcal{X}} \leqslant a(\beta(\|w\|_{\mathcal{W}})\|x\|_{\mathcal{X}})$, for all $(w, x, d) \in W \times \mathcal{X} \times D$.
Then (1.1) is non-uniformly RGAS.

The following lemma provides a characterization of non-uniform RGAS in terms of K_∞ functions. Such estimates are essential for the establishment of converse Lyapunov results for dynamical systems. Its proof is provided in the Appendix.

LEMMA 2.5 Suppose that (1.1) is RFC. System (1.1) is non-uniformly RGAS if and only if there exist functions $a_1, a_2 \in K_\infty, \beta \in K^+$ and a constant $c > 0$, such that the following estimate holds for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ and $t \in \mathbb{Z}^+$:

$$a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}) \leqslant \exp(-ct)a_2(\beta(\|w_0\|_{\mathcal{W}})\|x_0\|_{\mathcal{X}}). \tag{2.3}$$

The following robust stability notion is 'stronger' than non-uniform RGAS and is an extension of the notion of Global K-Exponential Stability introduced in Lefeber *et al.* (1999) for continuous-time systems.

DEFINITION 2.6 We say that (1.1) is non-uniformly Robustly Globally K-Exponentially Stable (RGK-ES) with constant $c > 0$ if (1.1) is non-uniformly RGAS and its solutions satisfy estimate (2.3) for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ and $t \in \mathbb{Z}^+$ with $a_1(s) = s$, namely,

$$\|x(t, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \exp(-ct)a_2(\beta(\|w_0\|_{\mathcal{W}})\|x_0\|_{\mathcal{X}}). \tag{2.4}$$

When the state space of system (1.1) $\mathcal{X}$ is finite-dimensional, i.e. $\mathcal{X} = \mathbb{R}^n$, and system (1.1) is non-uniformly RGAS, then we can always induce non-uniform RGK-ES after some coordinate change. This useful observation is implied by the following lemma and is going to be used in the subsequent sections. The proof of Lemma 2.7 is given in the Appendix.

LEMMA 2.7 Consider system (1.1) with $\mathcal{X} = \mathbb{R}^n$ and suppose that (1.1) is non-uniformly RGAS. Then there exists a homeomorphism $\Phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ with $\Phi(0) = 0$, such that the system (1.1) under the transformation $z = \Phi(x)$, namely,

$$\begin{gathered} z(t+1) = \Phi(f(w(t), \Phi^{-1}(z(t)), d(t))), \\ w(t+1) = g(w(t), \Phi^{-1}(z(t)), d(t)), \\ z(t) \in \mathbb{R}^n, \quad w(t) \in W \subseteq \mathcal{W}, \quad d(t) \in D, \quad t \in \mathbb{Z}^+, \end{gathered} \tag{2.5}$$

is non-uniformly RGK-ES. Particularly, if (2.3) is satisfied for the solutions of (1.1) then a possible selection for the transformation $z = \Phi(x)$ is given by $z_i = \operatorname{sgn}(x_i)a_1(|x_i|)$, for $i = 1, \ldots, n$.

The following lemma for non-uniform RGK-ES plays the same role as Lemma 2.4 for non-uniform RGAS and it is an essential tool for establishing non-uniform RGK-ES for (1.1). Its proof is given in the Appendix.

LEMMA 2.8 Suppose that (1.1) is RFC and satisfies the Robust Global Exponential Attractivity Property with constant $\mu > 0$ (property P4 of Definition 2.3). Then there exists a function $a \in K^+$ such that the following estimate holds for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ and $t \in \mathbb{Z}^+$:

$$\|x(t, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \exp(-\mu t)a(\|w_0\|_{\mathcal{W}} + \|x_0\|_{\mathcal{X}}). \tag{2.6}$$

Moreover, if, in addition, system (1.1) satisfies hypothesis (H1), then for every $c \in (0, \mu)$, system (1.1) is non-uniformly RGK-ES with constant c.

Under the assumption of Robust Forward Completeness for (1.1), hypothesis (H1) is implied by the following 'stronger' hypothesis concerning the continuity of the dynamics of (1.1):

(H2) For every bounded set $S \subset \mathcal{X} \times W$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup\{\| f(w, x, d) - f(w_0, x_0, d)\|_{\mathcal{X}} + \|g(w, x, d) - g(w_0, x_0, d)\|_{\mathcal{W}}; d \in D\} < \varepsilon$, for all $(x, w) \in S$, $(x_0, w_0) \in S$ with $\|x - x_0\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} < \delta$.

Under the assumption of Robust Forward Completeness for (1.1), the proof of the implication (H2) $\Rightarrow$ (H1) is made by defining the following function:

$$a(T, s) := \sup\{\| f(w, x, d)\|_{\mathcal{X}}; w \in W, \|w\|_{\mathcal{W}} \leqslant T, \|x\|_{\mathcal{X}} \leqslant s, d \in D\},$$

which is well-defined for all $T, s \geqslant 0$ (by virtue of statement (ii) of Proposition 2.2). Moreover, for every $T, s \geqslant 0$, the functions $a(\cdot, s)$ and $a(T, \cdot)$ are non-decreasing and since $f(w, 0, d) = 0 \in \mathcal{X}$ for all $(w, d) \in W \times D$, we also obtain $a(T, 0) = 0$ for all $T \geqslant 0$. Finally, let $\varepsilon > 0$ and $T \geqslant 0$. It can be shown that hypothesis (H2) guarantees the existence of $\delta := \delta(\varepsilon, T) > 0$ such that $a(T, \delta(\varepsilon, T)) < \varepsilon$ and, consequently, we have $\lim_{s \to 0^+} a(T, s) = 0$ for all $T \geqslant 0$. It turns out from Lemma 2.3 in Karafyllis & Tsinias (2003a) that there exist functions $\zeta \in K_\infty, \beta \in K^+$, such that $a(T, s) \leqslant \zeta(\beta(T) s)$, for all $T, s \geqslant 0$, and, consequently, hypothesis (H1) is satisfied.

The following fact is an immediate consequence of hypothesis (H2) for system (1.1).

Fact: Suppose that (1.1) is RFC. System (1.1) under hypothesis (H2) satisfies the property of *continuous dependence with respect to the initial conditions*, i.e. for every bounded set $S \subset W \times \mathcal{X}, \varepsilon > 0$ and $N \in \mathbb{Z}^+$, there exists $\delta := \delta(\varepsilon, N, S) > 0$ such that:

$$\begin{aligned} &\|x - x_0\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} \leqslant \delta, (w_0, x_0) \in S, (w, x) \in S \\ &\Rightarrow \sup_{0 \leqslant t \leqslant N, d \in M_D} \{\|x(t, w, x; d) - x(t, x_0, w_0; d)\|_{\mathcal{X}} + \|w(t, w, x; d) - w(t, x_0, w_0; d)\|_{\mathcal{W}}\} \leqslant \varepsilon. \end{aligned}$$

Finally, we end this section with the statement of two converse Lyapunov theorems for non-uniform RGAS and RGK-ES for discrete-time systems. Both theorems are proved in the next section.

THEOREM 2.9 Suppose that (1.1) is RFC and satisfies hypothesis (H1). Then the following statements are equivalent:

(i) System (1.1) is non-uniformly RGAS.

(ii) There exist functions $V: W \times \mathcal{X} \to \mathbb{R}^+,\ a_1, a_2 \in K_\infty,\ \beta \in K^+$ and a constant $\lambda \in (0, 1)$ such that the following inequalities are satisfied for all $(w, x, d) \in W \times \mathcal{X} \times D$:

$$a_1(\|x\|_{\mathcal{X}}) \leqslant V(w, x) \leqslant a_2(\beta(\|w\|_{\mathcal{W}})\|x\|_{\mathcal{X}}), \tag{2.7a}$$

$$V(g(w, x, d),\ f(w, x, d)) \leqslant \lambda V(w, x). \tag{2.7b}$$

Moreover, if hypothesis (H2) is satisfied, then V is continuous on $W \times X$ and is uniformly continuous on every bounded set $S \subset W \times \mathcal{X}$.

(iii) There exist constants $M \geqslant 0$, functions $V, Q: W \times \mathcal{X} \to \mathbb{R}^+,\ a_1, a_2, a_3 \in K_\infty$ with $a_3(s) \leqslant s$ for all $s \geqslant 0,\ \beta, q \in K^+$ with $\lim_{t \to +\infty} q(t) = 0$ and a continuous function $\varphi: \mathbb{R}^+ \times \mathbb{R}^+ \to$

(0, 1], satisfying the following inequalities for all $(w, x, d) \in W \times \mathcal{X} \times D$:

$$a_1(\|x\|_{\mathcal{X}}) \leqslant V(w, x) \leqslant a_2(\beta(\|w\|_{\mathcal{W}})\|x\|_{\mathcal{X}}) + \beta(\|w\|_{\mathcal{W}}), \tag{2.8a}$$

$$V(g(w, x, d),\ f(w, x, d)) \leqslant V(w, x) - a_3(V(w, x)) + Mq(Q(w, x)), \tag{2.8b}$$

$$Q(g(w, x, d),\ f(w, x, d)) \geqslant Q(w, x) + \min\{1, M\}\varphi(\|x\|_{\mathcal{X}}, Q(w, x)). \tag{2.8c}$$

In addition, if $M > 0$, then for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ the sequence $Q(t) = Q(w(t), x(t))$, i.e. the value of the function $Q(w, x)$ along the trajectories of (1.1) with initial condition $(x_0, w_0) \in \mathcal{X} \times W$ and corresponding to $d \in M_D$ is increasing and satisfies $\lim Q(t) = +\infty$.

THEOREM 2.10 Suppose that (1.1) is RFC and satisfies hypothesis (H1). Then the following statements are equivalent:

(i) System (1.1) is non-uniformly RGK-ES.

(ii) There exist functions $V: W \times \mathcal{X} \to \mathbb{R}^+$, $a_2 \in K_\infty$, $\beta \in K^+$ and a constant $\lambda \in (0, 1)$ such that inequalities (2.7a,b) are satisfied for all $(w, x, d) \in W \times \mathcal{X} \times D$ with $a_1(s) = s$. Moreover, if hypothesis (H2) is satisfied, then V is continuous on $W \times \mathcal{X}$ and is uniformly continuous on every bounded set $S \subset W \times \mathcal{X}$.

(iii) There exist constants $M, K \geqslant 0$, $p, r > 0$, $\lambda \in (0, 1)$, functions $V, Q: W \times \mathcal{X} \to \mathbb{R}^+$, $a \in K_\infty$, $\beta \in K^+$ and $\varphi: W \times \mathcal{X} \times D \to \mathbb{R}^+$, satisfying the following inequalities for all $(w, x, d) \in W \times \mathcal{X} \times D$:

$$\|x\|_{\mathcal{X}}^p \leqslant V(w, x) \leqslant a(\beta(\|w\|_{\mathcal{W}})\|x\|_{\mathcal{X}}) + \beta(\|w\|_{\mathcal{W}}), \tag{2.9a}$$

$$\begin{aligned} V(g(w, x, d),\ f(w, x, d)) \leqslant{}& \lambda V(w, x) + MK\varphi(w, x, d) \\ & \times \exp(-Q(g(w, x, d),\ f(w, x, d))), \end{aligned} \tag{2.9b}$$

$$Q(g(w, x, d),\ f(w, x, d)) \geqslant Q(w, x) + rM + M\varphi(w, x, d). \tag{2.9c}$$

REMARK 2.11 If statement (ii) of Theorem 2.10 holds, then system (1.1) is non-uniformly RGK-ES with constant $c := -\log(\lambda) > 0$. Similarly, if statement (iii) of Theorem 2.10 holds with $M > 0$, then system (1.1) is non-uniformly RGEAS with constant $\mu = \min\{-\frac{\log(\lambda)}{p}, \frac{rM}{p}\}$ and, consequently, by Lemma 2.8, it follows that for every $c \in (0, \mu)$, (1.1) is non-uniformly RGK-ES with constant c.

EXAMPLE 2.12 Consider the finite-dimensional discrete-time system:

$$\begin{aligned} x(t+1) &= f(w(t), x(t), d(t)) := \left(\frac{1 + x^2(t) + 2d(t)\exp(-w(t))}{2 + 2x^2(t)}\right)^{\frac{1}{2}} x(t), \\ w(t+1) &= g(w(t), x(t), d(t)) := w(t) + \frac{1}{1 + w(t) + |x(t)|}, \\ & x(t) \in \mathbb{R}, \quad w(t) \in \mathbb{R}^+ \subset \mathbb{R}, \quad d(t) \in [-\mu, \mu] \subset \mathbb{R}, \end{aligned} \tag{2.10}$$

where $\mu \geqslant 0$ is a constant. Let $V(w, x) := x^2$ and $Q(w, x) := w$, both defined on $\mathbb{R}^+ \times \mathbb{R}$. We obtain for all $(w, x, d) \in \mathbb{R}^+ \times \mathbb{R} \times [-\mu, \mu]$:

$$\begin{aligned} V(g(w, x, d),\ f(w, x, d)) &\leqslant V(w, x) - \frac{1}{2}V(w, x) + (1 + \mu)\exp(-Q(w, x)), \\ Q(g(w, x, d),\ f(w, x, d)) &\geqslant Q(w, x) + \frac{1}{1 + |x| + Q(w, x)}. \end{aligned} \tag{2.11}$$

It follows that inequalities (2.8 a,b,c) are satisfied with $a_1(s) = a_2(s) := s^2$, $\beta(t) \equiv 1$, $a_3(s) := \frac{1}{2}s$, $r := 0$, $M := 1 + \mu$, $q(t) := \exp(-t)$ and $\varphi(s,t) := \frac{1}{1+s+t}$. Using the equivalence of statements (i) and (iii) of Theorem 2.9 (and since hypothesis (H2) is satisfied), we conclude that system (2.10) is non-uniformly RGAS. Moreover, since $M := 1 + \mu > 0$, we guarantee that every solution of (2.10) satisfies $\lim w(t) = +\infty$.

