A posteriori error control for the Allen-Cahn problem

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Abstract

1 Introduction

2 Preliminaries

We denote by $L^r(\Omega)$, $1 \le r \le \infty$, the standard Lebesgue spaces, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with corresponding norms $\| \cdot \|_r$. The norm of $L^2(\Omega)$ will be denoted by $\|\cdot\|$ for brevity, and the corresponding inner product by $\langle \cdot, \cdot \rangle$. We also denote by $W_r^k(\Omega)$ the k-th order Sobolev space based on $L^r(\Omega)$ and by $H^k(\Omega) := W_r^k(\Omega)$, the standard Hilbertian Sobolev space of index $k \ge 0$ of real-valued functions defined on $\Omega \subset \mathbb{R}^d$, along with the corresponding norms $\| \cdot \|_{W_r^k(\Omega)}$ and $\| \cdot \|_{H^K(\Omega)}$, respectively.

2.1 Model problem

Let $T > 0$ and Ω a bounded open polygonal domain in $\mathbb{R}^2$, and let $\partial\Omega$ denote its boundary. We consider the Allen-Cahn equation

$$u_t - \Delta u + \epsilon^{-2} f(u) = 0 \quad \text{in } (0, T] \times \Omega, \tag{1}$$

where $f(u) := u^3 - u$, along with the initial and boundary conditions

$$\begin{aligned} u(0, x) &= u_0(x) && \text{for } x \in \Omega, \\ \nabla u(t, x) \cdot \mathbf{n} &= 0 && \text{for } x \in \partial\Omega, t \in (0, T], \end{aligned} \tag{2}$$

where $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$.

2.2 Meshes, finite element spaces and trace operators

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular (tetrahedral if $d = 3$) or quadrilateral (hexahedral if $d = 3$) elements $\kappa \in \mathcal{T}$. We define $h_\kappa := \text{diam}(\kappa)$ and we collect them into the element-wise constant function $\mathbf{h} : \Omega \to \mathbb{R}$, with $\mathbf{h}|_\kappa = h_\kappa$, $\kappa \in \mathcal{T}$. We assume that the subdivision $\mathcal{T}$ is shape-regular (see, e.g., p.124 in [2]) and that it is constructed via

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affine element mappings F_κ, where $F_\kappa : \hat{\kappa} \to \kappa$, with non-singular Jacobian, where $\hat{\kappa}$ is the reference triangle or quadrilateral. The above mappings are assumed to be constructed so as to ensure $\bar{\Omega} = \cup_{\kappa\in\mathcal{T}}\bar{\kappa}$.

Also, for a nonnegative integer r, we denote by $\mathcal{P}_r(\hat{\kappa})$, the set of all polynomials of total degree at most r, if $\hat{\kappa}$ is the reference triangle, or the set of all tensor-product polynomials on $\hat{\kappa}$ of degree at most r in each coordinate direction, if $\hat{\kappa}$ is the reference quadrilateral. For $r \geq 1$, we consider the finite element space

$$S_h^r := \{v \in H^1(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{P}_r(\hat{\kappa}),\ \kappa \in \mathcal{T}\}.$$

We shall assume throughout that the families of meshes considered are locally quasiuniform, i.e., there exists constant $c \geq 1$, independent of $\mathbf{h}$, such that, for any pair of elements κ^+ and κ^- in $\mathcal{T}$ which share an edge,

$$c^{-1} \leq h_{\kappa^+}/h_{\kappa^-} \leq c.$$

2.3 Finite element method

We consider the (semidiscrete) finite element method for the model problem (1), (2):

$$\text{find } u_h \in S_h^r \text{ such that } \quad \langle (u_h)_t, v_h\rangle + B(u_h, v_h) + \langle f(u_h), v_h\rangle = 0 \quad \forall v_h \in S_h^r, \tag{3}$$

where the bilinear form is given by

$$B(w, v) = \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x. \tag{4}$$

3 A posteriori error bounds in the $L^\infty(L^p)$-norm

Definition 3.1 *We define the elliptic reconstruction $w \in H^1(\Omega)$ of u_h by*

$$B(w, v) = B(u_h, v) + \epsilon^{-2}\langle f(u_h) - \Pi f(u_h), v\rangle, \quad \textit{for all } v \in H^1(\Omega),$$

such that

$$\int_\Omega w \,\mathrm{d}x = 0.$$

We can now split the error as follows

$$e := u - u_h = \theta - \rho,$$

where $\theta := w - u_h$ and $\rho := w - u$.

