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Robust output feedback stabilization and nonlinear observer design

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Abstract

Necessary and sufficient conditions for the solution of robust output feedback stabilization (ROFS) problem are given. The obtained results are applied to the problem of Global Identity Observer Design for forward complete systems without inputs. It is shown that a necessary and sufficient condition for the existence of a Global Identity Observer is the existence of an observer Lyapunov function (OLF).

Keywords: Output stability; Observer design; Lyapunov functions; Time-varying systems

1. Introduction

In this paper we consider the problem of robust global stabilization of the output

$$Y = H(t, x) \tag{1.1a}$$

of a time-varying affine system:

$$\begin{aligned} &\dot{x} = f(t, d, x) + g(t, d, x)u, \\ &x \in \Re^n, t \geqslant 0, d \in D, u \in U \end{aligned} \tag{1.1b}$$

by means of a time-varying output feedback law $u = k(t, y)$, where

$$y = h(t, x). \tag{1.1c}$$

For obvious reasons the output $y = h(t, x)$ is called the "measured output" and the output $Y = H(t, x)$ is called the "stabilized output". This problem is called the Robust Output Feedback Stabilization (ROFS) problem and sufficient as well as necessary conditions for the solvability of this problem are given in this paper. The ROFS problem includes as a special case the problem of static output feedback stabilization, where the "stabilized output" is the whole state, i.e., $Y = x$. Sufficient conditions for uniform (static or dynamic) output feedback stabilization were given in many papers (see for instance [3–5,19,22,23] and the references therein). However, in this paper we are

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concerned with non-uniform in time output feedback stabilization of the output given by (1.1a). The notions of non-uniform in time Robust Global Asymptotic Stability (RGAS) and Robust Global Asymptotic Output Stability (RGAOS) have been developed in [6–11] and have been proved useful for the solution of many problems of Mathematical Control Theory.

The ROFS problem arises naturally when one considers the problem of Global Identity Observer Design for a forward complete time-varying nonlinear system:

$$\begin{aligned} &\dot{x} = f(t,x),\\ &y = h(t,x),\\ &x \in \Re^n,\ t \geqslant 0. \end{aligned} \tag{1.2}$$

For this problem, the aim is to design a vector field $k(\cdot)$ such that system (1.2) with $\dot{z} = k(t,z,y)$ has the property of Global Asymptotic Output Stability for the "stabilized output" $Y = H(t,x,z) := z - x$. Thus the problem of Global Identity Observer Design is actually a special case of the ROFS problem. The problem of Global Identity Observer Design has a long history in Mathematical Control Theory (see [16] and the references in [18]) and its solvability for nonlinear systems is addressed in many papers (see for instance [12–15,20,21]). Recently, the notions of State Independent Error Lyapunov Function and Observer Lyapunov Function (OLF) were introduced in [17]. In this paper we prove that the existence of a time-varying OLF is a necessary and sufficient condition for the existence of a global identity observer for (1.2) (Theorem 3.5).

Notations. Throughout this paper we adopt the following notations:

- By M_D we denote the set of all measurable functions from $\Re^+$ to D, where $D \subset \Re^m$ is a given compact set.
- By $C^j(A)(C^j(A;\Omega))$, where $j \geqslant 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
- For $x \in \Re^n$, x' denotes its transpose and $|x|$ its usual Euclidean norm.
- By $B[x,r]$ where $x \in \Re^n$ and $r \geqslant 0$, we denote the closed sphere in $\Re^n$ of radius r, centered at $x \in \Re^n$.
- K^+ denotes the class of positive C^0 functions $\phi : \Re^+ \to (0,+\infty)$. By K we denote the class of strictly increasing C^0 functions $a : \Re^+ \to \Re^+$ with $a(0)=0$. By K_∞ we denote the class of strictly increasing C^0 functions $a : \Re^+ \to \Re^+$ with $a(0)=0$ and $\lim_{s\to+\infty} a(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s,t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geqslant 0$ the mapping $\sigma(\cdot,t)$ is of class K; (ii) for each $s \geqslant 0$, the mapping $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty}\sigma(s,t)=0$. By K_{con} we denote the class of functions $a : \Re^+ \to \Re^+$ with the properties: (i) $a \in C^1(\Re^+) \cap K$; (ii) the function $\frac{\mathrm{d}a}{\mathrm{d}s}(s)$ is non-decreasing (see [6]).
- We denote by $\mathbf{E}$ the class of functions $\mu \in C^0(\Re^+;\Re^+)$ that satisfy $\int_0^{+\infty}\mu(t)\,\mathrm{d}t < +\infty$ and $\lim_{t\to+\infty}\mu(t) = 0$.
- We say that the function $H : \Re^+ \times \Re^n \to \Re^k$, is locally Lipschitz with respect to $x \in \Re^n$ if for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geqslant 0$ such that $|H(t,x) - H(t,y)| \leqslant L|x-y|$ for all $(t,x,y) \in I \times S \times S$.
- We say that the function $f : \Re^+ \times D \times \Re^n \to \Re^k$ where $D \subset \Re^m$ is a compact set, is locally Lipschitz with respect to $x \in \Re^n$, uniformly in $d \in D$, if for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geqslant 0$ such that $|f(t,d,x) - f(t,d,y)| \leqslant L|x-y|$ for all $(t,x,y,d) \in I \times S \times S \times D$. Notice that for the dynamical system $\dot{x} = f(t,d,x)$, $x \in \Re^n$, $d \in D$, the assumption that the dynamics $f(\cdot)$ are continuous everywhere and locally Lipschitz with respect to $x \in \Re^n$, uniformly in $d \in D$, with $f(\cdot,0)=0$, guarantees that the system has a unique local Caratheodory solution (see [2]) for all $d(\cdot) \in M_D$.

Moreover, throughout this paper we assume the following:

H1. Concerning system (1.1), we assume that $D \subset \Re^l$ is a compact set and $U \subseteq \Re^m$ is a non-empty convex set with $0 \in U$. Moreover, the dynamics of system (1.1), $f(\cdot)$, $g(\cdot)$ are C^0 vector fields, which are locally Lipschitz with respect to $x \in$

$\Re^n$, uniformly in $d \in D$, with $f(t,d,0)=0$ for all $(t,d) \in \Re^+ \times D$.

H2. The output map $h : \Re^+ \times \Re^n \to \Re^k$ involved in (1.1c), is a C^0 function, locally Lipschitz with respect to $x \in \Re^n$, with $h(t,0)=0$ for all $t \geqslant 0$ that satisfies:

(1) There exists a region $S \subseteq \Re^k$ such that $S = h(t, \Re^n)$ for all $t \geqslant 0$.
(2) There exists a continuous function $a : \Re^+ \times S \to \Re^+$, such that for every $(t,y) \in \Re^+ \times S$ there exists $x \in \Re^n$ with $y = h(t,x)$ and $|x| \leqslant a(t,y)$.

H3. The output map $H : \Re^+ \times \Re^n \to \Re^l$, involved in (1.1a), is a C^0 function with $H(t,0)=0$ for all $t \geqslant 0$.

It should be emphasized at this point that assumption H2 is not restrictive. Since the ROFS problem allows output transformations of the form $\tilde{h}(t,x) = p(t, h(t,x))$ for all $t \geqslant 0$, $x \in \Re^n$, where $p : \Re^+ \times \Re^k \to \Re^k$ is a locally Lipschitz function with $p(t,0)=0$ for all $t \geqslant 0$, we may redefine a given output map so that part 1 of hypothesis H2 is satisfied. For example, the output map $h(t,x) = \exp(-t)\sin(x)$ does not satisfy (1) of hypothesis H2. However, we may consider the ROFS problem for the output map $\tilde{h}(t,x) = \sin(x)$, which satisfies part 1 of hypothesis H2 and is related to the given output map by the transformation $\tilde{h}(t,x) = \exp(t)h(t,x)$. We also note that part 1 of hypothesis H2 is automatically satisfied for the important case of time-invariant output maps, i.e., for the case $h(t,x) \equiv h(x)$. Concerning part 2 of hypothesis H2, we notice that this assumption is satisfied for output maps that are commonly used in the literature. For example, when $x = (x_1, \ldots, x_n) \in \Re^n$ and $y = h(t,x) := x_1 \in \Re$, we may select $a(t,y) := |y|$ for all $(t,y) \in \Re^+ \times \Re$.

