2. Bivariate Cases

Case 2.1

Suppose the data follows a Normal distribution with unknown mean μ and unknown variance σ^2: $X|\mu,\sigma^2 \sim \text{Normal}(\mu,\sigma^2)$. Let us consider a ***conjugate prior*** distribution where $\mu|\sigma^2$ follows a Normal distribution with hyperparameters $\mu_0, \sigma^2/k_0$ with k_0 some known constant: $\mu|\sigma^2 \sim \text{N}(\mu_0, \sigma^2/k_0)$ and the prior distribution of the parameter σ^2 is scaled-Inverse-X² with hyperparameters $\nu_0, {\sigma_0}^2$: $\sigma^2 \sim \text{scaled-Inverse-X}^2(\nu_0, {\sigma_0}^2)$.

__Data__: $p(x|\mu,\sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Leftrightarrow X|\mu,\sigma^2 \sim \text{Normal}(\mu,\sigma^2)$.

__Prior distribution__:

$$p(\mu|\sigma^2) = \frac{1}{\sigma/\sqrt{k_0}\,\sqrt{2\pi}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma^2/k_0}} \Leftrightarrow \mu|\sigma^2 \sim \text{N}(\mu_0, \sigma^2/k_0).$$

$$p(\sigma^2) = \frac{\left(\frac{\nu_0}{2}{\sigma_0}^2\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)} (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} e^{-\frac{\nu_0 {\sigma_0}^2}{2\sigma^2}} \Leftrightarrow \sigma^2 \sim \text{scaled-Inverse-X}^2(\nu_0, {\sigma_0}^2).$$

In brief: $p(\mu,\sigma^2) = p(\mu|\sigma^2)\cdot p(\sigma^2) \sim \text{Normal-Inv-}X^2\left(\mu,\sigma^2 \middle| \mu_0, \sigma^2/k_0, \nu_0, {\sigma_0}^2\right)$.

Likelihood function:

$$p\left(\mathbf{x}^{(n)}\middle|\mu,\sigma^2\right)=(2\pi)^{-n/2}(\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$\propto(\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\overbrace{\sum_{i=1}^{n}(x_i-\overline{x})^2}^{(n-1)S^2}\right\}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\overline{x}-\mu)^2\right\}$$

$$\propto(\sigma^2)^{-n/2}\exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\}\exp\left\{-\frac{n(\overline{x}-\mu)^2}{2\sigma^2}\right\}$$

$$\propto(\sigma^2)^{-1/2}\exp\left\{-\frac{(\overline{x}-\mu)^2}{2\sigma^2/n}\right\}\cdot(\sigma^2)^{-\frac{n-1}{2}}\exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\},\qquad \text{where } S^2=\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n-1}.$$

Posterior distribution:

$$p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)\propto(\sigma^2)^{-1/2}(\sigma^2)^{-n/2}(\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)}\exp\left\{-\frac{n(\overline{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\}\exp\left\{-\frac{k_0(\mu-\mu_0)^2+\nu_0\sigma_0^{\,2}}{2\sigma^2}\right\}$$

$$\propto\left(\sigma^2\right)^{-1/2}(\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)}\exp\left\{-\frac{(n+k_0)\mu^2+(-2n\overline{x}-2k_0\mu_0)\mu+\nu_0\sigma_0^{\,2}+(n-1)S^2+k_0\mu_0^{\,2}+n\overline{x}^2}{2\sigma^2}\right\}$$

$$\propto(\sigma^2)^{-1/2}(\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)}\exp\left\{-\frac{\nu_0\sigma_0^{\,2}+(n-1)S^2+(k_0+n)\left[\mu^2-2\left(\frac{n\overline{x}+k_0\mu_0}{k_0+n}\right)\mu\pm\left(\frac{n\overline{x}+k_0\mu_0}{k_0+n}\right)^2\right]+k_0\mu_0^{\,2}+n\overline{x}^2}{2\sigma^2}\right\}$$

$$\propto(\sigma^2)^{-1/2}(\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)}\exp\left\{-\frac{\nu_0\sigma_0^{\,2}+(n-1)S^2+(k_0+n)\left[\mu-\frac{n\overline{x}+k_0\mu_0}{k_0+n}\right]^2-\frac{(n\overline{x}+k_0\mu_0)^2}{k_0+n}+\left(n\overline{x}^2+k_0\mu_0^{\,2}\right)}{2\sigma^2}\right\}$$

A new identity is formed using the two last quantities of the exponential:

$$-\frac{(n\overline{x}+k_0\mu_0)^2}{k_0+n}+\left(\frac{k_0+n}{k_0+n}\right)\left(n\overline{x}^2+k_0\mu_0^{\,2}\right)=$$

$$= \frac{-(n\bar{x})^2 - (k_0\mu_0)^2 - 2n\bar{x}k_0\mu_0 + (k_0\mu_0)^2 + nk_0\mu_0{}^2 + nk_0\bar{x}^2 + (n\bar{x})^2}{k_0 + n}$$

$$= \frac{nk_0(-2\bar{x}\mu_0 + \mu_0{}^2 + \bar{x}^2)}{k_0 + n} = \frac{nk_0(\bar{x} - \mu_0)^2}{k_0 + n}.$$

Therefore, the posterior distribution can be written in a well-known form:

$$p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right) \propto (\sigma^2)^{-1/2} (\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)} \exp\left\{ -\frac{\nu_0\sigma_0{}^2 + (n-1)S^2 + \frac{nk_0(\bar{x}-\mu_0)^2}{k_0+n} + (k_0+n)\left[\mu - \frac{n\bar{x}+k_0\mu_0}{k_0+n}\right]^2}{2\sigma^2} \right\}$$

$$\propto (\sigma^2)^{-1/2} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{ -\frac{\nu_n\sigma_n{}^2 + k_n(\mu-\mu_n)^2}{2\sigma^2} \right\} \Leftrightarrow$$

$$p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right) = p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) \cdot p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right) = \text{Normal-Inv-}X^2\left(\mu, \sigma^2 \middle| \mu_n, \frac{\sigma^2}{k_n}, \nu_n, \sigma_n{}^2\right),$$

where $\mu_n = \frac{k_0}{k_0+n}\mu_0 + \frac{n}{k_0+n}\bar{x}$,

$k_n = k_0 + n$, $\nu_n = \nu_0 + n$,

$\nu_n\sigma_n{}^2 = \nu_0\sigma_0{}^2 + (n-1)S^2 + \frac{nk_0(\bar{x}-\mu_0)^2}{k_0+n}$.

