Available online at www.sciencedirect.com
SCIENCE DIRECT
Systems & Control Letters 54 (2005) 939–951
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Applications of non-uniform in time robust global asymptotic output stability to robust partial state feedback stabilization

Iasson Karafyllis*

Division of Mathematics, Department of Economics, University of Athens, 8 Pesmazoglou Str., 10559 Athens, Greece

Received 24 January 2003; received in revised form 14 October 2003; accepted 12 January 2005
Available online 17 March 2005

Abstract

The problem of adding an integrator is considered for time-varying control systems. Sufficient conditions for the solution of this problem are given, which are weaker than the corresponding conditions given in the literature. To this end, the notion of non-uniform in time robust global asymptotic output stability (RGAOS) is used. Applications to problems of partial state feedback global stabilization are considered.

Keywords: Output stability; Lyapunov functions; Time-varying systems; Adding an integrator

1. Introduction

The problems of adding an integrator are considered in many papers (see for instance the "classical" papers [1,7,22] as well as [17,21] and the references therein) and the literature about this subject is vast. In this paper we give sufficient conditions for the solution of these problems that are weaker than the corresponding conditions given in the literature (Theorems 2.2 and 2.6). To this end, we use the notion of non-uniform in time robust global asymptotic output stability (RGAOS) and its Lyapunov characterizations, given in [10]. Particularly, we consider the problems of "adding an integrator" for the system

$$\dot{x} = F(t, \theta, x, y), \tag{1.1a}$$

$$\dot{y} = f(t, \theta, x, y) + g(t, \theta, x, y)u, \quad x \in \Re^n, \quad y \in \Re, \quad t \geqslant 0, \quad u \in \Re, \quad \theta \in \Omega \tag{1.1b}$$

and we assume that $\Omega \subset \Re^m$ is a compact set and that the dynamics F, f, g are C^0 on $\Re^+ \times \Omega \times \Re^n \times \Re$ and locally Lipschitz with respect to (x, y), uniformly in $\theta \in \Omega$, with $F(t, \theta, 0, 0) = 0$, $f(t, \theta, 0, 0) = 0$ for all $(t, \theta) \in \Re^+ \times \Omega$. There exist two different problems of "adding an integrator", exactly as in the time-invariant case, which roughly speaking can be stated as follows:

1st Problem (output feedback problem): Suppose that there exists a stabilizing feedback law $y = k(t, x)$ for system (1.1a). Is there a stabilizing feedback law of the form $u = \tilde{k}(t, k(t, x), y)$ for system (1.1a,b)?

* Tel.: +30 2109942608; fax: +30 2109942611.
E-mail address: ikarafil@econ.uoa.gr.

0167-6911/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2005.02.003

2nd Problem (state feedback problem): Suppose that there exists a stabilizing feedback law $y = k(t, x)$ for system (1.1a). Is there a stabilizing feedback law $u = \tilde{k}(t, x, y)$ for system (1.1a,b)?

In this paper we develop two different results that provide solutions for the above problems (Theorems 2.2 and 2.6, respectively). The proofs of the presented results are constructive. However, since we consider time-varying uncertain systems, the proofs are more technical than in the autonomous case without uncertainties.

Results that provide solutions to the above problems have been used as tools for the application of backstepping procedures in triangular systems (see for instance [3]). However, there is a clear difference between the two problems, since the 1st problem requires a feedback law that depends only on the values of the "virtual" feedback stabilizer of subsystem (1.1a) and the state y of the one-dimensional subsystem (1.1b), while the 2nd problem requires a feedback law that depends on the whole state of the composite system (1.1a,b). This is exactly the reason that theorems provide solutions to the 1st problem are useful for the construction of "partial state" feedback stabilizers, while theorems that provide solutions to the 2nd problem lead to state feedback laws (see for instance [1,3,7,21,22] and the references therein). Moreover, it is clear that solvability of the 1st problem implies the solvability of the 2nd problem, but the converse is not true in general.

The solution of the problem of "adding an integrator" by means of output feedback that is presented in this paper is combined with recent results given in the literature concerning non-uniform in time global asymptotic stability (see [8–14]) in order to find sufficient conditions for the partial state feedback global stabilization of the system

$$
\begin{aligned}
&\dot{z} = f(z, x),\\
&\dot{x}_i = x_{i+1}, \quad i = 1, \ldots, n-1,\\
&\dot{x}_n = a(z, x) + b(z, x)u,\\
&x = (x_1, \ldots, x_n) \in \Re^n, \quad z \in \Re^l, \quad u \in \Re,
\end{aligned}
\tag{1.2}
$$

where the mappings f, a, b are locally Lipschitz with respect to (z, x), with $f(0, 0) = 0$ and $a(0, 0) = 0$. System (1.2) is important, because under mild conditions a general affine control system, can take the form (1.2) after an appropriate change of coordinates (see [4]). The problem of the partial state feedback stabilization of system (1.2) has attracted the interest of current research (see [4–6,15,19] and the references therein).

Notations. Throughout this paper we adopt the following notations:

- By M_D we denote the set of all measurable functions from $\Re^+$ to D, where $D \subset \Re^m$ is a given compact set.
- By $C^j(A)$ ($C^j(A; \Omega)$), where $j \geqslant 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
- For $x \in \Re^n$, x' denotes its transpose and $|x|$ its usual Euclidean norm.
- $L^\infty_{\text{loc}}(A)$ denotes the set of all measurable functions $u : A \to \Re^m$ that are essentially bounded on any non-empty compact subset of A.
- By $B[x, r]$, where $x \in \Re^n$ and $r \geqslant 0$, we denote the closed sphere in $\Re^n$ of radius r, centered at $x \in \Re^n$.
- K^+ denotes the class of positive C^0 functions $\phi : \Re^+ \to (0, +\infty)$. K^* denotes the class of non-decreasing C^∞ functions $\phi : \Re^+ \to [1, +\infty)$ with $\lim_{t\to+\infty} [\dot{\phi}(t)/\phi^r(t)] = 0$ for some constant $r \geqslant 1$ (see [8]). For the definitions of classes K, K_∞ see [16]. By KL we denote the set of all continuous functions $\sigma = \sigma(s, t) : \Re^+ \times \Re^+ \to \Re^+$ with the properties: (i) for each $t \geqslant 0$ the mapping $\sigma(\cdot, t)$ is of class K; (ii) for each $s \geqslant 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s, t) = 0$.
- We say that the function $H : \Re^+ \times \Re^n \to \Re^k$, is locally Lipschitz with respect to $x \in \Re^n$ if for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geqslant 0$ such that
$$
\begin{aligned}
&|H(t, x) - H(t, y)| \leqslant L|x - y|\\
&\quad \forall t \in I \ \ \forall (x, y) \in S \times S.
\end{aligned}
$$
- We say that the mapping $f : \Re^+ \times \Omega \times \Re^n \to \Re^k$ where $\Omega \subset \Re^m$ is a compact set, is locally Lipschitz with respect to $x \in \Re^n$, uniformly in $\theta \in \Omega$, if for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geqslant 0$ such that
$$
\begin{aligned}
&|f(t, \theta, x) - f(t, \theta, y)| \leqslant L|x - y|\\
&\quad \forall t \in I \ \ \forall (x, y) \in S \times S \ \ \forall \theta \in \Omega.
\end{aligned}
$$

Notice that for the dynamical system $\dot{x} = f(t, \theta, x)$, $x \in \Re^n$, $\theta \in \Omega$, the assumption that the dynamics of $f(\cdot)$ are continuous everywhere and locally Lipschitz with respect to $x \in \Re^n$, uniformly in $\theta \in \Omega$, with $f(\cdot, 0) = 0$, guarantees that the system has a unique local Caratheodory solution (see [2]) for every $\theta(\cdot) \in M_\Omega$.

2. Adding an integrator

In this section we present the solutions of the two different problems of adding an integrator for the time-varying case (1.1). We first give the definitions of the notions of robust forward completeness (RFC) and RGAOS, introduced in [10]. By virtue of Lemmas 2.3 and 3.3 in [10], the reader can immediately verify that the simplified definitions given here are equivalent to $\varepsilon - \delta$ definitions given in [10]. Consider the system

$$\begin{aligned} &\dot{x} = f(t, d, x),\\ &Y = H(t, x),\\ &x \in \Re^n, \quad t \geqslant 0, \quad d \in D, \end{aligned} \tag{2.1}$$

where $H \in C^0(\Re^+ \times \Re^n; \Re^k)$, $f \in C^0(\Re^+ \times D \times \Re^n; \Re^n)$ is a locally Lipschitz vector field with respect to $x \in \Re^n$, uniformly with respect to $d \in D$, with $f(t, d, 0) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \Re^+ \times D$.