2. Robost Output Feedback Stabilazation

In this section we define the Robust Output Feedback Stabilization problem and we give necessary and sufficient conditions for its solvability. We first give the definitions of the notions of Robust Forward Completeness (RFC) and Robust Global Asymptotic Output Stability (RGAOS), introduced in [8]. By virtue of Lemmas 2.3 and 3.3 in [8], the reader can immediately verify that the simplified definitions given here are equivalent to the $\varepsilon - \delta$ definitions given in [8]. Consider the system:

$$\begin{aligned}
&\dot{x} = f(t,d,x),\\
&Y = H(t,x),\\
&x \in \Re^n,\ t \geqslant 0,\ d \in D,
\end{aligned} \tag{2.1}$$

where $H \in C^0(\Re^+ \times \Re^n; \Re^k)$, $f \in C^0(\Re^+ D \times \Re^n; \Re^n)$ is a locally Lipschitz vector field with respect to $x \in \Re^n$, uniformly with respect to $d \in D$, with $f(t,d,0)=0$, $H(t,0)=0$ for all $(t,d) \in \Re^+ \times D$.

Definition 2.1. We say that (2.1) is Robustly Forward Complete (RFC) if there exist functions $q \in K^+$ and $a \in K_\infty$ such that for every $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ the solution $x(t)$ of (2.1) with initial condition $x(t_0) = x_0$ and corresponding to input $d \in M_D$ exists for all $t \geqslant t_0$ and satisfies

$$|x(t)| \leqslant q(t)a(|x_0|), \quad \forall t \geqslant t_0. \tag{2.2}$$

We say that (2.1) is Robustly Globally Asymptotically Output Stable (RGAOS) if system (2.1) is RFC and there exist functions $\beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ the solution $x(t)$ of (2.1) with initial condition $x(t_0) = x_0$ and corresponding to input $d \in M_D$ exists for all $t \geqslant t_0$ and satisfies

$$|H(t, x(t))| \leqslant \sigma(\beta(t_0)|x_0|, t - t_0), \quad \forall t \geqslant t_0. \tag{2.3}$$

We are now ready to provide the definition of the ROFS problem for systems with outputs.

Definition 2.2. Consider system (1.1b) with output maps given by (1.1a) and (1.1c). The output $Y = H(t,x)$ is called the "stabilized output" while the output $y = h(t,x)$ is called the "measured output". The ROFS problem for (1.1) with measured output $y = h(t,x)$ and stabilized output $Y = H(t,x)$ is said to be

(1) *Semi-globally solvable* if for every $R > 0$, there exists a function $k \in C^0(\Re^+ \times S; U)$, which is locally Lipschitz with respect to $y \in S$, such that if we denote by $x(t)$ the solution of the closed-loop system (1.1) with $u = k(t, h(t,x))$

corresponding to $d(\cdot) \in M_D$ and initiated from $x_0 \in \Re^n$ at time $t_0 \geqslant 0$, the following properties hold:

P1. For every $T \geqslant 0$ we have: $\sup\{|x(t_0+s)|;\ |x_0| \leqslant R,\ t_0 \in [0,T],\ s \in [0,T],\ d(\cdot) \in M_D\} < +\infty$.

P2. For every $\varepsilon > 0$, $T \geqslant 0$ and $0 \leqslant r \leqslant R$, there exists $\tau := \tau(\varepsilon, T, r) \geqslant 0$, such that: $|x_0| \leqslant r,\ t_0 \in [0,T] \Rightarrow |Y(t)| \leqslant \varepsilon,\ \forall t \geqslant t_0 + \tau,\ \forall d(\cdot) \in M_D$.

(2) *Globally solvable* if there exists a function $k \in C^0(\Re^+ \times S; U)$, which is locally Lipschitz with respect to $y \in S$, such that the closed-loop system (1.1a,b) with $u = k(t, h(t,x))$ is non-uniformly in time RGAOS. Particularly, we say that the feedback function $k \in C^0(\Re^+ \times S; U)$ *globally solves* the ROFS problem for (1.1) with measured output $y = h(t,x)$ and stabilized output $Y = H(t,x)$.

(3) *Globally strongly solvable* if there exists a feedback function $k \in C^0(\Re^+ \times S; U)$, which globally solves the ROFS problem for (1.1) with measured output $y = h(t,x)$ and stabilized output $Y = H(t,x)$ and the set $H^{-1}(t,0)$ is positively invariant. Particularly, we say that the feedback function $k \in C^0(\Re^+ \times S; U)$ *globally strongly solves* the ROFS problem for (1.1) with measured output $y = h(t,x)$ and stabilized output $Y = H(t,x)$.

For convenience, we will next call the ROFS problem for (1.1) with measured output $y = h(t,x)$ and stabilized output $Y = H(t,x)$ as Problem (1).

Remark 2.3. If properties P1 and P2 hold then it can be shown (as in proof of Lemma 3.3 in [8]) that the following property is also satisfied:

P3. For every $\varepsilon > 0$, $T \geqslant 0$, it holds that $\sup\{|Y(t)|;\ t \geqslant t_0,\ |x_0| \leqslant R,\ t_0 \in [0,T],\ d(\cdot) \in M_D\} < +\infty$ and there exists $\delta := \delta(\varepsilon, T) > 0$ such that: $|x_0| \leqslant \delta,\ t_0 \in [0,T] \Rightarrow |Y(t)| \leqslant \varepsilon,\ \forall t \geqslant t_0$, $\forall d(\cdot) \in M_D$ (robust Lyapunov output stability).

The following examples show that recent research results in [9,6] can be regarded as solutions of the ROFS problem for appropriate systems.

Example 2.4. Consider the following system:

$$\dot{z} = Az, \quad z \in \Re^{n-j} \tag{2.4a}$$

$$\begin{aligned} &\dot{\xi}_i = f_i(z, \xi_1, \ldots, \xi_i) + \xi_{i+1}, \quad 1 \leqslant i \leqslant j \\ &u := \xi_{j+1} \in \Re,\ \xi := (\xi_1, \ldots, \xi_j) \in \Re^j \end{aligned} \tag{2.4b}$$

with state $x := (z, \xi) \in \Re^n$ and output maps

$$Y = y = H(t,x) = h(t,x) := \xi, \tag{2.5}$$

where A is a matrix of appropriate dimensions and $f_i(\cdot)$ ($i = 1, \ldots, j$) are continuous mappings, locally Lipschitz with respect to (z, ξ) that satisfy

$$\begin{aligned} &|f_i(z, \xi_1, \ldots, \xi_i)| \leqslant K(1 + |z|^a)|(\xi_1, \ldots, \xi_i)|, \\ &\quad \forall z \in \Re^{n-j},\ (\xi_1, \ldots, \xi_i) \in \Re^i \end{aligned} \tag{2.6}$$

for some constants $a, K \geqslant 1$. Clearly, we have for the solution of (2.4a) initiated from $z_0 \in \Re^{n-j}$ at time $t_0 \geqslant 0$:

$$|z(t)| \leqslant \exp(|A|t)|z_0|, \tag{2.7}$$

where $|A| := \sup\{Az; |z| = 1\}$. Inequality (2.6) in conjunction with (2.7), implies

$$\begin{aligned} &|f_i(z(t), \xi_1, \ldots, \xi_i)| \\ &\quad \leqslant K(1 + |z_0|^a)\exp(rt)|(\xi_1, \ldots, \xi_i)|, \\ &\qquad \forall z_0 \in \Re^{n-j}, (\xi_1, \ldots, \xi_i) \in \Re^i, \end{aligned} \tag{2.8}$$

where $r := a|A|$. Moreover, by virtue of inequality (2.8) and Corollary 3.2 in [9], there exist a C^∞ mapping $k : \Re^+ \to \Re^{1\times j}$, a C^0 non-decreasing function $\rho : \Re^+ \to \Re^+$, a constant $\bar{r} > 0$ and a C^0, non-negative function $D : \Re^+ \times \Re^+ \to \Re^+$, with the property that for each $t \geqslant 0$ the mapping $D(\cdot, t)$ is non-decreasing and

$$\lim_{t \to +\infty} \exp\{\varepsilon t\} D(s,t) = 0, \quad \forall \varepsilon \geqslant 0, s \geqslant 0 \tag{2.9}$$

such that the solution $\xi(t)$ of the closed-loop system (2.4b) with $u = k(t)\xi$, initiated from $\xi_0 \in \Re^j$ at time t_0 and corresponding to $z(t) = \exp(A(t-t_0))z_0$, satisfies

$$\begin{aligned} &|\xi(t)| \leqslant D(|z_0|, t)\exp\{\rho(|z_0|)\exp(\bar{r}t_0)\}|\xi_0|, \\ &\quad \forall t \geqslant t_0. \end{aligned} \tag{2.10}$$

It follows that the linear time-varying feedback law $u = k(t)\xi$, *globally strongly solves* the ROFS problem for (2.4) with measured output $y = h(t,x) = \xi$ and stabilized output $Y = H(t,x) = \xi$.