Normalizing constant of the posterior distribution:

$$K = \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}\sigma_n{}^2\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)}$$

Full Conditional posterior distributions:

- $p\left(\sigma^2 \middle| \mu, \mathbf{x}^{(n)}\right) = \frac{p\left(\sigma^2, \mu \middle| \mathbf{x}^{(n)}\right)}{p\left(\mu \middle| \mathbf{x}^{(n)}\right)} \propto p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)$

$$\propto (\sigma^2)^{-\left(\frac{\nu_0+n+1}{2}+1\right)} \exp\left\{-\frac{\nu_0{\sigma_0}^2 + k_0(\mu-\mu_0)^2 + \overbrace{\sum_{i=1}^{n}(x_i-\mu)^2}^{n\cdot S_\mu^2}}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\left(\frac{\nu_0+n+1}{2}+1\right)} \exp\left\{-\frac{\nu_0{\sigma_0}^2 + k_0(\mu-\mu_0)^2 + n{S_\mu}^2}{2\sigma^2}\right\} \quad \Leftrightarrow$$

$$p\left(\sigma^2 \middle| \mu, \mathbf{x}^{(n)}\right) = \text{scaled-Inv-}X^2\left(\sigma^2 \middle| \nu_0 + n + 1, \frac{\nu_0{\sigma_0}^2 + k_0(\mu-\mu_0)^2 + n{S_\mu}^2}{\nu_0 + n + 1}\right)$$

- $$p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) = \frac{p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)}{p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right)} \propto p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)$$

$$\propto \exp\left\{-\frac{(k_0+n)}{2\sigma^2}\left(\mu - \frac{k_0\mu_0 + n\overline{x}}{k_0+n}\right)^2\right\} \Leftrightarrow$$

$$p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) = \text{Normal}\left(\mu \middle| \frac{k_0\mu_0 + n\overline{x}}{k_0+n}, \sigma^2 / (k_0+n)\right) = \text{N}\left(\mu \middle| \mu_n, \sigma^2 / k_n\right)$$

<u>Marginal posterior distributions</u>:

- $$p\left(\mu \middle| \mathbf{x}^{(n)}\right) = \int_0^{+\infty} p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right) \mathrm{d}\sigma^2$$

$$= \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}{\sigma_n}^2\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} \cdot \int_0^{+\infty} (\sigma^2)^{-1/2} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{\nu_n{\sigma_n}^2 + k_n(\mu-\mu_n)^2}{2\sigma^2}\right\} \mathrm{d}\sigma^2$$

$$= \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}{\sigma_n}^2\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} \cdot \int_0^{+\infty} (\sigma^2)^{-\left(\frac{\nu_n+1}{2}+1\right)} \exp\left\{-\frac{\nu_n{\sigma_n}^2}{2\sigma^2}\left[1 + \frac{k_n(\mu-\mu_n)^2}{\nu_n{\sigma_n}^2}\right]\right\} \mathrm{d}\sigma^2$$

We set $z = \frac{\nu_n \sigma_n^{\ 2}}{2\sigma^2} \Rightarrow \quad \sigma^2 = \frac{\nu_n \sigma_n^{\ 2}}{2z} \quad \Leftrightarrow \quad \left(\sigma^2\right)^{-1} = \frac{2z}{\nu_n \sigma_n^{\ 2}}$

and $d\sigma^2 = -\frac{\nu_n \sigma_n^{\ 2}}{2z^2} dz$,with the integral boundaries being reversed.

Additionally, we set $B = \left[1 + \frac{k_n(\mu-\mu_n)^2}{\nu_n \sigma_n^{\ 2}}\right]$. Thus, we have:

$$p\left(\mu \middle| \mathbf{x}^{(n)}\right) = \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}\sigma_n^{\ 2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} \cdot \int_{+\infty}^{0} \left(\frac{2z}{\nu_n \sigma_n^{\ 2}}\right)^{\frac{\nu_n+1}{2}+1} \left(-\frac{\nu_n \sigma_n^{\ 2}}{2z^2}\right) \exp\{-B\cdot z\}\, \mathrm{d}z$$

$$= \frac{\sqrt{k_n}}{\sqrt{\pi}} \cdot \frac{\left(\nu_n\right)^{\frac{\nu_n}{2}} \left(\sigma_n^{\ 2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} \cdot \frac{2^{-\frac{1}{2}+\frac{1}{2}+\frac{\nu_n}{2}-\frac{\nu_n}{2}+1-1}}{\left(\nu_n \sigma_n^{\ 2}\right)^{\frac{\nu_n+1}{2}+1-1}} \int_{0}^{+\infty} z^{\frac{\nu_n+1}{2}-1} \exp\{-B\cdot z\}\, \mathrm{d}z$$

$$= \frac{\sqrt{k_n}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)} \left(\sigma_n^{\ 2}\right)^{-\frac{1}{2}} \left(\nu_n\right)^{-\frac{1}{2}} \cdot \left[1 + \frac{k_n(\mu-\mu_n)^2}{\nu_n \sigma_n^{\ 2}}\right]^{-\frac{\nu_n+1}{2}}$$

$$= \frac{1}{\sqrt{\nu_n \cdot \pi}} \cdot \frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)} \sqrt{k_n} \left(\sigma_n^{\ 2}\right)^{-\frac{1}{2}} \cdot \left[1 + \frac{1}{\nu_n} \frac{(\mu-\mu_n)^2}{\sigma_n^{\ 2}/k_n}\right]^{-\frac{\nu_n+1}{2}} \quad \Leftrightarrow$$

$$p\left(\mu \middle| \mathbf{x}^{(n)}\right) = \mathrm{Student}\left(\mu \middle| \mu_n, \frac{\sigma_n^{\ 2}}{k_n}, \nu_n\right)$$

- $p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right) = \int_{-\infty}^{+\infty} p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right) \mathrm{d}\mu$

$$= \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}\sigma_n^{\ 2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{\nu_n \sigma_n^{\ 2}}{2\sigma^2}\right\} \cdot \int_{0}^{+\infty} \underbrace{(\sigma^2)^{-1/2} \exp\left\{-\frac{k_n(\mu-\mu_n)^2}{2\sigma^2}\right\}}_{\text{Normal-kernel}} \mathrm{d}\mu$$

$$= \frac{\sqrt{k_n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{\nu_n}{2}\sigma_n^{\ 2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{\nu_n \sigma_n^{\ 2}}{2\sigma^2}\right\} \cdot \left(\frac{\sqrt{k_n}}{\sqrt{2\pi}}\right)^{-1}$$

$$= \frac{\left(\frac{\nu_n}{2}\sigma_n^{\ 2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{\nu_n \sigma_n^{\ 2}}{2\sigma^2}\right\} \quad \Leftrightarrow$$

$$p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right) = \text{scaled-Inv-Gamma}\left(\sigma^2 \middle| \frac{\nu_n}{2}, \frac{\nu_n \sigma_n^{\ 2}}{2}\right) = \text{scaled-Inv-}X^2\left(\sigma^2 \middle| \nu_n, \sigma_n^{\ 2}\right)$$

Predictive distribution:

We must use the relation $p\left(y \middle| \mathbf{x}^{(n)}\right) = \int p\left(y \middle| \mathbf{x}^{(n)}, \sigma^2\right) p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right) d\sigma^2$. Therefore, we initially have to induce the distribution of $Y \middle| \mathbf{x}^{(n)}, \sigma^2$ which remains unknown up to this point.