2.1 *Specialization of the main results for the time-varying case (1.2)*

For the time-varying case (1.2), under the following hypothesis (which is the analogue of hypothesis (H1)):

(A1) There exist functions $a \in K_\infty$, $\beta \in K^+$ such that $\| f(t,x,d)\|_{\mathcal{X}} \leqslant a(\beta(t)\|x\|_{\mathcal{X}})$, for all $(t,x,d) \in \mathbb{Z}^+ \times \mathcal{X} \times D$.

we notice that by virtue of Proposition 2.2, system (1.2) is automatically RFC. Moreover, it is immediate to verify that for the time-varying case (1.2) under hypothesis (A1), the notion of non-uniform in time RGAS of the equilibrium point $0 \in \mathcal{X}$ for (1.2) as defined in Karafyllis (2004) coincides with the notion of non-uniform RGAS for system (1.3). Thus, if we consider the following hypothesis (which is the analogue of hypothesis (H2)):

(A2) For every bounded set $S \subset \mathbb{Z}^+ \times \mathcal{X}$ and for every $\varepsilon > 0$ there exists $\delta > 0$, such that $\sup\{\| f(t,x,d) - f(t,y,d)\|_{\mathcal{X}}; d \in D\} < \varepsilon$, for all $(t,x) \in S$, $(t,y) \in S$, with $\|x - y\|_{\mathcal{X}} < \delta$.

then the following corollary is a direct application of Theorem 2.9 to the time-varying case (1.2):

COROLLARY 2.13 Consider system (1.2) under hypothesis (A1). Then the following statements are equivalent:

(i) $0 \in \mathcal{X}$ is non-uniformly in time RGAS for (1.2).

(ii) There exist functions $V: \mathbb{Z}^+ \times \mathcal{X} \to \mathbb{R}^+$, $a_1, a_2 \in K_\infty$, $\beta \in K^+$ and a constant $\lambda \in (0,1)$ such that the following inequalities are satisfied for all $(t,x,d) \in \mathbb{Z}^+ \times X \times D$:

$$a_1(\|x\|_{\mathcal{X}}) \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_{\mathcal{X}}), \tag{2.12a}$$

$$V(t+1, f(t,x,d)) \leqslant \lambda V(t,x). \tag{2.12b}$$

Moreover, if hypothesis (A2) is satisfied, then V is continuous on $\mathbb{Z}^+ \times \mathcal{X}$ and is uniformly continuous on every bounded set $S \subset \mathbb{Z}^+ \times \mathcal{X}$.

(iii) There exist functions $V: \mathbb{Z}^+ \times \mathcal{X} \to \mathbb{R}^+$, $a_1, a_2, a_3 \in K_\infty$ with $a_3(s) \leqslant s$ for all $s \geqslant 0$, $\beta, q \in K^+$ with $\lim_{t\to+\infty} q(t) = 0$, satisfying the following inequalities for all $(t,x,d) \in \mathbb{Z}^+ \times \mathcal{X} \times D$:

$$a_1(\|x\|_{\mathcal{X}}) \leqslant V(t,x) \leqslant a_2(\beta(t)\|x\|_{\mathcal{X}}) + \beta(t), \tag{2.13a}$$

$$V(t+1, f(t,x,d)) \leqslant V(t,x) - a_3(V(t,x)) + q(t). \tag{2.13b}$$

Similarly, the definition of RGK-ES may be specified for the case (1.2) in the following way:

DEFINITION 2.14 We say that $0 \in \mathcal{X}$ is non-uniformly in time RGK-ES for (1.2) with constant $c > 0$ if $0 \in \mathcal{X}$ is non-uniformly in time RGAS for (1.2) and there exist functions $a_2 \in K_\infty$, $\beta \in K^+$ and a constant $c > 0$ such that for all $(t_0, x_0, d) \in \mathbb{Z}^+ \times \mathcal{X} \times M_D$, the solution $x(t)$ of (1.2) with initial condition $x(t_0) = x_0$ and corresponding to input $d \in M_D$ satisfies the following estimate:

$$\|x(t)\|_{\mathcal{X}} \leqslant \exp(-c(t-t_0))a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}), \quad \forall t \geqslant t_0. \tag{2.14}$$

Again, it can be verified for the time-varying case (1.2) under hypothesis (A1) that the notion of non-uniform in time RGK-ES with constant $c > 0$ of the equilibrium point $0 \in \mathcal{X}$ for (1.2) as defined above coincides with the notion of non-uniform RGK-ES with the same constant $c > 0$ for system (1.3). Thus, by virtue of Theorem 2.10, we obtain the following corollary for the case (1.2):

COROLLARY 2.15 Consider system (1.2) under hypothesis (A1). Then the following statements are equivalent:

(i) $0 \in \mathcal{X}$ is non-uniformly in time RGK-ES for (1.2).

(ii) There exist functions $V : \mathbb{Z}^+ \times \mathcal{X} \to \mathbb{R}^+$, $a_2 \in K_\infty$, $\beta \in K^+$ and a constant $\lambda \in (0, 1)$ such that inequalities (2.12a,b) are satisfied for all $(t, x, d) \in \mathbb{Z}^+ \times \mathcal{X} \times D$ with $a_1(s) = s$. Moreover, if hypothesis (A2) is satisfied, then V is continuous on $\mathbb{Z}^+ \times \mathcal{X}$ and is uniformly continuous on every bounded set $S \subset \mathbb{Z}^+ \times \mathcal{X}$.

(iii) There exist constants $M, K \geqslant 0$, $p > 0$, $\lambda \in (0, 1)$, functions $V : \mathbb{Z}^+ \times \mathcal{X} \to \mathbb{R}^+$, $a \in K_\infty$, satisfying the following inequalities for all $(t, x, d) \in \mathbb{Z}^+ \times \mathcal{X} \times D$:

$$\|x\|_{\mathcal{X}}^p \leqslant V(t, x) \leqslant a(\beta(t)\|x\|_{\mathcal{X}}) + \beta(t), \tag{2.15a}$$

$$V(t + 1, f(t, x, d)) \leqslant \lambda V(t, x) + K \exp(-Mt). \tag{2.15b}$$

Particularly, if statement (iii) of Corollary 2.15 holds, it follows that for every $c \in (0, \mu)$, $0 \in \mathcal{X}$ is non-uniformly in time RGK-ES for (1.2) with constant c, where $\mu = \min\{-\frac{\log(\lambda)}{p}, \frac{M}{2p}\}$.

3. Proofs of Theorems 2.9 and 2.10

This section is devoted to the proof of Theorems 2.9 and 2.10. Notice that the implication (ii) $\Rightarrow$ (iii) is obvious for both theorems (select $M = 0$, $Q(w, x) \equiv 0$, $\varphi(t, s) \equiv 1$, $a_3(s) := (1-\lambda)s$, $q(t) := \exp(-t)$ for Theorem 2.9 and select $M = 0$, $p = r = K = 1$, $\varphi(w, x, d) \equiv 1$, $Q(w, x) \equiv 0$, $a(s) := a_2(s)$ for Theorem 2.10). Moreover, the implication (ii) $\Rightarrow$ (i) in both theorems is immediate since for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ and $t \in \mathbb{Z}^+$ we obtain by induction and use of inequality (2.7b):

$$V(w(t, x_0, w_0; d), x(t, x_0, w_0; d)) \leqslant \exp(-ct)V(w_0, x_0),$$

where $c := -\log(\lambda)$. The previous estimate in conjunction with inequality (2.7a) implies (2.3) (or (2.4) in the case of Theorem 2.10). Thus, Lemma 2.5 (or Definition 2.6 in the case of Theorem 2.10) implies that (1.1) is non-uniformly RGAS (RGK-ES with constant c).

In order to prove implication (iii) $\Rightarrow$ (i) of Theorem 2.9, notice that by virtue of Lemma 2.4 it suffices to prove that (1.1) satisfies the Property of Robust Uniform Attractivity on bounded sets of initial data (property P3 of Definition 2.3). The proof of the Property of Robust Uniform Attractivity on bounded sets of initial data is based on the following technical lemmas.

LEMMA 3.1 Let $M > 0$, $a \in K_\infty$ with $a(s) \leqslant s$ for all $s \geqslant 0$ and consider a sequence $\{V(t) \in \mathbb{R}^+; t \in \mathbb{Z}^+\}$ that satisfies the following inequality:

$$V(t + 1) \leqslant V(t) - a(V(t)) + M, \quad \forall t \geqslant t_0 \in \mathbb{Z}^+. \tag{3.1}$$

Then the following inequalities hold:

$$V(t) \leqslant V(t_0) + a^{-1}(M) + M, \quad \forall t \geqslant t_0 \in \mathbb{Z}^+, \tag{3.2}$$

$$V(t) < a^{-1}(2M) + M, \quad \forall t \geqslant t_0 + \frac{V(t_0)}{M}. \tag{3.3}$$

Proof. We first prove (3.2) by induction. Notice that (3.2) holds for $t = t_0$. Suppose that (3.2) holds for some $t \in \mathbb{Z}^+$ with $t \geqslant t_0$. Consider the cases:

* if $a(V(t)) \geqslant M$, then (3.1) implies $V(t+1) \leqslant V(t)$ and, consequently, (3.2) holds for $t+1$.
* if $a(V(t)) < M$ or equivalently if $V(t) < a^{-1}(M)$, then (3.1) implies $V(t+1) \leqslant V(t) + M < a^{-1}(M) + M$ and, consequently, (3.2) holds for $t+1$.

Next we prove the following claim: if (3.3) holds for some $t = T \in \mathbb{Z}^+$ with $T \geqslant t_0$, then (3.3) holds for all $t \geqslant T$. Consider the cases:

* if $a(V(t)) \geqslant M$, then (3.1) implies $V(t+1) \leqslant V(t)$ and, consequently, (3.3) holds for $t+1$.
* if $a(V(t)) < M$ or equivalently if $V(t) < a^{-1}(M)$, then (3.1) implies $V(t+1) \leqslant V(t) + M < a^{-1}(M) + M \leqslant a^{-1}(2M) + M$ and, consequently, (3.3) holds for $t+1$.

The proof of inequality (3.3) is made by contradiction. Suppose that there exists $T \in \mathbb{Z}^+$ with $T \geqslant t_0 + \frac{V(t_0)}{M}$ such that $V(T) \geqslant a^{-1}(2M) + M$. By virtue of the previous claim, this implies that $V(t) \geqslant a^{-1}(2M)+M$ for all $t = t_0, \ldots, T$. Consequently, we have $-a(V(t))+M \leqslant -M$ for all $t = t_0, \ldots, T$. Thus, we obtain from (3.1)

$$V(t+1) \leqslant V(t) - M, \quad \text{for all} \quad t = t_0, \ldots, T. \tag{3.4}$$

Clearly, inequality (3.4) implies that $V(T) \leqslant V(t_0) - M(T - t_0)$ and this estimate in conjunction with our assumption $T \geqslant t_0 + \frac{V(t_0)}{M}$ gives $V(T) \leqslant 0$. Clearly, this implication is in contradiction with the assumption $V(T) \geqslant a^{-1}(2M) + M > 0$. The proof is complete. □

LEMMA 3.2 Let $\varphi \in K^+$ and consider a sequence $\{Q(t) \in \mathbb{R}^+; t \in \mathbb{Z}^+\}$ that satisfies the following inequality:

$$Q(t+1) \geqslant Q(t) + \varphi(Q(t)), \quad \forall t \in \mathbb{Z}^+. \tag{3.5}$$

Then for every $L > 0$, it holds that

$$Q(t) > L, \quad \forall t > 1 + \frac{L}{\min\{\varphi(s); 0 \leqslant s \leqslant L\}}. \tag{3.6}$$

It follows that $\lim Q(t) = +\infty$.

