Sweedler theory for double categories

Christina Vasilakopoulou

National Technical University of Athens, Greece

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Outline

1. Motivation and background
2. Sweedler theory for monoidal double categories
3. Oplax monoidal double structure
4. Further directions

Fibered versus enriched categories

• A $\mathcal{V}$-enriched category has hom-objects that belong to $\mathcal{V}$, and composition rule is a morphism in $\mathcal{V}$, for a monoidal category $\mathcal{V}$

• A category fibered over a category $\mathcal{B}$ is an ordinary category, whose objects and morphisms lie over specified objects and morphisms in $\mathcal{B}$, with certain (cartesian) liftings

... a common pattern in what follows is

$$\begin{array}{ccc} \mathcal{C} & \overset{\text{enriched}}{\dashrightarrow} & \mathcal{D} \\ {\scriptstyle\text{fibered}}\downarrow & & \downarrow{\scriptstyle\text{opfibered}} \\ \mathcal{A} & \underset{\text{enriched}}{\dashrightarrow} & \mathcal{B} \end{array}$$

⋆ Bunge / Shulman: *enriched indexed categories*, over a fixed base. Above picture? Theory of *enriched fibrations*, over different bases.

Monoids and comonoids

Suppose $(\mathcal{V}, \otimes, I)$ is monoidal category.

- A *monoid* is an object A together with maps $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ which are associative and unital:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \\ {\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} I \otimes A & \xrightarrow{\eta\otimes 1} & A \otimes A & \xleftarrow{1\otimes\eta} & A \otimes I \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & A & & \end{array}$$

- Dually, a *comonoid* is an object C together with maps $\delta\colon C \to C \otimes C$ and $\epsilon\colon C \to I$ which are coassociative and counital.

$\star$ These form categories Mon and Comon, with maps preserving structure.

When $(\mathcal{V}, \otimes, I, \sigma)$ is braided, Mon and Comon are monoidal, with I and

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\sigma\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu\otimes\mu} A \otimes B$$

■ If $\mathcal{V}$ is also closed, induced $[\text{-},\text{-}]\colon \mathsf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathsf{Mon}(\mathcal{V}) \to \mathsf{Mon}(\mathcal{V})$ makes $[C,A]$ into a monoid via *convolution*

$$[C,A]\otimes[C,A]\otimes C \xrightarrow{1\otimes\delta} [C,A]\otimes[C,A]\otimes C\otimes C \xrightarrow{1\otimes\sigma\otimes 1} [C,A]\otimes C\otimes[C,A]\otimes C \xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} A\otimes A \xrightarrow{\mu} A$$

⋆ In Vect_k, linear dual $C^* = \mathrm{Hom}_k(C,k)$ for a k-coalgebra is a k-algebra – while A^* for a k-algebra is a k-coalgebra only if it is finite dimensional.

Sweedler dual 'fixes' that: $A^o = \{f \in A^* \mid \mathit{ker} f \text{ contains cofinite ideal}\}$ is a k-coalgebra, and $\mathsf{Alg}(A, C^*) \cong \mathsf{Coalg}(C, A^o)$ by adjunction

$$\mathsf{Coalg}^{\mathrm{op}} \underset{((-)^o)^{\mathrm{op}}}{\overset{(-)^*}{\rightleftarrows}} \mathsf{Alg} \qquad (\bot)$$

'Sweedler theory' for monoidal categories

- *Universal measuring k-coalgebra* $\mathrm{Alg}(A,[C,B]) \cong \mathrm{Coalg}(C,P(A,B))$, algebraically as terminal object in category of k-coalgebras that *measure*.

⋆ Moving to general context of braided monoidal closed categories, local presentability gives passage from Vect_k to (d)gVect_k, Mod_R &many more.