Example 2.5. Consider the following system:

$$\dot{z} = f_0(d, z), \quad z \in \Re^{n-j}, d \in D, \tag{2.11a}$$

$$\dot{\xi}_i = f_i(d, z, \xi_1, \ldots, \xi_i) + g_i(d, z, \xi_1, \ldots, \xi_i)\xi_{i+1}$$
$$1 \leqslant i \leqslant j,$$

$$u := \xi_{j+1} \in \Re, \xi := (\xi_1, \ldots, \xi_j) \in \Re^j,$$
$$d \in D, \tag{2.11b}$$

with state $x := (z, \xi) \in \Re^n$ and output maps

$$Y = y = H(t, x) = h(t, x) := \xi, \tag{2.12}$$

where $D \subset \Re^l$ is a compact set, $f_0(\cdot)$ and $f_i(\cdot), g_i(\cdot)$ $(i = 1, \ldots, j)$ are continuous mappings, locally Lipschitz with respect to (z, ξ), uniformly in $d \in D$, with $f_0(\cdot, 0) = 0$, that satisfy for $i = 1, \ldots, j$:

$$|f_i(d, z, \xi_1, \ldots, \xi_i)| \leqslant a(\rho(|z|)|(\xi_1, \ldots, \xi_i)|),$$
$$\forall z \in \Re^{n-j}, (\xi_1, \ldots, \xi_i) \in \Re^i, d \in D \tag{2.13}$$

for certain $a \in K_{\text{con}}$ (see notations for the definition of K_{con}) and certain C^0 non-decreasing function $\rho : \Re^+ \to \Re^+$. Suppose that (2.11a) is RFC. Then using Lemma 2.2 in [6] and Proposition 4.1 in [6] it can be shown, as in the previous example, that the ROFS problem for (2.11) with measured output $y = h(t, x) = \xi$ and stabilized output $Y = H(t, x) = \xi$ is *semi-globally solvable*.

The following result is an immediate consequence of Proposition 3.6 in [8].

Theorem 2.6. *Suppose that Problem* (1) *is globally solvable. Then there exist functions $V \in C^\infty(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$ and $\beta_1, \beta_2 \in K^+$ such that the following inequalities hold*:

$$a_1(|(\beta_1(t)x, H(t, x))|) \leqslant V(t, x) \leqslant a_2(\beta_2(t)|x|),$$
$$\forall (t, x) \in \Re^+ \times \Re^n, \tag{2.14a}$$

$$\inf_{u \in U} \sup_{\substack{x \in h^{-1}(t,y) \\ d \in D}} \left\{ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + \frac{\partial V}{\partial x}(t, x) g(t, d, x) u + V(t, x) \right\} \leqslant 0,$$
$$\forall (t, y) \in \Re^+ \times S. \tag{2.14b}$$

If the output map $H : \Re^+ \times \Re^n \to \Re^l$, involved in (1.1a), *is a C^1 function and Problem* (1) *is globally strongly solvable, then the following conditions are additionally satisfied*:

$$\inf_{u \in U^*(t,y)} \sup_{\substack{x \in h^{-1}(t,y) \\ d \in D}} \left\{ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + \frac{\partial V}{\partial x}(t, x) g(t, d, x) u + V(t, x) \right\} \leqslant 0,$$
$$\forall t \geqslant 0 \text{ and for all } y \in S \text{ for which}$$
$$H^{-1}(t, 0) \cap h^{-1}(t, y) \neq \emptyset, \tag{2.14c}$$

where

$$U^*(t, y) = \left\{ u \in U;\ \frac{\partial H}{\partial t}(t, x) + \frac{\partial H}{\partial x}(t, x) \times f(t, d, x) + \frac{\partial H}{\partial x}(t, x) g(t, d, x) u = 0 \right.$$
$$\text{for all } d \in D \text{ and}$$
$$\left. x \in H^{-1}(t, 0) \cap h^{-1}(t, y) \right\}. \tag{2.14d}$$

Finally, if the feedback function $k \in C^0(\Re^+ \times S; U)$ that globally solves Problem (1) *satisfies $k(t, 0) = 0$ for all $t \geqslant 0$, then the following condition is satisfied:*

$$\sup_{\substack{x \in h^{-1}(t,0) \\ d \in D}} \left\{ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + V(t, x) \right\}$$
$$\leqslant 0, \quad \forall t \geqslant 0. \tag{2.14e}$$

For the statement of our main result we need to define the notions of the "capturing region" and the "bounding map".

Definition 2.7. Consider system (2.1) with output map $y = h(t, x)$, where h satisfies hypothesis H2. Let $0 \in P(t, y) \subseteq \Re^n$ be a set-valued map with closed images. We say that $P(t, y) \subseteq \Re^n$ is a "*capturing region*" for (2.1) if the following property is satisfied:

(**P**). For every $R > 0$, $T \geqslant 0$ there exist $\tau := \tau(R, T) \geqslant 0$ and $\varepsilon := \varepsilon(R, T) \geqslant 0$ such that for every $d(\cdot) \in M_D$, $x_0 \in B[0, R]$ and $t_0 \in [0, T]$, the solution $x(\cdot)$ of system (2.1) corresponding to $d(\cdot) \in M_D$ and initiated from $x_0 \in \Re^n$ at time $t_0 \geqslant 0$, satisfies $\sup_{t_0 \leqslant t \leqslant t_0 + \tau} |x(t)| \leqslant \varepsilon$ and $x(t) \in P(t, h(t, x(t)))$ for all $t \geqslant t_0 + \tau$.

Roughly speaking, the capturing region is a closed set in the state space with the property that every solution of the system enters this set in finite time.

Definition 2.8. Let $h : \Re^+ \times \Re^n \to \Re^k$ a map, which satisfies hypothesis H2 and let $h^{-1}(t, y) := \{x \in \Re^n;\ y = h(t, x)\}$. Every set-valued map $P(t, y) \subseteq \Re^n$ defined on $\Re^+ \times S$ with $0 \in P(t, y)$ for all $(t, y) \in \Re^+ \times S$, such that the map $h^{-1}(t, y) \cap P(t, y)$ is upper semi-continuous, with non-empty, compact images for all $(t, y) \in \Re^+ \times S$ is called a "*bounding map*" for h.

Notice that under assumption H2 for the function h, we can explicitly construct bounding maps for h. For example, the set-valued map $P(t, y) := \{x \in \Re^n;\ |x| \leqslant a(t, y) + \beta(t)\}$, where $\beta(t)$ is any non-negative continuous function and $a : \Re^+ \times S \to \Re^+$ is the continuous function involved in hypothesis H2, is a bounding map for h. Specifically, we then have

$$h^{-1}(t, y) \cap P(t, y) = \{x \in \Re^n;\ y = h(t, x), |x| \leqslant a(t, y) + \beta(t)\}.$$

Moreover, notice that in general a bounding map for the function h actually depends on $y \in S$. This happens because the set-valued map $h^{-1}(t, y) \cap P(t, y)$ is required to be non-empty for all $(t, y) \in \Re^+ \times S$. For example, when $x = (x_1, \ldots, x_n) \in \Re^n$ and $y = h(t, x) := x_1 \in \Re$, in order to guarantee that the set-valued map $h^{-1}(t, y) \cap P(t, y)$ is non-empty for all $(t, y) \in \Re^+ \times \Re$, we must select $P(t, y) := \{x \in \Re^n;\ |x| \leqslant a(t, y)\}$, where the function a satisfies $a(t, y) \geqslant |y|$ for all $(t, y) \in \Re^+ \times \Re$.

The following theorem provides a set of sufficient Lyapunov-like conditions for the solvability of Problem (1).

Theorem 2.9. *Suppose that there exist functions* $V \in C^1(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$, $\beta_1, \beta_2 \in K^+$, $\phi \in \mathbf{E} \cap K^+$ (*see Notations for the definition of class* $\mathbf{E}$) *and a positive definite function* $\rho \in C^0(\Re^+; \Re^+)$, *such that the following inequalities hold*:

$$a_1(|(\beta_1(t)x, H(t, x))|) \leqslant V(t, x) \leqslant a_2(\beta_2(t)|x|), \quad \forall (t, x) \in \Re^+ \times \Re^n, \tag{2.15a}$$

$$\inf_{u\in U} \sup_{\substack{x\in h^{-1}(t,y)\\ d\in D}} \left\{ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + \frac{\partial V}{\partial x}(t, x) g(t, d, x) u + \rho(V(t, x)) \right\} \leqslant \phi(t), \quad \forall (t, y) \in \Re^+ \times S, \tag{2.15b}$$

$$\sup_{\substack{x\in h^{-1}(t,0)\\ d\in D}} \left\{ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + \rho(V(t, x)) \right\} \leqslant \phi(t), \quad \forall t \geqslant 0. \tag{2.15c}$$

Then Problem (1) *is semi-globally solvable. Particularly, for every bounding map* $P(t, y) \subseteq \Re^n$ *for h, there exists a function* $k \in C^\infty(\Re^+ \times S; U)$ *with* $k(t, 0) = 0$ *for all* $t \geqslant 0$, *such that*

$$\begin{aligned} &\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, d, x) + \frac{\partial V}{\partial x}(t, x) g(t, d, x) k(t, y) \\ &\quad \leqslant -\rho(V(t, x)) + 6\phi(t), \quad \forall (t, y) \in \Re^+ \times S, \ x \in h^{-1}(t, y) \cap P(t, y), d \in D. \end{aligned} \tag{2.16}$$

The proof of Theorem 2.9 is based on the following lemma, which provides the procedure for the construction of the feedback law. The result of the following lemma will be also used in the next section.