Note: $p\left(y \middle| \mu, \sigma^2, \mathbf{x}^{(n)}\right) = p\left(y \middle| \mu, \sigma^2\right)$ → conditional independence.

$$p\left(y \middle| \sigma^2, \mathbf{x}^{(n)}\right) = \int_{-\infty}^{+\infty} p\left(y \middle| \mu, \sigma^2, \mathbf{x}^{(n)}\right) p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) \mathrm{d}\mu = \int_{-\infty}^{+\infty} p\left(y \middle| \mu, \sigma^2\right) p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) \mathrm{d}\mu$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \frac{k_n}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{k_n}{2\sigma^2}(\mu-\mu_n)^2\right\} \mathrm{d}\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{(y-\mu)^2 + k_n(\mu-\mu_n)^2}{2\sigma^2}\right\} \mathrm{d}\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{y^2 + \mu^2 - 2y\mu + k_n\mu^2 + k_n\mu_n^2 - 2k_n\mu\mu_n}{2\sigma^2}\right\} \mathrm{d}\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{(k_n+1)\mu^2 - 2\mu(y + k_n\mu_n) + y^2 + k_n\mu_n^2}{2\sigma^2}\right\} \mathrm{d}\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(k_n+1)}{2\sigma^2} \left(\mu^2 - 2\mu\left(\frac{y+k_n\mu_n}{k_n+1}\right) + \frac{y^2}{k_n+1} + \frac{k_n\mu_n^2}{k_n+1} \right) \right\} d\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(k_n+1)}{2\sigma^2} \left(\mu^2 - 2\mu\left(\frac{y+k_n\mu_n}{k_n+1}\right) \pm \left(\frac{y+k_n\mu_n}{k_n+1}\right)^2 + \frac{y^2}{k_n+1} + \frac{k_n\mu_n^2}{k_n+1} \right) \right\} d\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2} \left(y^2 + k_n\mu_n^2 - \frac{(y+k_n\mu_n)^2}{k_n+1} \right) \right\} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(k_n+1)}{2\sigma^2} \left(\mu - \frac{y+k_n\mu_n}{k_n+1} \right)^2 \right\} d\mu$$

$$= \frac{\sqrt{k_n}}{2\pi\sigma^2} \cdot \frac{\sqrt{2\pi\sigma^2}}{\sqrt{k_n+1}} \cdot \exp\left\{ -\frac{1}{2\sigma^2} \left(\frac{k_n y^2 + y^2 + k_n{}^2\mu_n^2 + k_n\mu_n^2 - y^2 - k_n{}^2\mu_n^2 - 2yk_n\mu_n}{k_n+1} \right) \right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{\sqrt{k_n}}{\sqrt{k_n+1}} \cdot \exp\left\{ -\frac{1}{2\sigma^2} \cdot \frac{k_n}{k_n+1} \left(y^2 + \mu_n^2 - 2y\mu_n \right) \right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{\sqrt{k_n}}{\sqrt{k_n+1}} \cdot \exp\left\{ -\frac{1}{2\sigma^2} \cdot \frac{k_n}{k_n+1} \left(y - \mu_n \right)^2 \right\} \quad \Leftrightarrow$$

$$p\left(y \middle| \mathbf{x}^{(n)}, \sigma^2\right) = \text{Normal}\left(y \middle| \mu_n, \ \sigma^2 \Big/ \frac{k_n}{k_n+1} \right)$$

We substitute the result to the initial formula:

$$p\left(y \middle| \mathbf{x}^{(n)}\right) = \int p\left(y \middle| \mathbf{x}^{(n)}, \sigma^2\right) p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right) d\sigma^2$$

$$= \int_0^{+\infty} (2\pi\sigma^2)^{-1/2} \left(\frac{k_n}{k_n+1}\right)^{1/2} \exp\left\{ -\frac{1}{2\sigma^2} \frac{k_n}{k_n+1} (y-\mu_n)^2 \right\} \frac{\left(\frac{\nu_n}{2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} (\sigma_n{}^2)^{\left(\frac{\nu_n}{2}\right)} (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{ -\frac{\nu_n\sigma_n{}^2}{2\sigma^2} \right\} d\sigma^2$$

$$= (2\pi)^{-1/2} \left(\frac{k_n}{k_n+1}\right)^{1/2} \frac{\left(\frac{\nu_n}{2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)} (\sigma_n{}^2)^{\left(\frac{\nu_n}{2}\right)} \int_0^{+\infty} (\sigma^2)^{-\left(\frac{\nu_n+1}{2}+1\right)} \exp\left\{ -\frac{1}{2\sigma^2} \left[\frac{k_n}{k_n+1} (y-\mu_n)^2 + \nu_n\sigma_n{}^2 \right] \right\} d\sigma^2$$

$$= (2\pi)^{-1/2}\left(\frac{k_n}{k_n+1}\right)^{1/2}\frac{\left(\frac{\nu_n}{2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)}(\sigma_n{}^2)^{\left(\frac{\nu_n}{2}\right)}\underbrace{\int_0^{+\infty}(\sigma^2)^{-\left(\frac{\nu_n+1}{2}+1\right)}\exp\left\{-\frac{\nu_n+1}{2\sigma^2}\frac{\left[\frac{k_n}{k_n+1}(y-\mu_n)^2+\nu_n\sigma_n{}^2\right]}{\nu_n+1}\right\}}_{\text{scaled-Inv}-X^2\text{ kernel}}\mathrm{d}\sigma^2$$