Suppose $\mathcal{V}$ is a braided monoidal closed and locally presentable category. There is a parameterized adjunction between

$$[\text{-},\text{-}]\colon \mathrm{Comon}^{\mathrm{op}} \times \mathrm{Mon} \to \mathrm{Mon} \quad \text{convolution}$$
$$P(\text{-},\text{-})\colon \mathrm{Mon}^{\mathrm{op}} \times \mathrm{Mon} \to \mathrm{Comon} \quad \text{universal measuring}$$

$\boxed{P(A,B) = \int^C \mathrm{Mon}(A,[C,B]) \cdot C}$ in Set is $\mathrm{Mon}(A,B)$; in Vect_k contains k-algebra maps as grouplike; in dgVect_k, relates to bar-cobar adjunction.

- Convolution [-, -] is an *action* of the monoidal Comon$^{(\mathrm{op})}$ on Mon.

Any parameterized adjoint of an action $\bullet\colon \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ gives rise to a $\mathcal{V}$-enriched structure on $\mathcal{C}$, and all tensored $\mathcal{V}$-categories arise this way.

If $\mathcal{V}$ further symmetric, Mon is enriched in symmetric monoidal Comon.

Digression: *semi-Hopf* $\mathcal{V}$-categories generalize bimonoids

· $H(x,y)\otimes H(y,z)\to H(z,x),\ I\to H(x,x)$ 'global' multipl

· $H(a,b)\to H(a,b)\otimes H(a,b),\ H(a,b)\to I$ 'local' comultipl

$$\begin{array}{ccc} H_{x,y}\otimes H_{y,z} & \xrightarrow{\delta_{xy}\otimes\delta_{yz}} & H_{x,y}\otimes H_{x,y}\otimes H_{y,z}\otimes H_{y,z} \\ {\scriptstyle\mu_{xyz}}\downarrow & & \downarrow{\scriptstyle(\mu_{xyz}\otimes\mu_{xyz})\circ(1\otimes\sigma\otimes 1)} \\ H_{x,z} & \xrightarrow{\delta_{xz}} & H_{x,z}\otimes H_{x,z} \end{array}$$

The category of monoids in $\mathcal{V}$ is a semi-Hopf $\mathcal{V}$-category.

(Co)modules enter the picture

▶ For monoid A, an A-module M comes with associative and unital $\mu\colon A\otimes M\to M$, and dually a C-comodule X comes with $\chi\colon X\to C\otimes X$. With maps preserving (co)actions, categories ${}_A\mathrm{Mod}$ and ${}_C\mathrm{Comod}$.

■ 'Global' categories Mod, Comod of (co)modules for any (co)monoid, maps for Mod are $g\colon {}_AM\to {}_BN$ in $\mathcal{V}$ with $f\colon A\to B$ in Mon that

$$\begin{array}{ccccc} A\otimes M & & \xrightarrow{\quad\mu\quad} & & M \\ {\scriptstyle 1\otimes g}\downarrow & & & & \downarrow{\scriptstyle g} \\ A\otimes N & \xrightarrow{f\otimes 1} & B\otimes N & \xrightarrow{\mu} & N \end{array}$$

⋆ Naturally form fibration Mod → Mon and opfibration Comod → Comon.

If $\mathcal{V}$ symmetric monoidal closed&locally presentable, adjunction between

$$[\text{-},\text{-}]\colon \mathrm{Comod}^{\mathrm{op}}\times\mathrm{Mod}\to\mathrm{Mod}\quad \text{convolution}$$

$$Q(\text{-},\text{-})\colon \mathrm{Mod}^{\mathrm{op}}\times\mathrm{Mod}\to\mathrm{Comod}\quad \text{universal measuring}$$

and Mod is enriched in the symmetric monoidal Comod.