Lemma 2.10. *Let* $w_1 \in C^0(\Re^+ \times D \times \Re^n; \Re)$, $w_2 \in C^0(\Re^+ \times D \times \Re^n; \Re^m)$ *where* $D \subset \Re^l$ *is a compact set,* $\phi \in \mathbf{E} \cap K^+$, $U \subseteq \Re^m$ *be a non-empty convex set with* $0 \in U$, $\mu \in C^0(\Re^+ \times S; \Re)$ *with* $\mu(t, 0) = 0$ *for all* $t \geqslant 0$, *and h be a map that satisfies hypothesis* H2. *Suppose that*

$$\inf_{u\in U} \sup\{w_1(t, d, x) + w_2(t, d, x)u;\ d \in D, x \in h^{-1}(t, y)\} \leqslant \phi(t), \quad \forall (t, y) \in \Re^+ \times S, \tag{2.17a}$$

$$\sup\{w_1(t, d, x);\ d \in D, x \in h^{-1}(t, y)\} \leqslant \phi(t), \quad \forall t \geqslant 0 \text{ and for all } y \in S \text{ with } \mu(t, y) = 0. \tag{2.17b}$$

Then for every bounding map $P(t, y) \subseteq \Re^n$ *for h, there exists a function* $k \in C^\infty(\Re^+ \times S; U)$ *with* $k(t, y) = 0$ *for all* $t \geqslant 0$ *and for all* $y \in S$ *with* $\mu(t, y) = 0$, *such that*

$$w_1(t,d,x)+w_2(t,d,x)k(t,y)\leqslant 6\phi(t),\quad \forall (t,y)\in\Re^+\times S,\ x\in h^{-1}(t,y)\cap P(t,y),\quad d\in D. \tag{2.18}$$

Proof. Let $P(t,y)\subseteq\Re^n$ be an arbitrary bounding map for h. Since the set-valued map $h^{-1}(t,y)\cap P(t,y)$ is upper semi-continuous, with non-empty, compact images for all $(t,y)\in\Re^+\times S$ and the function $w_1(t,d,x)+w_2(t,d,x)u$ is continuous with respect to $(t,x,d,u)\in\Re^+\times\Re^n\times D\times U$, by virtue of Theorem 1.4.16 in [1] it follows that for all $u\in U$, the map

$$(t,y,u)\to w^0(t,y;u) := \sup\{w_1(t,d,x)+w_2(t,d,x)u;\ d\in D,\ x\in h^{-1}(t,y)\cap P(t,y)\} \tag{2.19}$$

is upper semi-continuous. Moreover, the map $u\to w^0(t,y;u)$ is convex. We proceed by noticing the following fact.

Fact. *For all $(t_0,y_0)\in\Re^+\times S$, there exists $u_0\in U$ and a neighborhood $\mathbf{N}(t_0,y_0)\subset\Re^+\times S$, such that*

$$(t,y)\in\mathbf{N}(t_0,y_0)\Rightarrow w^0(t,y;u_0)\leqslant 6\phi(t). \tag{2.20}$$

Moreover, if $\mu(t_0,y_0)=0$ then we may select $u_0=0$.

Proof. By virtue of (2.17a) and the fact that $\phi(t_0)>0$, it follows that for all $(t_0,y_0)\in\Re^+\times S$ there exists $u_0\in U$ such that

$$w^0(t_0,y_0;u_0)\leqslant 2\phi(t_0). \tag{2.21}$$

Moreover, by virtue of (2.17b) we may select $u_0=0$ if $\mu(t_0,y_0)=0$. Since the map $(t,y)\to w^0(t,y;u)$ is upper semi-continuous, there exists a neighborhood $\mathbf{N}(t_0,y_0)\subset\Re^+\times S$ around (t_0,y_0) such that for all $(t,y)\in\mathbf{N}(t_0,y_0)$:

$$w^0(t,y;u_0)\leqslant w^0(t_0,y_0;u_0)+\phi(t_0);\quad \phi(t_0)\leqslant 2\phi(t). \tag{2.22}$$

Therefore, inequalities (2.21) and (2.22) imply (2.20) for all $(t,y)\in\mathbf{N}(t_0,y_0)$.

Let $\Omega := \{(t,y)\in\Re^+\times S : \mu(t,y)\neq 0\}$. There exists a family of open sets $(\Omega_j)_{j\in J}$ with $\Omega_j\subset\Omega$ for all $j\in J$, which consists a locally finite open covering of Ω and a family of points $(u_j)_{j\in J}$ with $u_j\in U$ for all $j\in J$, such that

$$(t,y)\in\Omega_j\Rightarrow w^0(t,y;u_j)\leqslant 6\phi(t). \tag{2.23}$$

Let also

$$A := \bigcup_{(t_0,y_0)\notin\Omega}\mathbf{N}(t_0,y_0), \tag{2.24}$$

where $\Omega := \{(t,y)\in\Re^+\times S : \mu(t,y)\neq 0\}$, $\mathbf{N}(t_0,y_0)\subset\Re^+\times S$ is the neighborhood of (t_0,y_0) that satisfies (2.20) for $u_0=0$. Clearly, by virtue of (2.24) the set $A\subset\Re^+\times S$ is open. By using standard partition of unity arguments, there exists a family of functions $\theta_0:\Re^+\times S\to[0,1]$, $\theta_j:\Re^+\times S\to[0,1]$, with $\theta_j(t,y)=0$ if $(t,y)\notin\Omega_j\subset\Omega$ and $\theta_0(t,y)=0$ if $(t,y)\notin A$, $\theta_0(t,y)+\sum_j\theta_j(t,y)$ being locally finite and $\theta_0(t,y)+\sum_j\theta_j(t,y)=1$ for all $(t,y)\in\Re^+\times S$. We set:

$$k(t,y) := \sum_j \theta_j(t,y)u_j. \tag{2.25}$$

Notice that if $(t,y)\notin\Omega$ then $(t,y)\notin\Omega_j$ for all $j\in J$ and consequently by (2.25) we have $k(t,y)=0$ for all $t\geqslant 0$ and for all $y\in S$ with $\mu(t,y)=0$. Since each u_j is a member of the convex set U and $0\in U$, it follows from (2.25) that $k(t,y)\in U$ for all $(t,y)\in\Re^+\times S$. It also follows from the fact that $w^0(t,y;u)$ is convex with respect to u and definition (2.25) that for all $(t,y)\in\Re^+\times S$ and $J'(t,y)=\{j\in J;\theta_j(t,y)\neq 0\}$:

$$w^0(t,y;k(t,y))\leqslant\sum_{i\in J'(t,y)}\theta_i(t,y)w^0(t,y;u_i). \tag{2.26}$$

Combining (2.23) with (2.26) and (2.19), we obtain the desired (2.18). The proof is complete. □

We are now in a position to prove Theorem 2.9.