Definition 2.1. We say that (2.1) is Robustly Forward Complete (RFC) if there exist functions $q \in K^+$ and $a \in K_\infty$ such that for every $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ the solution $x(t)$ of (2.1) with initial condition $x(t_0) = x_0$ and corresponding to input $d \in M_D$ exists for all $t \geqslant t_0$ and satisfies

$$|x(t)| \leqslant q(t) a(|x_0|) \quad \forall t \geqslant t_0. \tag{2.2}$$

We say that (2.1) is non-uniformly in time Robustly Globally Asymptotically Output Stable (RGAOS) if system (2.1) is RFC and there exist functions $\beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re^+ \times \Re^n \times M_D$ the solution $x(t)$ of (2.1) with initial condition $x(t_0) = x_0$ and corresponding to input $d \in M_D$ exists for all $t \geqslant t_0$ and satisfies

$$|H(t, x(t))| \leqslant \sigma(\beta(t_0)|x_0|, t - t_0) \quad \forall t \geqslant t_0. \tag{2.3}$$

We remind the readers that the notion of non-uniform in time Robust Global Asymptotic Stability (RGAS) is given in [13] and is actually equivalent to the notion of RGAOS given above when $H(t, x) := x$ (i.e. when the output is the whole state vector). Our first result is the following theorem, which guarantees the existence of a smooth time-varying output feedback stabilizer, under appropriate hypotheses.

Theorem 2.2. *Suppose that*:

(B1) *There exists a C^1 function $k : \Re^+ \times \Re^n \to \Re$, with $k(t, 0) = 0$ for all $t \geqslant 0$ a C^0 function $\gamma(t, s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\gamma(t, \cdot) \in K_\infty$ for each $t \geqslant 0$, such that the following system is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for the following system with input $(\theta, d) \in D := \Omega \times [-1, 1]$:*

$$\begin{aligned} &\dot{x} = F(t, \theta, x, k(t, x) + \mathrm{d}\gamma(t, |x|)),\\ &Y = k(t, x). \end{aligned} \tag{2.4}$$

(B2) *There exists a function φ of class K^+, such that*

$$\begin{aligned} &\varphi(t) g(t, \theta, x, y) \geqslant 1\\ &\quad \forall (t, \theta, x, y) \in \Re^+ \times \Omega \times \Re^n \times \Re. \end{aligned} \tag{2.5}$$

Then for every C^0 function $\bar{\gamma}(t, s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\bar{\gamma}(t, \cdot) \in K_\infty$ for each $t \geqslant 0$, there exists a C^∞ function $\tilde{k} : \Re^+ \times \Re \to \Re$, with $\tilde{k}(t, 0) = 0$ for all $t \geqslant 0$, such that zero is non-uniformly in time RGAS for the following system with input $(\theta, d) \in D := \Omega \times [-1, 1]$:

$$\begin{aligned} &\dot{x} = F(t, \theta, x, y),\\ &\dot{y} = f(t, \theta, x, y) + g(t, \theta, x, y) \quad (\tilde{k}(t, y - k(t, x))\\ &\qquad + d\bar{\gamma}(t, |(x, y)|)). \end{aligned} \tag{2.6}$$

Remark 2.3. Notice that when k is independent of t (time-independent feedback), then RGAS for (2.1) implies also RGAOS. However, this is no longer true for the case of time-varying feedback. We also notice that when hypothesis B1 holds with $k \equiv 0$ then Theorem 2.2 shows that the non-uniform in time ISS property can be propagated through an integrator via smooth output feedback that depends only on y, without any assumption concerning the dynamics or the type of convergence (compare with the corresponding results in [7,22] for the autonomous case).

The proof of Theorem 2.2 relies on the following lemma, which provides minimal Lyapunov-like requirements for robust global asymptotic stability of zero for a time-varying uncertain system. Its proof is provided in the Appendix.

Lemma 2.4. *Consider system* (2.1) *and suppose that there exist functions* $W \in C^1(\Re^+ \times \Re^n; \Re^+)$, $a \in K_\infty$, $p \in K^+ \cap C^1(\Re^+)$ *with* $\lim_{t\to+\infty} p(t) = 0$, $\dot{p}(t) \leqslant 0$ *for all* $t \geqslant 0$ *and a constant* $0 < c < 1$ *such that the following properties hold*:

$$a(|x|) \leqslant W(t,x) \quad \forall (t,x) \in \Re^+ \times \Re^n \tag{2.7}$$

$$\text{if } W(t,x) \geqslant cp(t) \text{ then } \left.\frac{\mathrm{d}}{\mathrm{d}t} W(t,x)\right|_{(2.1)} := \frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x) f(t,d,x) \leqslant \frac{\dot{p}(t)}{cp(t)} W(t,x) \quad \forall d \in D. \tag{2.8}$$

Then zero is non-uniformly in time RGAS for (2.1).

Proof of Theorem 2.2. Let a C^0 function $\bar{\gamma}(t,s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\bar{\gamma}(t,\cdot) \in K_\infty$ for each $t \geqslant 0$. The proof is devoted to the construction of functions $W \in C^1(\Re^+ \times \Re^n \times \Re; \Re^+)$, $a \in K_\infty$ and $\tilde{k} \in C^\infty(\Re^+ \times \Re; \Re)$ with $\tilde{k}(t,0) = 0$ for all $t \geqslant 0$, such that the following properties hold:

$$a(|(x,y)|) \leqslant W(t,x,y) \quad \forall (t,x,y) \in \Re^+ \times \Re^n \times \Re. \tag{2.9a}$$

$$\text{If } W(t,x,y) \geqslant \frac{1}{2} p(t) \text{ then } \left.\frac{\mathrm{d}}{\mathrm{d}t} W(t,x,y)\right|_{(2.6)} \leqslant -\frac{1}{2} W(t,x,y) + \exp(-t) \quad \forall (\theta,d) \in D := \Omega \times [-1,1], \tag{2.9b}$$

where

$$p(t) := \exp\left(-\frac{t}{4}\right)\left(11 - \frac{8}{3}\exp\left(-\frac{3t}{4}\right)\right). \tag{2.9c}$$

Clearly, it will then follow by Lemma 2.4, that zero is non-uniformly in time RGAS for (2.6).

By virtue of Proposition 3.5 in [10], assumption B1 guarantees the existence of a function $U(\cdot) \in C^\infty(\Re^+ \times \Re^n; \Re^+)$ and functions $a_1(\cdot), a_2(\cdot) \in K_\infty$, $\beta(\cdot) \in K^+$ such that

$$a_1(|(x,k(t,x))|) \leqslant U(t,x) \leqslant a_2(\beta(t)|x|) \quad \forall (t,x) \in \Re^+ \times \Re^n, \tag{2.10a}$$

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial x}(t,x) F(t,\theta,x,k(t,x) + \mathrm{d}\gamma(t,|x|)) \leqslant -U(t,x) \quad \forall (t,x,\theta,d) \in \Re^+ \times \Re^n \times \Omega \times [-1,1]. \tag{2.10b}$$

By virtue of Fact V in [14], Corollary 10 and Remark 11 in [18], there exists a function $\sigma(\cdot)$ of class K_∞ and a function $r(\cdot)$ of class $K^+ \cap C^\infty(\Re^+)$ such that the following inequality holds for all $(t,x,z) \in \Re^+ \times \Re^n \times \Re$:

$$\sup_{\theta\in\Omega}\left\{|g(t,\theta,x,k(t,x)+z)\bar{\gamma}(t,|(x,k(t,x)+z)|)| + \left|f(t,\theta,x,k(t,x)+z) - \frac{\partial k}{\partial t}(t,x) - \frac{\partial k}{\partial x}(t,x)F(t,\theta,x,k(t,x)+z)\right|\right\} \leqslant r(t)(\sigma(|x|) + \sigma(|z|)). \tag{2.11}$$