From one to many objects

Goal: generalize from monoids in $\mathcal{V}$, to $\mathcal{V}$-categories!

$$\begin{array}{ccccccc} \mathsf{Mod} & & \mathsf{Comod} & & \mathcal{V}\text{-}\mathsf{Mod} & & \mathcal{V}\text{-}\mathsf{Comod} \\ \downarrow & \text{enriched in} & \downarrow & \rightsquigarrow & \downarrow & \text{enriched in} & \downarrow \\ \mathsf{Mon} & & \mathsf{Comon} & & \mathcal{V}\text{-}\mathsf{Cat} & & \mathcal{V}\text{-}\mathsf{Cocat} \end{array}$$

Method: work 'bottom-up' (technical but direct), or identify general framework and work 'top-down'

- A $\mathcal{V}$-module M consists of $\{M(x)\}_{x\in X}$ in $\mathcal{V}$ with an action $(\sum_y)A(x,y)\otimes M(y)\to M(x)$ for A some $\mathcal{V}$-category ($A \nrightarrow \mathcal{I}$).
- For $\mathcal{V}$ with coproducts preserved by $\otimes$, a $\mathcal{V}$-cocategory C consists of $\{C(x,z)\}_X$ in $\mathcal{V}$ with $C(x,z)\to\sum_y C(x,y)\otimes C(y,z)$, $C(x,x)\to I$.
- For $\mathcal{V}$ with coproducts preserved by $\otimes$, a $\mathcal{V}$-comodule K consists of $\{K(x)\}_{x\in X}$ in $\mathcal{V}$ with a coaction $K(x)\to\sum_y C(x,y)\otimes K(y)$.

From monoidal to double categories

⋆ Opcategories=$\mathcal{V}^{\mathrm{op}}$-categories? Not as convenient, formally!

- A double category $\mathbb{D}$ has $\mathbb{D}_0$ (0-cells&vertical 1-cells), $\mathbb{D}_1$ (horizontal 1-cells&2-maps) and $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \overset{\mathfrak{s}}{\underset{\mathfrak{t}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$+coherent isos.
- A monad in a double category $\mathbb{D}$ is $A\colon X \nrightarrow X$ with associative, unital

$$\begin{array}{ccccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \| & & \Downarrow\mu & & \| \\ X & & \xrightarrow[A]{} & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \| & \Downarrow\eta & \| \\ X & \xrightarrow[A]{} & X \end{array}$$

Dually, comonad $C\colon X \nrightarrow X$. These form categories $\mathrm{Mnd}(\mathbb{D})$ & $\mathrm{Cmd}(\mathbb{D})$.

⋆ Morphisms are different than those for (co)monads in bicategories.

For $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ of sets, functions and $\mathcal{V}$-matrices $S\colon X \nrightarrow Y$ i.e. $\{S(x,y)\} \in \mathcal{V}$ with $(S \circ T)(x,z) = \sum_y T(x,y) \otimes S(y,z)$, $\mathrm{Mnd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{Cat}$ and $\mathrm{Cmd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{Cocat}$.

- A module in $\mathbb{D}$ is $M\colon Z \nrightarrow X$ with an action of a monad $A\colon X \nrightarrow X$

$$\begin{array}{ccccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ \| & & \Downarrow & & \| \\ Z & & \xrightarrow[M]{\qquad\qquad} & & X \end{array}$$

Dually, comodule $Q\colon W \nrightarrow X$ with globular $Q \Rightarrow C \circ Q$ for comonad $C\colon X \nrightarrow X$. These form categories $\mathsf{Mod}(\mathbb{D})$ & $\mathsf{Comod}(\mathbb{D})$, with maps

$$\begin{array}{ccc} Z & \xrightarrow{M_A} & X \\ g\downarrow & \Downarrow\beta & \downarrow f \\ T & \xrightarrow[N_B]{} & Y \end{array} \quad \text{s.t.} \quad \begin{array}{ccccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ \| & & \Downarrow & & \| \\ Z & & \xrightarrow{M} & & X \\ g\downarrow & & \Downarrow\beta & & \downarrow f \\ T & & \xrightarrow[N]{} & & Y \end{array} = \begin{array}{ccccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ g\downarrow & \Downarrow\beta & \downarrow f & \Downarrow\alpha & \downarrow f \\ T & \xrightarrow[N]{} & Y & \xrightarrow[B]{} & Y \\ \| & & \Downarrow & & \| \\ T & & \xrightarrow[N]{} & & Y \end{array}$$

$\star$ Subcategories of interest: fixed-monad ${}_A\mathsf{Mod}(\mathbb{D})$, fixed-dom ${}^Z\mathsf{Mod}(\mathbb{D})$, bicat modules ${}^Z_A\mathsf{Mod}(\mathbb{D}) = \mathsf{Mod}(\mathcal{H}(\mathbb{D}))$...several monadicity results.