Lemma 3.1 *For $r \geq 2$ even positive integer, and $C_r := 4(r-1)/r^2$ we have*

$$\begin{aligned} &\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_r^r + C_r\|\nabla(\rho^{r/2})\|^2 + \epsilon^{-2}\langle f'(u_h)\rho^{r/2}, \rho^{r/2}\rangle \\ &\leq \langle \theta_t, \rho^{r-1}\rangle + \epsilon^{-2}\big(\langle f'(u_h)\theta, \rho^{r-1}\rangle + 3\langle e^2 u_h, \rho^{r-1}\rangle + \langle \theta^3, \rho^{r-1}\rangle + 3\langle \theta, \rho^{r+1}\rangle\big). \end{aligned} \tag{5}$$

Proof The error equation

$$\langle \rho_t, v\rangle + B(\rho, v) + \epsilon^{-2}\langle f(u_h) - f(u), v\rangle = \langle \theta_t, v\rangle, \tag{6}$$

holds for all $v \in H^1(\Omega)$; this follows directly using the PDE, the finite element method and the definition of w. Setting $v = \rho^{r-1}$, we deduce

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_r^r + C_r\|\nabla(\rho^{r/2})\|^2 + \epsilon^{-2}\langle f(u_h) - f(u), \rho^{r-1}\rangle = \langle \theta_t, \rho^{r-1}\rangle, \tag{7}$$

upon observing that

$$\langle \nabla\rho, \nabla(\rho^{r-1})\rangle = (r-1)\langle \nabla\rho, \rho^{r-2}\nabla\rho\rangle = (r-1)\|\rho^{r/2-1}\nabla\rho\|^2 = C_r\|\nabla(\rho^{r/2})\|^2.$$

Using Taylor's theorem, we find

$$f(u_h) - f(u) = -ef'(u_h) - 3e^2 u_h - e^3,$$

and, observing the crucial property

$$\langle e^3, \rho^{r-1}\rangle = \langle \theta^3, \rho^{r-1}\rangle - 3\langle \theta^2\rho, \rho^{r-1}\rangle + 3\langle \theta\rho^2, \rho^{r-1}\rangle - \langle \rho^3, \rho^{r-1}\rangle \leq \langle \theta^3, \rho^{r-1}\rangle + 3\langle \theta, \rho^{r+1}\rangle,$$

we arrive to

$$\begin{aligned}\langle f(u_h) - f(u), \rho^{r-1}\rangle \geq &- \langle f'(u_h)\theta, \rho^{r-1}\rangle + \langle f'(u_h)\rho^{r/2}, \rho^{r/2}\rangle \\ &- 3\langle e^2 u_h, \rho^{r-1}\rangle - \langle \theta^3, \rho^{r-1}\rangle - 3\langle \theta, \rho^{r+1}\rangle,\end{aligned}$$

which, in conjunction with (7), yields the result.

□

We now assume that there exists $\lambda > 0$ such that

$$-\lambda\|v\|^2 \leq \|\nabla v\|^2 + \epsilon^{-2}\langle f'(u_h)v, v\rangle, \quad \text{for all } v \in H^1(\Omega). \tag{8}$$

We can approximate this λ numerically as done in [1]. We Note that, in eigenvalue computations, the approximate eigenvalues converge twice as fast compared to the approximate eigenvectors. Therefore, the approximate eigenvalues λ are unlikely to challenge the practical applicability of the method. To verify that λ is independent of ϵ, we can solve numerically the corresponding eigenvalue problem as done in [1]. The following result is inspired by and extends [1].

Lemma 3.2 *Let $r \geq 2$ be an even positive integer. Let also*

$$\begin{aligned}
&A(t) := \frac{1}{r}\|\rho(0)\|_r^r + \frac{1}{r}\Big(\|\theta_t\|_r + \epsilon^{-2}\|f'(u_h)\|_\infty\|\theta\|_r + 6\epsilon^{-2}\|u_h\|\|\theta\|_{2r}^2 + \epsilon^{-2}\|\theta\|_{3r}^3\Big)^r,\\
&B(t) := 3\epsilon^{-2}\big(2\|u_h\|_\infty + \|\theta\|_\infty\big), \qquad \bar{B}(t) := \sup_{s\in[0,t]} B(s), \qquad \bar{B} := B(T),\\
&C(t) := rC_r\big(\lambda(1-\epsilon^2) + \|f'(u_h)\|_\infty\big) + r - 1, \qquad \bar{C}(t) := \sup_{s\in[0,t]} C(s), \qquad \bar{C} := C(T),
\end{aligned}$$