$$= (2\pi)^{-1/2}\left(\frac{k_n}{k_n+1}\right)^{1/2}\frac{\left(\frac{\nu_n}{2}\right)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)}(\sigma_n{}^2)^{\left(\frac{\nu_n}{2}\right)}\Gamma\left(\frac{\nu_n+1}{2}\right)\left(\frac{\nu_n+1}{2}\right)^{-\frac{\nu_n+1}{2}}\left[\frac{\frac{k_n}{k_n+1}(y-\mu_n)^2+\nu_n\sigma_n{}^2}{\nu_n+1}\right]^{-\frac{\nu_n+1}{2}}$$

$$= \frac{2^{-\frac{1}{2}+\frac{1}{2}+\frac{\nu_n}{2}-\frac{\nu_n}{2}}}{\sqrt{\pi}}\left(\frac{k_n}{k_n+1}\right)^{1/2}\frac{(\nu_n)^{\frac{\nu_n}{2}}}{\Gamma\left(\frac{\nu_n}{2}\right)}(\sigma_n{}^2)^{\left(\frac{\nu_n}{2}\right)}\Gamma\left(\frac{\nu_n+1}{2}\right)\frac{(\nu_n+1)^{-\frac{\nu_n+1}{2}}}{(\nu_n+1)^{-\frac{\nu_n+1}{2}}}\left(\nu_n\sigma_n{}^2\right)^{-\frac{\nu_n+1}{2}}\left[\frac{k_n}{k_n+1}\frac{(y-\mu_n)^2}{\nu_n\sigma_n{}^2}+1\right]^{-\frac{\nu_n+1}{2}}$$

$$= \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)}\left(\frac{k_n}{k_n+1}\right)^{1/2}(\nu_n)^{\frac{\nu_n}{2}-\frac{\nu_n}{2}-\frac{1}{2}}(\sigma_n{}^2)^{\left(\frac{\nu_n}{2}-\frac{\nu_n}{2}-\frac{1}{2}\right)}\left[1+\frac{k_n}{k_n+1}\frac{(y-\mu_n)^2}{\nu_n\sigma_n{}^2}\right]^{-\frac{\nu_n+1}{2}}$$

$$= \frac{1}{\sqrt{\nu_n\cdot\pi}}\frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)}\left(\frac{k_n}{k_n+1}\right)^{1/2}(\sigma_n{}^2)^{-\frac{1}{2}}\left[1+\frac{1}{\nu_n}\frac{(y-\mu_n)^2}{\sigma_n{}^2\Big/\left(\frac{k_n}{k_n+1}\right)}\right]^{-\frac{\nu_n+1}{2}} \Leftrightarrow$$

$$p\left(y\middle|\mathbf{x}^{(n)}\right) = \text{Student}\left(y\middle|\mu_n,\left(\frac{k_n}{k_n+1}\right)^{-1}\sigma_n{}^2,\nu_n\right)$$

Prior predictive distribution:

Usually, in order to find the a-priori predictive distribution (i.e. marginal likelihood), we use the relation $p(x)=\int_\Theta p(x|\theta)\,p(\theta)\,d\theta$. As to the particular biparametric case, we choose one of the parameters arbitrarilly and finally the distribution in question will be determined through the relation $p(x)=\int p(x|\sigma^2)\,p(\sigma^2)\,d\sigma^2$. Therefore, one must first calculate the distribution of $X\left|\sigma^2\right.$ (conditioning and then integrating over μ). The results are similar to those of the previous procedure for the predictive distribution.

$$p\left(x\left|\sigma^2\right.\right)=\int_{-\infty}^{+\infty} p\left(x\left|\mu,\sigma^2\right.\right)p\left(\mu\left|\sigma^2\right.\right)d\mu$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\frac{\sqrt{k_0}}{\sigma\sqrt{2\pi}}e^{-\frac{k_0(\mu-\mu_0)^2}{2\sigma^2}}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(x-\mu)^2+k_0(\mu-\mu_0)^2}{2\sigma^2}\right\}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{x^2+\mu^2-2\mu x+k_0\mu^2+k_0\mu_0^{\ 2}-2k_0\mu\mu_0}{2\sigma^2}\right\}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(k_0+1)\mu^2-2(x+k_0\mu_0)\mu+x^2+k_0\mu_0^{\ 2}}{2\sigma^2}\right\}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(k_0+1)}{2\sigma^2}\left[\mu^2-2\left(\frac{x+k_0\mu_0}{k_0+1}\right)\mu+\frac{x^2}{k_0+1}+\frac{k_0\mu_0^{\ 2}}{k_0+1}\right]\right\}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(k_0+1)}{2\sigma^2}\left[\left(\mu-\frac{x+k_0\mu_0}{k_0+1}\right)^2-\left(\frac{x+k_0\mu_0}{k_0+1}\right)^2+\frac{x^2}{k_0+1}+\frac{k_0\mu_0^{\ 2}}{k_0+1}\right]\right\}d\mu$$

$$=(2\pi\sigma^2)^{-1}k_0^{1/2}\exp\left\{-\frac{1}{2\sigma^2}\left[-\frac{(x+k_0\mu_0)^2}{k_0+1}+x^2+k_0\mu_0^{\ 2}\right]\right\}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(k_0+1)}{2\sigma^2}\left[\left(\mu-\frac{x+k_0\mu_0}{k_0+1}\right)^2\right]\right\}d\mu$$

$$=\frac{k_0^{1/2}}{(2\pi\sigma^2)}\frac{(2\pi\sigma^2)^{1/2}}{(k_0+1)^{1/2}}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{-x^2-(k_0\mu_0)^2-2xk_0\mu_0+k_0x^2+x^2+(k_0\mu_0)^2+k_0\mu_0^{\ 2}}{k_0+1}\right]\right\}$$

$$=(2\pi\sigma^2)^{-1/2}\left(\frac{k_0}{k_0+1}\right)^{1/2}\exp\left\{-\frac{1}{2\sigma^2}\frac{k_0}{k_0+1}(x-\mu_0)^2\right\}\ \Leftrightarrow$$

$$p\left(x\left|\sigma^2\right.\right)=\text{Normal}\left(x\left|\mu_0,\left(\frac{k_0}{k_0+1}\right)^{-1}\sigma^2\right.\right)$$