${}^{\{*\}}\mathsf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Mod}$ and ${}^{\{*\}}\mathsf{Comod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathsf{Comod}$.
$\mathsf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at})$ has $\{M(x,z)\}_{X\times Z}$ with $A(x,x') \otimes M(x',z) \to M(x,z)$.

For $\mathbb{D}$ a double category, $\mathrm{Mnd}(\mathbb{D})$ is enriched in $\mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Mod}(\mathbb{D})$ is enriched in $\mathrm{Comod}(\mathbb{D})$, *under certain conditions*.

- Fibrant: vertical 1-cells f turn to horizontal, companion $\hat{f}$ & conjoint $\check{f}$.

In $\mathcal{V}$-$\mathbb{M}$at, $f\colon X \to Y$ gives matrices $\hat{f}(x,y) = \check{f}(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

$\mathrm{Mod}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$ is a fibration, with reindexing $\check{f}\circ$-: ${}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$.
$\mathrm{Comod}(\mathbb{D}) \to \mathrm{Cmd}(\mathbb{D})$ is an opfibration, with $\hat{f}\circ$-: ${}_C\mathrm{Comod} \to {}_D\mathrm{Comod}$.

- Monoidal: $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal, $(M\otimes N)\circ(M'\otimes N')\cong(M\circ M')\otimes(N\circ N')$.

In $\mathcal{V}$-$\mathbb{M}$at, $(X\otimes Y)=X\times Y$ & $(S\otimes T)((x,y),(z,w))=S(x,z)\otimes T(y,w)$.

$\mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Comod}(\mathbb{D})$ are monoidal, with $Q_C\otimes P_D = (Q\otimes P)_{C\otimes D}$ via $Q\otimes P \to (C\circ Q)\otimes(D\circ P)\cong(C\otimes D)\circ(Q\otimes P)$. Symmetry is inherited.

■ Locally closed monoidal: lax double functor $H\colon \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ such that $\mathbb{D}_0$ & $\mathbb{D}_1$ monoidal closed, $\mathfrak{s}, \mathfrak{t}$ maps of adjunctions.

In $\mathcal{V}$-$\mathbb{M}$at, $[X, Y] = Y^X$ and $H(S, T)(f, g) = \prod_{x,y}[S(x, y), T(fx, gy)]$.

Lax double H induce functors $\mathrm{Cmd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$ and $\mathrm{Comod}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{D}) \to \mathrm{Mod}(\mathbb{D})$ which are actions. *convolution*

■ Locally presentable: $\mathbb{D}_0$ & $\mathbb{D}_1$ locally presentable, $\mathfrak{s}, \mathfrak{t}$ accessible right adjoints, $\mathbf{1}$ accessible, $- \circ -$ accessible in each variable.

In $\mathcal{V}$-$\mathbb{M}$at, Set is locally presentable & $\mathcal{V}$-$\mathbb{M}\mathrm{at}_1$ is too (Limit theorem. . .)

Induced functors H have adjoints $\mathrm{Mnd}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mnd}(\mathbb{D}) \to \mathrm{Cmd}(\mathbb{D})$ and $\mathrm{Mod}(\mathbb{D})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{D}) \to \mathrm{Comod}(\mathbb{D})$. *universal measuring*

◇ Obtain enrichment of $\mathrm{Mnd}(\mathbb{D})$ in $\mathrm{Cmd}(\mathbb{D})$ & of $\mathrm{Mod}(\mathbb{D})$ in $\mathrm{Comod}(\mathbb{D})$.

Moving to other contexts

⋆ For $\mathcal{V}$ symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ is a fibrant, locally closed monoidal & locally presentable double category.

- Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories
- Enrichment of $\mathcal{V}$-modules in $\mathcal{V}$-comodules

Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$

operads

- objects are sets $X, Y, \ldots$
- vertical 1-cells are functions $f, g, \ldots$
- horizontal 1-cells are coloured symmetric sequences $M\colon SX \times Y \to \mathcal{V}$
- 2-maps are $M(x_1, \ldots, x_n; y) \to N(fx_1, \ldots, fx_n; gy)$

Horizontal composition is generalization of substitution for species...

■ $\mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$ is fibrant; but not monoidal double anymore!

Oplax monoidal double structure

- A double category is *oplax* monoidal with comparison maps

$$(N \circ M) \otimes (N' \circ M') \to (M \otimes M') \circ (N \otimes N'), \quad 1_X \otimes 1_{X'} \to 1_{X \otimes X'}$$
$$I_1 \to I_1 \circ I_1, \quad I_1 \to 1_I$$

making $\otimes \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, $I \colon \mathbf{1} \to \mathbb{D}$ into oplax double functors.

An oplax monoidal double category with a single object and vertical arrow is precisely a duoidal category.

- *Normality* condition (I pseudo, $\otimes$ pseudo in each variable) reduces to normal duoidal structure & gives passage to 'oplax monoidal bicategories'.

Idea: show that $\mathcal{V}$-$\mathbb{S}$ym (in fact $\mathcal{V}$-$\mathbb{C}$at$\mathbb{S}$ym...) is normal oplax monoidal , by expressing it as a *Kleisli double category* of monoidal double $\mathcal{V}$-$\mathbb{P}$rof.

Monoidal double monads

- A *vertical double monad* is a double functor $T\colon \mathbb{D} \to \mathbb{D}$ with vertical transformations $m\colon TT \Rightarrow T$, $e\colon 1 \Rightarrow T$ with

$$
\begin{array}{ccc}
TTX & \xrightarrow{TTM} & TTY \\
{\scriptstyle m_X}\downarrow & \Downarrow m_M & \downarrow{\scriptstyle m_Y} \\
TX & \xrightarrow[TM]{} & TY
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{M} & Y \\
{\scriptstyle e_X}\downarrow & \Downarrow e_M & \downarrow{\scriptstyle e_Y} \\
TX & \xrightarrow[TM]{} & TY
\end{array}
$$

- It is *pseudomonoidal* when T lax monoidal and m, e pseudomonoidal

e.g.
$$
\begin{array}{ccc}
TTX_1 \otimes TTX_2 & \xrightarrow{1} & TTX_1 \otimes TTX_2 \\
\downarrow & & \downarrow{\scriptstyle m\otimes m} \\
TT(X_1 \otimes X_2) & \Downarrow & TX_1 \otimes TX_2 \\
{\scriptstyle m}\downarrow & & \downarrow \\
T(X_1 \otimes X_2) & \xrightarrow{1} & T(X_1 \otimes X_2)
\end{array}
$$
is invertible in vertical 2-category

- A vertical double monad is *special* when m_X, e_X have companions in $\mathbb{D}$, and each 2-cell transpose of m_M, e_M is invertible , e.g.

$$\begin{array}{ccccc} X & \xrightarrow{M} & Y & \xrightarrow{\hat{e}_Y} & TY \\ \| & & \Downarrow \hat{e}_M & & \| \\ X & \xrightarrow[\hat{e}_X]{} & TX & \xrightarrow[TM]{} & TY \end{array} = \begin{array}{ccccccc} X & \xrightarrow{1} & X & \xrightarrow{M} & Y & \xrightarrow{\hat{e}_Y} & TY \\ \| & \Downarrow & {\scriptstyle e_X}\downarrow & \Downarrow e_M & \downarrow{\scriptstyle e_Y} & \Downarrow & \| \\ X & \xrightarrow[\hat{e}_X]{} & TX & \xrightarrow[TM]{} & TY & \xrightarrow[1]{} & TY \end{array}$$

$\star$ 'Free symmetric strict monoidal 2-category monad' $S\colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$

$$\text{with } S_n(C)((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{\sigma} \prod_{1 \leq i \leq n} C(x_{\sigma(i)}, y_i)$$

extends to vertical double monad on $\mathcal{V}$-$\mathbb{P}$rof, special and pseudomonoidal:

$$((x_1, \ldots, x_m), (y_1, \ldots, y_n)) \mapsto ((x_1, y_1), (x_1, y_2), \ldots, (x_m, y_n)) := \vec{x} \boxtimes \vec{y}$$

Note: $\mathcal{V}$ needs to be cartesian monoidal.