and assume that, when $\bar{B} \neq 0$, there exist constant $c_1 > 0$ and a value $\eta \geq 0$ such that we have simultaneously

$$\eta \leq (c_1 r)^{1/r}\big(4\mathrm{e}^{\bar{C}T}\big)^{-1-1/r}\big(\tilde{c}C_\Omega\bar{B}\max\{rT, 1/(C_r\epsilon^2)\}\big)^{-1} \quad \textit{and} \quad \int_0^T A(t)\mathrm{d}t \leq c_1\eta^r,$$

where $\tilde{c}$ is the constant in the imbedding $\|w\|_4^2 \leq \tilde{c}\|w\|_{H^1(\Omega)}^2$, for all $w \in H^1(\Omega)$, and $C_\Omega := 1$ if $r = 2$, or $C_\Omega := |\Omega|^{r/(r-2)}$ if $r > 2$. Then, the following bound holds

$$\frac{1}{r}\|\rho\|_{L^\infty(0,T;L^r(\Omega))}^r + C_r\epsilon^2\|\nabla(\rho^{r/2})\|_{L^2(0,T;L^2(\Omega))}^2 \leq 4c_1\eta^r\mathrm{e}^{\bar{C}T}. \tag{9}$$

Proof We shall estimate the terms appearing in the error bound (5). Using the spectral estimate (8), we deduce

$$\begin{aligned}\|\nabla(\rho^{r/2})\|^2+\epsilon^{-2}\langle f'(u_h)\rho^{r/2},\rho^{r/2}\rangle =&(1-\epsilon^2)\big(\|\nabla(\rho^{r/2})\|^2+\epsilon^{-2}\langle f'(u_h)\rho^{r/2},\rho^{r/2}\rangle\big)\\ &+\epsilon^2\|\nabla(\rho^{r/2})\|^2+\langle f'(u_h)\rho^{r/2},\rho^{r/2}\rangle\\ \ge&-\lambda(1-\epsilon^2)\|\rho\|_r^r+\epsilon^2\|\nabla(\rho^{r/2})\|^2+\langle f'(u_h)\rho^{r/2},\rho^{r/2}\rangle,\end{aligned}$$

upon observing that $\|\rho^{r/2}\|^2=\|\rho\|_r^r$. Also, denoting by $r':=r/(r-1)$ the dual conjugate of r, various versions of Hölder's inequality yield

$$\begin{aligned}\langle\theta_t,\rho^{r-1}\rangle&\le\|\theta_t\|_r\|\rho^{r-1}\|_{r'}=\|\theta_t\|_r\|\rho\|_r^{r-1},\\ \langle f'(u_h)\theta,\rho^{r-1}\rangle&\le\|f'(u_h)\|_\infty\|\theta\|_r\|\rho\|_r^{r-1},\\ \langle e^2u_h,\rho^{r-1}\rangle&\le2\|u_h\|_\infty\big(\|\theta\|_{2r}^2\|\rho\|_r^{r-1}+\|\rho\|_{r+1}^{r+1}\big),\\ \langle\theta^3,\rho^{r-1}\rangle&\le\|\theta^3\|_r\|\rho\|_r^{r-1}=\|\theta\|_{3r}^3\|\rho\|_r^{r-1},\\ \langle\theta,\rho^{r+1}\rangle&\le\|\theta\|_\infty\|\rho\|_{r+1}^{r+1},\\ |\langle f'(u_h)\rho^{r/2},\rho^{r/2}\rangle|&=|\langle f'(u_h),\rho^r\rangle|\le\|f'(u_h)\|_\infty\|\rho\|_r^r,\end{aligned}$$

respectively.

Now, inserting these bounds into (5) and collecting the various terms accordingly, we obtain

$$\begin{aligned}\frac1r\frac{\mathrm d}{\mathrm dt}\|\rho\|_r^r+C_r\epsilon^2\|\nabla(\rho^{r/2})\|^2\le&\Big(rA(t)-\|\rho(0)\|_r^r\Big)^{1/r}\|\rho\|_r^{r-1}+B(t)\|\rho\|_{r+1}^{r+1}\\ &+\frac1r\big(C(t)+1-r\big)\|\rho\|_r^r.\end{aligned}\tag{10}$$