Proof. The proof will be made by contradiction. Let arbitrary $L > 0$ and suppose that there exists $T \in \mathbb{Z}^+$ with $T > 1 + \frac{L}{\min\{\varphi(s);0\leqslant s\leqslant L\}}$ such that $Q(T) \leqslant L$. Notice that by virtue of (3.5) the sequence $\{Q(t) \in \mathbb{R}^+; k \in \mathbb{Z}^+\}$ is non-decreasing and thus we must have $Q(t) \leqslant L$ for all $t = 0, 1, \ldots, T$. Consequently, we obtain by (3.5)

$$Q(t+1) \geqslant Q(t) + \min\{\varphi(s); 0 \leqslant s \leqslant L\}, \quad \text{for all } t = 0, 1, \ldots, T. \tag{3.7}$$

Clearly, inequality (3.7) implies that $Q(T) \geqslant Q(0) + T \min\{\varphi(s); 0 \leqslant s \leqslant L\}$ and this estimate in conjunction with our hypothesis $T > 1 + \frac{L}{\min\{\varphi(s);0\leqslant s\leqslant L\}}$ implies $Q(T) > L$, which contradicts the assumed inequality $Q(T) \leqslant L$. The proof is complete. □

We are now ready to provide the proof of the Property of Robust Uniform Attractivity on bounded sets of initial data for (1.1). Let arbitrary $\varepsilon > 0$, $R \geqslant 0$, $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ with $\|x_0\|_{\mathcal{X}} \leqslant R$, $\|w_0\|_{\mathcal{W}} \leqslant R$ and let $V(t) = V(w(t), x(t))$, $Q(t) = Q(w(t), x(t))$. Consider first the case $M = 0$. Let $s(\varepsilon) > 0$ denote the unique solution of the equation

$$a_3^{-1}(2s) + s = a_1(\varepsilon). \tag{3.8}$$

Inequality (2.8b) with $M = 0$ implies the following estimate for all $t \in \mathbb{Z}^+$:

$$V(t+1) \leqslant V(t) - a_3(V(t)) + s(\varepsilon). \tag{3.9}$$

Thus, using Lemma 3.1, definition (3.8) and inequality (3.9), we obtain that

$$V(t) < a_1(\varepsilon), \quad \forall t \geqslant \tau(\varepsilon, R) := 1 + \left[\frac{a_2\left(R \max\limits_{0\leqslant t\leqslant R} \beta(t)\right) + \max\limits_{0\leqslant t\leqslant R} \beta(t)}{s(\varepsilon)} \right] \tag{3.10}$$

and, consequently, by virtue of inequalities (2.8a) and (3.10) we obtain:

$$\|x(t)\|_{\mathcal{X}} < \varepsilon, \quad \forall t \geqslant \tau(\varepsilon, R). \tag{3.11}$$

Thus, the Property of Robust Uniform Attractivity on bounded sets of initial data for (1.1) is satisfied for the case $M = 0$.

Consider the case $M > 0$ and let $q_{\max} := \max\{q(t); t \geqslant 0\}$. Inequality (2.8b) implies that

$$V(t+1) \leqslant V(t) - a_3(V(t)) + Mq_{\max}, \quad \forall t \in \mathbb{Z}^+. \tag{3.12}$$

Thus, using Lemma 3.1 and inequality (3.12), we obtain that

$$V(t) \leqslant V(0) + a_3^{-1}(Mq_{\max}) + Mq_{\max}, \quad \forall t \in \mathbb{Z}^+. \tag{3.13}$$

It follows by inequality (2.8a) and estimate (3.13) that (1.1) is robustly Lagrange stable and satisfies

$$\|x(t)\|_{\mathcal{X}} \leqslant p(R), \quad \forall t \in \mathbb{Z}^+, \tag{3.14}$$

where $p(R) := a_1^{-1}(a_2(R \max_{0\leqslant t\leqslant R} \beta(t)) + \max_{0\leqslant t\leqslant R} \beta(t) + a_3^{-1}(Mq_{\max}) + Mq_{\max})$ is a continuous positive function. Let $\tau_1 := \tau_1(\varepsilon, R) \in \mathbb{R}^+$ such that

$$Mq(t) \leqslant s(\varepsilon), \quad \forall t \geqslant \tau_1(\varepsilon, R), \tag{3.15}$$

where $s(\varepsilon) > 0$ denotes the unique solution of (3.8). Let also

$$\bar{\varphi}_R(s) := \min\{1, M\} \min_{0\leqslant t\leqslant p(R)} \varphi(t, s). \tag{3.16}$$

Clearly, since $M > 0$, it follows that $\bar{\varphi}_R(\cdot) \in K^+$ for all fixed $R \geqslant 0$. It follows from (2.8c), (3.14) and definition (3.16) that the following inequality is satisfied:

$$Q(t+1) \geqslant Q(t) + \bar{\varphi}_R(Q(t)), \quad \forall t \in \mathbb{Z}^+. \tag{3.17}$$

Lemma 3.2 in conjunction with inequality (3.17) implies that $\lim Q(t) = +\infty$ and

$$Q(t) > \tau_1(\varepsilon, R), \quad \forall t > \tau_2(\varepsilon, R) := 1 + \frac{\tau_1(\varepsilon, R)}{\min\{\bar{\varphi}_R(s); 0 \leqslant s \leqslant \tau_1(\varepsilon, R)\}}. \tag{3.18}$$

Clearly, by virtue of (2.8b), (3.14), (3.15) and (3.18) we obtain that inequality (3.9) holds for all $t \geqslant t_0(\varepsilon, R) := 1 + [\tau_2(\varepsilon, R)]$. Thus, using Lemma 3.1, definition (3.8) and inequalities (2.8a), (3.9) and (3.13), we obtain that

$$\begin{aligned} V(t) &< a_1(\varepsilon), \quad \forall\, t \geqslant \tau(\varepsilon, R) := t_0(\varepsilon, R) + 1 \\ &+ \left[\frac{a_2\left(R \max\limits_{0\leqslant t\leqslant R} \beta(t)\right) + \max\limits_{0\leqslant t\leqslant R} \beta(t) + a_3^{-1}(Mq_{\max}) + Mq_{\max}}{s(\varepsilon)} \right] \geqslant t_0 + \frac{V(t_0)}{s(\varepsilon)} \end{aligned} \tag{3.19}$$

and, consequently, by virtue of inequalities (2.8a) and (3.19) we obtain (3.11). Thus, the Property of Robust Uniform Attractivity on bounded sets of initial data for (1.1) is satisfied for the case $M > 0$.

In order to prove implication (iii) $\Rightarrow$ (i) of Theorem 2.10, notice that by virtue of Lemma 2.8 it suffices to prove that (1.1) satisfies the Robust Global Exponential Attractivity Property (property P4 of Definition 2.3). The case $M = 0$ is considered by the implication (ii) $\Rightarrow$ (i) and, consequently, we are left with the case $M > 0$. Let arbitrary $\varepsilon > 0$, $R \geqslant 0$, $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ with $\|x_0\|_{\mathcal{X}} \leqslant R$, $\|w_0\|_{\mathcal{W}} \leqslant R$ and let $V(t) = V(w(t), x(t))$, $Q(t) = Q(w(t), x(t))$, $\varphi(t) = \varphi(w(t), x(t), d(t))$. Define $\mu := \min\{\frac{rM}{p}, -\frac{\log(\lambda)}{p}\} > 0$ and notice that by virtue of (2.9b,c) we obtain for all $t \in \mathbb{Z}^+$

$$V(t+1) \leqslant \exp(-p\mu)V(t) + KM\varphi(t)\exp(-Q(t+1)),$$

$$Q(t+1) \geqslant Q(t) + p\mu + M\varphi(t).$$

Define the sequence $R(t) = \exp(p\mu t)\, V(t)$. Using the above inequalities, it can be inductively proved that the sequences $R(t)$ and $Q(t)$ satisfy the following inequalities for all $t \in \mathbb{Z}^+$:

$$\begin{aligned} R(t) &\leqslant R(0) + KM \sum_{\tau=0}^{t-1} \varphi(\tau) \exp\left(-M \sum_{s=0}^{\tau} \varphi(s)\right) \\ &\leqslant R(0) + K \sum_{\tau=0}^{t-1} \int_{M\sum_{s=0}^{\tau-1}\varphi(s)}^{M\sum_{s=0}^{\tau}\varphi(s)} \exp(-u)\,\mathrm{d}u \leqslant R(0) + K \int_0^{M\sum_{s=0}^{t-1}\varphi(s)} \exp(-u)\,\mathrm{d}u \\ &\leqslant R(0) + K \\ Q(t) &\geqslant p\mu t + M \sum_{\tau=0}^{t-1} \varphi(\tau). \end{aligned} \tag{3.20}$$

Using the first inequality (3.20) in conjunction with definition $R(t) = \exp(p\mu t)V(t)$ and inequality (2.9a), we obtain

$$\|x(t)\|_{\mathcal{X}} \leqslant \exp(-\mu t) \left[a_2\left(R \max_{0\leqslant t\leqslant R} \beta(t)\right) + \max_{0\leqslant t\leqslant R} \beta(t) + K \right]^{\frac{1}{p}}. \tag{3.21}$$

Inequality (3.21) implies that property P4 of Definition 2.3 holds and, consequently, by virtue of Lemma 2.8, for every $c \in (0, \mu)$, (1.1) is RGK-ES with constant c.

We complete the proof of Theorem 2.6 by proving implication (i) $\Rightarrow$ (ii) for both theorems. Since (1.1) is RGAS (RGK-ES), by virtue of Lemma 2.5 (Definition 2.6), there exist $c > 0$ and functions

$a_1, a_2 \in K_\infty$ such that estimate (2.3) holds (with $a_1(s) := s$). Without loss of generality we may assume that $a_1 \in K_\infty$ is globally Lipschitz on $\mathbb{R}^+$ with unit Lipschitz constant, namely, $|a_1(s_1) - a_1(s_2)| \leqslant |s_1 - s_2|$ for all $s_1, s_2 \geqslant 0$. To see this, notice that we can always replace $a_1 \in K_\infty$ by the function $\tilde{a}_1(s) := \inf\{\min\{\frac{1}{2}y, a(y)\} + |y - s|; y \geqslant 0\}$, which is of class K_∞, globally Lipschitz on $\mathbb{R}^+$ with unit Lipschitz constant and satisfies $\tilde{a}_1(s) \leqslant a_1(s)$. For the case of Theorem 2.10, since $a_1(s) := s$, we can continue without replacing the function $a_1 \in K_\infty$. We define

$$V(w_0, x_0) := \sup\left\{\exp\left(\frac{ct}{2}\right) a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}); t \in \mathbb{Z}^+, d \in M_D\right\}. \tag{3.22}$$

Inequalities (2.7a,b) are immediate consequences of definition (3.22) and estimate (2.3). For the case of Theorem 2.10 we also obtain that (2.7a) holds with $a_1(s) := s$. Moreover, inequality (2.7b) holds with $\lambda := \exp(-\frac{c}{2})$, since we have for all $(w_0, x_0, d') \in W \times \mathcal{X} \times D$:

$$\begin{aligned}
&V(g(w_0, x_0, d'),\ f(w_0, x_0, d')) \\
&\quad = \sup\left\{\exp\left(\frac{ct}{2}\right) a_1(\|x(t,\ f(w_0, x_0, d'), g(w_0, x_0, d'); d)\|_{\mathcal{X}}); t \in \mathbb{Z}^+, d \in M_D\right\} \\
&\quad = \sup\left\{\exp\left(\frac{c(t-1)}{2}\right) a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}); t \geqslant 1, d \in M_D \quad \text{with } d_0 = d'\right\} \\
&\quad \leqslant \exp\left(-\frac{c}{2}\right) \sup\left\{\exp\left(\frac{ct}{2}\right) a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}); t \geqslant 1, d \in M_D\right\} \\
&\quad \leqslant \exp\left(-\frac{c}{2}\right) V(w_0, x_0).
\end{aligned}$$

Next, we show that under hypothesis (H2), for every $R \geqslant r > 0$, V is uniformly continuous on the bounded set:

$$S_{r,R} := \{(w, x) \in W \times \mathcal{X}; \|w\|_{\mathcal{W}} \leqslant R, r \leqslant \|x\|_{\mathcal{X}} \leqslant R\}. \tag{3.23}$$

Let arbitrary $x \neq 0$, $R \geqslant r > 0$, and define:

$$N(w, x) := 1 + \left[\frac{2}{c}\log\left(\frac{a_2(\beta(\|w\|_{\mathcal{W}})\|x\|_{\mathcal{X}})}{a_1(\|x\|_{\mathcal{X}})}\right)\right], \tag{3.24a}$$

$$M(r, R) := 2 + \left[\frac{2}{c}\log\left(\frac{a_2(R\max\{\beta(s); 0 \leqslant s \leqslant R\})}{a_1(r)}\right)\right], \tag{3.24b}$$

where we remind that by $[x]$ we denote the integer part of a real number $x \in \mathbb{R}$ (see Notations). Moreover, notice that by virtue of definitions (3.23) and (3.24a,b) we obtain

$$(w, x) \in S_{r,R} \Rightarrow N(w, x) \leqslant M(r, R). \tag{3.25}$$

On the other hand, for all $x_0 \neq 0$ and $N \in \mathbb{Z}^+$, we obtain by virtue of inequality (2.7a), estimate (2.3) and definition (3.22)

$$\begin{aligned}
a_1(\|x_0\|_{\mathcal{X}}) &\leqslant V(w_0, x_0) \\
&= \max\left\{\sup\left\{\exp\left(\frac{ct}{2}\right) a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}); 0 \leqslant t \leqslant N, d \in M_D\right\},\right. \\
&\qquad \left.\sup\left\{\exp\left(\frac{ct}{2}\right) a_1(\|x(t, x_0, w_0; d)\|_{\mathcal{X}}); t > N, d \in M_D\right\}\right\}
\end{aligned}$$

$$\leqslant \max\left\{\sup\left\{\exp\left(\frac{ct}{2}\right)a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}});\, 0\leqslant t\leqslant N, d\in M_D\right\},\right.$$
$$\left.\sup\left\{\exp\left(-\frac{ct}{2}\right)a_2(\beta(\|w_0\|_{\mathcal{W}})\|x_0\|_{\mathcal{X}});\, t>N\right\}\right\}$$
$$=\max\left\{\sup\left\{\exp\left(\frac{ct}{2}\right)a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}});\, 0\leqslant t\leqslant N, d\in M_D\right\},\right.$$
$$\left.\exp\left(-\frac{c(N+1)}{2}\right)a_2(\beta(\|w_0\|_{\mathcal{W}})\|x_0\|_{\mathcal{X}})\right\}.$$