Clearly, we can find a function $\delta(\cdot)$ of class $K_\infty \cap C^1(\Re^+)$, with $\mathrm{d}\delta/\mathrm{d}s(s)$ being non-decreasing, such that:

$$\delta(a_1(s)) \geqslant 2(s+1)(\sigma(s))^2 \quad \forall s \geqslant 1, \tag{2.12a}$$

$$\delta(s) \leqslant \frac{\mathrm{d}\delta}{\mathrm{d}s}(s)s \quad \forall s \geqslant 0, \tag{2.12b}$$

where a_1 is the function involved in (2.10a) (for example, we may select $\delta(s) := [2/a_1(1)]\int_0^{2s}(a_1^{-1}(\xi)+1)(\sigma(a_1^{-1}(\xi)))^2\,\mathrm{d}\xi$ and notice that since $(\mathrm{d}\delta/\mathrm{d}s)(s)$ is increasing, inequality (2.12b) is automatically satisfied). Let $m : (0,+\infty) \to (0,+\infty)$ be the C^0 non-increasing function, defined as

$$m(t) := \max_{t \leqslant s} \frac{(\sigma(s))^2}{\delta(a_1(s))}. \tag{2.13}$$

We notice that since (2.12a) holds, the right-hand side of (2.13) never exceeds 1 for $t \geqslant 1$. Let θ_1 be a function of class $K^+ \cap C^\infty(\Re^+)$ that satisfies $\theta_1(t) \geqslant 1$, for all $t \geqslant 0$ as well as

$$\delta\left(a_2\left(\frac{\beta(t)}{\theta_1(t)}\right)\right) + \frac{1}{2\theta_1^2(t)} \leqslant \frac{1}{4} p(t) \quad \forall t \geqslant 0, \tag{2.14a}$$

where a_2, β are the functions involved in (2.10a) and p is defined by (2.9c). Let θ_2 be a function of class $K^+ \cap C^\infty(\Re^+)$ that satisfies:

$$m\left(\frac{1}{\theta_1(t)}\right) \leqslant \theta_2(t) \quad \forall t \geqslant 0. \tag{2.14b}$$

By virtue of Fact V in [14], Corollary 10 and Remark 11 in [18], there exists a function $a_3(\cdot)$ of class K_∞ and a function $q(\cdot)$ of class $K^+ \cap C^\infty(\mathfrak{R}^+)$ such that:

$$\sup\left\{\delta'(U(t,x))\left|\frac{\partial U}{\partial x}(t,x)\right||F(t,\theta,x,k(t,x)+z) - F(t,\theta,x,k(t,x))|, \theta\in\Omega, |z|\leqslant s, \gamma(t,|x|)\leqslant s\right\} \leqslant q(t)a_3(s), \tag{2.15a}$$

where $\delta'(\cdot)$ denotes the continuous derivative of δ. Without loss of generality, we may also assume that

$$a_3(s)\geqslant s\sigma(s) \quad \forall s\geqslant 0, \tag{2.15b}$$

$$q(t)\geqslant r(t) \quad \forall t\geqslant 0, \tag{2.15c}$$

where σ, r are the functions involved in (2.11). Then inequalities (2.10b), (2.12b) and (2.15a) imply the following inequality for all $(t,x,y,\theta)\in\mathfrak{R}^+\times\mathfrak{R}^n\times\mathfrak{R}\times\Omega$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\,\delta(U(t,x))\right|_{(1.1a)} \leqslant -\delta(U(t,x)) + q(t)a_3(|y-k(t,x)|). \tag{2.16}$$

Define

$$W(t,x,y) := \delta(U(t,x)) + \tfrac{1}{2}\,(y-k(t,x))^2 \tag{2.17}$$

and let $a(s) := \min\{\delta(a_1(\frac{1}{2}s)), \frac{1}{8}s^2\}$ (obviously, is a class K_∞ function). Clearly, by virtue of the left-hand side inequality (2.10a) and definition (2.17), we obtain that

$$\begin{aligned} W(t,x,y) &\geqslant \delta(a_1(|(x,k(t,x))|)) + \tfrac{1}{2}\,(y-k(t,x))^2 \\ &\geqslant a(2|(x,k(t,x))|) + a(2|y-k(t,x)|) \\ &\geqslant a(|(x,k(t,x))| + |y-k(t,x)|) \end{aligned}$$

which directly implies (2.9a). Moreover, let $\tilde{a}_3:\mathfrak{R}\to\mathfrak{R}$ a C^∞ strictly increasing function that satisfies: $z\tilde{a}_3(z)\geqslant a_3(|z|)$, for all $|z|\geqslant 1$, $z\tilde{a}_3(z)\geqslant 0$ for all $z\in\mathfrak{R}$ and $\tilde{a}_3(0)=0$. Let also ϕ a function of class $K^+\cap C^\infty(\mathfrak{R}^+)$ that satisfies: $\phi(t)a_3^{-1}(\exp(-t)/4q(t))\geqslant 1$ and $\phi(t)\geqslant 1$, for all $t\geqslant 0$.We next define the feedback function

$$\begin{aligned}\tilde{k}(t,z) := &-\varphi(t)((1+\tfrac{1}{2}\,q^2(t)\theta_2(t))z \\ &+ 3q(t)\phi(t)\tilde{a}_3(\phi(t)z)),\end{aligned} \tag{2.18}$$

where φ is the function of class K^+ that satisfies (2.5). Without loss of generality we may assume that φ is of class $K^+\cap C^\infty(\mathfrak{R}^+)$ and consequently $\tilde{k}\in C^\infty(\mathfrak{R}^+\times\mathfrak{R};\mathfrak{R})$ with $\tilde{k}(t,0)=0$ for all $t\geqslant 0$.

For the rest of proof we adopt the notation $z=y-k(t,x)$. Using inequalities (2.5), (2.11), (2.15b,c), (2.16) and definitions (2.17), (2.18), we obtain for all $(t,x,y,\theta,d)\in\mathfrak{R}^+\times\mathfrak{R}^n\times\mathfrak{R}\times\Omega\times[-1,1]$:

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\,W(t,x,y)\right|_{(2.6)} \leqslant &-W(t,x,y) - \tfrac{1}{2}\,q^2(t)\theta_2(t)|z|^2 \\ &+ 2q(t)(a_3(|z|) - \phi(t)z\tilde{a}_3(\phi(t)z)) \\ &+ q(t)(|z|\sigma(|x|) - \phi(t)z\tilde{a}_3(\phi(t)z)).\end{aligned} \tag{2.19}$$

When $\phi(t)|z|\geqslant 1$ we have, by virtue of the assumed properties for $\tilde{a}_3$ and (2.19), that

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\,W(t,x,y)\right|_{(2.6)} \leqslant &-W(t,x,y) - \tfrac{1}{2}\,q^2(t)\theta_2(t)|z|^2 \\ &+ q(t)(|z|\sigma(|x|) - \phi(t)z\tilde{a}_3(\phi(t)z)).\end{aligned} \tag{2.20a}$$

When $|z|\leqslant 1/\phi(t)$, by (2.19) and due to the fact that $1/\phi(t)\leqslant a_3^{-1}(\exp(-t)/4q(t))$ it follows that:

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\,W(t,x,y)\right|_{(2.6)} \leqslant &-W(t,x,y) - \tfrac{1}{2}\,q^2(t)\theta_2(t)|z|^2 \\ &+ q(t)(|z|\sigma(|x|) - \phi(t)z\tilde{a}_3(\phi(t)z)) \\ &+ \tfrac{1}{2}\,\exp(-t).\end{aligned} \tag{2.20b}$$

It follows that (2.20b) holds for all $(t,x,y,\theta,d)\in\mathfrak{R}^+\times\mathfrak{R}^n\times\mathfrak{R}\times\Omega\times[-1,1]$. If $|z|\geqslant|x|$, it follows by (2.15b) that

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\,W(t,x,y)\right|_{(2.6)} \leqslant &-W(t,x,y) - \tfrac{1}{2}\,q^2(t)\theta_2(t)|z|^2 \\ &+ q(t)(a_3(|z|) - \phi(t)z\tilde{a}_3(\phi(t)z)) + \tfrac{1}{2}\,\exp(-t)\end{aligned}$$

and similarly as above (distinguishing the cases $\phi(t)|z|\geqslant 1$ and $|z|\leqslant 1/\phi(t)\leqslant a_3^{-1}(\exp(-t)/4q(t))$), we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\,W(t,x,y)\right|_{(2.6)} \leqslant -W(t,x,y) - \tfrac{1}{2}\,q^2(t)\theta_2(t)|z|^2 + \exp(-t). \tag{2.20c}$$

If $|x| \geqslant 1/\theta_1(t)$ then by virtue of definition (2.13) and inequality (2.14b) we obtain

$$(\sigma(|x|))^2 \leqslant m\left(\frac{1}{\theta_1(t)}\right)\delta(a_1(|x|)) \leqslant \theta_2(t)\delta(a_1(|x|)).$$