Thus, after substituting to the initial formula, we get:

$$p(x)=\int p\left(x\middle|\sigma^2\right)p\left(\sigma^2\right)d\sigma^2=$$

$$=\int_0^{+\infty}(2\pi\sigma^2)^{-1/2}\left(\frac{k_0}{k_0+1}\right)^{1/2}\exp\left\{-\frac{1}{2\sigma^2}\frac{k_0}{k_0+1}(x-\mu_0)^2\right\}\cdot\frac{\left(\frac{\nu_0}{2}\sigma_0^{\ 2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)}(\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)}\exp\left\{-\frac{\nu_0\sigma_0^{\ 2}}{2\sigma^2}\right\}d\sigma^2$$

$$=(2\pi)^{-1/2}\left(\frac{k_0}{k_0+1}\right)^{1/2}\frac{\left(\frac{\nu_0}{2}\sigma_0^{\ 2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)}\int_0^{+\infty}(\sigma^2)^{-\left(\frac{\nu_0+1}{2}+1\right)}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{k_0}{k_0+1}(x-\mu_0)^2+\nu_0\sigma_0^{\ 2}\right]\right\}d\sigma^2$$

$$=(2\pi)^{-1/2}\left(\frac{k_0}{k_0+1}\right)^{1/2}\frac{\left(\frac{\nu_0}{2}\sigma_0^{\ 2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)}\int_0^{+\infty}(\sigma^2)^{-\left(\frac{\nu_0+1}{2}+1\right)}\exp\left\{-\frac{1}{2\sigma^2}(\nu_0+1)\frac{\left[\frac{k_0}{k_0+1}(x-\mu_0)^2+\nu_0\sigma_0^{\ 2}\right]}{(\nu_0+1)}\right\}d\sigma^2.$$

Inside the integral, the kernel of a scaled-Inv-X² distribution has been formed.

$$p(x)=(2\pi)^{-1/2}\left(\frac{k_0}{k_0+1}\right)^{1/2}\frac{\left(\frac{\nu_0}{2}\sigma_0^{\ 2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\left(\frac{\nu_0+1}{2}\right)^{\frac{\nu_0+1}{2}}}\left[\frac{k_0}{k_0+1}\frac{(x-\mu_0)^2+\nu_0\sigma_0^{\ 2}}{\nu_0+1}\right]^{-\frac{\nu_0+1}{2}}$$

$$=\frac{2^{-1/2}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\frac{k_0}{k_0+1}\right)^{1/2}\sigma_0^{\ \nu_0}\left(\frac{\nu_0}{2}\right)^{\frac{\nu_0}{2}}\left(\frac{\nu_0+1}{2}\right)^{-\frac{\nu_0+1}{2}}(\nu_0+1)^{\frac{\nu_0+1}{2}}\left[\nu_0\sigma_0^{\ 2}\left(1+\frac{k_0}{k_0+1}\frac{(x-\mu_0)^2}{\nu_0\sigma_0^{\ 2}}\right)\right]^{-\frac{\nu_0+1}{2}}$$

$$=\frac{2^{-1/2}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\frac{k_0}{k_0+1}\right)^{1/2}\sigma_0^{\ \nu_0}\left(\frac{\nu_0}{2}\right)^{\frac{\nu_0}{2}}2^{\frac{\nu_0+1}{2}}\left[\nu_0\sigma_0^{\ 2}\right]^{-\frac{\nu_0+1}{2}}\left(1+\frac{1}{\nu_0}\frac{(x-\mu_0)^2}{\sigma_0^{\ 2}\Big/\frac{k_0}{k_0+1}}\right)^{-\frac{\nu_0+1}{2}}$$

$$=\frac{2^{\frac{\nu_0}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\frac{k_0}{k_0+1}\right)^{1/2}\sigma_0^{\nu_0}\left(\frac{\nu_0}{2}\right)^{\frac{\nu_0}{2}}\nu_0^{-1/2}\nu_0^{-\nu_0/2}\left[\sigma_0^{2}\right]^{-\frac{\nu_0+1}{2}}\left(1+\frac{1}{\nu_0}\frac{(x-\mu_0)^2}{\sigma_0^2/\frac{k_0}{k_0+1}}\right)^{-\frac{\nu_0+1}{2}}$$

$$=\frac{2^{\frac{\nu_0}{2}}}{\sqrt{\pi\nu_0}}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\frac{k_0}{k_0+1}\right)^{1/2}2^{-\nu_0/2}(\sigma_0^2)^{\nu_0/2}\left[\sigma_0^{2}\right]^{-\frac{\nu_0+1}{2}}\left(1+\frac{1}{\nu_0}\frac{(x-\mu_0)^2}{\sigma_0^2/\frac{k_0}{k_0+1}}\right)^{-\frac{\nu_0+1}{2}}$$

$$=\frac{1}{\sqrt{\pi\cdot\nu_0}}\frac{\Gamma\left(\frac{\nu_0+1}{2}\right)}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\frac{k_0}{k_0+1}\right)^{1/2}\left[\sigma_0^{2}\right]^{-\frac{1}{2}}\left(1+\frac{1}{\nu_0}\frac{(x-\mu_0)^2}{\sigma_0^2/\frac{k_0}{k_0+1}}\right)^{-\frac{\nu_0+1}{2}}\Leftrightarrow$$

$$p(x)=\text{Student}\left(x\,\middle|\,\mu_0,\ \sigma_0^2/\frac{k_0}{k_0+1},\nu_0\right)$$

Case 2.2

Suppose the data follows a Normal distribution with unknown mean μ and unknown variance σ^2: $X \left| \mu, \sigma^2 \sim \text{Normal}\left(\mu, \sigma^2\right)\right.$. Let us consider ***Jeffreys improper prior*** where $\mu \left| \sigma^2\right.$ is proportionate to $\frac{1}{\sigma^2}$.

<u>Data</u>: $p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

<u>Prior</u>: $p(\mu \mid \sigma^2) \propto \frac{1}{\sigma^2}$

<u>Note:</u> This particular prior distribution is improper but nevertheless sensible, as indicated below.

$$p(\varphi) = p\left(\log \sigma^2\right) = p\left(\sigma^2\right)\left|\frac{d\sigma^2}{d\phi}\right| \propto \sigma^2 \cdot \frac{1}{\sigma^2} = 1$$

<u>Likelihood function</u>:

$$p\left(\mathbf{x}^{(n)} \left| \mu, \sigma^2\right.\right) = (2\pi)^{-n/2}\left(\sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}$$

$$\propto \left(\sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \overline{x})^2\right\} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\overline{x} - \mu)^2\right\}$$

$$\propto \left(\sigma^2\right)^{-n/2} \exp\left\{-\frac{(n-1)S^2 + n(\overline{x} - \mu)^2}{2\sigma^2}\right\}, \qquad \text{where } S^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}.$$