Clearly, definition (3.24a) implies that $\exp(-\frac{c(N+1)}{2})a_2(\beta(\|w_0\|_{\mathcal{W}})\|x_0\|_{\mathcal{X}}) < a_1(\|x_0\|_{\mathcal{X}})$ for all $N\geqslant N(w_0,x_0)$. Thus, by virtue of (2.7a), we obtain for all $x_0\neq 0$ and $N\geqslant N(w_0,x_0)$

$$V(w_0,x_0)=\sup\left\{\exp\left(\frac{ct}{2}\right)a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}});\, 0\leqslant t\leqslant N, d\in M_D\right\}. \tag{3.26}$$

Let arbitrary $\varepsilon>0$, $R\geqslant r>0$, and consider the bounded set $S_{r,R}$ defined by (3.23). By virtue of the property of continuity with respect to the initial conditions (which holds under hypothesis (H2)), there exists $\delta:=\delta(\varepsilon,r,R)>0$ such that:

$$\|x-x_0\|_{\mathcal{X}}+\|w-w_0\|_{\mathcal{W}}\leqslant\delta,\ (w,x)\in S_{r,R},\ (w_0,x_0)\in S_{r,R}$$
$$\Rightarrow \sup\{\|x(t,w,x;d)-x(t,x_0,w_0;d)\|_{\mathcal{X}};\, 0\leqslant t\leqslant M(r,R), d\in M_D\}$$
$$\leqslant \varepsilon\exp\left(-\frac{cM(r,R)}{2}\right), \tag{3.27}$$

where $M(r,R)$ is defined by (3.24b). Let arbitrary $(w,x)\in S_{r,R}$, $(w_0,x_0)\in S_{r,R}$ with $\|x-x_0\|_{\mathcal{X}}+\|w-w_0\|_{\mathcal{W}}\leqslant\delta$. We have by virtue of properties (3.25) and (3.26)

$$|V(w,x)-V(w_0,x_0)|$$
$$=\left|\sup\left\{\exp\left(\frac{ct}{2}\right)a_1(\|x(t,x,w;d)\|_{\mathcal{X}});\, 0\leqslant t\leqslant M(r,R), d\in M_D\right\}\right.$$
$$\left.-\sup\left\{\exp\left(\frac{ct}{2}\right)a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}});\, 0\leqslant t\leqslant M(r,R), d\in M_D\right\}\right|$$
$$\leqslant\sup\left\{\exp\left(\frac{ct}{2}\right)|a_1(\|x(t,x,w;d)\|_{\mathcal{X}})-a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}})|;\, 0\leqslant t\leqslant M(r,R), d\in M_D\right\}$$
$$\leqslant\exp\left(\frac{cM(r,R)}{2}\right)\sup\{|a_1(\|x(t,x,w;d)\|_{\mathcal{X}})-a_1(\|x(t,x_0,w_0;d)\|_{\mathcal{X}})|;$$
$$0\leqslant t\leqslant M(r,R), d\in M_D\}.$$

Since $a_1\in K_\infty$ is globally Lipschitz on $\mathbb{R}^+$ with unit Lipschitz constant, namely, $|a_1(s_1)-a_1(s_2)|\leqslant |s_1-s_2|$ for all $s_1,s_2\geqslant 0$, we obtain

$$|V(w,x)-V(w_0,x_0)|\leqslant\exp\left(\frac{cM(r,R)}{2}\right)\sup\{\|x(t,x,w;d)-x(t,x_0,w_0;d)\|_{\mathcal{X}};$$
$$0\leqslant t\leqslant M(r,R), d\in M_D\}.$$

Consequently, by virtue of (3.27) we obtain $|V(w, x) - V(w_0, x_0)| \leqslant \varepsilon$ and this proves uniform continuity of V on the bounded set $S_{r,R}$.

In order to show that V is uniformly continuous on every bounded set $S \subset W \times \mathcal{X}$, it suffices to show that for every $R > 0$, V is uniformly continuous on the bounded set:

$$S_R := \{(w, x) \in W \times \mathcal{X}; \|w\|_{\mathcal{W}} \leqslant R, \|x\|_{\mathcal{X}} \leqslant R\}. \tag{3.28}$$

Let $\varepsilon > 0$, $R > 0$ and define

$$r := r(\varepsilon, R) = \min\left\{R, \frac{a_2^{-1}\left(\frac{\varepsilon}{4}\right)}{2\max\{\beta(s); 0 \leqslant s \leqslant R\}}\right\} > 0. \tag{3.29}$$

Let the set $S_{r,R}$ defined by (3.23). It follows that there exists $\delta := \delta(\varepsilon, r, R) > 0$ such that

$$\begin{aligned} &\|x - x_0\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} \leqslant \delta, (w, x) \in S_{r,R}, (w_0, x_0) \in S_{r,R} \\ &\qquad \Rightarrow |V(w, x) - V(w_0, x_0)| \leqslant \frac{\varepsilon}{2}. \end{aligned} \tag{3.30}$$

On the other hand, if $\|w\|_{\mathcal{W}} \leqslant R$, $\|x\|_{\mathcal{X}} \leqslant r$, and $\|w_0\|_{\mathcal{W}} \leqslant R$, $\|x_0\|_{\mathcal{X}} \leqslant r$, it follows by the right-hand side inequality (2.7a) and definition (3.29) that $|V(w, x) - V(w_0, x_0)| \leqslant \frac{\varepsilon}{2}$. Finally, we consider the case $\|w\|_{\mathcal{W}} \leqslant R$, $r < \|x\|_{\mathcal{X}} \leqslant R$, and $\|w_0\|_{\mathcal{W}} \leqslant R$, $\|x_0\|_{\mathcal{X}} \leqslant r$. Clearly, there exists $\lambda \in [0, 1)$ such that $\|x'\|_{\mathcal{X}} = r$, where $x' = x_0 + \lambda(x - x_0)$. Then we obtain by definition (3.29) and the right-hand side inequality (2.7a)

$$\begin{aligned} |V(w, x) - V(w_0, x_0)| &\leqslant |V(w, x) - V(w_0, x')| + |V(w_0, x') - V(w_0, x_0)| \\ &\leqslant |V(w, x) - V(w_0, x')| + \frac{\varepsilon}{2}. \end{aligned}$$

Clearly, if $\|x - x_0\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} \leqslant \delta(\varepsilon, r, R)$, then we have $\|x - x'\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} \leqslant \delta(\varepsilon, r, R)$ and, consequently, by (3.30) we have $|V(w, x) - V(w_0, x')| \leqslant \frac{\varepsilon}{2}$, which implies that $|V(w, x) - V(w_0, x_0)| \leqslant \varepsilon$. Thus, we conclude that

$$\|x - x_0\|_{\mathcal{X}} + \|w - w_0\|_{\mathcal{W}} \leqslant \delta, (w, x) \in S_R, (w_0, x_0) \in S_R \Rightarrow |V(w, x) - V(w_0, x_0)| \leqslant \varepsilon,$$

which shows that V is uniformly continuous on the bounded set S_R. The proof is complete.

4. Applications to numerical analysis

In this section we consider the explicit Euler method of time discretization with variable time step for the continuous-time finite-dimensional system (1.7). The time step h is used as a control input to stabilize the numerical approximation (1.8) and to this end we use the results obtained in the previous sections for non-uniform robust global asymptotic stability. The following theorem is the main result of the section. We remind the readers that (1.7) is said to be RFC if for every $T \geqslant 0$, $r \geqslant 0$, it holds that:

$$\sup\{|\phi(t_c, t_0, x_0; d')|; |x_0| \leqslant r, t_0 \in [0, T], \quad t_c \in [t_0, t_0 + T], d'(\cdot) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)\} < +\infty,$$

where $\phi(t_c, t_0, x_0; d')$ denotes the unique solution of (1.7) at time $t_c \geqslant t_0$, initiated from time $t_0 \geqslant 0$ at $x_0 \in \mathbb{R}^n$ and corresponding to input $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$ (see Karafyllis, 2004).

THEOREM 4.1 Consider the finite-dimensional continuous-time system (1.7) and assume that $\Omega \subset \mathbb{R}^m$ is a non-empty compact set and $\tilde{f}: \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a mapping with $\tilde{f}(t_c, 0, d') = 0$ for all $(t_c, d') \in \mathbb{R}^+ \times \Omega$ that satisfies the following hypotheses:

H1 The function $\tilde{f}(t_c, x, d')$ is continuous, for all $(t_c, x, d') \in \mathbb{R}^+ \times \mathbb{R}^n \times \Omega$.

H2 The function $\tilde{f}(t_c, x, d')$ is locally Lipschitz with respect to x, uniformly in $d' \in \Omega$, in the sense that for every bounded interval $I \subset \mathbb{R}^+$ and for every compact subset S of $\mathbb{R}^n$, there exists a constant $L \geqslant 0$ such that

$$|\tilde{f}(t_c, x, d') - \tilde{f}(t_c, y, d')| \leqslant L|x - y|,$$
$$\forall\, t_c \in I, \quad (x, y) \in S \times S, \quad d' \in \Omega.$$

Suppose that (1.7) is RFC and let $\phi(t_c, t_0, x_0; d')$ denote the solution of (1.7) at time $t_c \geqslant t_0$, initiated from time $t_0 \geqslant 0$ at $x_0 \in \mathbb{R}^n$ and corresponding to input $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$. Furthermore, consider the finite-dimensional discrete-time system (1.8), which corresponds to the difference equations of the explicit Euler method for (1.7) and the mapping F defined by (1.9). Then the following statements are equivalent:

(i) $0 \in \mathbb{R}^n$ is non-uniformly in time RGAS for (1.7).

(ii) There exists a constant $\mu > 0$, a positive continuous function $\varphi : \mathbb{R}^+ \times \mathbb{R}^n \to (0, 1]$ and a homeomorphism $\Phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ with $\Phi(0) = 0$, such that the following discrete-time system

$$\begin{aligned}
y(t+1) &= \exp\left(-\mu \frac{1+\theta(t)-h(t)}{1+\theta(t)}\right) \\
&\quad \times \frac{1}{z(t)} \Phi\left(F\left(t_c(t), \Phi^{-1}(z(t)y(t)), d'(t), \frac{h(t)}{1+\theta(t)}\right)\right), \\
t_c(t+1) &= t_c(t) + \frac{h(t)}{1+\theta(t)}, \\
z(t+1) &= \exp\left(\mu \frac{1+\theta(t)-h(t)}{1+\theta(t)}\right) z(t), \\
&(y(t), t_c(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}^+ \times [1, +\infty), \quad h(t) \in [0, 1], \quad t \in \mathbb{Z}^+, \\
d(t) &:= (d'(t), \theta(t)) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+,
\end{aligned} \tag{4.1}$$

with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$, is RGEAS with constant $\mu > 0$.

Moreover, if statement (ii) holds, then

(a) For every $r \geqslant 0$ the closed-loop system (1.8) with $h(t) = \varphi(t_c(t), x(t))$ and $\Theta = [0, r]$ is non-uniformly RGAS.

(b) For every $r \geqslant 0$, $(d'(t), \theta(t)) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times [0, r]$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ the corresponding solution of the closed-loop system (1.8) with $h(t) = \varphi(t_c(t), x(t))$, $\Theta = [0, r]$ and initial condition $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ satisfies $\lim_{t\to+\infty} t_c(t) = +\infty$.

The proof of Theorem 4.1 relies on the following technical lemmas. The first lemma provides estimates for the Lyapunov function of system (1.7) along the solutions of the discrete-time system (1.8). Its proof is given in the Appendix.

LEMMA 4.2 Consider system (1.7), where the vector field $\tilde{f}: \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ satisfies the hypotheses of Theorem 4.1. If $0 \in \mathbb{R}^n$ is non-uniformly in time RGAS for (1.7), then there exist functions $\tilde{V} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, $a_1, a_2 \in K_\infty$, $\tilde{\beta} \in K^+$ and a non-negative C^0 function $\gamma: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(t_c, 0) = 0$ for all $t_c \geqslant 0$ such that for each fixed $s \geqslant 0$ the mappings $\gamma(\cdot, s)$ and $\gamma(s, \cdot)$ are non-decreasing, with the following properties:

$$a_1(|x|) \leqslant \tilde{V}(t_c, x) \leqslant a_2(\tilde{\beta}(t_c)|x|), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{4.2a}$$

$$\tilde{V}(t_c + h, F(t_c, x, d', h)) \leqslant \exp(-h)\tilde{V}(t_c, x) + \gamma(t_c, |x|)h^2, \quad \forall (t_c, x, d', h) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times [0, 1], \tag{4.2b}$$

where the mapping F is defined by (1.9).