Using the elementary inequality $q(t)|z|\sigma(|x|) \leqslant [1/(2\theta_2(t))](\sigma(|x|))^2 + \frac{1}{2}q^2(t)\theta_2(t)|z|^2$, the left-hand side inequality (2.10a) and definition (2.17), we get

$$|x| \geqslant \frac{1}{\theta_1(t)} \Rightarrow q(t)|z|\sigma(|x|) \leqslant \tfrac{1}{2}W(t,x,y) + \tfrac{1}{2}q^2(t)\theta_2(t)|z|^2.$$

By virtue of (2.20b) we obtain for the case $|x| \geqslant 1/\theta_1(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t,x,y)\bigg|_{(2.6)} \leqslant -\tfrac{1}{2}W(t,x,y) + \exp(-t). \tag{2.21}$$

Thus we may conclude that (2.21) holds if $|z| \geqslant |x|$ or $|x| \geqslant 1/\theta_1(t)$. Finally, we consider the case: $|z| \leqslant |x| \leqslant 1/\theta_1(t)$. Then by virtue of the right-hand side inequality (2.10a), inequality (2.14a) and definition (2.17) it follows that $W(t,x,y) \leqslant \frac{1}{4}p(t) < \frac{1}{2}p(t)$. Clearly, this implies that if $W(t,x,y) \geqslant \frac{1}{2}p(t)$ then we must have $|z| \geqslant |x|$ or $|x| \geqslant 1/[\theta_1(t)]$ and consequently (2.9b) holds. The proof is complete. □

Notice that Theorem 2.2 is an existence result. Although the proof of Theorem 2.2 is based on a constructive strategy, it cannot be used for the design of the required output feedback law, since it requires knowledge of a Lyapunov function for system (2.4), which satisfies the differential inequality (2.10b). Such a Lyapunov function is guaranteed to exist but is, in general, difficult to find in practice. However, as the following example shows, Lyapunov functions that satisfy less demanding differential inequalities can be used for design purposes.

Example 2.5. The origin of the following system:

$$\begin{aligned}&\dot{x} = -x^3 + y,\\&\dot{y} = x + u,\\&(x,y) \in \Re^2, \quad u \in \Re\end{aligned} \tag{2.22}$$

cannot be made GAS with the use of *output feedback* $u = k(y)$ for any C^1 function $k(\cdot)$ (since the linearization of the closed-loop system has one eigenvalue with positive real part). Notice that the following subsystem is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for the following system with input $d \in D := [-1,1]$:

$$\begin{aligned}&\dot{x} = -x^3 + \mathrm{d}\gamma(t,|x|),\\&Y = 0\end{aligned} \tag{2.23}$$

with $\gamma(t,s) := \frac{1}{2}s^3$. Therefore by virtue of Theorem 2.2 there exists a C^∞ time-varying feedback of the form $k(t,y)$ with $k(t,0) = 0$ for all $t \geqslant 0$, such that zero for the following system:

$$\begin{aligned}&\dot{x} = -x^3 + y,\\&\dot{y} = x + k(t,y) + d|(x,y)|,\\&(x,y) \in \Re^2, \quad d \in [-1,1]\end{aligned} \tag{2.24}$$

is (non-uniformly in time) RGAS. Consider the Lyapunov function

$$V(t,x,y) := \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2. \tag{2.25}$$

Clearly, using the elementary Young inequalities $3|x||y| \leqslant \frac{1}{4}x^4 + \frac{27}{4}|y|^{4/3}$, $\frac{27}{4}|y|^{4/3} \leqslant \frac{9}{4}\exp(3t)y^4 + \frac{9}{2}\exp(-3t)$, we have for all $(t,x,y,d) \in \Re^+ \times \Re \times \Re \times [-1,1]$:

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,x,y) + \frac{\partial V}{\partial x}(t,x,y)(-x^3+y)\\&\quad + \frac{\partial V}{\partial y}(t,x,y)(x + u + d|(x,y)|)\\&\qquad = -x^4 + 2xy + \mathrm{d}y|(x,y)| + yu\\&\qquad \leqslant -x^4 + 3|x||y| + y^2 + yu\\&\qquad \leqslant -\tfrac{1}{2}x^4 + \tfrac{9}{4}\exp(3t)y^4\\&\qquad\quad + y^2 + yu + 5\exp(-3t).\end{aligned}$$

The latter inequality shows that the smooth output feedback law:

$$u = k(t,y) := -\tfrac{9}{4}\exp(3t)y^3 - y - \tfrac{1}{2}y^3$$

guarantees that the following inequality holds for all $(t,x,y,d) \in \Re^+ \times \Re \times \Re \times [-1,1]$:

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,x,y) + \frac{\partial V}{\partial x}(t,x,y)(-x^3+y)\\&\quad + \frac{\partial V}{\partial y}(t,x,y)(x + k(t,y) + d|(x,y)|)\\&\qquad \leqslant -V^2(t,x,y) + 5\exp(-3t).\end{aligned}$$

The latter inequality in conjunction with Theorem 3.1 in [13] guarantees that zero is RGAS for (2.24).

We may relax assumptions B1 and B2 of Theorem 2.2, by making use of state time-varying feedback. This possibility is further exploited by the following theorem. Notice that the regularity requirements imposed for the original feedback are minimal and thus generalize the non-smooth "adding an integrator" results given in [17,21].

Theorem 2.6. *Suppose that*:

(B3) *There exists a* C^0 *function* $k : \Re^+ \times \Re^n \to \Re$, *with* $k(t,0)=0$ *for all* $t \geqslant 0$, *such that the mapping* $F(t,\theta,x,k(t,x))$ *is locally Lipschitz with respect to* $x \in \Re^n$, *uniformly in* $\theta \in \Omega$, *such that the following system is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for the following system with input* $\theta \in \Omega$:

$$\dot{x} = F(t,\theta,x,k(t,x)),\qquad Y = k(t,x). \tag{2.26}$$

(B4) *There exists a* C^∞ *function* $\varphi : \Re^+ \times \Re^n \times \Re \to \Re$, *such that*

$$\varphi(t,x,y)g(t,\theta,x,y) \geqslant 1 \quad \forall (t,\theta,x,y) \in \Re^+ \times \Omega \times \Re^n \times \Re. \tag{2.27}$$

Then for every C^0 *function* $\bar{\gamma}(t,s) : \Re^+ \times \Re^+ \to \Re^+$, *which is locally Lipschitz with respect to* s, *with* $\bar{\gamma}(t,\cdot) \in K_\infty$ *for each* $t \geqslant 0$, *there exists a* C^∞ *function* $\tilde{k} : \Re^+ \times \Re^n \times \Re \to \Re$, *with* $\tilde{k}(t,0)=0$ *for all* $t \geqslant 0$, *such that zero is non-uniformly in time RGAS for the following system with input* $(\theta,d) \in D := \Omega \times [-1,1]$:

$$\begin{aligned}\dot{x} &= F(t,\theta,x,y),\\ \dot{y} &= f(t,\theta,x,y) + g(t,\theta,x,y)(\tilde{k}(t,x,y) + d\bar{\gamma}(t,|(x,y)|)).\end{aligned} \tag{2.28}$$

The proof of Theorem 2.6 is based on the following technical lemma. It shows that without loss of generality we may assume that the function $k : \Re^+ \times \Re^n \to \Re$ involved in hypothesis B3 is of class $C^\infty(\Re^+ \times \Re^n; \Re)$.

