ISS Property for Time-Varying Systems and Application to Partial-Static Feedback Stabilization and Asymptotic Tracking

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***Abstract*—A concept of input-to-state stability for time-varying control systems is introduced that constitutes extension of the well-known notion concerning the autonomous case. We use this concept to derive sufficient conditions for global stabilization of triangular systems by means of a time-varying smooth feedback and to achieve asymptotic tracking of unbounded signals.**

***Index Terms*— Asymptotic tracking, global feedback stabilization, input-to-state stability, time-varying systems.**

I. Introduction

Our goal is to provide a concept of input-to-state stability for time-varying control systems

$$\dot{x} = f(t, x, u), \qquad (x, u) \in \mathcal{R}^n \times \mathcal{R},\ t \geq 0 \tag{1.1}$$

and give sufficient conditions for global stabilization for a class of triangular systems by means of time-varying feedback $u = \varphi(t, x)$. The key concepts of our analysis are the notions of completeness, attractiveness, and asymptotic stability concerning the general case (1.1) under the restriction that each admissible input u satisfies $(x(t), u(t)) \in L_t$, where L_t is a time-varying subset of the space $\mathcal{R}^n \times \mathcal{R}$ and $x(t)$ denotes the trajectory of (1.1) that corresponds to u. These notions are extensions of those given in [1]–[7] for the autonomous case, and in Theorem 2.4 of the present paper are used to derive sufficient conditions for global stabilization for systems

$$\begin{aligned} &\dot{x} = f(t, x) + y_1 b(t) \\ &\dot{y}_i = y_{i+1} + g_i(t, x, y_1, \cdots, y_i), \qquad 1 \leq i \leq m \\ &u := y_{m+1}, \qquad (x; y_1, \cdots, y_m) \in \mathcal{R}^n \times \mathcal{R}^m \end{aligned} \tag{1.2}$$

with u as input, where only the y_i components of the whole state $(x, y_1, \cdots, y_m)$ are available. Theorem 2.4 can be extended for systems with unknown time-varying parameters, and is applied to achieve asymptotic tracking of a given unbounded trajectory. Particularly, in Section III, it is shown that, even for autonomous systems, the problem of asymptotic tracking of unbounded signals is reduced to feedback stabilization of a time-varying system under the assumptions of the main Theorem 2.4.

Notations and Facts: $|x|$ denotes the usual Euclidean norm of a vector $x \in \mathcal{R}^n$, and x' is its transpose. A function $a: \mathcal{R}^n \to \mathcal{R}^+$ is called positive definite if $a(x) > 0$ for $x \neq 0$ and $a(0) = 0$; a function $a: \mathcal{R}^+ \to \mathcal{R}^+$ is of class K if it is continuous (C^o), positive definite, and nondecreasing. By K_∞, we denote the subclass of K consisting of all $a \in K$ with $a(s) \to +\infty$ as $s \to +\infty$. We denote by Π the subclass of K_∞ consisting of all functions a having the polynomial form $a(s) = \sum_{i=0}^m a_i s^i$ for a certain integer m and constants $a_i \geq 0$ with $a_1 \neq 0$. The following facts are direct consequences of the previous definitions.

- If a is of class K, it holds that

$$a(s_1 + s_2) \leq a(2s_1) + a(2s_2), \qquad \forall\, s_1, s_2 \geq 0. \tag{1.3}$$

- If $a \in \Pi$, then for every $\theta \geq 1$, we have

$$a(\theta s) \leq \theta^m a(s), \qquad \forall\, s \geq 0 \tag{1.4}$$

for a certain integer m. Furthermore, there exists a pair of constants $c_2 > c_1 > 0$ such that

$$c_1 s \leq a(s) \leq c_2 s, \qquad s \text{ near zero.} \tag{1.5}$$

- For every $a_1, a_2 \in \Pi$, there exists a function a_3 of the same class with

$$a_1(s) \leq a_3(a_2(s)), \qquad \forall\, s \geq 0 \tag{1.6}$$

$$a_1, a_2 \in \Pi \Rightarrow a_1 + a_2,\ a_1 a_2 \in \Pi. \tag{1.7}$$

II. Main Results

A. Input-to-State Stability Properties

Consider the system (1.1), whose dynamics $f: \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^n$ are C^1 with $f(t, 0) = 0$ for all $t \geq 0$, and let us denote by $x(t, t_o, x_o, u)$ its solution at time t that corresponds to input u with initial condition $x(t_o, t_o, x_o, u) = x_o$. Consider a pair of continuous mappings $\gamma_1, \gamma_2: \mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}$ with $\gamma_1(t, x) \leq \gamma_2(t, x)$ for all $x \in \mathcal{R}^n$, $t \geq 0$ and $\gamma_i(t, 0) = 0$, $i = 1, 2$ for every $t \geq 0$. We define

$$L_t := \{(x, u) \in \mathcal{R}^n \times \mathcal{R}: \gamma_1(t, x) \leq u \leq \gamma_2(t, x)\}.$$

- We say that (1.1) is *complete with respect to* L_t (L_t *complete*) if, for every $t_o \geq 0, T > t_o, x_o \in \mathcal{R}^n$ and (essentially bounded measurable) input u such that the solution $x(t) = x(t, t_o, x_o, u)$ of (1.1) exists on $[t_o, T)$ and satisfies

$$(x(t), u(t)) \in L_t \tag{2.1}$$

for $t_o \leq t < T$, it follows that $\overline{\lim}_{t \to T} |x(t)| < +\infty$.
- We say that the origin $0 \in \mathcal{R}^n$ is *stable with respect to* L_t (L_t *stable*) if, for any integer $N \geq 0$, positive $T < +\infty$, and $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, T, N) > 0$ such that

$$t^N |x(t, t_o, x_o, u)| \leq \varepsilon, \quad \forall\, |x_o| \leq \delta,\ t_o \in [0, T),\ t \geq t_o \tag{2.2}$$

and input u for which $x(t)$ exists and satisfies (2.1). Particularly, if the property above holds for $T = +\infty$, zero is called L_t uniformly stable (L_t-US).
- The origin $0 \in \mathcal{R}^n$ is called a *global attractor with respect to* $L_t(L_t$-$GA)$ if (1.1) is L_t complete, and for any compact sphere S_R of radius R centered at $0 \in \mathcal{R}^n$ and for any integer $N \geq 0$, $\varepsilon > 0$, and $0 < T < +\infty$ there is a $\tau = \tau(R, \varepsilon, T, N) > 0$ such that

$$t^N |x(t, t_o, x_o, u)| \leq \varepsilon, \quad \forall x_o \in S_R, t_o \in [0, T), t \geq t_0 + \tau \tag{2.3}$$

and input u for which (2.1) holds for all $t \geq t_o$. In the case where this property holds with $T = +\infty$, we say that zero is a global uniform attractor (L_t-UGA).