We observe that

$$\|\rho\|_{r+1}^{r+1}=\int_\Omega\rho\big(\rho^{r/2}\big)^2\mathrm dx\le\|\rho\|\|\rho^{r/2}\|_4^2\le\tilde c\|\rho\|\|\rho^{r/2}\|_{H^1(\Omega)}^2=\tilde c\|\rho\|\big(\|\rho\|_r^r+\|\nabla(\rho^{r/2})\|^2\big),$$

which, in conjunction with the (trivial) bound $\|\rho\|\le C_\Omega\|\rho\|_r$, implies

$$\|\rho\|_{r+1}^{r+1}\le\tilde cC_\Omega\|\rho\|_r\big(\|\rho\|_r^r+\|\nabla(\rho^{r/2})\|^2\big).$$

Using this bound along with a discrete Young's inequality of the form $\alpha\beta\le\alpha^r/r+\beta^{r'}/r'$ on the first term on the right hand side of (10) we arrive to

$$\frac1r\frac{\mathrm d}{\mathrm dt}\|\rho\|_r^r+C_r\epsilon^2\|\nabla(\rho^{r/2})\|^2\le A(t)-\frac1r\|\rho(0)\|_r^r+B(t)\tilde cC_\Omega\|\rho\|_r\big(\|\rho\|_r^r+\|\nabla(\rho^{r/2})\|^2\big)+\frac1rC(t)\|\rho\|_r^r.$$

An integration with respect to the time variable between 0 and τ, for $\tau\in[0,T]$, yields, respectively,

$$\begin{aligned}&\frac1r\|\rho(\tau)\|_r^r+C_r\epsilon^2\int_0^\tau\|\nabla(\rho^{r/2})\|^2\mathrm dt\\ \le&\int_0^\tau A(t)\mathrm dt+\bar C(\tau)\int_0^\tau\frac1r\|\rho\|_r^r\mathrm dt\\ &+\tilde cC_\Omega\bar B(\tau)\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|_r\Big(\tau\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|_r^r+\int_0^\tau\|\nabla(\rho^{r/2})\|^2\mathrm dt\Big)\\ \le&\int_0^T A(t)\mathrm dt+\bar C\int_0^\tau\frac1r\|\rho\|_r^r\mathrm dt\\ &+r^{1/r}\tilde cC_\Omega\bar B\max\{rT,1/(C_r\epsilon^2)\}\Big(\frac1r\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|_r^r+C_r\epsilon^2\int_0^\tau\|\nabla(\rho^{r/2})\|^2\mathrm dt\Big)^{1+1/r}\end{aligned}$$

We now consider the set

$$I := \{\tau \in [0,T] : \frac{1}{r}\operatorname{ess\,sup}_{[0,\tau]} \|\rho\|_r^r + C_r\epsilon^2 \int_0^\tau \|\nabla(\rho^{r/2})\|^2 \mathrm{d}t \le 4c_1\eta^r \mathrm{e}^{\bar{C}T}\},$$

which is non-empty due to the continuity of the left-hand side (we need maybe to assume that $\rho(0) = 0$ here). We set $\tau^* = \max I$ and we suppose that $\tau^* < T$. Hence, for $\tau \le \tau^*$, using also the hypotheses on c_1 and η, we deduce

$$\frac{1}{r}\|\rho(\tau)\|_r^r + C_r\epsilon^2 \int_0^\tau \|\nabla(\rho^{r/2})\|^2 \mathrm{d}t \le 2c_1\eta^r + \bar{C}\int_0^\tau \frac{1}{r}\|\rho\|_r^r \mathrm{d}t.$$

Gronwall's Lemma then implies

$$\frac{1}{r}\|\rho(\tau^*)\|_r^r + C_r\epsilon^2 \int_0^{\tau^*} \|\nabla(\rho^{r/2})\|^2 \mathrm{d}t \le 2c_1\eta^r \mathrm{e}^{\bar{C}T}, \tag{11}$$

setting $\tau = \tau^*$, which contradicts the hypothesis $\tau^* < T$, due to the continuity of the left-hand side of (11). Hence, $I = [0,T]$ and we conclude that (9) holds.

□

Before presenting the main a-posteriori bound, we recall a useful non-linear estimate (cf. [4] or Theorem 1.26 in [5]).

Theorem 3.1 *If $r \ge 2$ and $\delta > 0$, there exists a constant $\hat{c} > 0$ such that*

$$\|v\|^r_{W_r^{2/r-\delta}(\Omega)} \le \hat{c}\Big(\|v\|_r^r + \int_\Omega |v|^{r-2}|\nabla v|^2 \mathrm{d}x\Big), \tag{12}$$

for all $v \in L^r(\Omega)$ for which the second term on the right-hand side is finite.

We are now ready to present the main result of this section.