Lemma 2.7. *Suppose that hypothesis* B3 *holds. Then there exists a function* $\tilde{k} \in C^\infty(\Re^+ \times \Re^n; \Re)$, *with* $\tilde{k}(t,0)=0$ *for all* $t \geqslant 0$, *such that the following system is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for the following system with input* $\theta \in \Omega$:

$$\dot{x} = F(t,\theta,x,\tilde{k}(t,x)),\qquad Y = \tilde{k}(t,x). \tag{2.29}$$

Proof. By virtue of Proposition 3.5 in [10], assumption B3 guarantees the existence of a function $U(\cdot) \in C^\infty(\Re^+ \times \Re^n; \Re^+)$ and functions $a_1(\cdot), a_2(\cdot) \in K_\infty$, $\beta(\cdot) \in K^+$ such that

$$a_1(|(x,k(t,x))|) \leqslant U(t,x) \leqslant a_2(\beta(t)|x|) \quad \forall (t,x) \in \Re^+ \times \Re^n, \tag{2.30a}$$

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial x}(t,x)F(t,\theta,x,k(t,x)) \leqslant -U(t,x), \quad \forall (t,x,\theta) \in \Re^+ \times \Re^n \times \Omega. \tag{2.30b}$$

Following exactly the same procedure with the proof of Lemma 2.7 in [14], we can prove that there exist functions $a_3(\cdot), a_4(\cdot)$ of class K_∞ and $\kappa(\cdot)$ of class K^+, such that

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial x}(t,x)F(t,\theta,x,k(t,x)+v) \leqslant -U(t,x) + \exp(-2t)a_3(|x|)a_4(\kappa(t)|v|) \quad \forall (t,\theta,x,v) \in \Re^+ \times \Omega \times \Re^n \times \Re. \tag{2.30c}$$

Since $k(\cdot)$ is continuous with $k(t,0)=0$ for all $t \geqslant 0$, by virtue of Fact V in [14], there exists $\tilde{\zeta} \in K_\infty$ and $p \in K^+$ such that

$$|k(t,x)| \leqslant \tilde{\zeta}(p(t)|x|) \quad \forall (t,x) \in \Re^+ \times \Re^n$$

which implies that

$$|k(t,x)| \leqslant \tilde{\zeta}(\sqrt{|x|}) \quad \forall t \geqslant 0 \quad \text{and} \quad |x| \leqslant \frac{1}{p^2(t)}.$$

Define the C^0 function $\omega : \Re^+ \times \Re^+ \to \Re^+$:

$$\omega(t,s) := \min\left(2\tilde{\zeta}(\sqrt{s}), \frac{1}{\kappa(t)}a_4^{-1}\left(\frac{1}{1+a_3(s)}\right)\right) \tag{2.31}$$

which clearly satisfies

$$\omega(t,s) := 2\tilde{\zeta}(\sqrt{s}) \ \ \forall t \geqslant 0 \quad \text{and}$$
$$s \leqslant \min\left(1, \left(\tilde{\zeta}^{-1}\left(\frac{R}{2\kappa(t)}\right)\right)^2\right)$$
$$\text{where } R := a_4^{-1}\left(\frac{1}{1+a_3(1)}\right).$$

Consequently, there exists a function $\tilde{k} \in C^\infty(\Re^+ \times \Re^n; \Re)$, such that

$$|\tilde{k}(t,x) - k(t,x)| \leqslant \omega(t,|x|) \leqslant 2\tilde{\zeta}(\sqrt{|x|}) \quad \forall (t,x) \in \Re^+ \times \Re^n, \tag{2.32a}$$

$$\tilde{k}(t,x) = 0 \ \ \forall t \geqslant 0 \quad \text{and} \quad |x| \leqslant \frac{1}{2}\min\left(1, \frac{1}{p^2(t)}, \left(\tilde{\zeta}^{-1}\left(\frac{R}{2\kappa(t)}\right)\right)^2\right). \tag{2.32b}$$

Notice that by virtue of (2.32a) and the following elementary inequalities:

$$a_1(|(x,k(t,x))|) \geqslant \tfrac{1}{2}a_1(|x|) + \tfrac{1}{2}a_1(|k(t,x)|);$$
$$\delta(2|k(t,x)|) + \delta(4\tilde{\zeta}(\sqrt{|x|})) \geqslant \delta(|\tilde{k}(t,x)|),$$

$$\delta(|\tilde{k}(t,x)|) + \delta(|x|) \geqslant \delta(\tfrac{1}{2}|(x,\tilde{k}(t,x))|),$$

where

$$\delta(s) := \frac{1}{4}\min\left(a_1\left(\frac{s}{2}\right), a_1\left(\left(\tilde{\zeta}^{-1}\left(\frac{s}{4}\right)\right)^2\right)\right)$$

is a function of class K_∞, we obtain

$$\tilde{a}_1(|(x,\tilde{k}(t,x))|) \leqslant U(t,x) \leqslant a_2(\beta(t)|x|), \tag{2.33a}$$

where $\tilde{a}_1(s) := \delta(s/2)$, is a function of class K_∞. Moreover, inequalities (2.30c), (2.32a) and definition (2.31) imply that:

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial x}(t,x)F(t,\theta,x,\tilde{k}(t,x)) \leqslant -U(t,x) + \exp(-2t) \quad \forall (t,x,\theta) \in \Re^+ \times \Re^n \times \Omega. \tag{2.33b}$$

The rest of proof is an immediate consequence of inequalities (2.33a,b) and Proposition 3.6 in [10]. □

We are now in a position to provide the proof of Theorem 2.6.

Proof of Theorem 2.6. Let a C^0 function $\bar{\gamma}(t,s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\bar{\gamma}(t,\cdot) \in K_\infty$ for each $t \geqslant 0$. By virtue of Proposition 3.5 in [10], assumption B3 guarantees the existence of a function $U(\cdot) \in C^\infty(\Re^+ \times \Re^n; \Re^+)$ and functions $a_1(\cdot), a_2(\cdot) \in K_\infty, \beta(\cdot) \in K^+$ such that inequalities (2.30 a,b) hold. By virtue of Fact V in [14], Corollary 10 and Remark 11 in [18], there exists a function $\sigma(\cdot)$ of class K_∞ and a function $r(\cdot)$ of class $K^+ \cap C^\infty(\Re^+)$ such that (2.11) holds for all $(t,x,z) \in \Re^+ \times \Re^n \times \Re$. Due to the Lipschitz assumption for the dynamics, it follows that there exists a C^0 function $L : \Re^+ \times \Re^+ \to \Re^+$ such that the mappings $L(\cdot,t), L(t,\cdot)$ are non-decreasing for all $t \geqslant 0$ and such that

$$\left|\frac{\partial U}{\partial x}(t,x)\right| |F(t,\theta,x,k(t,x)+z) - F(t,\theta,x,k(t,x))| \leqslant |z|L(t,|x|+|z|) \quad \forall (t,x,\theta,z) \in \Re^+ \times \Re^n \times \Omega \times \Re. \tag{2.34}$$

Define

$$W(t,x,y) := U(t,x) + \tfrac{1}{2}(y - k(t,x))^2 \tag{2.35}$$

and let

$$a(s) := \min\{a_1(\tfrac{1}{2}s), \tfrac{1}{8}s^2\}$$

(obviously is a class K_∞ function). Clearly, by virtue of (2.30a) we have

$$\begin{aligned} W(t,x,y) &\geqslant a_1(|(x,k(t,x))|) + \tfrac{1}{2}(y-k(t,x))^2 \\ &\geqslant a(2|(x,k(t,x))|) + a(2|y-k(t,x)|) \\ &\geqslant a(|(x,k(t,x))| + |y-k(t,x)|) \end{aligned}$$

which implies

$$a(|(x,y)|) \leqslant W(t,x,y). \tag{2.36}$$

For the rest of proof we adopt the notation $z = y - k(t,x)$. Using inequalities (2.11), (2.30b), (2.34) and definition (2.35), we obtain for all $(t,x,y,\theta,d) \in \Re^+ \times \Re^n \times \Re \times \Omega \times [-1,1]$:

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}W(t,x,y)\right|_{(1.1),u=v+\mathrm{d}\bar{\gamma}(t,|(x,y|)} &\leqslant -W(t,x,y) + \tfrac{1}{2}z^2 + zg(t,\theta,x,y)v \\ &\quad + |z|L(t,|x|+|z|) + r(t)|z|(\sigma(|x|) \\ &\quad + \sigma(|z|)). \end{aligned} \tag{2.37}$$

Inequalities (2.27) and (2.37) enable us to construct, using standard partition of unity arguments (see

[13,20]), a C^∞ function $\tilde{k} : \Re^+ \times \Re^n \times \Re \to \Re$, with $\tilde{k}(t,0)=0$ for all $t \geqslant 0$, such that

$$\frac{d}{dt} W(t,x,y)\bigg|_{(2.28)} \leqslant -W(t,x,y) + \exp(-t)$$
$$\forall (t,x,y,\theta,d) \in \Re^+ \times \Re^n \times \Re \times \Omega \times [-1,1]. \tag{2.38}$$

The proof is complete, since inequalities (2.36) and (2.38) in conjunction with Theorem 3.1 in [13] guarantee that zero is RGAS for (2.28). □

Remark 2.8. Conditions B1 and B2 (B3 and B4) do not in general guarantee that the feedback stabilizer $\tilde{k}(\cdot)$ satisfies the same property B1 (B3) imposed for the original feedback $k(\cdot)$. This case arises only for time-varying feedback and especially when the feedback map $\tilde{k}(t, x(t), y(t))$ evaluated along the solution of the closed-loop system (2.6) ((2.28)) does not converge to zero as $t \to +\infty$. This is a drawback for the achievement of robust feedback stabilization for higher dimensional triangular time-varying systems by applying backstepping design. Therefore, some additional conditions should be imposed, concerning the dynamics of system (1.1), the rate of convergence of the solution of (2.4) ((2.26)) to zero and the original feedback map $k(\cdot)$ in order to propagate B1 (B3) to the new feedback map $\tilde{k}(\cdot)$. This possibility is exploited in [8], where the notion of ϕ-robust global asymptotic stability (ϕ-RGAS) is introduced.