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- Zero is called *globally asymptotically stable with respect to* $L_t(L_t\text{-}GAS)$ if it is L_t *stable* and $L_t\text{-}GA$; it is uniformly $L_t\text{-}GAS(L_t\text{-}UGAS)$ if, in addition, it is $L_t\text{-}US$ and $L_t\text{-}UGA$ (an analogous definition can be given for the concept of $L_t\text{-}local$ asymptotic stability).
- Finally, if M_t is a time-varying subset of $\mathcal{R}^n$, we say that M_t is L_t*-invariant,* if $x(t, t_o, x_o, u) \in M_t$ for every $t \geq t_o$ and input u such that (2.1) holds and for every initial (t_o, x_o) with $x_o \in M_{t_o}$.

Remark 2.1: i) Note that both (2.2) and (2.3) hold if zero is exponentially stable for (1.1) and those u for which (2.1) is satisfied. They are also fulfilled under the weaker hypothesis that there exists a function C: $\mathcal{R}^2 \to \mathcal{R}^+$ and positive constants a and l such that, for all $x_0 \in \mathcal{R}^n$, $t \geq t_0$, and u for which (2.1) holds, we have that $|x(t, t_0, x_0, u)| \leq C(t_0, |x_0|)t^a \exp(-lt)$. ii) If, in addition, we assume that the functions γ_i, $i = 1, 2$ satisfy $|\gamma_i(t, x)| \leq (1+t^\nu)r(|x|)$ for some $\nu \geq 0$ and $r \in K$, then as a consequence of the definitions of $L_t\text{-}AS$ given above, it follows that both inequalities (2.2) and (2.3) are satisfied if we replace $x(t, \cdot)$ by $(x(t, \cdot), u(t))$, provided that (2.1) holds. Particularly, the notions of L_t stability and L_t attractiveness are strengthened as follows:

- $\forall N \geq 0$, $0 < T < +\infty$, $\varepsilon > 0 \Rightarrow \exists \delta > 0$: $t^N|(x(t, t_0, x_0, u), u(t))| \leq \varepsilon, \forall |x_o| \leq \delta, t_o \in [0, T), t \geq t_o$
- $\forall N \geq 0, 0 < T < +\infty$, $\varepsilon > 0, R > 0 \Rightarrow \exists \tau > 0$: $t^N|(x(t, t_o, x_o, u), u(t))| \leq \varepsilon, \forall x_o \in S_R, t_o \in [0, T), t \geq t_0 + \tau$ for any u for which (2.1) holds.

A criterion for asymptotic stability used in the present work is given by the following proposition.

Proposition 2.2: Suppose that there exist a C^1 function V: $\mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}^+$ and positive constants a, c_1, c_2, and c_3 such that

$$c_1(1+t^a)^{-1}|x|^2 \leq V(t, x) \leq c_2|x|^2$$

$$\frac{d}{dt}V(t, x) := \frac{\partial V}{\partial x} f(t, x) + \frac{\partial V}{\partial t}(t, x) \leq -c_3|x|^2, \quad \forall t \geq 0, (x, u) \in L_t, x \text{ near zero.}$$

Then the origin is L_t *locally* AS; it is $L_t\text{-}GAS$ if these conditions hold for every $x \in \mathcal{R}^n$.

Proof: Suppose that the inequalities above hold for all $x \in \mathcal{R}^n$. It follows that

$$|x(t, t_o, x_o, u)|^2 \leq c_2 c_1^{-1}(1+t^a) \exp(-c_3 c_2^{-1}(t-t_o))|x_o|^2$$

for any $t \geq t_o$, x_o, and u such that $(x(t), u(t)) \in L_t$ for $t \geq t_o$. The statement is a direct consequence of the previous inequality. □

Remark: It should be noticed that the conditions of Proposition 2.2 do not, in general, imply L_t*-uniform* asymptotic stability.

B. Partial Static Feedback Stabilization

In this section, we derive sufficient conditions for global stabilization for triangular time-varying systems. The corresponding results extend those in [1] and [6]. Particularly, the methodology generalizes the design scheme proposed in [6] and [7], although, for the time-varying case, a more technical analysis is required. The following is the key technical result of our backstepping design.

Lemma 2.3: Consider the time-varying control system

$$\dot{x} = f(t, x) + yb(t) \tag{2.4a}$$

$$\dot{y} = u + g(t, x, y) \tag{2.4b}$$

where b: $\mathcal{R}^+ \to \mathcal{R}^n$, f: $\mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}^n$, and g: $\mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}$ are C^1 with $f(t, 0) = 0$ and $g(t, 0) = 0$ for all $t \geq 0$. Moreover, we assume the following.

A1) There exist C^1 mappings V, φ: $\mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}$ with $V(t, 0) = \varphi(t, 0) = 0$ for all $t \geq 0$, positive constants c_i, $1 \leq i \leq 5$ and an integer ν in such a way that

$$c_1(1+t^v)^{-1}|x|^2 \leq V(t, x) \leq c_2|x|^2 \tag{2.5a}$$

$$\left|\frac{\partial V}{\partial x}(t, x)\right| \leq c_3|x| \tag{2.5b}$$

$$\frac{d}{dt}V(t, x) := \frac{\partial V}{\partial x}(t, x)(f(t, x) + \varphi(t, x)b(t)) + \frac{\partial V}{\partial t}(t, x) \leq -c_4|x|^2 \tag{2.5c}$$

for every $t \geq 0$ and x (near zero) for which

$$|x| \leq c_5(1+t^\nu)^{-1}. \tag{2.5d}$$

A2) There exist functions $r_o, r_1 \in \Pi$ and a constant $C > 0$ such that

$$|b(t)| \leq C(1+t^\nu) \tag{2.6a}$$

$$|f(t, x)| + |\varphi(t, x)| + \left|\frac{\partial \varphi}{\partial t}(t, x)\right| \leq (1+t^\nu)r_o(|x|) \tag{2.6b}$$

$$\left|\frac{\partial \varphi}{\partial x}(t, x)\right| \leq (1+t^\nu)r_o(|x|+1) \tag{2.6c}$$

$$|g(t, x, y)| \leq (1+t^\nu)r_1(|(x', y)|) \tag{2.6d}$$

for every $t \geq 0$ and $(x, y) \in \mathcal{R}^n \times \mathcal{R}$.