Theorem 3.2 *With the same assumptions and notation as in Lemma 3.2, the following bounds hold*

$$\|e\|_{L^\infty(0,T;L^r(\Omega))} \le \mathcal{E}_r^\infty(\theta), \tag{13}$$

where

$$\mathcal{E}_r^\infty(\theta) := \big(4rc_1\mathrm{e}^{\bar{C}T}\big)^{1/r}\eta + \|\theta\|_{L^\infty(0,T;L^r(\Omega))}$$

and

$$\epsilon^{2/r}\|e\|_{L^r(0,T;W_r^{2/r-\delta}(\Omega))} \le \mathcal{E}_r^W(\theta), \tag{14}$$

for $\delta > 0$, where

$$\mathcal{E}_r^W(\theta) := \Big(4c_1\Big(\frac{\hat{c}}{r-1} + rT\Big)\mathrm{e}^{\bar{C}T}\Big)^{1/r}\eta + \epsilon^{2/r}\|\theta\|_{L^r(0,T;W_r^{2/r-\delta}(\Omega))},$$

where $\hat{c}$ is the constant in (12).

Proof The bound (13) follows directly from the use triangle inequality along with (9).
To show (14), we begin again with the triangle inequality:

$$\epsilon^{2/r}\|e\|_{L^r(0,T;W_r^{2/r-\delta}(\Omega))} \le \epsilon^{2/r}\|\rho\|_{L^r(0,T;W_r^{2/r-\delta}(\Omega))} + \epsilon^{2/r}\|\theta\|_{L^r(0,T;W_r^{2/r-\delta}(\Omega))},$$

whose first term on the right-hand side remains to be estimated. From (9) and (12), we have, respectively,

$$\begin{aligned}4C_r^{-1}\epsilon^{-2}c_1\eta^2\mathrm{e}^{\bar{C}T} \geq \|\nabla(\rho^{r/2})\|^2_{L^2(0,T;L^2(\Omega))} &= \frac{r^2}{4}\int_0^T\int_\Omega \rho^{r-2}|\nabla\rho|^2\mathrm{d}x\mathrm{d}t\\ &\geq \frac{r^2}{4\hat{c}}\int_0^T \big(\|\rho\|^r_{W_r^{2/r-\delta}(\Omega)} - \|\rho\|_r^r\big)\mathrm{d}t,\end{aligned}$$

which implies that

$$\epsilon^2\|\rho\|^r_{L^r(0,T;W_r^{2/r-\delta}(\Omega))} \leq \frac{4c_1\hat{c}}{r-1}\eta^r\mathrm{e}^{\bar{C}T} + T\|\rho\|^r_{L^\infty(0,T;L^r(\Omega))},$$

resulting to (14), by bounding the last term on the right-hand side using (9).

□

Remark 3.1 *It is possible to deduce a-posteriori bounds for the case r being any real positive number (not just even integer) using the Interpolation Theorem for L^r-spaces. This can be done in the future.*

We now present an attempt to derive $L^\infty(L^\infty)$-type a posteriori bounds, i.e., cover also the case $r=\infty$.

Theorem 3.3 *With the same assumptions and notation as in Lemma 3.2, the following bound holds*

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^\infty(\Omega))} \leq{}& \|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\rho(\cdot,0)\|_\infty + \|\theta_t\|_{L^1(0,T;L^\infty(\Omega))}\\ &+ C(d)\epsilon^{-2}\Big(\|f'(u_h)\|_{L^\infty(0,T;L^\infty(\Omega))}\mathcal{E}_2^\infty(\theta)\\ &\qquad + 3\|u_h\|_{L^\infty(0,T;L^\infty(\Omega))}\big(\mathcal{E}_4^\infty(\theta)\big)^2 + \big(\mathcal{E}_6^\infty(\theta)\big)^3\Big).\end{aligned}$$

Proof Using Lemma 2.1 from [3] in (6), we deduce

$$\rho(x,t) = \int_\Omega G(x,t;y,s)\rho(y,0)\mathrm{d}y + \int_0^t\int_\Omega G(x,t;y,s)\big(\theta_t + \epsilon^{-2}(ef'(u_h) + 3e^2u_h + e^3)\big)(y,s)\mathrm{d}y\mathrm{d}s,$$

for $s<t$, which implies

$$\begin{aligned}|\rho(x,t)| \leq{}&\|G(x,t;\cdot,s)\|_1\|\rho(\cdot,0)\|_\infty\\ &+\int_0^t\Big(\|G(x,t;\cdot,s)\|_1\|\theta_t\|_\infty + \epsilon^{-2}\|f'(u_h)\|_\infty\|G(x,t;\cdot,s)\|_2\|e\|_2\\ &+3\epsilon^{-2}\|u_h\|_\infty\|G(x,t;\cdot,s)\|_2\|e^2\|_2 + \epsilon^{-2}\|G(x,t;\cdot,s)\|_2\|e^3\|_2\Big)\mathrm{d}s.\end{aligned}$$