3. Robust partial state feedback stabilization of autonomous control systems

In this section we consider the problem of robust partial state feedback stabilization of autonomous control systems. The notions of non-uniform in time ϕ-RGAS introduced in [8] as well as the notion of non-uniform in time RGAOS introduced in [10] are used extensively.

Consider the system

$$\begin{aligned}
&\dot{z} = f_0(z,x),\\
&\dot{x}_i = f_i(\theta, x_1,\dots,x_i) + g_i(\theta, x_1,\dots,x_i)x_{i+1},\\
&\qquad i = 1,\dots,n-1,\\
&\dot{x}_n = f_n(\theta,z,x) + g_n(\theta,z,x)u,\\
&x = (x_1,\dots,x_n) \in \Re^n,\ z \in \Re^l,\ u \in \Re,\ \theta \in \Omega,
\end{aligned} \tag{3.1}$$

where $\Omega \subset \Re^p$ is a compact set, the mappings f_0, f_i, g_i $(i=1,\dots,n)$ are continuous and locally Lipschitz with respect to (z,x), uniformly in $\theta \in \Omega$, with $f_i(\theta,0,\dots,0)=0$ $(i=1,\dots,n-1)$, $f_0(0,0)=0$ and $f_n(\theta,0,0)=0$ for all $\theta \in \Omega$. We make the following assumptions:

(H1) *There exists a constant $c>0$ such that for all $(z,x,\theta) \in \Re^l \times \Re^n \times \Omega$ it holds that*:

$$c \leqslant g_i(\theta, x_1,\dots,x_i), \quad i=1,\dots,n-1, \tag{3.2a}$$

$$c \leqslant g_n(\theta,z,x). \tag{3.2b}$$

(H2) $0 \in \Re^l$ *is GAS for the system*: $\dot{z} = f_0(z,0)$.

(H3) *The subsystem $\dot{z} = f_0(z,x)$ is forward complete with x as input.*

Under the above hypotheses we can prove the following theorem.

Theorem 3.1. *Consider system* (3.1) *and suppose that hypotheses* H1, H2 *and* H3 *are fulfilled. Then for every C^0 function $\bar{\gamma}(t,s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\bar{\gamma}(t,\cdot) \in K_\infty$ for each $t \geqslant 0$, there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$, with $k(\cdot,0)=0$, such that $0 \in \Re^l \times \Re^n$ is RGAS for the system* (3.1) *with $u = k(t,x) + d\bar{\gamma}(t,|(z,x)|)$ and input $(\theta,d) \in D := \Omega \times [-1,1]$.*

Proof. By virtue of Theorem 2.2 it suffices to prove that there exists a C^∞ function $\bar{k} : \Re^+ \times \Re^{n-1} \to \Re$, with $\bar{k}(t,0)=0$ for all $t \geqslant 0$, and a C^0 function $\gamma(t,s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\gamma(t,\cdot) \in K_\infty$ for each $t \geqslant 0$, such that the following system is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for the following system, with input $(\theta,d) \in D := \Omega \times [-1,1]$:

$$\begin{aligned}
&\dot{z} = f_0(z,x),\\
&\dot{x}_i = f_i(\theta,x_1,\dots,x_i) + g_i(\theta,x_1,\dots,x_i)x_{i+1},\\
&\qquad i=1,\dots,n-1,\\
&x_n = \bar{k}(t,x_1,\dots,x_{n-1}) + d\gamma(t,|(z,x_1,\dots,x_{n-1})|),\\
&Y = \bar{k}(t,x_1,\dots,x_{n-1}).
\end{aligned} \tag{3.3}$$

For the rest of proof we denote $\xi = (x_1,\dots,x_{n-1}) \in \Re^{n-1}$. The proof is divided in two parts.

Part I: *Construction of partial state feedback*. By virtue of Hypotheses H2 and H3 and Proposition 3.7

in [14], the subsystem $\dot{z}=f_0(z,x)$ is non-uniformly in time ISS with x as input and there exist functions $\sigma \in KL$, $\rho \in K_\infty$, $\bar{\phi} \in K^+$ such that the unique solution of $\dot{z}=f_0(z,x)$ initiated from $z_0 \in \Re^l$ at time $t_0 \geqslant 0$ and corresponding to input $x(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0,+\infty))$, satisfies the estimate:

$$|z(t)| \leqslant \sigma(|z_0|, t-t_0) + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(\rho(\bar{\phi}(\tau)|x(\tau)|), t-\tau) \quad \forall t \geqslant t_0. \tag{3.4}$$

Lemma 2.2 in [8] guarantees the existence of a function $\phi \in K^*$ (see the Notations for the definition of the class K^*) such that $\bar{\phi}(t) \leqslant \phi(t)$, for all $t \geqslant 0$. Furthermore, Corollary 5.1 in [8], asserts that for every $\Gamma \in K_\infty$ being locally Lipschitz, there exists a C^∞ mapping $\bar{k}: \Re^+ \times \Re^{n-1} \to \Re$, with $\bar{k}(\cdot, 0)=0$, a function $\eta \in K_\infty$ and a constant $p \geqslant 0$, with

$$\phi^r(t)|\bar{k}(t,\xi)| \leqslant \eta(\phi^{p+r}(t)|\xi|) \quad \forall (t,\xi) \in \Re^+ \times \Re^{n-1}, \quad r \geqslant 0 \tag{3.5}$$

such that $0 \in \Re^{n-1}$ is ϕ-RGAS for the following system with $(\theta, d) \in D := \Omega \times [-1,1]$ as input:

$$\begin{aligned} \dot{x}_i &= f_i(\theta, x_1, \ldots, x_i) + g_i(\theta, x_1, \ldots, x_i)x_{i+1}, \\ &\quad i = 1, \ldots, n-1, \\ x_n &= \bar{k}(t,\xi) + \mathrm{d}\Gamma(|\xi|). \end{aligned} \tag{3.6}$$

Part II: *Stability analysis for the closed-loop system*. By virtue of Lemma 2.4 in [8] and the fact that $0 \in \Re^{n-1}$ is ϕ-RGAS for (3.6), we obtain that for every $p \geqslant 0$, there exist functions $\tilde{\sigma} \in KL$ and $\beta \in K^+$ such that the following estimate holds for the solution of (3.6):

$$|\xi(t)| \leqslant \frac{1}{\phi^p(t)} \tilde{\sigma}(\beta(t_0)|\xi_0|, t-t_0) \quad \forall t \geqslant t_0 \ \ \forall (\theta, d) \in M_D. \tag{3.7}$$

Estimation (3.7) in conjunction with (3.5) and the fact that $\phi(t) \geqslant 1$ for all $t \geqslant 0$, implies that the following system is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS, with input $(\theta, d) \in D := \Omega \times [-1,1]$:

$$\begin{aligned} \dot{x}_i &= f_i(\theta, x_1, \ldots, x_i) + g_i(\theta, x_1, \ldots, x_i)x_{i+1}, \\ &\quad i = 1, \ldots, n-1, \\ x_n &= \bar{k}(t,\xi) + \mathrm{d}\Gamma(|\xi|), \\ Y &= \phi(t)|(\xi, \bar{k}(t,\xi))|. \end{aligned} \tag{3.8}$$