A3) There exists a function $\gamma \in \Pi$ such that $0 \in \mathcal{R}^n$ is $L_t\text{-}GAS$ for the subsystem (2.4a) with y as input, where

$$L_t := \{(x, y): |y - \varphi(t, x)| \leq \gamma(|x|)\}. \tag{2.7}$$

Then there exist a C^1 map Ξ: $\mathcal{R}^+ \times \mathcal{R} \to \mathcal{R}$ and a linear function Γ: $\mathcal{R}^+ \to \mathcal{R}^+$ such that, if we define

$$\Phi(t, x, y) := \Xi(t, y - \varphi(t, x)) \tag{2.8}$$

$$\hat{L}_t := \{(x, y, u): |u - \Phi(t, x, y)| \leq \Gamma(|(x, y)|)\} \tag{2.9}$$

the origin $0 \in \mathcal{R}^{n+1}$ is $\hat{L}_t\text{-}GAS$ for the system (2.4) with u as input. Furthermore, there exist a C^1 function W: $\mathcal{R}^+ \times \mathcal{R}^{n+1} \to \mathcal{R}^+$, positive constants $\bar{c}_i$, $1 \leq i \leq 5$ and $\overline{C}$, a function $\bar{r}_o \in \Pi$, and an integer $\bar{\nu} \geq \nu$ such that (2.5) and (2.6a)–(2.6c) hold with W, Φ, $(f' + yb', g)'$, $(0, 1)'$, $\bar{c}_i$, and $\bar{\nu}$, instead of V, φ, f, b, c_i, and ν, respectively.

Proof: Notice, first, that from (1.3), (1.6), (1.7), (2.6b), and (2.6d), we get

$$\begin{aligned} |g(t, x, y)| &\leq (1+t^\nu)r_1(|(x, y)|) \\ &\leq (1+t^\nu)(r_1(4|x|) + r_1(4|\varphi(t, x)|) \\ &\quad + r_1(4|y - \varphi(t, x)|)) \\ &\leq (1+t^\nu)(r_1(4|x|) + r_1(4(1+t^\nu)r_o(|x|)) \\ &\quad + r_1(4|y - \varphi(t, x)|)) \\ &\leq (1+t^{\nu_1})(r_2(|x|) + r_3(|y - \varphi(t, x)|)), \\ &\qquad \forall t \geq 0, (x, y) \in \mathcal{R}^{n+1} \end{aligned} \tag{2.10}$$

for a certain integer $\nu_1 \geq \nu$ and functions $r_2, r_3 \in \Pi$. Moreover, from (2.6b), we obtain

$$\begin{aligned} |(x, y)| &\leq |x| + |\varphi(t, x)| + |y - \varphi(t, x)| \\ &\leq |x| + (1+t^\nu)r_o(|x|) + |y - \varphi(t, x)| \end{aligned} \tag{2.11}$$

for every x, y, and $t \geq 0$. Likewise, by (2.6a)–(2.6c) and (2.10), we can find an integer $\nu_2 \geq \nu_1$ and functions $p, q \in \Pi$ such that

$$|g(t, x, y)| + \left(\left|\frac{\partial \varphi}{\partial x}(t, x)\right| + \frac{d\gamma}{ds}(|x|)\right)|f(t, x) + yb(t)| + \left|\frac{\partial \varphi}{\partial t}(t, x)\right| \leq (1+t^{\nu_2})(p(|x|) + q(|y - \varphi(t, x)|)) \tag{2.12}$$

for all $t \geq 0$, x, and y. Next, recall (1.6) and (1.7), from which it follows that, for every constant $\xi > 0$, a function $\sigma \in \Pi$ can be found with

$$\xi(s + r_o(s)) + (\xi + 1)\gamma(s) + p(s) + q(\gamma(s)) \leq \sigma\left(\tfrac{1}{2}\gamma(s)\right), \qquad \forall s \geq 0. \tag{2.13}$$

Let ν_3 be an integer such that

$$\nu_3 \geq \max\{\nu_2, 3\nu\}. \tag{2.14}$$

By (2.11)–(2.14), a constant $E_o > 0$ can be determined in such a way that the following hold for all $E \geq E_o$:

$$\begin{aligned}
&\xi|(x, y)| + (1 + t^{\nu_2})q(|y - \varphi(t, x)|) + |g(t, x, y)| \\
&\quad + \left(\left|\frac{\partial \varphi}{\partial x}(t, x)\right| + \frac{d\gamma}{ds}(|x|)\right)|f(t, x) + yb(t)| + \left|\frac{\partial \varphi}{\partial t}(t, x)\right| \\
&\leq \xi(|x| + (1 + t^{\nu})r_o(|x|) + \gamma(|x|)) + (1 + t^{\nu_2})p(|x|) \\
&\quad + 2(1 + t^{\nu_2})q(\gamma(|x|)) \\
&\leq E(1 + t^{\nu_3})\sigma\left(\tfrac{1}{2}\gamma(|x|)\right) \\
&\leq E(1 + t^{\nu_3})\sigma(|y - \varphi(t, x)|), \\
&\qquad \text{for } \tfrac{1}{2}\gamma(|x|) \leq |y - \varphi(t, x)| \leq \gamma(|x|) \text{ and } t \geq 0;
\end{aligned} \tag{2.15a}$$

$$\begin{aligned}
&\xi|(x, y)| + \left(\left|\frac{\partial \varphi}{\partial x}(t, x)\right| + \frac{d\gamma}{ds}(|x|)\right)|f(t, x) + yb(t)| \\
&\quad + \left|\frac{\partial \varphi}{\partial t}(t, x)\right| + |y - \varphi(t, x)| + |g(t, x, y)| \\
&\quad - (1 + t^{\nu_2})q(|y - \varphi(t, x)|) \\
&\leq \xi(|x| + (1 + t^{\nu})r_o(|x|) + |y - \varphi(t, x)|) \\
&\quad + (1 + t^{\nu_2})p(|x|) + |y - \varphi(t, x)| \\
&\leq E(1 + t^{\nu_3})\sigma(|y - \varphi(t, x)|), \\
&\qquad \text{for } |y - \varphi(t, x)| \geq \tfrac{1}{2}\gamma(|x|) \text{ and } t \geq 0.
\end{aligned} \tag{2.15b}$$

Let σ_o, q_o: $\mathcal{R} \to \mathcal{R}$ be a pair of C^1 odd functions, whose restrictions on $\mathcal{R}^+$ coincide with σ and q, respectively. Let $\xi > 0$ be an arbitrary constant satisfying (2.13) and (2.15), and let

$$\begin{aligned}
\Xi(t, s) &:= -((E(1 + t^{\nu_3})\sigma_o(s) + (1 + t^{\nu_2})q_o(s)), \\
\Gamma(s) &:= \xi s, \qquad s \geq 0.
\end{aligned} \tag{2.16}$$

We establish that, for appropriate choice of the constant E, the origin is $\hat{L}_t$-GAS for the system (2.4) with $\hat{L}_t$, Φ, and Γ, as defined in (2.7)–(2.9) and (2.16). Particularly, we show that (2.1) and (2.2) hold for the trajectories of (2.4), with u satisfying (2.9). For simplicity, we deal only with the case $N = 0$. The general case follows similarly, and is left to the reader. The proof is divided into two parts.