Now, recalling that $\|G(x,t;\cdot,s)\|_1\leq 1$ and that $\int_0^t\|G(x,t;\cdot,s)\|_2\mathrm{d}s \leq C(d)$, for some constant $C(d)>0$, depending only on the dimension d, along with the trivial identities $\|e^2\|_2 = \|e\|_4^2$ and $\|e^3\|_2 = \|e\|_6^3$, we deduce

$$\begin{aligned}|\rho(x,t)| \leq{}&\|\rho(\cdot,0)\|_\infty + \int_0^t\|\theta_t\|_\infty\mathrm{d}s + C(d)\epsilon^{-2}\Big(\|f'(u_h)\|_{L^\infty(0,T;L^\infty(\Omega))}\|e\|_{L^\infty(0,T;L^2(\Omega))}\\ &+3\|u_h\|_{L^\infty(0,T;L^\infty(\Omega))}\|e\|^2_{L^\infty(0,T;L^4(\Omega))} + \|e\|^3_{L^\infty(0,T;L^6(\Omega))}\Big).\end{aligned}$$

Using now (13) for $r=2$, $r=4$ and $r=6$, respectively, the result follows using triangle inequality.

□

Remark 3.2 (Notes)

1. *I only did the semidiscrete case for simplicity, but I don't think it would be impossible to do the fully-discrete one. Bartels did the fully discrete one I think (along with an algorithm for the efficient solution of the corresponding eigenvalue problem), using the direct approach, of course (i.e., no elliptic reconstruction).*

2. *I think we could also include convection, as we are in fact calculating the spectral estimate explicitly, so there is no need to rely on known analytical results.*

3. *Everything should be extendable to discontinuous Galerkin in space, but I don't think that the corresponding elliptic a-posteriori bounds for the various norms of θ (especially the L^∞ one are available.*

4. *I haven't checked if the bounds above are "optimal" in terms of the mesh-size.*

5. *Another thing I would like to check is if we can get improved bounds for the $L^\infty(L^2)$-norm using the $L^\infty(L^\infty)$-bounds, i.e., some form of bootstrapping. I am looking into this.*

A A posteriori error bounds for the simpler case of $L^\infty(L^2)$-norm – just to draw the basic ideas

Definition A.1 *We define the elliptic reconstruction $w \in H^1(\Omega)$ of u_h by*

$$B(w, v) = B(u_h, v) + \epsilon^{-2}\langle f(u_h) - \Pi f(u_h), v\rangle, \quad \text{for all } v \in H^1(\Omega),$$

such that

$$\int_\Omega w \mathrm{d}x = 0.$$

We can now split the error as follows

$$e := u - u_h = \theta - \rho,$$

where $\theta := w - u_h$ and $\rho := w - u$.

Lemma A.1 *We have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2 + \|\nabla\rho\|^2 + \epsilon^{-2}\langle f'(u_h)\rho, \rho\rangle \le \langle\theta_t, \rho\rangle + \epsilon^{-2}\big(\langle f'(u_h)\theta, \rho\rangle + 3\langle e^2 u_h, \rho\rangle + \langle\theta^3, \rho\rangle + 3\langle\theta, \rho^3\rangle\big). \tag{15}$$

Proof The error equation

$$\langle\rho_t, v\rangle + B(\rho, v) + \epsilon^{-2}\langle f(u_h) - f(u), v\rangle = \langle\theta_t, v\rangle, \tag{16}$$

holds for all $v \in H^1(\Omega)$; this follows directly using the PDE, the finite element method and the definition of w. Setting $v = \rho$, we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2 + \|\nabla\rho\|^2 + \epsilon^{-2}\langle f(u_h) - f(u), \rho\rangle = \langle\theta_t, \rho\rangle. \tag{17}$$

Using Taylor's theorem, we find

$$f(u_h) - f(u) = -ef'(u_h) - 3e^2 u_h - e^3,$$

and, observing the crucial property

$$\langle e^3, \rho\rangle = \langle \theta^3, \rho\rangle - 3\langle \theta^2\rho, \rho\rangle + 3\langle \theta\rho^2, \rho\rangle - \langle \rho^3, \rho\rangle \le \langle \theta^3, \rho\rangle + 3\langle \theta, \rho^3\rangle,$$

we arrive to

$$\langle f(u_h) - f(u), \rho\rangle \ge -\langle f'(u_h)\theta, \rho\rangle + \langle f'(u_h)\rho, \rho\rangle - 3\langle e^2 u_h, \rho\rangle - \langle \theta^3, \rho\rangle - 3\langle \theta, \rho^3\rangle,$$

which, in conjunction with (17), yields the result.