Posterior distribution:

$$p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)\propto(\sigma^2)^{-\left(\frac{n+2}{2}\right)}\exp\left\{-\frac{n(\bar{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\}$$

$$\propto(\sigma^2)^{-\left(\frac{1}{2}\right)}\exp\left\{-\frac{(\bar{x}-\mu)^2}{2\cdot\sigma^2/n}\right\}(\sigma^2)^{-\left(\frac{n+1}{2}\right)}\exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\}$$

$$\propto(\sigma^2)^{-\left(\frac{1}{2}\right)}\exp\left\{-\frac{(\bar{x}-\mu)^2}{2\cdot\sigma^2/n}\right\}(\sigma^2)^{-\left(\frac{n-1}{2}+1\right)}\exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\}\quad\Leftrightarrow$$

$$p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)=\text{Normal-scaled-Inv-}X^2\left(\mu,\sigma^2\middle|\bar{x},\sigma^2/n,n-1,S^2\right)$$

Normalizing constant of the posterior distribution:

$$K=\frac{\sqrt{n}}{\sqrt{2\pi}}\cdot\frac{\left(\frac{n-1}{2}S^2\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$$

Note: The following equation holds and can be confirmed by the results below: $p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)=p\left(\mu\middle|\sigma^2,\mathbf{x}^{(n)}\right)\cdot p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right)$.

Full Conditional posterior distributions:

- $$p\left(\mu\middle|\sigma^2,\mathbf{x}^{(n)}\right)=\frac{p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)}{p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right)}\propto p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)\propto\exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right\}\quad\Leftrightarrow$$

  $$p\left(\mu\middle|\sigma^2,\mathbf{x}^{(n)}\right)=\mathrm{N}\left(\mu\middle|\bar{x},\sigma^2/n\right)$$

- $$p\left(\sigma^2\middle|\mu,\mathbf{x}^{(n)}\right)=\frac{p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)}{p\left(\mu\middle|\mathbf{x}^{(n)}\right)}\propto p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right)$$

$$\propto (\sigma^2)^{-\left(\frac{n+2}{2}\right)} \exp\left\{-\frac{n(\bar{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\} = (\sigma^2)^{-\left(\frac{n}{2}+1\right)} \exp\left\{-\frac{n(\bar{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\} \Leftrightarrow$$

$$p\left(\sigma^2 \middle| \mu, \mathbf{x}^{(n)}\right) = \text{scaled-Inv-}X^2\left(\sigma^2 \middle| n, \frac{n(\bar{x}-\mu)^2+(n-1)S^2}{n}\right)$$

<u>Marginal posterior distributions</u>:

- $$p\left(\mu \middle| \mathbf{x}^{(n)}\right) = \int_0^{+\infty} p\left(\mu,\sigma^2 \middle| \mathbf{x}^{(n)}\right) d\sigma^2 = K \cdot \int_0^{+\infty} (\sigma^2)^{-\left(\frac{n+2}{2}\right)} \exp\left\{-\frac{n(\bar{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\} d\sigma^2$$

  We set $A = n(\bar{x}-\mu)^2+(n-1)S^2$, for easier calculations.

  We also set $z = A/2\sigma^2 \Rightarrow d\sigma^2 = -\frac{A}{2z^2}dz$ and $(\sigma^2)^{-1} = \frac{2z}{A}$

  and substitute to the above integral (with reversed boundaries):

$$p\left(\mu \middle| \mathbf{x}^{(n)}\right) = K \cdot \int_{+\infty}^{0} A^{-\left(\frac{n+2}{2}\right)} (2z)^{\left(\frac{n+2}{2}\right)} \exp\{-z\}(-A)(2z)^{-2} dz$$

$$= 2^{\left(\frac{n}{2}-1\right)} K \cdot A^{-\left(\frac{n}{2}\right)} \cdot \int_0^{+\infty} \underbrace{z^{\left(\frac{n}{2}-1\right)} \exp\{-z\}}_{\text{Gamma}(n/2,1)-\text{kernel}} dz$$

$$= 2^{\left(\frac{n}{2}-1\right)} \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot \frac{\left(\frac{n-1}{2}S^2\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{1^{\frac{n}{2}}} A^{-\left(\frac{n}{2}\right)}$$

$$= \frac{\sqrt{n}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)\left((n-1)S^2\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \left[n(\bar{x}-\mu)^2+(n-1)S^2\right]^{-\frac{n}{2}}$$

$$= \frac{\sqrt{n}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left((n-1)S^2\right)^{\frac{n-1}{2}} \left[(n-1)S^2\right]^{-\frac{n}{2}} \left[1+\frac{n(\bar{x}-\mu)^2}{(n-1)S^2}\right]^{-\frac{n}{2}}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\sqrt{n}\cdot\left(S^2\right)^{-\frac{1}{2}}}{\sqrt{\pi(n-1)}} \left[1+\frac{1}{(n-1)}\frac{(\bar{x}-\mu)^2}{S^2/n}\right]^{-\frac{n}{2}} \quad \Leftrightarrow$$

$$p\left(\mu\middle|\mathbf{x}^{(n)}\right) = \text{Student}\left(\mu\middle| n-1, \bar{x}, S^2\big/n\right).$$

- $$p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right) = \int_{-\infty}^{+\infty} p\left(\mu,\sigma^2\middle|\mathbf{x}^{(n)}\right) d\mu = K\cdot\int_{-\infty}^{+\infty}(\sigma^2)^{-\left(\frac{n+2}{2}\right)} \exp\left\{-\frac{n(\bar{x}-\mu)^2+(n-1)S^2}{2\sigma^2}\right\} d\mu$$

$$= K\cdot(\sigma^2)^{-\left(\frac{n+2}{2}\right)} \exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\} \int_{-\infty}^{+\infty} \exp\left\{-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right\} d\mu$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}}\cdot\frac{\left(\frac{n-1}{2}S^2\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\cdot(\sigma^2)^{-\left(\frac{n+2}{2}\right)} \exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\} \frac{\sqrt{2\pi}}{\sqrt{n}} (\sigma^2)^{\left(\frac{1}{2}\right)}$$

$$= \frac{\left(\frac{n-1}{2}S^2\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (\sigma^2)^{-\left(\frac{n+1}{2}\right)} \exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\} \quad \Leftrightarrow$$

$$p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right) = \text{scaled-Inv-}X^2\left(\sigma^2\middle| n-1, S^2\right)$$

This last relation entails $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}{}^2$.