Part I—Local Stabilization: From (1.5), (2.6b), (2.6c), (2.11), (2.12), and the definition of functions σ_o and q_o, it follows that there exist positive constants M, $\overline{M}$, and $R \leq c_5$, c_5 being the constant defined in (2.5d), such that

$$|(x, y)| \leq M(1 + t^{\nu})(|x| + |y - \varphi(t, x)|) \tag{2.17a}$$

$$\begin{aligned}
\overline{M}|y - \varphi(t, x)| &\leq \sigma_o(|y - \varphi(t, x)|) + q_o(|y - \varphi(t, x)|) \\
&\leq M|y - \varphi(t, x)| \\
&\qquad \text{for all } t \geq 0 \text{ and } x, y \\
&\qquad \text{with } |(x, y)| \leq R(1 + t^{\nu})^{-1}
\end{aligned} \tag{2.17b}$$

$$|f(t, x)| + |\varphi(t, x)| + \left|\frac{\partial \varphi}{\partial t}(t, x)\right| \leq M(1 + t^{\nu})|x| \tag{2.17c}$$

$$\left|\frac{\partial \varphi}{\partial x}(t, x)\right| \leq M(1 + t^{\nu}), \qquad \forall t \geq 0 \text{ and } |x| \leq R(1 + t^{\nu})^{-1}. \tag{2.17d}$$

Define

$$\begin{aligned}
W(t, x, y) &:= V(t, x) + a(t)(y - \varphi(t, x))^2 \\
a(t) &:= \frac{c_1\varepsilon^2}{2M^2(1 - \varepsilon^2)(1 + t^{\nu})^3}
\end{aligned} \tag{2.18}$$

for a certain constant $0 < \varepsilon < 1/\sqrt{2}$. By (2.5a), (2.17c), and (2.18), we get

$$\begin{aligned}
&W(t, x, y) \\
&\geq c_1(1 + t^{\nu})^{-1}|x|^2 - 2a(t)|\varphi(t, x)|\,|y| + a(t)y^2 \\
&\geq \left(c_1(1 + t^{\nu})^{-1}|x|^2 - \frac{1}{\varepsilon^2}a(t)\varphi^2(t, x)\right) + y^2a(t)(1 - \varepsilon^2) \\
&\geq \frac{c_1(1 - 2\varepsilon^2)}{2(1 - \varepsilon^2)}(1 + t^{\nu})^{-1}|x|^2 + \frac{c_1\varepsilon^2}{2M^2}(1 + t^{\nu})^{-3}y^2 \\
&\geq \overline{c}_1(1 + t^{3\nu})^{-1}|(x, y)|^2 \qquad \forall t \geq 0,\ |(x, y)| \leq R(1 + t^{\nu})^{-1}
\end{aligned} \tag{2.19a}$$

for some $\overline{c}_1 > 0$. Likewise, by (2.5a), (2.5b), (2.17c), (2.17d), and (2.19), a pair of positive constants $\overline{c}_2$ and $\overline{c}_3$ can be found in such a way that

$$\begin{aligned}
&W(t, x, y) \\
&\quad \leq V(t, x) + 2a(t)\varphi^2(t, x) + 2a(t)y^2 \leq \overline{c}_2|(x, y)|^2
\end{aligned} \tag{2.19b}$$

$$\begin{aligned}
&\left|\frac{\partial W}{\partial(x, y)}(t, x, y)\right| \\
&\quad \leq \left|\frac{\partial V}{\partial x}(t, x)\right| + 2a(t)\left|\frac{\partial \varphi}{\partial x}(t, x) + 1\right||y - \varphi(t, x)| \\
&\quad \leq \overline{c}_3|(x, y)|
\end{aligned} \tag{2.20}$$

for all $t \geq 0$ and $|(x, y)| \leq R(1 + t^{\nu})^{-1}$. Next, we evaluate the derivative $(d/dt)W$ of W along the trajectories of (2.4) with $u := \Phi + u$, namely, with respect to

$$\begin{aligned}
\dot{x} &= f(t, x) + yb(t) \\
\dot{y} &= \Phi(t, x, y) + u + g(t, x, y)
\end{aligned}$$

with

$$|u| \leq \Gamma(|(x, y)|) \tag{2.21}$$

where Φ and Γ are defined in (2.8) and (2.16). We find

$$\begin{aligned}
&\frac{d}{dt}W(t, x, y) \\
&\quad \leq \frac{\partial V}{\partial x}(t, x)(f(t, x) + \varphi(t, x)b(t)) + \frac{\partial V}{\partial t}(t, x) \\
&\qquad + \left|\frac{\partial V}{\partial x}(t, x)\right||b(t)||y - \varphi(t, x)| \\
&\qquad + 2a(t)|y - \varphi(t, x)|\left|\frac{\partial \varphi}{\partial x}(t, x)\right| \\
&\qquad \cdot (|f(t, x)| + |y - \varphi(t, x)||b(t)| + |\varphi(t, x)||b(t)|) \\
&\qquad + 2a(t)|y - \varphi(t, x)|\Bigg(|g(t, x, y)| + (1 + t^{\nu_2}) \\
&\qquad \cdot q_o(|y - \varphi(t, x)|) + \left|\frac{\partial \varphi}{\partial t}(t, x)\right|\Bigg) \\
&\qquad + |\dot{a}(t)|(y - \varphi(t, x))^2 - 2Ea(t)(1 + t^{\nu_3}) \\
&\qquad \cdot |y - \varphi(t, x)|\sigma_o(|y - \varphi(t, x)|) \\
&\qquad + 2|u|a(t)|y - \varphi(t, x)|, \qquad \text{for } |u| \leq \xi|(x, y)|
\end{aligned} \tag{2.22}$$