□

We now assume that there exists $\lambda > 0$ such that

$$-\lambda\|v\|^2 \le \|\nabla v\|^2 + \epsilon^{-2}\langle f'(u_h)v, v\rangle, \quad \text{for all } v \in H^1(\Omega). \tag{18}$$

(We can actually approximate this λ numerically.) To verify that λ is independent of ϵ, we can solve numerically the corresponding eigenvalue problem [1].

Theorem A.1 *Let*

$$\begin{aligned}
A(t) &:= \frac{1}{2}\|\rho(0)\|^2 + \frac{1}{2}\Big(\|\theta_t\| + \epsilon^{-2}\|f'(u_h)\|_\infty\|\theta\| + 6\epsilon^{-2}\|u_h\|\,\|\theta\|_4^2 + \epsilon^{-2}\|\theta\|_6^3\Big)^2,\\
B(t) &:= 3\epsilon^{-2}\big(2\|u_h\|_\infty + \|\theta\|_\infty\big), \qquad \bar{B}(t) := \sup_{s\in[0,t]} B(s), \qquad \bar{B} := B(T),\\
C(t) &:= 2\lambda(1-\epsilon^2) + 2\|f'(u_h)\|_\infty + 1, \qquad \bar{C}(t) := \sup_{s\in[0,t]} C(s), \qquad \bar{C} := C(T),
\end{aligned}$$

and assume that, when $\bar{B} \neq 0$, there exist constant $c_1 > 0$ and a value $\eta \in \mathbb{R}$ such that we have simultaneously

$$\eta \le \big(512 c_1 \mathrm{e}^{-3\bar{C}T}\big)^{1/2}\tilde{c}\big(\bar{B}\max\{T, \epsilon^{-2}\}\big)^{-1} \quad \text{and} \quad \int_0^T A(t)\mathrm{d}t \le c_1\eta^2,$$

where $\tilde{c}$ is the constant in the Gagliardo-Niremberg inequality $\|w\|_3^3 \le \tilde{c}\|w\|\,\|w\|_{H^1(\Omega)}^2$, for all $w \in H^1(\Omega)$. Then, the following bound holds

$$\frac{1}{2}\|\rho\|^2_{L^\infty(0,T;L^2(\Omega))} + \epsilon^2\|\nabla\rho\|^2_{L^2(0,T;L^2(\Omega))} \le 4c_1\eta^2\mathrm{e}^{\bar{C}T}, \tag{19}$$

implying the a posteriori bound

$$\frac{1}{2}\|e\|^2_{L^\infty(0,T;L^2(\Omega))} + \epsilon^2\|\nabla e\|^2_{L^2(0,T;L^2(\Omega))} \le \big(\mathcal{E}(\theta)\big)^2, \tag{20}$$

where

$$\mathcal{E}(\theta) := \Big(8c_1\eta^2\mathrm{e}^{\bar{C}T} + \|\theta\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\epsilon^2\|\nabla\theta\|^2_{L^2(0,T;L^2(\Omega))}\Big)^{1/2}.$$

Proof We shall estimate the terms appearing in the error bound (15). Using the spectral estimate (18), we deduce

$$\|\nabla\rho\|^2+\epsilon^{-2}\langle f'(u_h)\rho,\rho\rangle=(1-\epsilon^2)\big(\|\nabla\rho\|^2+\epsilon^{-2}\langle f'(u_h)\rho,\rho\rangle\big)+\epsilon^2\|\nabla\rho\|^2+\langle f'(u_h)\rho,\rho\rangle$$
$$\geq -\lambda(1-\epsilon^2)\|\rho\|^2+\epsilon^2\|\nabla\rho\|^2+\langle f'(u_h)\rho,\rho\rangle.$$

Also, various versions of Hölder's inequality yield

$$\langle\theta_t,\rho\rangle\leq\|\theta_t\|\|\rho\|,\qquad \langle f'(u_h)\theta,\rho\rangle\leq\|f'(u_h)\|_\infty\|\theta\|\|\rho\|,$$
$$\langle e^2u_h,\rho\rangle\leq 2\|u_h\|_\infty\big(\|\theta\|_4^2\|\rho\|+\|\rho\|_3^3\big),\qquad \langle\theta^3,\rho\rangle\leq\|\theta\|_6^3\|\rho\|,$$
$$\langle\theta,\rho^3\rangle\leq\|\theta\|_\infty\|\rho\|_3^3,\qquad |\langle f'(u_h)\rho,\rho\rangle|\leq\|f'(u_h)\|_\infty\|\rho\|^2,$$

respectively.