Proof of Theorem 2.9. Applying Lemma 2.10 for $w_1(t,d,x)=\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,d,x)+\rho(V(t,x))$, $w_2(t,d,x)=\frac{\partial V}{\partial x}(t,x)g(t,d,x)$ and $\mu(t,y)=|y|$, we conclude that for every bounding map $P(t,y)\subseteq\Re^n$ there exists a function $k\in C^\infty(\Re^+\times S;U)$ with $k(t,0)=0$ for all $t\geqslant 0$, such that (2.16) holds. We next prove that Problem (1) is semi-globally solvable. Notice that by virtue of

Lemma 3.2 in [10], there exist a function $\sigma \in KL$ and a constant $M>0$, such that for every absolutely continuous function $w : \Re^+ \to \Re^+$, that satisfies $\dot{w}(t) \leqslant -\rho(w(t)) + 6\phi(t)$ a.e. for $t \geqslant t_0$, we have:

$$w(t) \leqslant \sigma(w(t_0)+M, t-t_0), \quad \forall t \geqslant t_0. \tag{2.27}$$

Let $R, T>0$ be arbitrary and let a be the continuous non-negative function involved in hypothesis H2 for h. Define the following bounding map for h:

$$P(t,y) := \left\{ x \in \Re^n : |x| \leqslant a(t,y) + R + \frac{1}{\beta_1(t)} a_1^{-1}\left(\sigma\left(M + a_2\left(R \max_{0 \leqslant \tau \leqslant t} \beta_2(\tau)\right), 0\right)\right)\right\}, \tag{2.28}$$

where $a_1, a_2 \in K_\infty$, $\beta_1, \beta_2 \in K^+$ are the functions involved in (2.15a). Let $k \in C^\infty(\Re^+ \times S; U)$ with $k(t,0)=0$ for all $t \geqslant 0$, be the function that satisfies (2.16) for $P(t,y) \subset \Re^n$ as defined by (2.28). Consider the trajectory $x(t)$ of the solution of the closed-loop system (1.1) with $u=k(t,y)$ and initial condition $x(t_0)=x_0 \in B[0,R]$, $t_0 \in [0,T]$, corresponding to input $d(\cdot) \in M_D$, namely

$$\dot{x} = f(t,d,x) + g(t,d,x)k(t,y). \tag{2.29}$$

By continuity of the solution $x(t)$, there exists time $t_1 > t_0$ such that $|x(t)| - a(t,h(t,x(t))) < R + \frac{1}{\beta_1(t)} a_1^{-1}(\sigma(M + a_2(R \max_{0 \leqslant \tau \leqslant t} \beta_2(\tau)), 0))$ for all $t \in [t_0, t_1)$, i.e., $x(t) \in P(t, h(t,x(t)))$ for all $t \in [t_0, t_1)$. It follows from (2.15a), (2.16) and (2.27) that the following inequality holds:

$$|(\beta_1(t)x, H(t,x(t)))| \leqslant a_1^{-1}(\sigma(M + a_2(\beta_2(t_0)|x_0|), t-t_0)), \quad \forall t \in [t_0, t_1). \tag{2.30}$$

Clearly, using continuity of the solution, inequality (2.30) and the fact that $x_0 \in B[0,R]$, it follows that inequality $|x(t)| - a(t,h(t,x(t))) < R + \frac{1}{\beta_1(t)} a_1^{-1}(\sigma(M + a_2(R \max_{0 \leqslant \tau \leqslant t} \beta_2(\tau)), 0))$ holds also for $t = t_1$. Thus (2.30) holds for all $t \geqslant t_0$. This shows that Problem (1) is semi-globally solvable. The proof is complete. □

The following corollary provides useful criteria for the semi-global solution of Problem (1) obtained by Theorem 2.9 to be a global solution.

Corollary 2.11. *Suppose that there exist functions* $V \in C^1(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$, $\beta_1, \beta_2 \in K^+$ *that satisfy* (2.15a) *and there exist* $\phi \in \mathbf{E} \cap K^+$, *a positive definite function* $\rho \in C^0(\Re^+; \Re^+)$, *a function* $k \in C^0(\Re^+ \times S; U)$, *with* $k(t,0)=0$ *for all* $t \geqslant 0$, *which is locally Lipschitz with respect to* $y \in S$, *and a set-valued map* $P(t,y) \subseteq \Re^n$, *such that inequality* (2.16) *holds. Moreover, suppose that the set-valued map* $P(t,y) \subseteq \Re^n$ *is a capturing region for the closed-loop system* (1.1) *with* $u=k(t,h(t,x))$. *Then the feedback function* $k \in C^0(\Re^+ \times S; U)$ *globally solves Problem* (1).

Proof. Notice that by virtue of Lemma 3.2 in [10], there exist a function $\sigma \in KL$ and a constant $M>0$, such that every absolutely continuous function $w : \Re^+ \to \Re^+$, which satisfies $\dot{w}(t) \leqslant -\rho(w(t)) + 6\phi(t)$ a.e. for $t \geqslant t_0$, satisfies also estimate (2.27). Let $R, T > 0$ be arbitrary and consider the trajectory $x(t)$ of the solution of the closed-loop system (1.1) with $u=k(t,y)$ and initial condition $x(t_0)=x_0 \in B[0,R]$, $t_0 \in [0,T]$, corresponding to input $d(\cdot) \in M_D$. If the set-valued map $P(t,y) \subseteq \Re^n$ is a capturing region for the closed-loop system (1.1) with $u=k(t,y)$, then inequalities (2.15a) and (2.16) guarantee that

$$|(\beta_1(t)x, H(t,x(t)))| \leqslant a_1^{-1}(\sigma(M + a_2(\beta_2(t_0+\tau)\varepsilon), t-t_0-\tau)), \quad \forall t \geqslant t_0+\tau,$$

where $a_1, a_2 \in K_\infty$, $\beta_1, \beta_2 \in K^+$ are the functions involved in (2.15a) and $\tau := \tau(R,T) \geqslant 0$, $\varepsilon := \varepsilon(R,T) \geqslant 0$, are the quantities involved in property (P) of Definition 2.7. The latter inequality, in conjunction with the fact that $\sup_{t_0 \leqslant t \leqslant t_0+\tau} |x(t)| \leqslant \varepsilon$ (as guaranteed by property (P) of Definition 2.7), shows that system (2.29) is RFC. Using Lemma 3.5 in [8], it can also be shown that the closed-loop system (1.1) with $u=k(t,y)$, with output given by (1.1a) is non-uniformly in time RGAOS. Thus, Problem (1) is globally solvable. □

3. The problem of observer design for forward complete systems with no inputs

Throughout this section we consider system (1.2) and we suppose that the mapping $f \in C^0(\Re^+ \times$

$\Re^n; \Re^n)$ is locally Lipschitz with respect to $x \in \Re^n$, with $f(t,0)=0$, while the output $h \in C^0(\Re^+ \times \Re^n; \Re^k)$ satisfies assumption H2. We also assume that system (1.2) is RFC.

Definition 3.1. Consider system (1.2) and suppose that system (1.2) is RFC. The system

$$\begin{aligned} &\dot{z} = k(t,z,y), \quad t \geqslant 0, z \in \Re^m, \\ &\bar{x} = \Psi(t,z,y), \quad \bar{x} \in \Re^n, \end{aligned} \tag{3.1}$$

where $\Psi : \Re^+ \times \Re^m \times \Re^k \to \Re^n$ is a continuous map with $\Psi(t,0,0)=0$, $k \in C^0(\Re^+ \times \Re^m \times \Re^k; \Re^m)$ is locally Lipschitz with respect to $(z,y) \in \Re^n \times \Re^k$, with $k(t,0,0)=0$, is called a Global Observer for (1.2) if system (1.2) with (3.1) is RFC and the following properties are satisfied:

(1) *Asymptotic property*: There exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $(x_0,z_0) \in \Re^n \times \Re^m$, $t_0 \geqslant 0$, the solution $(x(\cdot),z(\cdot))$ of system (1.2) with (3.1) initiated from $(x_0,z_0) \in \Re^n \times \Re^m$ at time $t_0 \geqslant 0$, satisfies the following estimate:

$$|\bar{x}(t) - x(t)| \leqslant \sigma(\beta(t_0)|(x_0,z_0)|, t-t_0), \quad \forall t \geqslant t_0 \tag{3.2}$$

(2) *Consistent initialization property*: For every $(t_0,x_0) \in \Re^+ \times \Re^n$ there exists $z_0 \in \Re^m$ such that the solution $(x(\cdot),z(\cdot))$ of system (1.2) with (3.1) initiated from $(x_0,z_0) \in \Re^n \times \Re^m$ at time $t_0 \geqslant 0$, satisfies $x(t) = \bar{x}(t)$ for all $t \geqslant t_0$.

If there exists a Global Observer for (1.2), we say that system (1.2) admits a Global Observer or that the observer design (OD) problem is globally solvable for (1.2). The continuous map $\Psi : \Re^+ \times \Re^m \times \Re^k \to \Re^n$ is called the reconstruction map of the observer. Particularly, if $\Psi(t,z,y) \equiv z$ with $z \in \Re^n$, we say that (3.1) is a Global Identity Observer.