for every $t \geq 0$ and x, y with $|(x, y)| \leq R(1 + t^{\nu})^{-1}$. By taking into account (2.5c), (2.5d), (2.12), (2.14), (2.17), and (2.18), and then completing the squares on the right-hand side expression in (2.22), we can find a constant $E \geq E_o$ such that

$$\begin{aligned}\frac{d}{dt}W(t,\,x,\,y) &\leq -\frac{1}{2}c_4|x|^2 - \frac{1}{2}a(t)E(1+t^{\nu_3})(y-\varphi(x,\,t))^2\\ &\leq -\overline{c}_4 W(t,\,x,\,y) \qquad (2.23a)\end{aligned}$$

$$\forall\, t \geq 0,\ |(x,\,y)| \leq \frac{R}{1+t^{\nu}}, \quad |u| \leq \Gamma(|(x,\,y)|) \qquad (2.23b)$$

for some $\overline{c}_4 > 0$. By virtue of (2.19a), (2.19b), and (2.23), it follows that the solution $\omega(t,\,t_o,\,\omega_o,\,u) := (x(t,\,t_o,\,x_o,\,u),\,y(t,\,t_o,\,x_o,\,u)),\,\omega_o := (x_o,\,y_o)$ of (2.21) satisfies

$$|\omega(t)|K(1+t^{3\nu})^{1/2}\exp(-\ell(t-t_o))|\omega_o|, \qquad \forall\, t \geq t_o \qquad (2.24)$$

for certain positive constants K and ℓ and as long as (2.23b) holds. Notice next that there exists an appropriate small positive constant ρ such that

$$\rho\,\frac{\exp\,\ell t_o}{1+t_0^{3\nu}} \leq \frac{\exp\,\ell t}{(1+t^{\nu})(1+t^{3\nu})^{1/2}}, \qquad \forall\, t \geq t_o. \qquad (2.25)$$

Without loss of generality, assume that $\rho K^{-1} < 1$. We define

$$\begin{aligned}A_t &:= \left\{(x,\,y)\colon |(x,\,y)| \leq \frac{\rho R K^{-1}}{1+t^{3\nu}}\right\}\\ B_t &:= \left\{(x,\,y)\colon |(x,\,y)| \leq \frac{R}{1+t^{\nu}}\right\}. \qquad (2.26)\end{aligned}$$

Obviously, then, $A_t \subset B_t$ for all $t \geq 0$, and the following holds.

Property 1: "Local Attractiveness"—For any positive ε and T, there is a constant $\tau = \tau(\varepsilon,\,T) > 0$ such that

$$|\omega(t,\,t_o,\,\omega_o,\,u)| \leq \varepsilon, \qquad \forall\, t \geq \tau,\ t_o \in [0,\,T],\ \omega_o \in A_{t_o} \qquad (2.27)$$

and for any input u for which

$$(\omega(t,\,t_o,\,\omega_o,\,u),\,u(t)) \in L := \{(x,\,y,\,u)\colon |u| \leq \Gamma(|(x,\,y)|)\}. \qquad (2.28)$$

"Local Stability"—For every initial $t_o \geq 0,\,\omega_o \in A_{t_o}$ and input u such that (2.23b) holds, the corresponding trajectory $\omega(t)$ exists and satisfies $\omega(t) \in B_t$ for all $t \geq t_0$.

In order to establish this property, we take into account (2.24)–(2.26), from which we get $|\omega(t,\,t_o,\,\omega_o,\,u)|R(1+t^{\nu})^{-1}$ for every $t \geq t_o > 0,\,\omega_o \in A_{t_o}$ and u such that (2.28) holds. The desired property is then a direct implication of the previous inequality.

Part II—Global Part: We establish that the origin $0 \in \mathcal{R}^{n+1}$ for the system (2.4) is $\hat{L}_t$*-GAS*, where $\hat{L}_t$ is defined by (2.9) or, equivalently, $0 \in \mathcal{R}^{n+1}$ is L*-GAS* for (2.21) with L as defined in (2.28). Let

$$M_t := \{(x,\,y)\colon |y-\varphi(t,\,x)| \leq \gamma(|x|)\}. \qquad (2.29)$$

Property 2: The set M_t is positively invariant, namely, $\omega(t) \in M_t$ for all $t \geq t_o$ with $\omega_o \in M_{t_o}$, provided that the corresponding input u satisfies (2.28).

To establish this property, it suffices to show that

$$\dot{y}(t) = \Phi(t,\,x(t),\,y(t)) + u(t) + g(t,\,x(t),\,y(t))$$
$$\begin{cases} \leq \dfrac{\partial\varphi}{\partial x}(f(t,\,x(t))+y(t)b(t)) + \dfrac{\partial\varphi}{\partial t} + \dfrac{d\gamma}{ds}(|x|)\,\dfrac{d|x(t)|}{dt}, \\ \quad \text{for } \tfrac{1}{2}\gamma(|x(t)|) \leq y(t)-\varphi(t,\,x(t)) < \gamma(|x(t)|) \\ \geq \dfrac{\partial\varphi}{\partial x}(f(t,\,x(t))+y(t)b(t)) + \dfrac{\partial\varphi}{\partial t} + \dfrac{d\gamma}{ds}(|x|)\,\dfrac{d|x(t)|}{dt}, \\ \quad \text{for } \tfrac{1}{2}\gamma(|x(t)|) \leq \varphi(t,\,x(t))-y(t) < \gamma(|x(t)|) \end{cases}$$

but this is a direct consequence of (2.15a) and the definition of Φ.

For each $\delta > 0$, let us denote by S_δ the closed sphere of radius δ centered at $0 \in \mathcal{R}^{n+1}$. The following property is a direct consequence of Property 2, our Hypothesis A3, (2.17c), and Remark 2.1ii).

Property 3:

I) For every positive constant $\delta,\,\varepsilon$, and T^*, there exists a $\tau = \tau(\delta,\,\varepsilon,\,T^*) > 0$ such that

$$|\omega(t,\,t_o,\,\omega_o,\,u)| \leq \varepsilon, \quad \forall\, t \geq \tau,\ t_o \in [0,\,T^*],\ \omega_o \in M_{t_o} \cap S_\delta \qquad (2.30)$$

for every u for which (2.28) holds. Moreover, $\omega(t)$ exists and satisfies $\omega(t) \in M_t,\ \forall\, t \geq t_0$.