We define: $f(t,\theta,\xi,v) = (f_1(\theta,x_1)+g(\theta,x_1)x_2, \ldots, f_{n-1}(\theta,\xi)+g_{n-1}(\theta,\xi)\bar{k}(t,\xi)+g_{n-1}(\theta,\xi)v)$, where $v \in \Re$. Since system (3.8) is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for (3.8), by virtue of Proposition 3.5 in [10] and the fact that $\phi(t) \geqslant 1$ for all $t \geqslant 0$, there exists a function $U(\cdot) \in C^\infty(\Re^+ \times \Re^{n-1}; \Re^+)$ and functions $a_1(\cdot), a_2(\cdot) \in K_\infty$, $\bar{\beta}(\cdot) \in K^+$ such that

$$a_1(\phi(t)|(\xi, \bar{k}(t,\xi))|) \leqslant U(t,\xi) \leqslant a_2(\bar{\beta}(t)|\xi|) \quad \forall (t,\xi) \in \Re^+ \times \Re^{n-1}, \tag{3.9a}$$

$$|v| \leqslant \Gamma(|\xi|) \Rightarrow \frac{\partial U}{\partial t}(t,\xi) + \frac{\partial U}{\partial \xi}(t,\xi) f(t,\theta,\xi,v) \leqslant -U(t,\xi) \quad \forall (t,\theta,\xi) \in \Re^+ \times \Omega \times \Re^{n-1}. \tag{3.9b}$$

Define $a(t,s) := \sup\{|(\partial U/\partial x)(t,\xi)(f(t,\theta,\xi,v) - f(t,\theta,\xi,0))|;\ |v| \leqslant s, \theta \in \Omega, \Gamma(|\xi|) \leqslant s\}$. Clearly, this function is continuous with $a(t,0)=0$ and consequently, by Fact V in [14] there exist functions $a_3(\cdot) \in K_\infty$ and $\mu(\cdot) \in K^+$ such that: $a(t,s) \leqslant a_3(\mu(t)s)$, for all $t, s \geqslant 0$. Thus by virtue of (3.9b) (and by distinguishing the cases $|v| \leqslant \Gamma(|\xi|)$ and $\Gamma(|\xi|) \leqslant |v|$), we obtain

$$\frac{\partial U}{\partial t}(t,\xi) + \frac{\partial U}{\partial \xi}(t,\xi) f(t,\theta,\xi,v) \leqslant -U(t,\xi) + a_3(\mu(t)|v|) \quad \forall (t,\theta,\xi,v) \in \Re^+ \times \Omega \times \Re^{n-1} \times \Re. \tag{3.9c}$$

Inequalities (3.9a) and (3.9c), imply that the following estimate holds for the solution of $\dot{\xi} = f(t,\theta,\xi,v)$ initiated from $\xi_0 \in \Re^{n-1}$ at time $t_0 \geqslant 0$ and corresponding to input $(\theta(\cdot), v(\cdot)) \in M_\Omega \times \mathrm{L}^\infty_{\mathrm{loc}}([t_0,+\infty))$:

$$\phi(t)|(\xi(t), \bar{k}(t,\xi(t)))| \leqslant \bar{\sigma}(\bar{\beta}(t_0)|\xi_0|, t-t_0) + \sup_{t_0 \leqslant \tau \leqslant t} a_1^{-1}(2a_3(\mu(\tau)|v(\tau)|)) \quad \forall t \geqslant t_0, \tag{3.10}$$

where $\bar{\sigma}(s,t) := a_1^{-1}(2\exp(-t)a_2(s))$.

Consider the system

$$\begin{aligned} \dot{z} &= f_0(z, \xi, \bar{k}(t,\xi)+v), \\ \dot{\xi} &= f(t,\theta,\xi,v), \\ z &\in \Re^l, \quad \xi \in \Re^{n-1}, \quad v \in \Re, \quad \theta \in \Omega. \end{aligned} \tag{3.11}$$

Using estimate (3.4) in conjunction with estimate (3.10) for $v \equiv 0$ and the fact $\bar{\phi}(t) \leqslant \phi(t)$ for all $t \geqslant 0$, we obtain

$$|z(t)| \leqslant \sigma(|z_0|, t-t_0) + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(\rho(\bar{\sigma}(\bar{\beta}(t_0)|\xi_0|, \tau - t_0)),\ t-\tau) \quad \forall t \geqslant t_0. \tag{3.12}$$

By virtue of Fact VI in [14], there exists a KL function $R : \Re^+ \times \Re^+ \to \Re^+$ such that

$$\sup_{t_0 \leqslant \tau \leqslant t} \sigma(\rho(\bar{\sigma}(s, \tau - t_0)), t-\tau) \leqslant R(s, t-t_0) \quad \forall t \geqslant t_0.$$

The latter inequality in conjunction with estimate (3.12) for the case $v \equiv 0$ implies that $0 \in \Re^l \times \Re^{n-1}$ is RGAS for system (3.11) with $v \equiv 0$. Moreover, using estimates (3.4) and (3.10) and the fact $\bar{\phi}(t) \leqslant \phi(t)$, for all $t \geqslant 0$, we obtain that for all $t \geqslant t_0$ and $(\theta(\cdot), v(\cdot)) \in M_\Omega \times \mathsf{L}^\infty_{\mathrm{loc}}([t_0, +\infty))$ the solution of (3.11) satisfies

$$\begin{aligned} |z(t)| &\leqslant \sigma(|z_0|, 0) + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(\rho(2\phi(\tau)|(\xi(\tau), \bar{k}(t, \xi(\tau)))|) + \rho(2\phi(\tau)|v(\tau)|), 0) \\ &\leqslant \sigma(|z_0|, 0) + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(2\rho(2\phi(\tau)|(\xi(\tau), \bar{k}(t, \xi(\tau)))|), 0) + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(2\rho(2\phi(\tau)|v(\tau)|), 0) \\ &\leqslant \sigma(|z_0|, 0) + \sigma(4\rho(4\bar{\sigma}(\bar{\beta}(t_0)|\xi_0|, 0)), 0) \\ &\quad + \sup_{t_0 \leqslant \tau \leqslant t} \sigma\left(4\rho\left(4 \sup_{t_0 \leqslant s \leqslant \tau} a_1^{-1} \times (2a_3\ (\mu(s)|v(s)|))\right), 0\right) \\ &\quad + \sup_{t_0 \leqslant \tau \leqslant t} \sigma(2\rho(2\phi(\tau)|v(\tau)|), 0). \end{aligned}$$

The latter inequality combined with estimate (3.10) and the fact that $\phi(t) \geqslant 1$, for all $t \geqslant 0$, implies that

$$|(z(t), \xi(t), \bar{k}(t, \xi(t)))| \tag{3.13}$$

$$\leqslant a(\tilde{\beta}(t_0)|(z_0, \xi_0)|) + \sup_{t_0 \leqslant \tau \leqslant t} \zeta(\delta(\tau)|v(\tau)|) \ \ \forall t \geqslant t_0. \tag{3.13}$$

for certain functions $a, \zeta \in K_\infty$ and $\tilde{\beta}, \delta \in K^+$ and for all $(\theta(\cdot), v(\cdot)) \in M_\Omega \times \mathsf{L}^\infty_{\mathrm{loc}}([t_0, +))$. We complete the proof by noticing that since estimates (3.10) and (3.13) hold for the solutions of system (3.11) and $0 \in \Re^l \times \Re^{n-1}$ is RGAS for system (3.12) with $v \equiv 0$, it follows by virtue of Proposition 4.2 in [10] that there exists a C^0 function $\gamma(t, s) : \Re^+ \times \Re^+ \to \Re^+$, which is locally Lipschitz with respect to s, with $\gamma(t, \cdot) \in K_\infty$ for each $t \geqslant 0$, such that system (3.3) is non-uniformly in time RGAOS and zero is non-uniformly in time RGAS for (3.3). The proof is complete. □

Remark 3.2. (i) Notice that in the proof of Theorem 3.1 the result of Theorem 2.2 is explicitly used. If Theorem 2.6 was used instead of Theorem 2.2, we would have obtained a feedback law that depends on the whole state (z, x) of system (3.1). This application clearly shows that the two different solutions of the problems of "adding an integrator" given in the previous section lead to different feedback forms, as already remarked in the Introduction.

(ii) Notice that Theorem 3.1 is an existence result. Its proof cannot be used for the design of the required partial state feedback law, since it involves the existence result of Theorem 2.2.

We are now in a position to address the problem of partial state feedback stabilization for (1.2). The proof of the following corollary is an immediate consequence of Theorem 3.1 and will be omitted.

Corollary 3.3. *Consider system* (1.2), *where the mappings f, a, b are locally Lipschitz with respect to* (z, x), *with* $f(0,0)=0$ *and* $a(0,0)=0$. *We make the following assumptions*:

(A1) *There exists a* C^∞ *mapping* $r : \Re^n \to (0, +\infty)$, *such that for all* $(z, x) \in \Re^l \times \Re^n$ *it holds that*: $1 \leqslant b(z, x)r(x)$.