The second lemma is an immediate consequence of the consistency of the explicit Euler method. Its proof is given in the Appendix.

LEMMA 4.3 Suppose that (1.7) is RFC and let $\phi(t_c, t_0, x_0; \tilde{d})$ denote the unique solution of (1.7) with initial condition $\phi(t_0, t_0, x_0; \tilde{d}) = x_0$ and corresponding to input $\tilde{d}$, for some $(t_0, x_0, \tilde{d}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$. Define the Euler arc for the solution $(t_c(t), x(t))$ of (1.8) with initial condition $(t_c(0), x(0)) = (t_0, x_0)$ and corresponding to sequences $(d'(t), \theta(t), h(t)) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+ \times (0, 1]$ with $\tilde{d} = d'(0) = d'(1) = d'(2) = \cdots$ by the following recursive formula:

$$\phi_h(t_c, t_0, x_0; \tilde{d}, \theta) := x(t) + \int_{t_c(t)}^{t_c} \tilde{f}(\tau, x(t), \tilde{d}(\tau))\, \mathrm{d}\tau, \quad \text{for } t_c(t) \leqslant t_c < t_c(t+1), \tag{4.3}$$

as well as the global discretization error of the explicit Euler method for $t_c \in [t_0, \lim t_c(t))$:

$$e(t_c) := \phi(t_c, t_0, x_0; \tilde{d}) - \phi_h(t_c, t_0, x_0; \tilde{d}, \theta). \tag{4.4}$$

Suppose furthermore that there exist functions $\mu \in K^+$, $a \in K_\infty$ and constant $R \geqslant 0$ such that for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and for all sequences $(d'(t), \theta(t), h(t)) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+ \times (0, 1]$, the solution $(t_c(t), x(t))$ of (1.8) with initial condition $(t_c(0), x(0)) = (t_0, x_0)$ and corresponding to sequences $(d'(t), \theta(t), h(t)) \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+ \times (0, 1]$ satisfies for all $t \in \mathbb{Z}^+$:

$$|x(t)| \leqslant \mu(t_c(t))a(|x_0| + R). \tag{4.5}$$

Then there exists a function $g: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to (0, +\infty)$ such that for all $(t_0, x_0, \tilde{d}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$ and for all sequences $(\theta(t), h(t)) \in \mathbb{R}^+ \times (0, 1]$, the following estimate of the global discretization error holds for all $t_c \in [t_0, \lim t_c(t))$:

$$|e(t_c)| \leqslant g(t_c, t_0, |x_0|) \sup_{t \geqslant 0} \left(\frac{h(t)}{1 + \theta(t)} \right). \tag{4.6}$$

We are now in a position to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii). Since $0 \in \mathbb{R}^n$ is non-uniformly in time RGAS for (1.7), it follows from Lemma 4.2 that there exist functions $\tilde{V} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, $a_1, a_2 \in K_\infty$, $\tilde{\beta} \in K^+$ and a non-negative C^0 function $\gamma\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(t_c, 0) = 0$ for all $t_c \geqslant 0$ such that inequalities (4.2a,b) hold. We define

$$\varphi(t_c, x) := \frac{\exp(-2(t_c+1))}{\exp(-2(t_c+1)) + \gamma(t_c, |x|)}, \tag{4.7}$$

which is clearly continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ with values in $(0, 1]$, with $\varphi(t_c, 0) = 1$ for all $t_c \geqslant 0$. Notice that by virtue of (4.2b) and definition (4.7) it follows that the solution of the closed-loop system (1.8) with $\Theta := \mathbb{R}^+$ and $h(k) = \varphi(t(k), x(k))$ satisfies the following inequality for all $k \in \mathbb{Z}^+$:

$$\tilde{V}(t_c(t+1), x(t+1)) \leqslant \exp\left(-\frac{\varphi(t_c(t), x(t))}{1+\theta(t)}\right)\tilde{V}(t_c(t), x(t)) + \frac{\varphi(t_c(t), x(t))}{1+\theta(t)}\exp(-2t_c(t)-2),$$
$$\forall\,(t_c(t), x(t), d'(t), \theta(t)) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+. \tag{4.8}$$

Let the homeomorphism $\Phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ with $\Phi(0) = 0$, $\Phi(x) := (\phi_1(x), \ldots, \phi_n(x))$ with

$$\phi_i(x) = \frac{\operatorname{sgn}(x_i)a_1(|x_i|)}{\sqrt{n}}, \qquad \text{for } i = 1, \ldots, n. \tag{4.9}$$

System (1.8) is extended by the following equation:

$$z(t+1) = \exp\left(\mu\frac{1+\theta(t)-\varphi(t_c(t), x(t))}{1+\theta(t)}\right)z(t), \qquad z(t) \in [1, +\infty) \subset \mathbb{R}, \tag{4.10}$$

which is solved explicitly:

$$z(t) = z(0)\exp\left(\mu t - \mu\sum_{\tau=0}^{t-1}\frac{\varphi\,(t_c(\tau), x(\tau))}{1+\theta(\tau)}\right). \tag{4.11}$$

Next, the following transformation is applied:

$$y = \frac{1}{z}\Phi(x). \tag{4.12}$$

Then in y coordinates, the closed-loop system (1.8) with $\Theta := \mathbb{R}^+$ and $h(t) = \varphi(t_c(t), x(t))$ is given by the closed-loop system (4.1) with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$. Define the Lyapunov function:

$$V(t_c, z, y) := \frac{1}{z}\tilde{V}(t_c, \Phi^{-1}(zy)), \quad \forall\,(t_c, z, y) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n. \tag{4.13}$$

It follows from inequalities (4.2a) and (4.8), equation (4.10) with $\mu = 1$, definitions (4.9) and (4.13) and the trivial inequality $t_c(t+1) \leqslant t_c(t) + 1$ that the following inequalities hold:

$$|y| \leqslant V(t_c, z, y) \leqslant a_2(\tilde{\beta}(t_c)\sqrt{n}a_1^{-1}(z\sqrt{n}|y|)), \quad \forall\,(t_c, z, y) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n, \tag{4.14a}$$

$$V(t_c(t+1), z(t+1), y(t+1))$$
$$\leqslant \exp(-1)V(t_c(t), z(t), y(t)) + \frac{\varphi(t_c(t), \Phi^{-1}(z(t)y(t)))}{z(t+1)(1+\theta(t))}\exp(-2t_c(t+1)),$$
$$\forall\,(t_c(t), z(t), y(t), d'(t), \theta(t)) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \times \mathbb{R}^+. \tag{4.14b}$$

By virtue of Lemma 2.3 in Karafyllis & Tsinias (2003a) there exists $a \in K_\infty$ and $\beta \in K^+$ such that

$$a_2(\tilde{\beta}(t_c)\sqrt{n}a_1^{-1}(z\sqrt{n}|y|)) \leqslant a(\beta(|(t_c, z)|)|y|), \quad \forall (t_c, z, y) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n. \tag{4.15}$$

Notice that the closed-loop system (4.1) with $\mu = 1$, $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$ takes the form of system (1.1) with

$$w(t) = (w_1(t), w_2(t)) := (t_c(t), z(t)) \in W := \mathbb{R}^+ \times [1, +\infty) \subset \mathcal{W} := \mathbb{R}^2,$$

$$f(w, y, d) := \Phi\left(F\left(w_1, \Phi^{-1}(w_2 y), d', \frac{\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta}\right)\right), \quad \mathcal{X} = \mathbb{R}^n,$$

$$g(w, y, d) := \left(w_1 + \frac{\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta}, \exp\left(\frac{1+\theta-\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta}\right)w_2\right).$$

Moreover, by continuity of the maps f and g we obtain that the closed-loop system (4.1) with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$ is RFC. Furthermore, it follows from inequalities (4.14a,b) and (4.15) that the following inequalities hold for all $(w, x, d) \in W \times \mathcal{X} \times D$:

$$|y| \leqslant V(w, y) \leqslant a(\beta(|w|)|y|), \tag{4.16a}$$

$$\begin{aligned} V(g(w, y, d),\ f(w, y, d)) \leqslant \exp(-1)V(w, y) + \frac{\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta} \\ \times \exp(-Q(g(w, y, d),\ f(w, y, d))), \end{aligned} \tag{4.16b}$$

$$Q(g(w, y, d),\ f(w, y, d)) = Q(w, y) + 1 + \frac{\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta}, \tag{4.16c}$$

where $Q(w, y) := 2w_1 + \log(w_2)$. Thus, by virtue of (4.16a–c), it follows that statement (iii) of Theorem 2.10 holds with $K = r = M = 1$, $\lambda = \exp(-1)$ and $\varphi(w, y, d) := \frac{\varphi(w_1, \Phi^{-1}(w_2 y))}{1+\theta}$. We conclude that the closed-loop system (4.1) with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$ is non-uniformly RGEAS with constant $\mu = 1$.

(ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (a) and (ii) $\Rightarrow$ (b). Since the closed-loop system (4.1) with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$ is non-uniformly RGEAS with constant $\mu > 0$, it follows from Lemma 2.8 that there exists $a \in K^+$ such that the following estimate holds for all $(t_0, z_0, y_0, d) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n \times M_D$ and $t \in \mathbb{Z}^+$:

$$|y(t)| \leqslant \exp(-\mu t)a(|y_0| + t_0 + z_0). \tag{4.17}$$

Next, the following transformation is applied:

$$x = \Phi^{-1}(zy). \tag{4.18}$$

Then in x coordinates, the closed-loop system (4.1) with $h(t) = \varphi(t_c(t), \Phi^{-1}(z(t)y(t)))$ is given by the extended closed-loop system (1.8) with (4.10), $\Theta := \mathbb{R}^+$ and $h(t) = \varphi(t_c(t), x(t))$. Notice that since $\Phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ is a homeomorphism with $\Phi(0) = 0$, there exist functions $a_1, a_2 \in K_\infty$ such that

$$|\Phi(x)| \leqslant a_1(|x|), |\Phi^{-1}(x)| \leqslant a_2(|x|), \quad \forall x \in \mathbb{R}^n. \tag{4.19}$$

Using estimate (4.17) in conjunction with the explicit solution (4.11), definition (4.18) and inequalities (4.19), we obtain the following estimate for all $(t_0, z_0, x_0, d) \in \mathbb{R}^+ \times [1, +\infty) \times \mathbb{R}^n \times M_D$ and $t \in \mathbb{Z}^+$:

$$|x(t)| \leqslant a_2\left(z_0 \exp\left(-\mu \sum_{\tau=0}^{t-1} \frac{\varphi(t_c(\tau), x(\tau))}{1+\theta(\tau)}\right) a(a_1(|x_0|) + t_0 + z_0)\right). \tag{4.20}$$

Since the x component of the solution of the closed-loop system (1.8) with (4.10), $\Theta := \mathbb{R}^+$ and $h(t) = \varphi(t_c(t), x(t))$ does not depend on z_0, we conclude that estimate (4.20) holds with $z_0 = 1$. Moreover, we have

$$t_c(t) - t_0 = \sum_{\tau=0}^{t-1} \frac{\varphi(t_c(\tau), x(\tau))}{1+\theta(\tau)}. \tag{4.21}$$

Combining estimates (4.20) (with $z_0 = 1$) and (4.21), we obtain that the following estimate holds for the solution of the closed-loop system (1.8) with $\Theta := \mathbb{R}^+$ and $h(t) = \varphi(t_c(t), x(t))$ for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathbb{R}^n \times M_D$ and $t \in \mathbb{Z}^+$:

$$|x(t)| \leqslant a_2(\exp(-\mu(t_c(t) - t_0))a(a_1(|x_0|) + t_0 + 1)). \tag{4.22}$$

Notice that if $\theta(t) \in [0, r]$ for all $t \in \mathbb{Z}^+$ and some $r \geqslant 0$, inequality (4.22) gives

$$t_c(t+1) \geqslant t_c(t) + \frac{1}{1+r} \min\{\varphi(t_c(t), x(t)); |x(t)| \leqslant a_2(a(a_1(|x_0|) + t_0 + 1))\}, \tag{4.23}$$

and the latter inequality in conjunction with Lemma 3.2 shows that $\lim t_c(t) = +\infty$ and this proves statement (b). Next we prove statement (a). It follows from Lemma 2.4 that it suffices to prove that the Property of Robust Uniform Attractivity on bounded sets of initial data is satisfied for system (1.8) with $h(t) = \varphi(t_c(t), x(t))$ and $\Theta = [0, r]$. Let $\varepsilon > 0$, $R \geqslant 0$, $|x_0| \leqslant R$, $t_0 \in [0, R]$ and $d \in M_D$. Let $L(\varepsilon, R) := \frac{1}{\mu}\log(1 + \frac{a(a_1(R)+R+1)}{a_2^{-1}(\varepsilon)})$. Lemma 3.2 in conjunction with inequality (4.23) guarantees the existence of $\tau := \tau(\varepsilon, R) \in \mathbb{Z}^+$, such that $t_c(t) \geqslant R + L(\varepsilon, R)$ for all $t \geqslant \tau(\varepsilon, R)$. Hence, it follows from (4.22) that $|x(t)| \leqslant \varepsilon$ for all $t \geqslant \tau\ (\varepsilon, R)$. This proves that the Property of Robust Uniform Attractivity on bounded sets of initial data is satisfied for system (1.8) with $h(t) = \varphi(t_c(t), x(t))$ and $\Theta = [0, r]$.