II) For every positive ε and T^*, there exists a $\delta = \delta(\varepsilon,\,T^*) > 0$ with

$$|\omega(t,\,t_o,\,\omega_o,\,u)| \leq \varepsilon, \quad \forall\, t \geq t_o,\ t_o \in [0,\,T^*],\ \omega_o \in M_{t_o} \cap S_\delta \qquad (2.31)$$

and for every u for which (2.28) holds.

Using (2.15b), we can also establish the following property.

Property 4: For every positive δ' and t_o and for any initial $\omega_o \in S_{\delta'}\backslash(M_{t_0} \cup A_{t_o})$, there is a $T' = T'(t_o,\,\omega_o,\,u) > 0$ such that

$$\omega(t,\,t_o,\,\omega_o,\,u) \notin M_t \cup A_t \qquad \text{for } t_o \leq t < T' \qquad (2.32a)$$
$$\omega(T',\,t_o,\,\omega_o,\,u) \in M_{T'} \cup A_{T'} \qquad (2.32b)$$

provided that (2.28) holds. Moreover, the trajectory $\omega(t)$ is defined for every $t_o \leq t \leq T'$.

Indeed, by (2.15b) and evaluating the time derivative of $(1/2)(y-\varphi(t,\,x))^2$ along the trajectories of (2.21), we obtain

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}(y-\varphi(t,\,x))^2\\ &\quad= (y-\varphi(t,\,x))\Bigg(\Phi(t,\,x,\,y)+u+g(t,\,x,\,y)\\ &\qquad\qquad - \frac{\partial\varphi}{\partial x}(f(t,\,x)+yb(t)) - \frac{\partial\varphi}{\partial t}\Bigg)\\ &\quad\leq -E(1+t^{\nu_3})|y-\varphi(t,\,x)|\sigma(|y-\varphi(t,\,x)|)\\ &\qquad - (1+t^{\nu_2})|y-\varphi(t,\,x)|q(|y-\varphi(t,\,x)|)\\ &\qquad + |y-\varphi(t,\,x)|\Bigg(\xi|(x,\,y)| + |g(t,\,x,\,y)|\\ &\qquad\qquad + \left|\frac{\partial\varphi}{\partial x}(f(t,\,x)+yb(t))\right| + \left|\frac{\partial\varphi}{\partial t}\right|\Bigg)\\ &\quad\leq -(y-\varphi(t,\,x))^2.\end{aligned}$$

Thus, by (2.6b), it follows that

$$\begin{aligned}|y(t)-\varphi(t,\,x(t))| &\leq \exp(-(t-t_o))|y_o-\varphi(t_o,\,x_o)|\\ &\leq \exp(-(t-t_o))(1+t_o^{\nu})\overline{r}(|(x_o,\,y_o)|) \qquad (2.33)\end{aligned}$$

for certain $\overline{r} \in \Pi$ and for every $t \geq t_o$ for which the solution $\omega(t)$ exists on $[t_o,\,t]$ and satisfies both (2.28) and

$$\tfrac{1}{2}\gamma(|x(t)|) \leq |y(t)-\varphi(t,\,x(t))|. \qquad (2.34)$$

Assume that $(1/2)\rho R K^{-1} < \overline{r}(\delta')$, K being the constant defined in (2.24), and define

$$s(t) := \gamma^{-1}\left(\frac{\rho R K^{-1}}{1+t^{3\nu}}\right). \qquad (2.35)$$

Then for every $t \geq t_o$ for which $\omega(t)$ exists on $[t_o,\,t]$ and satisfies (2.28), and as long as $\omega(t) \notin S_{\delta'}\backslash(M_t \cup A_t)$, we have by (2.26), (2.34), and (2.35)

$$|y(t)-\varphi(t,\,x(t))| \geq \frac{1}{2}\gamma(|x(t)|) \geq \frac{1}{2}\gamma(s(t)) = \frac{1}{2}\,\frac{\rho R K^{-1}}{1+t^{3\nu}}. \qquad (2.36)$$

For each δ' and t_o, consider a positive constant $T = T(t_o, \delta')$ satisfying

$$\frac{1/2\rho RK^{-1}}{1+T^{3\nu}} = \exp(-(T-t_o))(1+t_o^\nu)\bar{r}(\delta'). \tag{2.37}$$

This, in conjunction with (2.26), (2.33), and (2.36), implies that, for each t_o and $\omega_o \in S_{\delta'}\backslash(M_{t_o} \cup A_{t_0})$, there exists a time $T' = T'(t_o, \omega_o, u) \le T$ such that (2.32) hold. We now establish that each solution $\omega(t)$, starting from ω_o at time t_o, exists for every $t \ge t_o$ as long as it remains outside $M_t \cup A_t$. Suppose, on the contrary, that $\omega(t)$ exists on $[t_0, T'')$ for some $T'' \le T'$ in such a way that (2.28) holds and $\omega(t) \notin M_t$ for $t_o \le t < T''$, but $\overline{\lim}_{t\to T''}|\omega(t)| = \infty$. Assume, first, that $\overline{\lim}_{t\to T''}|x(t)| < \infty$. Then (2.6b) and (2.34) imply that $\overline{\lim}_{t\to T''}|y(t)| \le \overline{\lim}_{t\to T''}|y(t) - \varphi(t, x(t))| + \overline{\lim}_{t\to T''} r_o(|x(t)|)(1+t^\nu) < \infty$; hence, $\overline{\lim}_{t\to T''}|\omega(t)| < \infty$, a contradiction. The other case is $\overline{\lim}_{t\to T''}|x(t)| = \infty$. The left-hand inequality in (2.33), (2.34), and the fact that $\gamma \in K_\infty$ imply $\infty = (1/2)\gamma(\overline{\lim}_{t\to T''}|x(t)|) \le \overline{\lim}_{t\to T''}|y(t) - \varphi(t, x(t))| < \infty$, a contradiction. Hence, $\omega(t)$ is defined for every $t_o \le t \le T'$, and since, by Property 1, $\omega(t) \in M_t \cup B_t$ for $t \ge T'$, is defined for every $t \ge t_o$. This, in particular, means that (2.21) is L complete. To complete the proof, we also need to establish the following property, which is a consequence of Property 4.

Property 5: For every $\tilde{T} > 0$ and $\delta > 0$, there exists a positive $\delta' < \delta$ such that $|\omega(T'(t_o, \omega_o, u), t_o, \omega_o, u)| \le \delta$, $\forall \omega_o \in S_{\delta'}\backslash(M_{t_o} \cup A_{t_o})$, for any $t_o \in [0, \tilde{T}]$ and input u such that (2.28) holds for $t_o \le t \le T' = T'(t_o, \omega_o, u)$, where the latter is defined in Property 4.