__Predictive distribution__:

Similar to the conjugate case, we will use the formula $p\left(y\middle|\mathbf{x}^{(n)}\right) = \int p\left(y\middle|\mathbf{x}^{(n)},\sigma^2\right) p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right) d\sigma^2$, thus it is fundamental to primarily derive the distribution of $Y\middle|\mathbf{x}^{(n)},\sigma^2$.

$$p\left(y\middle|\mathbf{x}^{(n)},\sigma^2\right) = \int_{-\infty}^{+\infty} p\left(y\middle|\mu,\sigma^2,\mathbf{x}^{(n)}\right) p\left(\mu\middle|\sigma^2,\mathbf{x}^{(n)}\right) d\mu = \int_{-\infty}^{+\infty} p\left(y\middle|\mu,\sigma^2\right) p\left(\mu\middle|\sigma^2,\mathbf{x}^{(n)}\right) d\mu$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}\frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{n}{2\sigma^2}(\overline{x}-\mu)^2\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(y-\mu)^2+n(\overline{x}-\mu)^2}{2\sigma^2}\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{y^2+\mu^2-2y\mu+n\overline{x}^2+n\mu^2-2n\overline{x}\mu}{2\sigma^2}\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(n+1)\mu^2-2\mu(y+n\overline{x})+y^2+n\overline{x}^2}{2\sigma^2}\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(n+1)}{2\sigma^2}\left[\mu^2-2\mu\left(\frac{y+n\overline{x}}{n+1}\right)+\frac{y^2+n\overline{x}^2}{n+1}\right]\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(n+1)}{2\sigma^2}\left[\mu^2-2\mu\left(\frac{y+n\overline{x}}{n+1}\right)\pm\left(\frac{y+n\overline{x}}{n+1}\right)^2+\frac{y^2+n\overline{x}^2}{n+1}\right]\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left[y^2+n\overline{x}^2-\frac{(y+n\overline{x})^2}{n+1}\right]\right\}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(n+1)}{2\sigma^2}\left[\mu-\frac{y+n\overline{x}}{n+1}\right]^2\right\}d\mu$$

$$=\frac{\sqrt{n}}{2\pi\sigma^2}\frac{\sqrt{2\pi\sigma^2}}{\sqrt{n+1}}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{ny^2+n^2\overline{x}^2+y^2+n\overline{x}^2-y^2-n^2\overline{x}^2-2yn\overline{x}}{n+1}\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\frac{\sqrt{n}}{\sqrt{n+1}}\exp\left\{-\frac{1}{2\sigma^2\Big/\frac{n}{n+1}}(y-\overline{x})^2\right\}\Leftrightarrow$$

$$p\left(y\middle|\mathbf{x}^{(n)},\sigma^2\right)=\mathrm{N}\left(y\middle|\overline{x},\ \sigma^2\Big/\frac{n}{n+1}\right)$$

Substituting to the initial integral, we get:

$$p\left(y\middle|\mathbf{x}^{(n)}\right)=\int p\left(y\middle|\mathbf{x}^{(n)},\sigma^2\right)p\left(\sigma^2\middle|\mathbf{x}^{(n)}\right)d\sigma^2$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{n}{n+1}\right)^{\frac{1}{2}} \exp\left\{-\left(\frac{n}{n+1}\right)\frac{(y-\overline{x})^2}{2\sigma^2}\right\} \cdot \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{\frac{n-1}{2}} (\sigma^2)^{-\left(\frac{n-1}{2}+1\right)} \exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\} d\sigma^2$$

$$= (2\pi)^{-\frac{1}{2}} \sqrt{\frac{n}{n+1}} \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{\frac{n-1}{2}} \cdot \int_0^{+\infty} (\sigma^2)^{-\left(\frac{n-1+1}{2}+1\right)} \exp\left\{-\frac{1}{2\sigma^2}\left[\left(\frac{n}{n+1}\right)(y-\overline{x})^2 + (n-1)S^2\right]\right\} d\sigma^2$$

$$= (2\pi)^{-\frac{1}{2}} \sqrt{\frac{n}{n+1}} \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{\frac{n-1}{2}} \cdot \underbrace{\int_0^{+\infty} (\sigma^2)^{-\left(\frac{n}{2}+1\right)} \exp\left\{-\frac{1}{2\sigma^2} \cdot n \cdot \frac{\left[\left(\frac{n}{n+1}\right)(y-\overline{x})^2 + (n-1)S^2\right]}{n}\right\}}_{\text{scaled-Inv}-X^2 \text{ kernel}} d\sigma^2$$

$$= (2\pi)^{-\frac{1}{2}} \sqrt{\frac{n}{n+1}} \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{\frac{n-1}{2}} \cdot \Gamma\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^{-\left(\frac{n}{2}\right)} \cdot \left[\left(\frac{n}{n+1}\right)\frac{(y-\overline{x})^2 + (n-1)S^2}{n}\right]^{-\frac{n}{2}}$$

$$= \frac{2^{-\frac{1}{2}}}{\sqrt{\pi}} \sqrt{\frac{n}{n+1}} \frac{2^{-\frac{n-1}{2}} (n-1)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{\frac{n-1}{2}} \cdot \Gamma\left(\frac{n}{2}\right) \frac{\left(\frac{n}{2}\right)^{-\left(\frac{n}{2}\right)}}{(n)^{-\left(\frac{n}{2}\right)}} \cdot \left[(n-1)S^2\right]^{-\left(\frac{n}{2}\right)} \left[1 + \frac{(y-\overline{x})^2}{(n-1)S^2 \Big/ \left(\frac{n}{n+1}\right)}\right]^{-\frac{n}{2}}$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\frac{n}{n+1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} (n-1)^{\frac{n-1}{2}-\frac{n}{2}} (S^2)^{\frac{n-1}{2}-\frac{n}{2}} \cdot \left[1 + \frac{(y-\overline{x})^2}{(n-1)S^2 \Big/ \left(\frac{n}{n+1}\right)}\right]^{-\frac{n}{2}}$$

$$= \frac{1}{\sqrt{(n-1)\pi}} \sqrt{\frac{n}{n+1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} (S^2)^{-\frac{1}{2}} \cdot \left[1 + \frac{1}{n-1} \frac{(y-\bar{x})^2}{S^2 \Big/ \left(\frac{n}{n+1}\right)} \right]^{-\frac{n}{2}} \Leftrightarrow$$

$$p\left(y \middle| \mathbf{x}^{(n)}\right) = \text{Student}\left(y \middle| n-1, \bar{x}, \; S^2 \Big/ \frac{n}{n+1} \right)$$

Prior predictive distribution:

* The marginal distribution $p(x)$ cannot be written in closed form!