Finally, we prove statement (i). Let $(t_0, x_0, \tilde{d}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$ be arbitrary. Consider the parameterized family of the solutions of the closed-loop system (1.8) with $\Theta := \mathbb{R}^+$ and $h(t) = \varphi(t_c(t), x(t))$ initiated from $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and corresponding to $d' = (d'(0), d'(1), d'(2), \ldots) \in M_D$ with $\tilde{d} = d'(0) = d'(1) = \cdots$ and $2^\mu = \theta(0) = \theta(1) = \cdots$ with parameter $\mu \geqslant 0$. Let arbitrary $T \geqslant t_0$. By virtue of Lemma 4.3 and letting $\mu \to +\infty$, the Euler polygonal arc converges uniformly on $[t_0, T]$ to the unique solution $x(t_c)$ of system (1.7) with initial condition $x(t_0) = x_0$ and corresponding to $\tilde{d} \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$. It follows from (4.22) that the solution $x(t_c)$ of (1.7) satisfies the following inequality for all $t_c \in [t_0, T]$:

$$|x(t_c)| \leqslant a_2(\exp(-\mu(t_c - t_0))a(a_1(|x_0|) + t_0 + 1)). \tag{4.24}$$

Estimate (4.24) in conjunction with Proposition 2.5 in Karafyllis & Tsinias (2003a) shows that zero is non-uniformly in time RGAS for (1.7) and thus statement (i) is proved. The proof is complete. □

EXAMPLE 4.4 Zero is (uniformly) globally asymptotically stable for the scalar system $\dot{x} = -x^3$. As already pointed out in Section 1 there is no $h > 0$, such that its Euler discrete-time approximation with constant step size $x(t+1) = (1 - hx^2(t))x(t)$, $t_c(t) = ht$, is globally asymptotically stable. On the other hand, we claim that for every $r \geqslant 0$, the Euler discrete-time approximation with variable step size

$$\begin{aligned} x(t+1) &= f(t_c(t), x(t), \theta(t)) := x(t) - \frac{h(t)}{1+\theta(t)} x^3(t), \\ t_c(t+1) &= g(t_c(t), x(t), \theta(t)) := t_c(t) + \frac{h(t)}{1+\theta(t)}, \\ x(t) &\in \mathbb{R}, \quad t_c(t) \in \mathbb{R}^+, \quad \theta(t) \in [0, r], \quad t \in \mathbb{Z}^+, \end{aligned} \tag{4.25}$$

with

$$h(t) = \varphi(t_c(t), x(t)) := \frac{1}{1 + x^2(t)}, \tag{4.26}$$

is RGAS. To prove this claim, we consider the Lyapunov function $V(t_c, x) := |x|$ and notice that we obtain for all $(t_c, x, \theta) \in \mathbb{R}^+ \times \mathbb{R} \times [0, r]$

$$V(g(t_c, x, \theta),\ f(t_c, x, \theta)) \leqslant V(t_c, x) - a_3(V(t_c, x)), \tag{4.27}$$

where $a_3(s) := \frac{s^3}{(1+r)(1+s^2)} \in K_\infty$. Consequently, statement (iii) of Theorem 2.9 is satisfied with $a_1(s) = a_2(s) := s$, $M = 0$, $\beta(t) \equiv 1$ and $Q(t_c, x) \equiv 0$. By virtue of the equivalence of statements (i) and (iii) of Theorem 2.9 we conclude that (4.25) is non-uniformly RGAS. Notice that in this example the time step policy suggested by the feedback law (4.26) is to apply sufficiently small time steps in the beginning of the simulation and larger time steps when the solution has approached zero ($\lim h(t) = 1$).

REMARK 4.5

(i) As already remarked in Section 1, the obtained result implies that the global discretization error between the solution of explicit Euler discrete-time approximation of (1.7) with $0 < h(t) = \phi(t_c(t), x(t))$ and the exact solution of (1.7) has to be bounded on the positive semi-axis. This implication is important for numerical analysis.

(ii) Notice that the explicit formula (4.7) for time step feedback control depends on $t_c(t)$ even if the original system (1.7) is autonomous and the Lyapunov function provided by Lemma 4.2 is time-invariant. In this case, Example 4.4 shows the possible existence of time-invariant feedback laws that induce RGAS for the closed-loop discrete-time Euler approximation of (1.7). Hence, it should be emphasized that in practice it could happen that there exist different feedback laws that suggest time step policies different from the policy suggested by the explicit formula (4.7).

(iii) The analysis presented in this section can be directly extended to discrete-time systems obtained via higher order explicit Runge–Kutta discretization schemes for continuous-time systems with smooth dynamics.

(iv) It should be emphasized that Theorem 4.1 provides a novel way to prove that zero is non-uniformly in time RGAS for (1.7), namely, by verifying that the closed-loop discrete-time system (4.1) is RGEAS for appropriate homeomorphism $\Phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ with $\Phi(0) = 0$ and constant $\mu > 0$. This is clearly shown by the following example.

EXAMPLE 4.6 Consider the linear planar continuous-time system

$$\begin{aligned}
&\frac{\mathrm{d}x_1}{\mathrm{d}t_{\mathrm{c}}}(t_{\mathrm{c}}) = -4x_1(t_{\mathrm{c}}),\\
&\frac{\mathrm{d}x_2}{\mathrm{d}t_{\mathrm{c}}}(t_{\mathrm{c}}) = -2x_2(t_{\mathrm{c}}) + \exp(t_{\mathrm{c}})x_1(t_{\mathrm{c}}),\\
&(x_1, x_2) \in \mathbb{R}^2, \quad t_{\mathrm{c}} \geqslant 0.
\end{aligned} \tag{4.28}$$

There are many ways to prove that zero is non-uniformly in time RGAS for (4.28). Here for illustration purposes we employ Theorem 4.1 and we consider the discrete-time system

$$\begin{aligned}
y_1(t+1) &= \exp\left(-\frac{1+\theta(t)-h(t)}{1+\theta(t)}\right)\left(1-\frac{4h(t)}{1+\theta(t)}\right)y_1(t),\\
y_2(t+1) &= \exp\left(-\frac{1+\theta(t)-h(t)}{1+\theta(t)}\right)\left[\left(1-\frac{2h(t)}{1+\theta(t)}\right)y_2(t)\right.\\
&\quad \left. + \left(\exp\left(\frac{h(t)}{1+\theta(t)}\right)-1\right)\exp\left(t_{\mathrm{c}}(t)\right)y_1(t)\right],\\
z(t) &= \exp\left(\frac{1+\theta(t)-h(t)}{1+\theta(t)}\right)z(t),\\
t_{\mathrm{c}}(t+1) &= t_{\mathrm{c}}(t) + \frac{h(t)}{1+\theta(t)},\\
y &:= (y_1, y_2) \in \mathbb{R}^2, \quad z \in [1, +\infty), \quad t_{\mathrm{c}} \in \mathbb{R}^+, \quad \theta(t) \in \mathbb{R}^+, \quad t \in \mathbb{Z}^+.
\end{aligned} \tag{4.29}$$

Clearly, the discrete-time system (4.29) corresponds to system (4.1) for constant $\mu = 1$ and homeomorphism $\Phi(x) := x \in \mathbb{R}^2$ for the continuous-time system (4.28). Consider the Lyapunov function $V(t_{\mathrm{c}}, y_1, y_2) := \exp(t_{\mathrm{c}})|y_1| + |y_2|$ and the feedback law $\varphi(t_{\mathrm{c}}, zy) := \frac{1}{4}$. Evaluating the Lyapunov function along the solution of (4.29) with $h(t) = \frac{1}{4}$ and making use of the inequalities $0 \leqslant 1 - \frac{2h(t)}{1+\theta(t)} \leqslant \exp(-\frac{2h(t)}{1+\theta(t)})$, $2\exp(\frac{2h(t)}{1+\theta(t)}) - \frac{4h(t)}{1+\theta(t)}\exp(\frac{2h(t)}{1+\theta(t)}) - \exp(\frac{h(t)}{1+\theta(t)}) \leqslant 1$ (which hold for $h(t) = \frac{1}{4}$ and $\theta(t) \in \mathbb{R}^+$), we obtain the following inequality for all $(y_1(t), y_2(t)) \in \mathbb{R}^2$, $t_{\mathrm{c}}(t) \in \mathbb{R}^+$, $\theta(t) \in \mathbb{R}^+$:

$$V(t_{\mathrm{c}}(t+1), y_1(t+1), y_2(t+1)) \leqslant \exp(-1)V(t_{\mathrm{c}}(t), y_1(t), y_2(t)). \tag{4.30}$$

Consequently, statement (ii) of Theorem 2.10 is satisfied with $a_2(s) := 2s$, $\beta(t) := \exp(t)$ and $\lambda := \exp(-1)$. Hence, by Remark 2.11 it follows that system (4.29) is RGK-ES and RGEAS with constant $\mu = 1$. Using Theorem 4.1 we may conclude that zero is non-uniformly in time RGAS for the original continuous-time system (4.28).

5. Conclusions

The notion of non-uniform robust global asymptotic output stability (RGAS) presented in this paper generalizes the notion of non-uniform in time RGAS for finite- or infinite-dimensional discrete-time systems. Lyapunov characterizations for this stability notion are provided. The results are applied to finite-dimensional discrete-time systems obtained by time discretization of continuous-time systems by the explicit Euler method. It is shown that if zero is non-uniformly in time robustly globally asymptotically stable for the continuous-time system then there exists a continuous function $\phi\colon \mathbb{R}^+ \times \mathbb{R}^n \to (0, 1]$

such that if the integration step size satisfies $h(t) \leqslant \phi(t_c(t), x(t))$, then the discrete-time numerical approximation is non-uniformly robustly globally asymptotically stable. Moreover, we explicitly construct the continuous function $\phi\colon \mathbb{R}^+ \times \mathbb{R}^n \to (0, 1]$ based on the knowledge of a Lyapunov function for the continuous-time system. The obtained result implies that the global discretization error between the solution of explicit Euler discrete-time approximation and the exact solution is bounded on the positive semi-axis.

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Appendix

A.1 *Proof of Lemma 2.4*

Proof. It suffices to show that (1.1) under hypothesis (H1) is robustly Lagrange and Lyapunov stable, i.e. it satisfies properties P1 and P2 of Definition 2.3. First we show that (1.1) is robustly Lagrange stable, by showing that $a(T,s) < +\infty$ for all $T \geqslant 0$, $s \geqslant 0$, where

$$a(T,s) := \sup\{\|x(t,x_0,w_0;d)\|_{\mathcal{X}} : w_0 \in W, t \in \mathbb{Z}^+, \|x_0\|_{\mathcal{X}} \leqslant s, \|w_0\|_{\mathcal{W}} \leqslant T, d \in M_D\}$$

Notice that by virtue of the Property of Robust Uniform Attractivity on bounded sets of initial data, we have for every $\varepsilon > 0$,

$$\begin{aligned} a(T,s) \leqslant \varepsilon + \sup\{&\|x(t,x_0,w_0;d)\|_{\mathcal{X}} : d \in M_D, 0 \leqslant t \leqslant \tau(\varepsilon, T+s), \\ &\|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\}, \end{aligned}$$

where $\tau := \tau(\varepsilon, R) \in \mathbb{Z}^+$ is the time involved in the Property of Robust Uniform Attractivity on compact sets of initial data (property P3 of Definition 2.3). Moreover, notice that by virtue of Robust Forward Completeness (which implies that the set $\{x(k,x_0,w_0;d); 0 \leqslant k \leqslant \tau, \|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T, d \in M_D\}$ is bounded), we obtain

$$\begin{aligned} \sup\{&\|x(k,x_0,w_0;d)\|_{\mathcal{X}} : d \in M_D, 0 \leqslant k \leqslant \tau(\varepsilon, T+s), \\ &\|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\} < +\infty. \end{aligned}$$

Combining the previous inequalities, we obtain that $a(T,s) < +\infty$ for all $T \geqslant 0$, $s \geqslant 0$, or equivalently that (1.1) is robustly Lagrange stable.

Next we show that (1.1) under hypothesis (H1) is Robustly Lyapunov Stable. We proceed by noticing the following fact.