Remark 3.2. Concerning Definition 3.1 the following points must be made:

(i) By virtue of Definition 3.1, if zero is non-uniformly in time GAS for system (1.2) then the system $\dot{z} = f(t,z)$, $\bar{x} = z$ is a Global Identity Observer for (1.2). To prove this fact, notice that if zero is non-uniformly in time GAS, then by virtue of Proposition 2.2 in [10], there exist functions $\sigma \in KL$ and $\beta \in K^+$, such that for every $x_0 \in \Re^n$, $t_0 \geqslant 0$, the solution $x(\cdot)$ of system (1.2) initiated from $x_0 \in \Re^n$ at time $t_0 \geqslant 0$, satisfies the estimate $|x(t)| \leqslant \sigma(\beta(t_0)|x_0|, t-t_0)$, for all $t \geqslant t_0$. This implies that for every $(x_0,z_0) \in \Re^n \times \Re^n$, $t_0 \geqslant 0$, the solution $(x(\cdot),z(\cdot))$ of system (1.2) with $\dot{z} = f(t,z)$, initiated from $(x_0,z_0) \in \Re^n \times \Re^n$ at time $t_0 \geqslant 0$, satisfies the estimate $|z(t)-x(t)| \leqslant 2\sigma(\beta(t_0)|(x_0,z_0)|, t-t_0)$, for all $t \geqslant t_0$. For example, this fact can be used in order to design a global observer for the system $\dot{x}_1 = -x_1 + x_2^2$, $\dot{x}_2 = -x_1 x_2$, for which zero is (uniformly in time) GAS. This system is given as an example in [15], where a local observer for this system is constructed.

(ii) Definition 3.1 is different from Definition 7.1.3 in [18], even when we consider autonomous systems without inputs and full order autonomous observers because Definition 3.1 does not guarantee the "Lyapunov stability" property for the error $e = \bar{x} - x$: i.e., if the initial value for the error $e_0 = \bar{x}_0 - x_0$ is "sufficiently small", we cannot guarantee that all future values of the error will be "small". This stronger property would be satisfied if instead of (3.2) the following estimate were satisfied for the solution $(x(\cdot),z(\cdot))$ of system (1.2) with (3.1):

$$\begin{aligned} &|\bar{x}(t) - x(t)| \leqslant \sigma(\beta(t_0)|\bar{x}_0 - x_0|, t-t_0), \\ &\quad \forall t \geqslant t_0. \end{aligned} \tag{3.3}$$

Instead, our asymptotic property (3.2) guarantees that if the initial condition (z_0,x_0) is "sufficiently small", then all future values of the error will be "small". It should be emphasized that if the definition of the observer were based on (3.3) instead of (3.2) then the system $\dot{z} = f(t,z)$, $\bar{x} = z$ would not be a Global Identity Observer for system (1.2) for the case of non-uniform in time global asymptotic stability of zero for (1.2), unless system (1.2) had special structure (e.g., linear systems).

An immediate consequence of Definitions 3.1 and 3.2 is the following fact.

Fact 3.3. *System* (1.2) *admits a Global Identity Observer if and only if the ROFS problem for the system:*

$$\begin{aligned} &\dot{x} = f(t,x), \quad t \geqslant 0, x \in \Re^n, \\ &\dot{z} = u, z \in \Re^n, u \in \Re^n, \end{aligned} \tag{3.4}$$

with measured output $\tilde{y} = \tilde{h}(t,x,z) := (z,y) = (z,h(t,x))$ *and stabilized output* $Y = H(t,x,z) := z - x$, *is globally strongly solvable*.

For convenience, we will next call the ROFS problem for system (3.4) with measured output $\tilde{y} = \tilde{h}(t,x,z) := (z,y) = (z,h(t,x))$ and stabilized output $Y = H(t,x,z) := z - x$ as *Problem* (2).

We next provide the definition of an Observer Lyapunov Function (OLF). Notice that our definition is significantly different from the corresponding definition given in [17].

Definition 3.4. We say that system (1.2) admits an Observer Lyapunov Function (OLF) if there exist functions $V \in C^1(\Re^+ \times \Re^n \times \Re^n; \Re^+)$ (called the OLF), $a_1, a_2 \in K_\infty$, $\beta \in K^+$, $\phi \in \mathbf{E}$, a mapping $w \in C^0(\Re^+ \times \Re^n; \Re^+)$ with $w(\cdot, 0) \leqslant 0$ such that the set-valued map $Q(t) := \{x \in \Re^n : w(t,x) \leqslant 0\}$ is a capturing region for (1.2), and a positive definite function $\rho \in C^0(\Re^+; \Re^+)$, such that the following inequalities hold:

$$a_1(|z-x|) \leqslant V(t,z,x) \leqslant a_2(\beta(t)|(z,x)|), \quad \forall (t,z,x) \in \Re^+ \times \Re^n \times \Re^n, \tag{3.5a}$$

$$\begin{aligned}\inf_{u\in\Re^n} \sup_{x \in h^{-1}(t,y)} \bigg\{ &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &+ \frac{\partial V}{\partial z}(t,z,x)u + \rho(V(t,z,x)) \\ &- w(t,x)(1+V(t,z,x)) \bigg\} \leqslant \phi(t), \\ &\forall (t,z,y) \in \Re^+ \times \Re^n \times S,\end{aligned} \tag{3.5b}$$

$$\begin{aligned}\sup_{x \in h^{-1}(t,y)} \bigg\{ &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &+ \frac{\partial V}{\partial z}(t,z,x) f(t,z) + \rho(V(t,z,x)) \\ &- w(t,x)(1+V(t,z,x)) \bigg\} \leqslant \phi(t), \\ &\forall (t,y) \in \Re^+ \times S \text{ and } z \in h^{-1}(t,y).\end{aligned} \tag{3.5c}$$

Particularly, if $V \in C^1(\Re^+ \times \Re^n \times \Re^n; \Re^+)$ is an OLF, which depends only on $t \in \Re^+$ and $e := z - x$, then it is called a state-independent OLF.

We are now in a position to state and prove our main result concerning the existence of a global identity observer for (1.2).

Theorem 3.5. *Assume that* (1.2) *is RFC. Then the following statements are equivalent:*

(i) *System* (1.2) *admits a Global Identity Observer*.
(ii) *System* (1.2) *admits an OLF*.

Proof. (i)⇒(ii) By virtue of Fact 3.3 and Theorem 2.6, there exist functions $V \in C^\infty(\Re^+ \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$ and $\beta_1, \beta_2 \in K^+$ such that the following inequalities hold:

$$\begin{aligned}&a_1(|(\beta_1(t)(z,x), z-x)|) \\ &\quad \leqslant V(t,z,x) \leqslant a_2(\beta_2(t)|(z,x)|), \\ &\quad \forall (t,z,x) \in \Re^+ \times \Re^n \times \Re^n,\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\inf_{u\in\Re^n} \sup_{x \in h^{-1}(t,y)} \bigg\{ &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &+ \frac{\partial V}{\partial z}(t,z,x)u + V(t,x) \bigg\} \leqslant 0, \\ &\forall (t,z,y) \in \Re^+ \times \Re^n \times S,\end{aligned} \tag{3.6b}$$

$$\begin{aligned}\inf_{u\in U^*(t,\tilde{y})} \sup_{x \in h^{-1}(t,y)} \bigg\{ &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &+ \frac{\partial V}{\partial z}(t,z,x)u + V(t,x) \bigg\} \leqslant 0, \\ &\forall t \geqslant 0 \text{ and for all } \tilde{y} = (z,y) \in \Re^n \times S \\ &\quad \text{for which } H^{-1}(t,0) \cap \tilde{h}^{-1}(t,\tilde{y}) \neq \emptyset,\end{aligned} \tag{3.6c}$$

where

$$\begin{aligned}U^*(t,\tilde{y}) := \{&u \in \Re^n;\ f(t,x) - u = 0 \\ &\text{for all } (z,x) \in H^{-1}(t,0) \cap \tilde{h}^{-1}(t,\tilde{y})\}.\end{aligned} \tag{3.6d}$$

Clearly, the condition $H^{-1}(t,0) \cap \tilde{h}^{-1}(t,\tilde{y}) \neq \emptyset$ is equivalent to the condition $z \in h^{-1}(t,y)$ (notice that if $z \in h^{-1}(t,y)$ then $(z,z) \in H^{-1}(t,0) \cap \tilde{h}^{-1}(t,\tilde{y})$ and conversely if $(z,x) \in H^{-1}(t,0) \cap \tilde{h}^{-1}(t,\tilde{y})$ then $z = x \in h^{-1}(t,y)$). Thus, we obtain $U^*(t,\tilde{y}) = U^*(t,z,y) = \{f(t,z) \in \Re^n\}$ and consequently conditions (3.6c,d) can be combined to give the following condition:

$$\begin{aligned}\sup_{x \in h^{-1}(t,y)} \bigg\{ &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &+ \frac{\partial V}{\partial z}(t,z,x) f(t,z) + V(t,x) \bigg\} \leqslant 0, \\ &\forall t \geqslant 0 \text{ and for all } y \in S,\ z \in h^{-1}(t,y).\end{aligned} \tag{3.7}$$

Inequalities (3.5a,b,c) are immediate consequences of inequalities (3.6a,b) and (3.7) with $\beta(t) \equiv \beta_2(t)$, $\phi(t) \equiv 0$, $\rho(s) = s$ and $w(t,x) \equiv 0$. Moreover, the set-valued map $Q(t) := \{x \in \Re^n : w(t,x) \leqslant 0\} = \Re^n$ is (trivially) a capturing region for (1.2).