Case 2.3

Suppose the data follows a Normal distribution with unknown mean μ and unknown variance σ^2: $X\left|\mu,\sigma^2 \sim \text{Normal}\left(\mu,\sigma^2\right)\right.$. Let us consider a ***semi-conjugate prior*** distribution where $\mu\left|\sigma^2\right.$ follows a Normal distribution with hyperparameters μ_0,τ_0^2: $\mu\,|\,\sigma^2=\mu \sim \text{N}\left(\mu_0,\tau_0{}^2\right)$ and the prior distribution of the parameter σ^2 is scaled-Inverse-X² with hyperparameters $\nu_0,\sigma_0{}^2$: $\sigma^2 \sim \text{scaled-Inverse-X}^2\left(\nu_0,\sigma_0{}^2\right)$.

__Data__: $p\left(x\left|\mu,\sigma^2\right.\right)=\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Leftrightarrow X\left|\mu,\sigma^2 \sim \text{Normal}\left(\mu,\sigma^2\right)\right.$

__**Prior distribution**__:

$$p\left(\mu,\sigma^2\right)=(2\pi)^{-\frac{1}{2}}(\tau_0)^{-1/2}\exp\left\{-\frac{(\mu-\mu_0)^2}{2\tau_0{}^2}\right\}\cdot\frac{\left(\frac{\nu_0}{2}\right)^{\frac{\nu_0}{2}}\left(\sigma_0{}^2\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)}(\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)}\exp\left\{-\frac{\nu_0\sigma_0{}^2}{2\sigma^2}\right\} \Leftrightarrow$$

$$p\left(\mu,\sigma^2\right)=p\left(\mu\left|\sigma^2\right.\right)\cdot p\left(\sigma^2\right)\sim \text{Normal-scaled-Inv-}X^2\left(\mu,\sigma^2\left|\mu_0,\tau_0{}^2,\nu_0,\sigma_0{}^2\right.\right).$$

__**Likelihood function**__:

$$p\left(\mathbf{x}^{(n)}\left|\mu,\sigma^2\right.\right)=(2\pi)^{-n/2}(\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$\propto(\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\bar{x}-\mu)^2\right\}$$

$$\propto(\sigma^2)^{-n/2}\exp\left\{-\frac{(n-1)S^2+n(\bar{x}-\mu)^2}{2\sigma^2}\right\}, \qquad \text{where} \quad S^2=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}.$$

__**Posterior distribution**__:

The joint posterior distribution of the parameter vector $\left(\mu,\sigma^2\right)$ cannot be written in closed form.

Full Conditional posterior distributions:

- $$p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) = \frac{p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)}{p\left(\sigma^2 \middle| \mathbf{x}^{(n)}\right)} \propto p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)$$

$$\propto \exp\left\{-\frac{(\mu-\mu_0)^2}{2\tau_0^{\,2}}\right\}\exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(\mu-\mu_0)^2}{\tau_0^{\,2}} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\mu^2+\mu_0^{\,2}-2\mu\mu_0}{\tau_0^{\,2}} + \frac{n(\bar{x}^2+\mu^2-2\bar{x}\mu)}{\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\tau_0^{\,2}}+\frac{n}{\sigma^2}\right)\mu^2 - 2\mu\left(\frac{\mu_0}{\tau_0^{\,2}}+\frac{n\bar{x}}{\sigma^2}\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\underbrace{\left(\frac{1}{\tau_0^{\,2}}+\frac{n}{\sigma^2}\right)}_{\left(\tau_n^2\right)^{-1}}\left[\mu^2 - 2\mu\left(\frac{\frac{\mu_0}{\tau_0^{\,2}}+\frac{n\bar{x}}{\sigma^2}}{\left(\frac{1}{\tau_0^{\,2}}+\frac{n}{\sigma^2}\right)}\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2\tau_n^2}\left[\mu^2 - 2\mu\cdot\underbrace{\tau_n^2\left(\frac{\mu_0}{\tau_0^{\,2}}+\frac{n\bar{x}}{\sigma^2}\right)}_{\mu_n}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2\tau_n^2}[\mu-\mu_n]^2\right\} \Leftrightarrow$$

$$p\left(\mu \middle| \sigma^2, \mathbf{x}^{(n)}\right) \sim \mathrm{N}\left(\mu \middle| \mu_n, \tau_n^{\,2}\right) \qquad \text{where} \quad \mu_n = \frac{\frac{\mu_0}{\tau_0^{\,2}}+\frac{n}{\sigma^2}\bar{x}}{\frac{1}{\tau_0^{\,2}}+\frac{n}{\sigma^2}}, \quad \tau_n^{\,2} = \left(\frac{1}{\tau_0^{\,2}}+\frac{n}{\sigma^2}\right)^{-1}$$

- $$p\left(\sigma^2 \middle| \mu, \mathbf{x}^{(n)}\right) = \frac{p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)}{p\left(\mu \middle| \mathbf{x}^{(n)}\right)} \propto p\left(\mu, \sigma^2 \middle| \mathbf{x}^{(n)}\right)$$

$$\propto (\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \underbrace{\sum_{i=1}^{n} (x_i - \mu)^2}_{n \cdot S_\mu^2} \right\} (\sigma^2)^{-\left(\frac{\nu_0}{2}+1\right)} \exp\left\{ -\frac{\nu_0 {\sigma_0}^2}{2\sigma^2} \right\}$$

$$\propto (\sigma^2)^{-\left(\frac{n+\nu_0}{2}+1\right)} \exp\left\{ -\frac{nS_\mu^2 + \nu_0 {\sigma_0}^2}{2\sigma^2} \right\} = (\sigma^2)^{-\left(\frac{n+\nu_0}{2}+1\right)} \exp\left\{ -\left(n+\nu_0\right) \frac{\frac{nS_\mu^2 + \nu_0 {\sigma_0}^2}{n+\nu_0}}{2\sigma^2} \right\} \Leftrightarrow$$

$$p\left(\sigma^2 \middle| \mu, \mathbf{x}^{(n)}\right) \sim \text{scaled-Inv-}X^2\left(\sigma^2 \middle| \nu_0 + n, \frac{\nu_0 {\sigma_0}^2 + n{S_\mu}^2}{\nu_0 + n}\right), \quad \text{where } {S_\mu}^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}.$$

<u>Marginal posterior distributions</u>:

The posterior marginal distributions of μ and σ^2 cannot be written in closed form.

<u>Predictive and Prior predictive distributions</u>:

The same holds for the predictive and the prior predictive distributions.