Fact: Consider system (1.1) under hypothesis (H1) and suppose that (1.1) is RFC. Then for every $\varepsilon > 0$, $N \in \mathbb{Z}^+$ and $T \geqslant 0$ there exists $\bar{\delta} := \bar{\delta}(\varepsilon, N, T) \in (0, \varepsilon]$ such that:

$$\|x_0\|_{\mathcal{X}} \leqslant \bar{\delta}, \|w_0\|_{\mathcal{W}} \leqslant T, w_0 \in W \Rightarrow \sup\{\|x(t,x_0,w_0;d)\|_{\mathcal{X}}; 0 \leqslant t \leqslant N, d \in M_D\} \leqslant \varepsilon.$$

We prove this fact by induction on $N \in \mathbb{Z}^+$. First notice that the fact holds for $N = 0$ (by selecting $\bar{\delta}(\varepsilon, 0, T) = \varepsilon$). We next assume that the fact holds for some $N \in \mathbb{Z}^+$ and we prove it for the next integer $N+1$. In order to have $\|x(N+1, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \varepsilon$, by virtue of hypothesis (H1) it suffices to have $\|x(N, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \frac{a^{-1}(\varepsilon)}{\beta(\|w(N,x_0,w_0;d)\|_{\mathcal{W}})}$. Moreover, since (1.1) is RFC, by virtue of statement (iii) of Proposition 2.2 there exist functions $\mu \in K^+$, $a' \in K_\infty$ and a constant $R > 0$ such that $\|w(N, x_0, w_0; d)\|_{\mathcal{W}} \leqslant \mu(N)a'(R + \varepsilon + T)$. Combining the two previous observations, we conclude that in order to have $\|x(N+1, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \varepsilon$, it suffices to have

$$\|x(N, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \gamma(\varepsilon, N, T) := \frac{a^{-1}(\varepsilon)}{\max\{\beta(s); 0 \leqslant s \leqslant \mu(N)a'(R + \varepsilon + T)\}}.$$

It follows that the selection $\bar{\delta}(\varepsilon, N+1, T) := \min\{\bar{\delta}(\varepsilon, N, T), \bar{\delta}(\gamma(\varepsilon, N, T), N, T)\} > 0$ guarantees that $\sup\{\|x(t, x_0, w_0; d)\|_{\mathcal{X}}; 0 \leqslant t \leqslant N, d \in M_D\} \leqslant \min\{\varepsilon, \gamma(\varepsilon, N, T)\}$ and $\sup\{\|x(N+1, x_0, w_0; d)\|_{\mathcal{X}}; d \in M_D\} \leqslant \varepsilon$, for all $\|x_0\|_{\mathcal{X}} \leqslant \bar{\delta}$, $\|w_0\|_{\mathcal{W}} \leqslant T$, $w_0 \in W$. The proof of the Fact is complete.

Finally, let arbitrary $\varepsilon > 0$ and $T \geqslant 0$. By virtue of the Property of Robust Uniform Attractivity on bounded sets of initial data, there exists a $\tau := \tau(\varepsilon, T + \varepsilon) \in \mathbb{Z}^+$, such that:

$$\|x_0\|_{\mathcal{X}} \leqslant \varepsilon, \|w_0\|_{\mathcal{W}} \leqslant T, w_0 \in W \Rightarrow \|x(t, x_0, w_0; d)\|_{\mathcal{X}} \leqslant \varepsilon, \quad \forall t \geqslant \tau, \quad \forall d \in M_D. \tag{A.1}$$

By virtue of the above Fact, it follows that for every $\varepsilon > 0$ and $T \geqslant 0$ there exists $\bar{\delta} > 0$, such that

$$\begin{aligned} &\sup\{\|x(t, x_0, w_0; d)\|_{\mathcal{X}}; d \in M_D, 0 \leqslant t \leqslant \tau(\varepsilon, T + \varepsilon), \\ &\quad w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\} \leqslant \varepsilon \text{provided that } \|x_0\|_{\mathcal{X}} < \bar{\delta}, \end{aligned} \tag{A.2}$$

where $\tau := \tau(\varepsilon, T + \varepsilon) \in \mathbb{Z}^+$ is the time involved in (A.1). It is clear from (A.1) and (A.2), that the Robust Lyapunov Stability property is satisfied for $\delta(\varepsilon, T) = \min\{\varepsilon, \bar{\delta}\}$. The proof is complete. □

A.2 *Proof of Lemma 2.5*

Proof. As in the proof of Proposition 2.2 in Karafyllis & Tsinias (2003a), let $\xi \in \mathbb{Z}^+$, $T \geqslant 0$, $s \geqslant 0$ and define

$$a(T, s) := \sup\{\|x(t, x_0, w_0; d)\|_{\mathcal{X}} : d \in M_D, t \in \mathbb{Z}^+, \|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\}, \tag{A.3}$$

$$M(\xi, T, s) := \sup\{\|x(\xi, x_0, w_0, d)\|_{\mathcal{X}} : d \in M_D, \|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\}. \tag{A.4}$$

First notice that by virtue of Robust Lagrange Stability a is well-defined, i.e. $a(T, s) < +\infty$ for every $T \geqslant 0$, $s \geqslant 0$. Furthermore, notice that M is well-defined, since by definitions (A.3) and (A.4) the following inequality is satisfied for all $\xi \in \mathbb{Z}^+$, $T \geqslant 0$ and $s \geqslant 0$:

$$M(\xi, T, s) \leqslant a(T, s). \tag{A.5}$$

Moreover, a satisfies all hypotheses of the Lemma 2.3 in Karafyllis & Tsinias (2003a), namely, for each fixed $s \geqslant 0$, $a(\cdot, s)$ is non-decreasing and, for each fixed $T \geqslant 0$, $a(T, \cdot)$ is non-decreasing and satisfies $a(\cdot, 0) = 0$. In addition, Robust Lyapunov Stability asserts that for every $T \geqslant 0$, $\lim_{s \to 0^+} a(T, s) = 0$. It turns out from Lemma 2.3 in Karafyllis & Tsinias (2003a) that there exist functions $\zeta_1 \in K_\infty$ and $\gamma \in K^+$ such that

$$a(T, s) \leqslant \zeta_1(\gamma(T)s), \quad \forall (T, s) \in (\mathbb{R}^+)^2. \tag{A.6}$$

Next we proceed exactly as in the proof of Proposition 2.2 in Karafyllis & Tsinias (2003a) to establish that

$$M(\xi, T, s) \leqslant \mu(\xi)\theta(T, s), \quad \forall\, T, s \in \mathbb{R}^+, \ \ \xi \in \mathbb{Z}^+, \tag{A.7}$$

where $\theta(T, s) := p(T)g(\zeta_1(\gamma(T)s))$, $\mu \in K^+$ is a strictly decreasing function with $\lim_{\xi\to+\infty}\mu(\xi) = 0$, $p \in K^+$ is a non-decreasing function with $p(0) = 1$ and $\lim_{t\to+\infty} p(t) = +\infty$ and $g(s) := \sqrt{s} + s^2$. Applying again Lemma 2.3 in Karafyllis & Tsinias (2003a), we guarantee that there exist functions $\zeta_2 \in K_\infty$ and $\beta \in K^+$, such that

$$\theta(T, s) \leqslant \zeta_2(\beta(T)s), \quad \forall\,(T, s) \in (\mathbb{R}^+)^2. \tag{A.8}$$

Define the KL function $\sigma(s, t) := \mu(t)\zeta_2(s)$. By virtue of Proposition 7 in Sontag (1998b), there exist functions $a_1, a_2 \in K_\infty$ such that

$$\sigma(s, t) \leqslant a_1^{-1}(\exp(-t)a_2(s)), \quad \forall\, s, t \geqslant 0.$$

The desired (2.3) with $c := 1$ is a consequence of the previous inequality, (A.7), (A.8) and definition (A.4).

The converse statement, namely, if (2.3) holds for all $(w_0, x_0, d) \in W \times \mathcal{X} \times M_D$ and $t \in \mathbb{Z}^+$, then system (1.1) is non-uniformly RGAS and is an immediate consequence of (2.3) and the properties of KL functions. □

A.3 *Proof of Lemma 2.7*

Proof. Notice that the transformation $z_i = \operatorname{sgn}(x_i)a_1(|x_i|)$ for $i = 1, \ldots, n$ satisfies

$$a_1\left(\frac{|x|}{\sqrt{n}}\right) \leqslant |z| \leqslant \sqrt{n}\, a_1(|x|). \tag{A.9}$$

In addition, the solution $(z(t), w(t))$ of (2.5) with initial condition $(w_0, z_0) \in W \times \mathbb{R}^n$ and corresponding to input $d \in M_D$ satisfies $z_i(t) = \operatorname{sgn}(x_i(t))a_1(|x_i(t)|)$ for $i = 1, \ldots, n$, where $(x(t), w(t))$ is the solution of (1.1) with initial condition $(w_0, x_0) \in W \times \mathbb{R}^n$, $x_{0,i} = \operatorname{sgn}(z_{0,i})a_1^{-1}\left(\left|z_{0,i}\right|\right)$ and corresponding to the same input $d \in M_D$. Consequently, by virtue of (2.3) and (A.9), we obtain for all $(w_0, z_0, d) \in W \times \mathbb{R}^n \times M_D$ and $t \in \mathbb{Z}^+$

$$|z(t)| \leqslant \exp(-ct)\sqrt{n}\, a_2(\beta(\|w_0\|_{\mathcal{W}})\sqrt{n}\, a_1^{-1}(|z_0|)). \tag{A.10}$$

Applying Lemma 2.3 in Karafyllis & Tsinias (2003a), we guarantee that there exist functions $\zeta \in K_\infty$ and $\tilde{\beta} \in K^+$ such that

$$\sqrt{n}\, a_2(\beta(t)\sqrt{n}\, a_1^{-1}(s)) \leqslant \zeta(\tilde{\beta}(t)s), \quad \forall\,(t, s) \in (\mathbb{R}^+)^2. \tag{A.11}$$

Thus, from (A.10) and inequality (A.11), we obtain for all $(w_0, z_0, d) \in W \times \mathbb{R}^n \times M_D$ and $t \in \mathbb{Z}^+$

$$|z(t)| \leqslant \exp(-ct)\zeta(\tilde{\beta}(\|w_0\|_{\mathcal{W}})|z_0|),$$

and this establishes non-uniform RGK-ES for system (2.5). □

A.4 *Proof of Lemma 2.8*

Proof. Let $c \in (0, \mu)$ be arbitrary, where $\mu > 0$ is the constant involved in the Robust Global Exponential Attractivity Property for (1.1). As in the proof of Lemma 2.5 in Karafyllis & Tsinias (2003a), let $T \geqslant 0$, $s \geqslant 0$ and define

$$\begin{aligned}\gamma(T,s) := \sup\{&\exp(ct)\|x(t,x_0,w_0;d)\|_{\mathcal{X}} : d \in M_D, t \in \mathbb{Z}^+,\\ &\|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\}.\end{aligned} \tag{A.12}$$

First notice that by virtue of the Robust Global Exponential Attractivity Property for (1.1), γ is well-defined, i.e. $\gamma(T,s) < +\infty$ for every $T \geqslant 0$, $s \geqslant 0$. Furthermore, for each fixed $s \geqslant 0$, $\gamma(\cdot, s)$ is non-decreasing, and for each fixed $T \geqslant 0$, $\gamma(T, \cdot)$ is non-decreasing and satisfies $\gamma(\cdot, 0) = 0$. Let

$$\begin{aligned}B(R) := \sup\{&\exp(\mu t)\|x(t,x_0,w_0;d)\|_{\mathcal{X}}; t \in \mathbb{Z}^+, \|x_0\|_{\mathcal{X}} \leqslant R,\\ &\|w_0\|_{\mathcal{W}} \leqslant R, w_0 \in W, d \in M_D\} < +\infty\end{aligned} \tag{A.13}$$

and notice that definitions (A.12) and (A.13) imply that for every $N \in \mathbb{Z}^+$, $T, s \geqslant 0$, we have

$$\begin{aligned}\gamma(T,s) \leqslant \sup\{&\exp(ct)\|x(t,x_0,w_0;d)\|_{\mathcal{X}} : d \in M_D, 0 \leqslant t \leqslant N,\\ &\|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in \mathcal{W}, \|w_0\|_{W} \leqslant T\} + \exp(-(\mu - c)N)B(T+s).\end{aligned} \tag{A.14}$$

Let $a \in K^+$ be a function that satisfies $a(s) \geqslant B(s)$, for all $s \geqslant 0$. Clearly, inequality (2.6) is directly implied by definition (A.13) and the previous inequality.

Notice that by virtue of property (A.14) we have for every $\varepsilon > 0$

$$\begin{aligned}\gamma(T,s) \leqslant \frac{\varepsilon}{2} + \exp(c\tau(\varepsilon, T+s)) \sup\{&\|x(t,x_0,w_0;d)\|_{\mathcal{X}} : d \in M_D, 0 \leqslant t \leqslant \tau(\varepsilon, T+s),\\ &\|x_0\|_{\mathcal{X}} \leqslant s, w_0 \in W, \|w_0\|_{\mathcal{W}} \leqslant T\},\end{aligned} \tag{A.15}$$

where $\tau := \tau(\varepsilon, R) := 1 + [-\frac{1}{\mu - c}\log(\frac{\varepsilon}{\varepsilon + 2B(R)})]$.