Now, inserting these bounds into (15) and collecting the various terms accordingly, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2+\epsilon^2\|\nabla\rho\|^2\leq\Big(2A(t)-\|\rho(0)\|^2\Big)^{1/2}\|\rho\|+B(t)\|\rho\|_3^3+\frac{1}{2}\big(C(t)-1\big)\|\rho\|^2.$$

Using the Gagliardo-Niremberg inequality (and Young's inequality) we arrive to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2+\epsilon^2\|\nabla\rho\|^2\leq A(t)-\frac{1}{2}\|\rho(0)\|^2+B(t)\tilde{c}\|\rho\|\big(\|\rho\|^2+\|\nabla\rho\|^2\big)+\frac{1}{2}C(t)\|\rho\|^2.$$

An integration with respect to the time variable between 0 and τ, for $\tau\in[0,T]$, yields, respectively,

$$\begin{aligned}\frac{1}{2}\|\rho(\tau)\|^2+\epsilon^2\int_0^\tau\|\nabla\rho\|^2\mathrm{d}t&\leq\int_0^\tau A(t)\mathrm{d}t+\frac{1}{2}\bar{C}(\tau)\int_0^\tau\|\rho\|^2\mathrm{d}t\\&\quad+\tilde{c}\bar{B}(\tau)\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|\Big(\tau\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|^2+\int_0^\tau\|\nabla\rho\|^2\mathrm{d}t\Big)\\&\leq\int_0^T A(t)\mathrm{d}t+\frac{1}{2}\bar{C}\int_0^\tau\|\rho\|^2\mathrm{d}t\\&\quad+\sqrt{8}\tilde{c}\bar{B}\max\{T,\epsilon^{-2}\}\Big(\frac{1}{2}\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|^2+\epsilon^2\int_0^\tau\|\nabla\rho\|^2\mathrm{d}t\Big)^{3/2}\end{aligned}$$

We now consider the set

$$I:=\big\{\tau\in[0,T]:\frac{1}{2}\operatorname{ess\,sup}_{[0,\tau]}\|\rho\|^2+\epsilon^2\int_0^\tau\|\nabla\rho\|^2\mathrm{d}t\leq 4c_1\eta^2\mathrm{e}^{\bar{C}T}\big\},$$

which is non-empty due to the continuity of the left-hand side (we need maybe to assume that $\rho(0)=0$ here). We set $\tau^*=\max I$ and we suppose that $\tau^*<T$. Hence, for $\tau\leq\tau^*$, using also the hypotheses on c_1 and η, we deduce

$$\frac{1}{2}\|\rho(\tau)\|^2+\epsilon^2\int_0^\tau\|\nabla\rho\|^2\mathrm{d}t\leq 2c_1\eta^2+\frac{1}{2}\bar{C}\int_0^\tau\|\rho\|^2\mathrm{d}t.$$

Gronwall's Lemma then implies

$$\frac{1}{2}\|\rho(\tau^*)\|^2+\epsilon^2\int_0^{\tau^*}\|\nabla\rho\|^2\mathrm{d}t\leq 2c_1\eta^2\mathrm{e}^{\bar{C}T},\tag{21}$$

setting $\tau=\tau^*$, which contradicts the hypothesis $\tau^*<T$, due to the continuity of the left-hand side of (21). Hence, $I=[0,T]$ and we conclude that (19) holds. The a posteriori bound (20) now follows from triangle inequality.

□

References

[1] Bartels, S. A posteriori error analysis for time-dependent Ginzburg-Landau type equations. *Numer. Math. 99*, 4 (2005), 557–583.

[2] Ciarlet, P. G. *The finite element method for elliptic problems*, vol. 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[3] Demlow, A., Lakkis, O., and Makridakis, C. A posteriori error estimates in the maximum norm for parabolic problems. *ArXiv.org*.

[4] Málek, J., Pražák, D., and Steinhauer, M. On the existence and regularity of solutions for degenerate power-law fluids. *Differential Integral Equations 19*, 4 (2006), 449–462.

[5] Roubíček, T. *Nonlinear partial differential equations with applications*, vol. 153 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel, 2005.