(A2) $0 \in \Re^l$ *is GAS for the system*: $\dot{z} = f(z, 0)$.

(A3) *The subsystem* $\dot{z} = f(z, x)$ *is forward complete with x as input*.

Then for every C^0 *function* $\bar{\gamma}(t, s) : \Re^+ \times \Re^+ \to \Re^+$, *which is locally Lipschitz with respect to s, with* $\bar{\gamma}(t, \cdot) \in K_\infty$ *for each* $t \geqslant 0$, *there exists a* C^∞ *mapping* $k : \Re^+ \times \Re^n \to \Re$, *with* $k(\cdot, 0) = 0$, *such that system* (1.2) *with* $u = k(t, x) + d\bar{\gamma}(t, |(z, x)|)$ *is* (*non-uniformly in time*) *RGAS with input* $d \in [-1, 1]$.

System (1.2) is important because under mild conditions a general affine control system, can take the form (1.2) after an appropriate change of coordinates (see [4]). It should be emphasized that if $a(z,0)=0$ for all $z \in \mathfrak{R}^l$, then A2 is also a necessary condition for the stabilization of (1.2) by means of a locally Lipschitz mapping $k:\mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}$, with $k(\cdot,0)=0$.

4. Conclusions

We have considered two different problems of "adding an integrator" for time-varying systems and we have given sufficient conditions for the solution of these problems that are weaker of the corresponding conditions given in the literature (Theorems 2.2 and 2.6). To this end, we have used the notion of non-uniform in time Robust Global Asymptotic Output Stability (RGAOS). Applications to problems of partial state feedback global stabilization were given.

Appendix

Proof of Lemma 2.4. First notice that by virtue of (2.8) we obtain the following property:

$$\text{if } W(t,x) \geqslant cp(t) \text{ then } \left.\frac{\mathrm{d}}{\mathrm{d}t} W(t,x)\right|_{(2.1)} \leqslant \dot{p}(t) \quad \forall d \in D. \tag{A.1}$$

Next we claim that the set $L_t := \{x \in \mathfrak{R}^n : W(t,x) \leqslant p(t)\}$ is positively invariant for (2.1). To prove this suppose (the contrary) that there exists some initial condition $x_0 \in L_{t_0}$, some time $t_1 > t_0$ and an input $d(\cdot) \in M_D$ such that $W(t_1, x(t_1)) > p(t_1)$. Let

$$T := \max\{t;\, t_0 \leqslant t \leqslant t_1,\, W(t,x(t)) \leqslant p(t)\}$$

and since $W(t,x(t))$ is continuous with respect to t, we obtain $W(T,x(T))=p(T)$ and $W(t,x(t)) \geqslant p(t) > cp(t)$ for all $t \in [T,t_1]$. Since the solution of (2.1) $x(t)$ is absolutely continuous with respect to t, it follows that $W(t,x(t))$ is also absolutely continuous with respect to t. By virtue of (A.1), this implies:

$$W(t_1,x(t_1)) = W(T,x(T)) + \int_T^{t_1} \left.\frac{\mathrm{d}}{\mathrm{d}t} W(\tau,x(\tau))\right|_{(2.1)} \mathrm{d}\tau \leqslant p(t_1)$$

which obviously is a contradiction.

Let $(t_0,x_0,d) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times M_D$ be arbitrary and consider the solution $x(t)$ of (2.1) with initial condition $x(t_0)=x_0$ corresponding to input $d(\cdot) \in M_D$. Let also $t > t_0$ be sufficiently small such that the solution exists. If $x(t) \notin L_t$, then positive invariance of L_t implies that: $x(\tau) \notin L_\tau$ (or equivalently $W(\tau,x(\tau)) > p(\tau) > cp(\tau)$) for all $\tau \in [t_0,t]$. Hence, the comparison principle in conjunction with (2.8) implies for this case: $W(t,x(t)) \leqslant (p(t)/p(t_0))^{1/c} W(t_0,x_0)$. Due to the fact that $p(t)$ is decreasing and $0<c<1$, we may conclude that if $x(t) \notin L_t$ then $W(t,x(t)) \leqslant [p(t)/p(t_0)] \times W(t_0,x_0)$. On the other hand, if $x(t) \in L_t$, then we obtain that: $W(t,x(t)) \leqslant p(t)$. Thus in any case and using (2.7) we are led to the following estimate for the solution of (2.1):

$$a(|x(t)|) \leqslant W(t,x(t)) \leqslant p(t)\left(1 + \frac{1}{p(t_0)} W(t_0,x_0)\right). \tag{A.2}$$

Clearly, (A.2) implies that the solution of (2.1) exists for all $t \geqslant t_0$ and that (2.1) is robustly forward complete. Moreover, since $\lim_{t\to+\infty} p(t)=0$, (A.2) implies that the output attractivity property is satisfied for (2.1) with output $H(t,x) := x$. Consequently, (A.2) in conjunction with Lemma 3.5 in [10] implies that zero is non-uniformly in time RGAS for (2.1). The proof is complete. □

References

[1] J.M. Coron, L. Praly, Adding an integrator for the stabilization problem, Systems Control Lett. 17 (1991) 89–104.

[2] A.V. Fillipov, Differential Equations with Discontinuous Right-Hand Sides, Kluwer Academic Publishers, Dordrecht, 1988.

[3] R.A. Freeman, P.V. Kokotovic, Robust Nonlinear Control Design, State-Space and Lyapunov Techniques, Birkhauser, Boston, 1996.

[4] A. Isidori, Nonlinear Control Systems, second ed., Springer, London, 1989.

[5] A. Isidori, Nonlinear Control Systems II, Springer, London, 1999.

[6] A. Isidori, A tool for semiglobal stabilization of uncertain non-minimum-phase nonlinear systems via output feedback, IEEE Trans. Automat. Control 45 (10) (2000) 1817–1827.

[7] Z.P. Jiang, A. Teel, L. Praly, Small-Gain theorem for ISS systems and applications, Math. Control Signals Systems 7 (1994) 95–120.

[8] I. Karafyllis, Non-uniform stabilization of control systems, IMA J. Math. Control Inform. 19 (4) (2002) 419–444.

[9] I. Karafyllis, Necessary and sufficient conditions for the existence of stabilizing feedback for control systems, IMA J. Math. Control Inform. 20 (1) (2003) 37–64.
[10] I. Karafyllis, Non-uniform in time robust global asymptotic output stability, Systems Control Lett. 54 (3) (2005) 181–193.
[11] I. Karafyllis, J. Tsinias, Global stabilization and asymptotic tracking for a class of nonlinear systems by means of time-varying feedback, Internat. J. Robust Nonlinear Control 13 (6) (2003) 559–588.
[12] I. Karafyllis, J. Tsinias, Non-uniform in time stabilization for linear systems and tracking control for nonholonomic systems in chained form, Internat. J. Control 76 (15) (2003) 1536–1546.
[13] I. Karafyllis, J. Tsinias, A converse Lyapunov theorem for non-uniform in time global asymptotic stability and its application to feedback stabilization, SIAM J. Control Optim. 42 (3) (2003) 936–965.
[14] I. Karafyllis, J. Tsinias, Non-uniform in time ISS and the small-gain theorem, IEEE Trans. Automat. Control 49 (2) (2004) 196–216.
[15] D. Liberzon, A.S. Morse, E.D. Sontag, Output–input stability and minimum-phase nonlinear systems, IEEE Trans. Automat. Control 47 (2002) 422–436.
[16] Y. Lin, E.D. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability, SIAM J. Control Optim. 34 (1996) 124–160.
[17] L. Rifford, On the existence of non-smooth control Lyapunov functions in the sense of generalized gradients, ESAIM Control Optim. Calc. Var. 6 (2001) 593–611.
[18] E.D. Sontag, Comments on integral variants of ISS, Systems Control Lett. 34 (1998) 93–100.
[19] A. Teel, L. Praly, Tools for semiglobal stabilization by partial state and output feedback, SIAM J. Control Optim. 33 (5) (1995) 1443–1488.
[20] J. Tsinias, Sufficient Lyapunov-like conditions for stabilization, Math. Control Signals Systems 2 (1989) 343–357.
[21] J. Tsinias, An extension of Artstein's theorem on stabilization by using ordinary feedback integrators, Systems Control Lett. 20 (1993) 141–148.
[22] J. Tsinias, Versions of Sontag's input to state stability condition and output feedback stabilization, J. Math. Systems Estimation Control 6 (1) (1996) 113–126.