Next we show that for every $T \geqslant 0$, $\lim_{s \to 0^+} \gamma(T,s) = 0$. It suffices to show that for every $\varepsilon > 0$, $T \geqslant 0$ there exists $\bar{\delta} := \bar{\delta}(\varepsilon, T) > 0$ such that $\gamma(T, \bar{\delta}(\varepsilon, T)) \leqslant \varepsilon$. By virtue of the Fact stated in the proof of Lemma 2.4, we conclude that for every $\varepsilon > 0$ and $T \geqslant 0$ there exists $\delta := \delta(\varepsilon, T) > 0$ such that

$$\begin{aligned}\|x_0\|_{\mathcal{X}} \leqslant \delta, \|w_0\|_{\mathcal{W}} \leqslant T, w_0 \in W&\\ \Rightarrow \sup\{&\|x(t,x_0,w_0;d)\|_{\mathcal{X}}; d \in M_D, 0 \leqslant k \leqslant \tau(\varepsilon, T+1)\}\\ \leqslant &\frac{\varepsilon}{2}\exp(-c\tau(\varepsilon, T+1)),\end{aligned} \tag{A.16}$$

where $\tau := \tau(\varepsilon, R) := 1 + [-\frac{1}{\mu - c}\log(\frac{\varepsilon}{\varepsilon + 2B(R)})]$ is involved in (A.15). Let $\bar{\delta}(\varepsilon, T) = \min\{\delta(\varepsilon, T), 1\}$. Combining (A.15) with (A.16) and using the inequality $\tau(\varepsilon, T + \bar{\delta}) \leqslant \tau(\varepsilon, T+1)$, we obtain that $\gamma(T, \bar{\delta}(\varepsilon, T)) \leqslant \varepsilon$.

It follows that γ satisfies all hypotheses of Lemma 2.3 in Karafyllis & Tsinias (2003a) and, consequently, there exist functions $a_2 \in K_\infty$ and $\beta \in K^+$ such that

$$\gamma(T,s) \leqslant a_2(\beta(T)s), \quad \forall (T,s) \in (\mathbb{R}^+)^2. \tag{A.17}$$

Inequality (2.4) follows from inequality (A.17) in conjunction with definition (A.12). The proof is complete. □

A.5 *Proof of Lemma 4.2*

Proof. Since the dynamics of (1.7) are locally Lipschitz with respect to x, uniformly in $d' \in \Omega$, there exists a positive C^0 function $L\colon \mathbb{R}^+ \times \mathbb{R}^+ \to (0, +\infty)$ such that for each fixed $s \geqslant 0$ the mappings $L(\cdot, s)$and $L(s, \cdot)$ are non-decreasing and the following holds:

$$|\tilde{f}(t_c, x, d') - \tilde{f}(t_c, y, d')| \leqslant L(t_c, |x| + |y|)|x - y|, \quad \forall (t_c, x, y, d') \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega. \tag{A.18}$$

Furthermore, since $0 \in \mathbb{R}^n$ is RGAS for (1.7), it follows from Proposition 2.2 in Karafyllis & Tsinias (2003a) that there exist functions σ of class KL and β of class K^+ being non-decreasing, such that for every $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$ it holds:

$$|\phi(t_c, t_0, x_0; d')| \leqslant \sigma(\beta(t_0)|x_0|, t_c - t_0), \quad \forall t \geqslant t_0, \tag{A.19}$$

where $\phi(t_c, t_0, x_0; d')$ denotes the solution of (1.7) at time $t_c \geqslant t_0$, initiated from time $t_0 \geqslant 0$ at $x_0 \in \mathbb{R}^n$ and corresponding to input $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$. The following elementary property for the solution of (1.7) is an immediate consequence of the Gronwall–Bellman inequality, inequalities (A.18) and (A.19) and the fact that $\tilde{f}(t_c, 0, d') = 0$ for all $(t_c, d') \in \mathbb{R}^+ \times \Omega$:

$$\begin{aligned} |\phi(t_c, t_0, x_0; d') - x_0| \leqslant \left(\exp\left(\int_{t_0}^{t_c} \tilde{L}(s, |x_0|)\, ds\right) - 1\right)|x_0|,& \\ \forall (t_0, x_0, d') \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}^\infty(\mathbb{R}^+; \Omega) \text{ and } t_c \geqslant t_0,& \end{aligned} \tag{A.20}$$

where

$$\tilde{L}(t_c, s) := L(t_c, \sigma(\beta(t_c)s, 0)). \tag{A.21}$$

Since $0 \in \mathbb{R}^n$ is RGAS for (1.7), it follows from Theorem 3.1 in Karafyllis & Tsinias (2003a) that there exist functions $\tilde{V} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, $a_1, a_2 \in K_\infty$ and $\tilde{\beta} \in K^+$, such that for every $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_c, h \geqslant 0$ and $x \in \mathbb{R}^n$, it holds that

$$a_1(|x|) \leqslant \tilde{V}(t_c, x) \leqslant a_2(\tilde{\beta}(t_c)|x|), \tag{A.22}$$

$$\tilde{V}(t_c + h, \phi(t_c + h, t_c, x; d')) \leqslant \exp(-h)\tilde{V}(t_c, x). \tag{A.23}$$

Define the following continuous non-negative function:

$$L_V(t_c, s) := \sup\left\{\left|\frac{\partial \tilde{V}}{\partial x}(\tau, \xi)\right|; \tau \in [0, t_c + 1], |\xi| \leqslant s + \sigma(\beta(t_c)s, 0) + sL(t_c + 1, s)\right\}. \tag{A.24}$$

Clearly, for each fixed $s \geqslant 0$ the mappings $L_V(\cdot, s)$and $L_V(s, \cdot)$ are non-decreasing. Moreover, notice that for every $h \in [0, 1]$, for every $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_c \geqslant 0$ and $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned} &\tilde{V}(t_c + h, \phi(t_c + h, t_c, x; d')) - \tilde{V}(t_c + h, F(t_c, x, d', h)) \\ &\quad = \int_0^1 \frac{\partial \tilde{V}}{\partial x}(t_c + h, s\phi(t_c + h, t_c, x; d') \\ &\qquad + (1 - s)F(t_c, x, d', h))\, ds(\phi(t_c + h, t_c, x; d') - F(t_c, x, d', h)), \end{aligned} \tag{A.25}$$

where the mapping F is defined by (1.9). Notice that by virtue of (A.18) and the fact that $\tilde{f}(t_c, 0, d') = 0$ for all $(t_c, d') \in \mathbb{R}^+ \times \Omega$ we have $|F(t_c, x, d', h)| \leqslant |x| + hL(t_c + h, |x|)|x|$, for all $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$,

$t_c, h \geqslant 0$ and $x \in \mathbb{R}^n$. The previous inequality in conjunction with (A.19), (A.25) and definition (A.24) implies that the following inequality holds for every $h \in [0, 1]$, $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_c \geqslant 0$ and $x \in \mathbb{R}^n$:

$$\begin{aligned} &|\tilde{V}(t_c + h, \phi(t_c + h, t_c, x; d')) - \tilde{V}(t_c + h, F(t_c, x, d', h))| \\ &\quad \leqslant L_V(t_c, |x|)|\phi(t_c + h, t_c, x; d') - F(t_c, x, d', h)|. \end{aligned} \tag{A.26}$$

Since $\phi(t_c + h, t_c, x; d') = x + \int_{t_c}^{t_c+h} \tilde{f}(\tau, \phi(\tau, t_c, x; d'), d'(\tau))\, d\tau$, it follows from (A.18) and (A.19) that the following inequality holds for all $h \in [0, 1]$, $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_c \geqslant 0$ and $x \in \mathbb{R}^n$:

$$\begin{aligned} &|\phi(t_c + h, t_c, x; d') - F(t_c, x, d', h)| \\ &\quad \leqslant L(t_c + 1, |x| + \sigma(\beta(t_c)|x|, 0)) \int_{t_c}^{t_c+h} |\phi(\tau, t_c, x; d') - x|\, d\tau. \end{aligned} \tag{A.27}$$

Thus, putting together inequalities (A.20), (A.26) and (A.27), we conclude that the following inequality holds for all $h \in [0, 1]$, $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_c \geqslant 0$ and $x \in \mathbb{R}^n$:

$$|\tilde{V}(t_c + h, \phi(t_c + h, t_c, x; d')) - \tilde{V}(t_c + h, F(t_c, x, d', h))| \leqslant \gamma(t_c, |x|)h^2, \tag{A.28}$$

where

$$\gamma(t_c, |x|) := L(t_c + 1, |x| + \sigma(\beta(t_c)|x|, 0))\tilde{L}(t_c + 1, |x|) \exp(\tilde{L}(t_c + 1, |x|))L_V(t_c, |x|)|x|.$$

Inequalities (4.2a,b) are immediate consequences of inequalities (A.22), (A.23) and (A.28). □

A.6 *Proof of Lemma 4.3*

Proof. Since the dynamics of (1.7) are locally Lipschitz with respect to x, uniformly in $d' \in \Omega$, there exists a positive C^0 function $L\colon \mathbb{R}^+ \times \mathbb{R}^+ \to (0, +\infty)$ such that for each fixed $s \geqslant 0$ the mappings $L(\cdot, s)$ and $L(s, \cdot)$ are non-decreasing and in such a way that inequality (A.18) holds. Furthermore, since (1.7) is RFC, it follows from Lemma 3.5 in Karafyllis (2004) that there exist $R' \geqslant 0$, $\gamma \in K_\infty$ and $\beta \in K^+$ being non-decreasing, such that for every $d' \in \mathcal{L}^\infty(\mathbb{R}^+; \Omega)$, $t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$, it holds that

$$|\phi(t_c, t_0, x_0; d')| \leqslant \beta(t_c)\gamma(|x_0| + R'), \quad \forall\, t_c \geqslant t_0. \tag{A.29}$$

Moreover, by virtue of (A.29) we obtain that estimate (A.20) holds with $\tilde{L}(t_c, s) := L(t_c, \beta(t_c)\gamma(s + R'))$. Clearly, the global discretization error satisfies

$$\begin{aligned} e(t_c) = e(t_c(t)) + \int_{t_c(t)}^{t_c} (\tilde{f}(\tau, \phi(\tau, t_0, x_0; \tilde{d}), \tilde{d}(\tau)) - \tilde{f}(\tau, x(t), \tilde{d}(\tau)))\, d\tau,& \\ \text{for } t_c(t) \leqslant t_c < t_c(t + 1).& \end{aligned} \tag{A.30}$$

Without loss of generality, we may assume that the function $\mu \in K^+$ involved in (4.5) is non-decreasing. It follows from (4.5), (A.18), (A.29) and (A.30) that

$$|e(t_c)| \leqslant |e(t_c(t))| + \tilde{g}(t_c, t_0, |x_0|) \int_{t_c(t)}^{t_c} |x(t) - \phi(\tau, t_0, x_0; \tilde{d})|\, d\tau, \tag{A.31}$$

where $\tilde{g}(t_c, t_0, s) := L(t_c, \beta(t_c)\gamma(s + R') + \mu(t_c)a(s + R))$. Inequalities (A.20), (A.29) and (A.31) directly imply that

$$\begin{aligned} |e(t_c)| &\leqslant |e(t_c(t))| + |e(t_c(t))|\tilde{g}(t_c, t_0, |x_0|)\left(\frac{h(t)}{1+\theta(t)}\right) \\ &\quad + \tilde{g}(t_c, t_0, |x_0|)\int_{t_c(t)}^{t_c} |\phi(t_c(t), t_0, x_0; \tilde{d}) - \phi(\tau, t_0, x_0; \tilde{d})| \, d\tau \\ &\leqslant |e(t_c(t))| + |e(t_c(t))|\tilde{g}(t_c, t_0, |x_0|)\left(\frac{h(t)}{1+\theta(t)}\right) \\ &\quad + \tilde{g}(t_c, t_0, |x_0|)\beta(t_c)\gamma(|x_0| + R')\tilde{L}(t_c, |x_0|)\exp(\tilde{L}(t_c, |x_0|))\left(\frac{h(t)}{1+\theta(t)}\right)^2. \end{aligned} \tag{A.32}$$

The latter inequality in conjunction with the fact that all functions involved are non-decreasing with respect to t_c and the identity $t_c(t) - t_0 = \sum_{\tau=0}^{t-1} \frac{h(\tau)}{1+\theta(\tau)}$ can be used in order to prove inductively the following estimate:

$$|e(t_c(t))| \leqslant \upsilon(t_c(t), t_0, |x_0|) \sup_{\tau \geqslant 0}\left(\frac{h(\tau)}{1+\theta(\tau)}\right), \tag{A.33}$$

where $v(t_c, t_0, s) := (t_c - t_0)\tilde{g}(t_c, t_0, s)\exp(\tilde{g}(t_c, t_0, s)(t_c - t_0))\beta(t_c)\gamma(s + R')\tilde{L}(t_c, s)\exp(\tilde{L}(t_c, s))$ is non-decreasing with respect to t_c. Estimates (A.32) and (A.33) imply inequality (4.6) with

$$g(t_c, t_0, s) := v(t_c, t_0, s) + v(t_c, t_0, s)\tilde{g}(t_c, t_0, s) + \tilde{g}(t_c, t_0, s)\beta(t_c)\gamma(s + R')\tilde{L}(t_c, s)\exp(\tilde{L}(t_c, s)).$$

The proof is complete. □