The proof is similar to that given in Property 4 to establish completeness. Suppose, on the contrary, that there exist a $\delta > 0$, sequences of vectors $\mathcal{R}^{n+1} \ni \omega_{ok} \to 0$ as $k \to +\infty$, times $T'_k < T$, $t_{ok} < \tilde{T}$, and inputs u_k such that the corresponding solution $\omega_k(\cdot) = (x_k(\cdot), y_k(\cdot))$ of (2.21) satisfies $(\omega_k(t), u_k(t)) \in L$, $\omega_k(t) \notin M_t \cup A_t$ for $t_{ok} \le t < T'_k$, $\omega_k(T'_k) \in M_{T'_k} \cup A_{T'_k}$, but $|\omega_{k_i}(T'_k)| \ge \delta$ for all integers k. Assume, first, that $\overline{\lim}|x_k(T'_k)| = 0$. Then (2.6b) and the first inequality in (2.33) yield $\overline{\lim}|y_k(T'_k)| \le \overline{\lim}|y_k(T'_k) - \varphi(T'_k, x_k(T'_k))| + \overline{\lim}|\varphi(T'_k, x_k(T'_k))| = 0$, hence, $|\omega_k(T'_k)| \to 0$, a contradiction. The case $\overline{\lim}|x_k(T'_k)| > 0$ is also excluded since, otherwise, by (2.34) and the fact that $\omega_{ok} \to 0$, we would have $0 < (1/2)\gamma(\underline{\lim}|x_k(T'_k)|) \le \overline{\lim}|y_k(T'_k) - \varphi(T'_k, x_k(T'_k))| = 0$, a contradiction.

We are now in a position to establish that $0 \in \mathcal{R}^{n+1}$ is an L *attractor* for (2.21). Consider the constants δ, ε, T^*, and τ as defined by (2.30) of Property 3. Without loss of generality, we may assume that ε and τ coincide with the corresponding constants defined in (2.27) of Property 1 and $T^* > T$, where T is defined in (2.37). We also use Properties 4 and 5 with the same δ, appropriately small $\delta' > 0$, and $\tilde{T} := T^* - T$. We claim that

$$|\omega(t, t_o, \omega_o, u)| < \varepsilon, \qquad \forall t_o \in [0, \tilde{T}],\ t \ge T + \tau,\ \omega_o \in S_{\delta'} \tag{2.38}$$

for each u such that (2.28) holds. We distinguish the following two cases:

- $\omega(t) \notin A_t,\ \forall t \ge t_o$;
- $\omega(t) = \omega(t, T_o, \overline{\omega}, u)$ for some $T_o > 0$ and $\overline{\omega} \in A_{T_o}$.

For the first case, (2.38) is a consequence of Properties 2–5, whereas, for the second case we apply (2.27) of Property 1. Similarly, by using (2.31) and Properties 4 and 5, we can establish that $0 \in \mathcal{R}^{n+1}$ is L *stable*, and thus L-AS for (2.21). Finally, the rest of the proof is a consequence of definition (2.8) and (2.16) of Φ and (2.20). Details are left to the reader. □

As a direct consequence of Lemma 2.3, we obtain the following.

Theorem 2.4: For the system (1.2), where only $(y_1, \cdots, y_m)$ is available, assume that hypotheses (A1) and (A3) are satisfied for the subsystem $\dot{x} = f(t, x) + yb(t)$ with $\varphi \equiv 0$, and that there exist functions $r_i \in \Pi$, $i = 0, 1, \cdots, m$, and an integer ν such that (2.5a), (2.5d), and (2.6) hold, and further,

$$\begin{aligned} |f(t,x)| &\le (1+t^\nu) r_o(|x|), \\ |g_i(t, x, y_1, \cdots, y_i)| &\le (1+t^\nu) r_i(|(x, y_1, \cdots, y_i)|), \\ & \qquad i = 1, \cdots, m \quad \forall x, y_1, \cdots, y_i, t \ge 0. \end{aligned} \tag{2.39}$$

Then there exist C^1 mappings $\Xi_i: \mathcal{R}^+ \times \mathcal{R} \to \mathcal{R}$, $i = 1, \cdots, m$ with $\Xi_i(t, 0) = 0$ for $t \ge 0$ so that the feedback

$$u := \Xi_m(t, y_m - \Xi_{m-1}(t, \cdots, \Xi_2(t, y_2 - \Xi_1(t, y_1)) \cdots)) \tag{2.40}$$

globally asymptotically stabilizes (1.2) at $0 \in \mathcal{R}^{n+m}$ [namely, the origin of the closed-loop (1.2) with (2.40) is globally asymptotically stable].

The proof follows directly by the use of Lemma 2.3 and the induction procedure.

Remark 2.5: The result of Theorem 2.4 can directly be extended for systems (1.2) containing unknown parameters. Particularly, the result is valid under the same hypothesis for unknown dynamics f and g_i satisfying (2.39) for some known r_i. Another straightforward extension can be obtained for systems (1.2), where each y_i satisfies $\dot{y}_i = d_i(t) y_{i+1} + g_i(\cdot)$ where each d_i is everywhere strictly positive and satisfies $d_i(t) + |\dot{d}_i(t)| \le C(1+t^\nu)$ for every $t \ge 0$ for some $C > 0$ and integer ν.

Example 2.6: Consider the planar system $\dot{x} = y + p(t, x)$, $\dot{y} = u$, where p is a polynomial of t and x. It turns out that $|p(t, x)| \le (1+t^\nu) r_o(|x|)$, $\forall t, x$ for a certain integer ν and $r_o \in \Pi$. Let $q: \mathcal{R} \to \mathcal{R}$ be the odd extension of r_o, and let ξ_o be an arbitrary positive constant. Then we can easily verify that all conditions of Lemma 2.3 hold with $V(t, x) := x^2$, $\varphi(t, x) := -q(x)(1+t^\nu) - x - \xi_o x$, $\gamma(s) = \xi_o s$; thus, the system is globally feedback stabilizable. If, in addition, $xp(t, x) \le -\hat{q}(t)\hat{p}(x)|x|$ for certain $\hat{q}(\cdot) > 1$, $\hat{p}$ being positive definite, then all conditions of Lemma 2.3 are fulfilled with $\varphi \equiv 0$; hence, global stabilization can be achieved by means of a time-varying feedback being independent of x.