(ii)⇒(i) Suppose that there exist functions $V \in C^1(\Re^+ \times \Re^n \times \Re^n; \Re^+)$, $a_1, a_2 \in K_\infty$, $\beta \in K^+$, $\phi \in \mathbf{E}$, $w \in C^0(\Re^+ \times \Re^n; \Re^+)$ with $w(t,0) \leqslant 0$ for all $t \geqslant 0$ and a positive definite function $\rho \in C^0(\Re^+; \Re^+)$, such that inequalities (3.5a–c) hold and such that the set-valued map $Q(t) := \{x \in \Re^n : w(t,x) \leqslant 0\}$ is a capturing region for (1.2). We define:

$$u = f(t,z) + v, \quad v \in \Re^n \tag{3.8}$$

and notice that using v instead of u and the definition $\tilde{y} = \tilde{h}(t,x,z) := (z,y) = (z, h(t,x))$, we have

$$\inf_{v \in \Re^n} \sup_{(z,x) \in \tilde{h}^{-1}(t,\tilde{y})} \{w_1(t,z,x) + w_2(t,z,x)v\} \leqslant \phi(t),$$
$$\forall (t,z,y) \in \Re^+ \times \Re^n \times S, \tag{3.9a}$$

$$\sup_{(z,x) \in \tilde{h}^{-1}(t,\tilde{y})} \{w_1(t,z,x)\} \leqslant \phi(t),$$
$$\forall (t,y) \in \Re^+ \times S \text{ and } \mu(t,\tilde{y}) = 0, \tag{3.9b}$$

where

$$\begin{aligned} w_1(t,z,x) &:= \frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) \\ &\quad + \frac{\partial V}{\partial z}(t,z,x) f(t,z) + \rho(V(t,z,x)) \\ &\quad - w(t,x)(1 + V(t,z,x)), \end{aligned} \tag{3.9c}$$

$$\begin{aligned} w_2(t,z,x) &:= \frac{\partial V}{\partial z}(t,z,x); \\ \mu(t,\tilde{y}) &:= |y - h(t,z)|. \end{aligned} \tag{3.9d}$$

Since (1.2) is RFC, by virtue of Lemma 2.2 in [8], there exist functions $q(\cdot) \in K^+$, $a'(\cdot) \in K_\infty$, such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$, the unique solution $x(t)$ of (1.2) initiated from x_0 at time t_0 exists for all $t \geqslant t_0$ and satisfies

$$|x(t)| \leqslant q(t) a'(|x_0|), \quad \forall t \geqslant t_0. \tag{3.10}$$

We define the bounding map for $\tilde{h}$:

$$\begin{aligned} \tilde{P}(t,z,y) &:= \{z\} \times \{x \in \Re^n : |x| \leqslant a(t,y) \\ &\quad + q(t)a'(t+1) + q(t)a'(|z|)\} \subseteq \Re^n \times \Re^n, \end{aligned} \tag{3.11}$$

where $a : \Re^+ \times S \to \Re^+$ is the continuous positive function involved in hypothesis H2 for h. By virtue of Lemma 2.10 and (3.9a–d) there exists a function $k \in C^\infty(\Re^+ \times \Re^n \times S; \Re^n)$ with $k(t,z,y) = 0$ for all $t \geqslant 0$ and $z \in h^{-1}(t,y)$, such that

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,z,x) + \frac{\partial V}{\partial x}(t,z,x) f(t,x) + \frac{\partial V}{\partial z}(t,z,x) \\ &\quad \times (f(t,z) + k(t,z,y)) \leqslant -\rho(V(t,z,x)) \\ &\quad + 6\phi(t) + w(t,x)(1 + V(t,z,x)), \\ &\quad \forall (t,z,y) \in \Re^+ \times \Re^n \times S, \\ &\quad (z,x) \in \tilde{h}^{-1}(t,z,y) \cap \tilde{P}(t,z,y). \end{aligned} \tag{3.12}$$

We next show that for every $(t_0, z_0 x_0) \in \Re^+ \times \Re^n \times \Re^n$, the unique solution $(z(t), x(t))$ of the system

$$\begin{aligned} \dot{x} &= f(t,x), \\ \dot{z} &= f(t,z) + k(t,z,h(t,x)) \end{aligned} \tag{3.13}$$

initiated from (z_0, x_0) at time t_0 exists for all $t \geqslant t_0$ and that the set-valued map $\tilde{P}(t,z,y)$ is a capturing region for (3.13). Notice that since $k(t,z,y) = 0$ for all $t \geqslant 0$ and $z \in h^{-1}(t,y)$, we can guarantee that the set $H^{-1}(t,0) = \{(z,x) \in \Re^n \times \Re^n : z = x\}$ is positively invariant for (3.13). Clearly, there exists time $t_1 > t_0$ such that the solution of (3.13) exists on $[t_0, t_1)$. Notice that (3.11) in conjunction with inequality (3.10) implies that

$$\begin{aligned} &(z(t), x(t)) \notin \operatorname{int} \tilde{P}(t, z(t), h(t, x(t))) \\ &\quad \Rightarrow t_0 \leqslant t \leqslant t_0 + |x_0| \text{ and } |z(t)| \leqslant |x_0|, \end{aligned} \tag{3.14}$$

where int denotes the interior of a set. Moreover, by virtue of (3.12), we have

$$\begin{aligned} &(z(t), x(t)) \in \tilde{P}(t, z(t), h(t, x(t))) \\ &\quad \Rightarrow \dot{V}|_{(3.13)} \leqslant w(t, x(t)) V(t, z(t), x(t)) \\ &\quad + w(t, x(t)) + 6\phi(t). \end{aligned} \tag{3.15}$$

Consider the cases

(1) $(z(t), x(t)) \notin \operatorname{int} \tilde{P}(t, z(t), h(t, x(t)))$. In this case, by virtue of (3.10) and (3.14), the estimate $|(z(t), x(t))| \leqslant q(t)a'(|x_0|) + |x_0|$ holds.

(2) $(z(t), x(t)) \in \operatorname{int} \tilde{P}(t, z(t), h(t, x(t)))$. Define

$$\begin{aligned} t_2 := \inf\{\tau \in [t_0, t) : (z(s), x(s)) \\ \in \operatorname{int} \tilde{P}(s, z(s), h(s, x(s))) \\ \text{for all } \tau < s \leqslant t\} \end{aligned}$$

i.e., $(t_2, t) \subset (t_0, t)$ is the maximal interval that satisfies $(z(s), x(s)) \in \operatorname{int} \tilde{P}(s, z(s), h(s, x(s)))$

for all $s \in (t_2, t) \subset [t_0, t)$. Thus we either have $t_2 = t_0$ or $(z(t_2), x(t_2)) \notin \operatorname{int} \tilde{P}(t, z(t_2), h(t, x(t_2)))$. Using (3.15), we obtain for $M := 6\int_0^{+\infty} \phi(t)\,dt$

$$V(t, z(t), x(t)) \leqslant \exp\left(\int_{t_2}^{t} |w(\tau, x(\tau))|\,d\tau\right) \times (V(t_2, z(t_2), x(t_2)) + M + \int_{t_2}^{t} |w(\tau, x(\tau))|\,d\tau\bigg).$$

Using (3.5a) and estimate (3.10) in conjunction with the latter inequality, we obtain the estimate

$$\begin{aligned}|(z(t), x(t))| \leqslant{} & 2q(t)a'(|x_0|) + |x_0| \\ & + a_1^{-1}\left(\exp\left(\int_{t_2}^{t} |w(\tau, x(\tau))|\,d\tau\right)\right. \\ & \times \big(a_2(\beta(t_2)|(z(t_2), x(t_2))|) \\ & \left.+ M + \int_{t_2}^{t} |w(\tau, x(\tau))|\,d\tau\big)\right).\end{aligned}$$