Kleisli double category

- Each special vertical double monad $T\colon \mathbb{D} \to \mathbb{D}$ gives double $\mathbb{K}\mathrm{l}(T)$
  - $\mathbb{K}\mathrm{l}(T)_0$ is $\mathbb{D}_0$
  - $M\colon X \rightsquigarrow Y$ are horizontal $M\colon X \to\!\!\!\!\!\!+\;\, TY$ in $\mathbb{D}$
  - 2-morphisms $\begin{array}{ccc} X & \overset{M}{\rightsquigarrow} & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ Z & \underset{N}{\rightsquigarrow} & W \end{array}$ are $\begin{array}{ccc} X & \overset{M}{\longrightarrow} & TY \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle Tg} \\ Z & \underset{N}{\longrightarrow} & TW \end{array}$ in $\mathbb{D}$
  - horizontal composition is $X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{\hat{m}_Z} TZ$

> For $\mathbb{D}$ monoidal double category and T pseudomonoidal special vertical double monad, if structure maps $I \to TI$ and $TX \otimes TY \xrightarrow{\tau} T(X \otimes Y)$ have companions, then $\mathbb{K}\mathrm{l}(T)$ is an oplax monoidal double category.

⋆ Induced tensor is $M \boxtimes N = X \otimes Z \xrightarrow{M \otimes N} TY \otimes TW \xrightarrow{\hat{\tau}} T(Y \otimes W)$.

Arithmetic product of coloured symmetric sequences

⋆ Acquired $S\colon \mathcal{V}\text{-}\mathbb{P}\mathrm{rof} \to \mathcal{V}\text{-}\mathbb{P}\mathrm{rof}$ satisfies above conditions!

Kleisli double category is $\mathcal{V}\text{-}\mathbb{C}\mathrm{at}\mathbb{S}\mathrm{ym}$ of *categorical symmetric sequences* $M\colon SY^{\mathrm{op}} \times X \to \mathcal{V}$, with 'discrete' case $\mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$.

▶ Horizontal composition is many-object generalisation of substitution

$$(N \circ M)(\vec{z}, x) = \int^{SZ,SY} SZ[\vec{z}, \bigotimes_i \vec{w}^i] \times \prod N(\vec{w}^i, y_i) \times M(\vec{y}, x)$$

▶ Oplax monoidal structure is many-object generalisation of 'arithmetic product' of species

$$(M \boxtimes N)(\vec{a}, (x, z)) = \int^{\vec{y},\vec{w}} S(Y \times W)(\vec{a}, \vec{y} \boxtimes \vec{w}) \times M(\vec{y}, x) \times N(\vec{w}, z)$$

Some future directions

▶ Extend 'Sweedler theory' from monoidal double to oplax monoidal double categories: do we still obtain an enrichment of monads in comonads, and of modules in comodules?

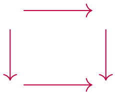
▶ Further explore structure of $\mathcal{V}$-$\mathbb{S}$ym: is it locally monoidal closed and locally presentable as a double category, for introduced definitions?

[One-object case] If $\mathcal{V}$ is symmetric monoidal closed and loc presentable,

- positive operads are enriched in positive cooperads, if $\mathcal{V}$ has biproducts;
- symmetric operads are enriched in symmetric cooperads, if $\mathcal{V}$ is cartesian.

▶ Boardman-Vogt tensor of bimodules of symmetric coloured operads and bimodules: abstract double categorical framework behind that?

Thank you for your attention!



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- *Gambino, Garner, Vasilakopoulou*, "Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences", arXiv:2206.06858