By extending the analysis of the previous example, we can easily establish by the induction procedure the following result concerning triangular systems:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + f_i(t, x_1, \cdots, x_i), \\ & \qquad i = 1, \cdots, n,\ x_i \in \mathcal{R},\ u := x_{i+1}. \end{aligned} \tag{2.41}$$

Corollary 2.7: Assume that each f_i is C^1, possibly unknown, and there exist some known $r_i \in \Pi$ and an integer ν such that $|f_i(t, x_1, \cdots, x_i)| \le (1 + t^\nu) r_i(|(x_1, \cdots, x_i)|)$, $\forall (x_1, \cdots, x_i)$, $t \ge 0$, $i = 1, \cdots, n$. Then (2.41) is globally asymptotically stabilizable by means of a time-varying feedback.

III. Asymptotic Tracking

We briefly discuss the applicability of the methodology developed in the previous section to asymptotic tracking. For reasons of simplicity, let us consider the autonomous case:

$$\begin{aligned} \dot{x} &= Ax + y_1 b \\ \dot{y}_1 &= y_2 + g_1(x, y_1) \\ \dot{y}_2 &= y_3 + g_2(x, y_1, y_2) \\ \dot{y}_3 &= u; \qquad x \in \mathcal{R}^n, \quad y_i \in \mathcal{R}, \quad i = 1, 2, 3 \end{aligned} \tag{3.1}$$

and assume that A and b are time invariant, A is Hurwitz, g_1 and g_2 are C^2 vanishing at zero, and the following holds globally:

$$|Dg_i(x, y_1, \cdots, y_i)| \le r_i(|(x, y_1, \cdots, y_i)|) \tag{3.2}$$

for $i = 1, 2$ for certain functions $r_i \in \Pi$ (Dg_i represents the derivative of g_i). Let us denote by S the class of smooth functions ξ: $\mathcal{R}^+ \to \mathcal{R}$ with the property that, for every $i = 0, 1, 2, \cdots$, there is a constant $C > 0$ and an integer ν such that

$$|\xi^{(i)}(t)| \le C(1 + t^\nu), \qquad \forall t \ge 0.$$

Consider a C^2 signal ξ: $\mathcal{R}^+ \to \mathcal{R}$, and assume that it is of class S. Our goal is to find a feedback law, being independent of x, in such a way that the y_1 component of the resulting closed-loop system satisfies

$$|y_1(t) - \xi(t)| \to 0 \qquad \text{as } t \to +\infty \tag{3.3}$$

for every initial state $(x_0, y_0) \in \mathcal{R}^{n+3}$ and time t_0. We proceed as follows.

Let us first denote by $\theta(t) := x(t, 0; \xi)$ the solution of the linear equation

$$\dot{x} = Ax + \xi b$$

starting from zero at time $t = 0$ and corresponding to the signal ξ. Obviously, since $\xi \in S$ and A is Hurwitz, θ is of class S as well.

We define

$$a := x - \theta, \quad w_1 := y_1 - \xi(t), \quad w_2 := y_2, \quad w_3 := y_3. \tag{3.4}$$

Then (3.1) takes the equivalent form:

$$\begin{aligned} \dot{a} &= Aa + w_1 b \\ \dot{w}_1 &= y_2 + g_1(a + \theta, y_1) - \xi^{(1)} = w_2 + F_1(t, a, w_1) \\ \dot{w}_2 &= y_3 + g_2(a + \theta, y_1, y_2) = w_3 + F_2(t, a, w_1, w_2) \\ \dot{w}_3 &= u \end{aligned} \tag{3.5a}$$

where

$$\begin{aligned} F_1(t, a, w_1) &:= g_1(a + \theta, w_1 + \xi(t)) - \xi^{(1)}(t) \\ F_2(t, a, w_1, w_2) &:= g_2(a + \theta, w_1 + \xi(t), w_2). \end{aligned} \tag{3.5b}$$

Notice that (3.5a) has the triangular structure of (1.2), but its dynamics do not, in general, vanish for $w = 0$ and $t \ge 0$. As in the case of bounded signals (see, for instance, [2]), we transform (3.5) into a system of the form (1.2) whose dynamics vanish at zero, so the problem is reduced to partial feedback stabilization of a system with a triangular structure. In our case, the resulting system will be time-varying and satisfies the hypothesis of Theorem 2.4. We apply the transformation

$$\begin{aligned} z_1 &:= w_1 \\ z_2 &:= w_2 + F_1(t, 0, 0) \\ z_3 &:= w_3 + F_2(t, 0, 0, -F_1(t, 0, 0)) + \frac{d}{dt} F_1(t, 0, 0). \end{aligned} \tag{3.6}$$

Using (3.6) and feedback

$$\begin{aligned} u &:= u + \phi_1(t), \\ \phi_1(t) &:= -\frac{d}{dt}\left(F_2(t, 0, 0, F_1(t, 0, 0)) + \frac{d}{dt} F_!(t, 0, 0)\right) \end{aligned} \tag{3.7}$$

the system (3.5) becomes

$$\begin{aligned} \dot{a} &= Aa + z_1 b \\ \dot{z}_1 &= z_2 + F_1(t, a, z_1) - F_1(t, 0, 0) \\ \dot{z}_2 &= z_3 + F_2(t, a, z_1, z_2 - F_1(t, 0, 0)) \\ &\quad - F_2(t, 0, 0, -F_1(t, 0, 0)) \\ \dot{z}_3 &= u. \end{aligned} \tag{3.8}$$

Since both θ and ξ belong to S, it follows that there exist functions $\overline{r}_1, \overline{r}_2 \in \Pi$, and an integer m such that

$$\begin{aligned} &|F_1(t, a, z_1) - F_1(t, 0, 0)| \le (1 + t^m)\overline{r}_1(|(a, z_1)|) \\ &|F_2(t, a, z_1, z_2 - F_1(t, 0, 0)) - F_2(t, 0, -F_1(t, 0, 0), 0)| \\ &\quad \le (1 + t^m)\overline{r}_2(|(a, z_1, z_2)|) \end{aligned}$$

for every a, z_1, z_2, θ, and $t \ge 0$. The previous inequalities and our assumption that the matrix A is Hurwitz assert that all hypotheses of Theorem 2.4 are fulfilled for (3.8); thus, there exists a time-varying C^1 feedback $u = \varphi_2(t, z_1, z_2)$ which globally asymptotically stabilizes (3.8) at the origin. This, by virtue of the first equations in (3.4) and (3.6), implies that, for the original system (3.1) with $u = \phi_1(t) + \varphi_2(t, y_1 + \xi(t), y_2 + F_1(t, 0, 0))$, the desired property (3.3) holds.

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