Since $|(z(t_2), x(t_2))| \leqslant |z(t_2)| + q(t_2)a'(|x_0|)$, the latter estimate implies for $\tilde{\beta}(t) := \max_{0 \leqslant \tau \leqslant t} \beta(\tau)$ and $\tilde{q}(t) := \max_{0 \leqslant \tau \leqslant t} q(\tau)$:

$$\begin{aligned}&|(z(t), x(t))| \\ &\quad \leqslant 2q(t)a'(|x_0|) + |x_0| + a_1^{-1} \\ &\qquad \times \left(\exp\left(\int_{t_0}^{t} |w(\tau, x(\tau))|\,d\tau\right)\right. \\ &\qquad \times \big(a_2(\tilde{\beta}(t)|z(t_2)| + \tilde{\beta}(t)\tilde{q}(t)a'(|x_0|)) \\ &\qquad \left.+ M + \int_{t_0}^{t} |w(\tau, x(\tau))|\,d\tau\big)\right).\end{aligned}$$

For the case $(z(t_2), x(t_2)) \notin \operatorname{int} \tilde{P}(t, z(t_2), h(t, x(t_2)))$ we have (using (3.14)) $|z(t_2)| \leqslant |x_0|$ and for the case $t_2 = t_0$ we have $|z(t_2)| = |z_0|$. Thus we obtain

$$\begin{aligned}&|(z(t), x(t))| \\ &\quad \leqslant 2q(t)a'(|x_0|) + |x_0| \\ &\qquad + a_1^{-1}\left(\exp\left(\int_{t_0}^{t} |w(\tau, x(\tau))|\,d\tau\right)\right. \\ &\qquad \times \big(a_2(\tilde{\beta}(t)|z_0| + \tilde{\beta}(t)|x_0| + \tilde{\beta}(t)\tilde{q}(t)a'(|x_0|)) \\ &\qquad \left.+ M + \int_{t_0}^{t} |w(\tau, x(\tau))|\,d\tau\big)\right).\end{aligned} \tag{3.16}$$

Notice that estimate (3.16) holds also if $(z(t), x(t)) \notin \operatorname{int} \tilde{P}(t, z(t), h(t, x(t)))$. We conclude that estimate (3.16) holds for all $t \in [t_0, t_1)$ and consequently it holds for all $t \geqslant t_0$. Thus system (3.13) satisfies $\sup\{|(z(t_0+s), x(t_0+s))|;\ |x_0| \leqslant r, t_0 \in [0, T], s \in [0, T]\} < +\infty$, for every $T \geqslant 0$, $r \geqslant 0$ and consequently by virtue of Lemma 2.3 in [8] system (3.13) is RFC. Moreover by (3.14)we have that $(z(t), x(t)) \in \tilde{P}(t, z(t), h(t, x(t)))$ for all $t \geqslant t_0 + |x_0|$ and thus the set-valued map $\tilde{P}(t, z, y)$ is a capturing region for (3.13). Since the set-valued map $Q(t) := \{x \in \Re^n : w(t, x) \leqslant 0\}$ is also a capturing region for (1.2), this implies that the set-valued map

$$\begin{aligned}\bar{P}(t, z, y) := \{z\} \times \{x \in Q(t) : |x| \leqslant{} & a(t, y) \\ & + \mu(t)a'(t+1) + \mu(t) \\ & \times a'(|z|)\} \subseteq \Re^n \times \Re^n\end{aligned} \tag{3.17}$$

is a capturing region for (3.13) and that the following inequality holds:

$$\begin{aligned}&\dot{V}|_{(3.13)} \leqslant -\rho(V(t, z, x)) + 6\phi(t), \\ &\quad \forall (t, z, y) \in \Re^+ \times \Re^n \times S, \\ &\qquad (z, x) \in \tilde{h}^{-1}(t, z, y) \cap \bar{P}(t, z, y).\end{aligned}$$

Exploiting the result of Corollary 2.11, we conclude that Problem (2) is globally strongly solvable. □

Remark 3.6. It should be emphasized that the construction of the observer in the proof of implication (ii)⇒(i) relies heavily on estimate (3.10) for the solutions of (1.2) and therefore on the assumption that system (1.2) is RFC. Moreover, although the proof of implication (ii)⇒(i) of Theorem 3.5 is constructive, it cannot be used for observer design purposes since it involves partition of unity arguments (it is the result of Lemma 2.10 that is used for the observer construction).

The following example shows that an autonomous system may not admit a time-invariant State Independent Error Lyapunov Function in the sense of [17], but it may admit a time-varying State Independent OLF in the sense of Definition 3.4 and consequently a time-varying global identity observer.

Example 3.7. We consider two-dimensional systems of the form

$$\begin{aligned}&\dot{x}_1 = f(x_1) + x_2, \quad \dot{x}_2 = g(x_1, x_2), \\ &y = x_1, (x_1, x_2) \in \Re^2,\end{aligned} \tag{3.18}$$

where f, g are locally Lipschitz mappings with $f(0)=g(0,0)=0$. In [17] it is shown that there exists a time-invariant "State Independent" Error Lyapunov Function, if and only if there exists a constant $M>0$ such that

$$(z-x)(g(y,z)-g(y,x))\leqslant M|z-x|^2,\quad \forall (y,z,x)\in\Re^3. \tag{3.19}$$

Here we suppose that system (3.18) is forward complete and that there exists a function $L(\cdot)\in C^0(\Re;\Re)$, such that the following inequality is satisfied:

$$(z-x)(g(y,z)-g(y,x))\leqslant L(y)|z-x|^2,\quad \forall (y,z,x)\in\Re^3. \tag{3.20}$$

Notice that, since (3.18) is RFC, there exist $\mu(\cdot)\in K^+$ and $a\in K_\infty$ such that the unique solution $(x_1(t),x_2(t))$ of system (3.18) initiated from $x_0\in\Re^2$ at time $t_0\geqslant 0$ satisfies

$$|(x_1(t),x_2(t))|\leqslant \mu(t)a(|x_0|),\quad \forall t\geqslant t_0.$$

Moreover, we can find a function $a'\in K_\infty$ and a constant $R>0$ such that

$$2L(x_1)\leqslant R+a'(|x_1|),\quad \forall x_1\in\Re. \tag{3.21}$$

The above inequalities imply that the set-valued map

$$Q(t):=\{(x_1,x_2)\in\Re^2 : 2L(x_1)-M(t)\leqslant 0\}, \tag{3.22a}$$

where $M(t)$ is any C^1 function that satisfies

$$M(t)\geqslant 1+R+a'(\mu(t)a(1+t)),\quad \forall t\geqslant 0 \tag{3.22b}$$

is a capturing region for (3.18). Particularly, by virtue of (3.21) and (3.22a,b), we obtain that $(x_1(t),x_2(t))\in Q(t)$ for all $t\geqslant t_0+|x_0|$. Next define

$$V(t,e_1,e_2):=\beta(t)e_1^2+(e_2-M(t)e_1)^2, \tag{3.23}$$

where $\beta(t):=4M^2(t)+1$, $e_1:=z_1-x_1$, $e_2:=z_2-x_2$. Notice that this particular selection for $\beta(t)$ guarantees that

$$\tfrac{1}{2}(e_1^2+e_2^2)\leqslant V(t,e_1,e_2)\leqslant 2(8M^2(t)+1)(e_1^2+e_2^2).$$

We introduce the observer:

$$\dot z_1=f(y)+z_2+u_1;$$
$$\dot z_2=g(y,z_2-M(t)(z_1-y))+u_2 \tag{3.24a}$$

$$u_1=-\frac{3}{2}M(t)e_1-\frac{\dot\beta(t)}{2\beta(t)}e_1;$$
$$u_2=M(t)u_1-\beta(t)e_1+M^2(t)e_1+\dot M(t)e_1 \tag{3.24b}$$

and evaluate the derivative

$$\begin{aligned}\dot V &:= \frac{\partial V}{\partial t}(t,e_1,e_2)+\frac{\partial V}{\partial e_1}(t,e_1,e_2)(e_2+u_1)\\ &\quad+\frac{\partial V}{\partial e_2}(t,e_1,e_2)(g(x_1,x_2+e_2-M(t)e_1)\\ &\quad-g(x_1,x_2)+u_2)\end{aligned}$$

along the trajectories of system (3.18) with (3.24). We obtain

$$\begin{aligned}\dot V &\leqslant -M(t)V(t,e_1,e_2)\\ &\quad+\max\{2L(x_1)-M(t),0\}V(t,e_1,e_2).\end{aligned} \tag{3.25}$$

Since the set-valued map $Q(t)$ as defined by (3.22) is a capturing region, we conclude from (3.25) that (3.24) is a global identity observer for (3.18) and the function V defined in (3.23) is a state independent OLF.

4. Conclusions

We have considered the Robust Output Feedback Stabilization problem for time-varying systems we have given necessary and sufficient conditions for the solution of this problem (Theorems 2.6 and 2.9). The results were applied to the problem of the Global Identity Observer Design for forward complete systems without inputs. We have showed that the existence of an Observer Lyapunov Function is a necessary and sufficient condition for the solvability of this problem (Theorem 3.5).

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