Hopf categories as Hopf monads in enriched matrices

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Outline

1. Hopf monads in monoidal bicategories
2. Hopf categories in enriched matrices
3. Monoidal fibrant double context
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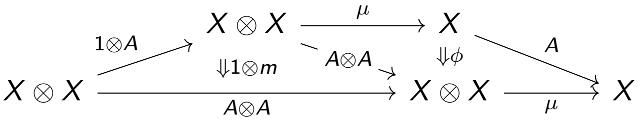
Hopf monads in $(\mathcal{K}, \otimes, I)$

- A *monad* $A\colon X \to X$ in any $\mathcal{K}$ is a monoid in $(\mathcal{K}(X,X), \circ, 1_X)$.
- A *pseudomonoid* or *monoidale* is $X \otimes X \xrightarrow{\mu} X \xleftarrow{\eta} I$ associative and unital up to coherent iso. Together with *oplax 1-cells* (f, ϕ, ϕ_0) and *2-cells* $\alpha\colon f \Rightarrow g$, they form a bicategory $\mathrm{PsMon}_{\mathrm{opl}}(\mathcal{K})$.

For $\mathcal{K} = (\mathbf{Cat}, \times, \mathbf{1})$, $\mathrm{PsMon}_{\mathrm{opl}}(\mathbf{Cat}) = \mathbf{MonCat}_{\mathrm{opl}}$ of monoidal categories, opmonoidal functors and monoidal nat. transformations.

- A *bimonad* or *opmonoidal monad* is a monad in $\mathrm{PsMon}_{\mathrm{opl}}(\mathcal{K})$: a monad (A, m, j) on (X, μ, η) where (A, ϕ, ϕ_0) is oplax 1-cell and m, j are oplax 2-cells.
- A *Hopf monad* is a bimonad for which the *Hopf map* is iso:



Special case: duoidal endohom-categories

[Böhm-Lack] Let X be an *opmap monoidale*, i.e. $\delta \dashv \mu$, $\epsilon \dashv \eta$. Then $(\mathcal{K}(X,X), \circ, 1_X, \bullet, J)$ is a *duoidal* category, with

$$S \bullet T = X \xrightarrow{\delta} X \otimes X \xrightarrow{S \otimes T} X \otimes X \xrightarrow{\mu} X$$
$$J = X \xrightarrow{\epsilon} I \xrightarrow{\eta} X$$

⋆ A $\circ$-monoid is monad on X, a $\bullet$-comonoid is oplax 1-cell $X \to X$.

⋆ A $(\circ, \bullet)$-bimonoid is a bimonad on X.

⋆ Hopf structure in duoidal context can take different expressions, equivalent to invertibility of Hopf map under certain conditions:

- Existence of antipode $\sigma \colon A \Rightarrow A^{-}$ & properties (*nat. Frobenius*)
- Galois map invertibility (*well-pointed*): for A-comodule (N, ψ)

$$N \circ A \xrightarrow{\psi \circ 1} (A \bullet N) \circ (A \bullet J) \xrightarrow{\xi} (A \circ A) \bullet (N \circ J) \xrightarrow{m \bullet 1} A \bullet (N \circ J).$$

$\mathcal{V}$-matrices

Let $\mathcal{V}$ braided monoidal with coproducts such that $\otimes$ preserves them.

Monoidal bicategory $\mathcal{V}$-**Mat**:

- 0-cells are sets
- 1-cells $S\colon X \nrightarrow Y$ are functors $S\colon X \times Y \to \mathcal{V}$, i.e. $\{S_{x,y}\} \in \mathcal{V}$
- 2-cells $\alpha\colon S \Rightarrow T$ are natural transformations $\alpha_{xy}\colon S_{x,y} \to T_{x,y}$
- Composition $X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z$ by $(S \circ T)_{x,z} = \sum_y S_{x,y} \otimes T_{y,z}$
- Tensor $(X \overset{S}{\nrightarrow} Y) \otimes (Z \overset{P}{\nrightarrow} W)$ by $(S \otimes P)_{(x,z),(y,w)} = S_{x,y} \otimes P_{z,w}$

⋆ Every X is canonically an opmap monoidale with

$$X \underset{\mu}{\overset{\delta}{\rightleftarrows}} X \times X \quad \text{by } \delta_{x,y,z} = \mu_{y,z,x} = \begin{cases} I, & \text{if } x{=}y{=}z \\ 0, & \text{otherwise} \end{cases}$$

$$X \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} 1 \quad \text{by } \epsilon_x = \eta_x = I, \qquad \delta \dashv \mu \text{ and } \epsilon \dashv \eta.$$

Hopf monads in $\mathcal{V}$-**Mat**

Each $\mathcal{V}$-**Mat**(X, X) has duoidal structure $([X\times X, \mathcal{V}], \circ, 1_X, \bullet, J)$ with

$$(S \bullet T)_{x,y} = S_{x,y} \otimes T_{x,y} \text{ and } J_{x,y} = I$$

⋆ a $\circ$-monoid is a $\mathcal{V}$-category with objects X (monad in $\mathcal{V}$-**Mat**)

$$m_{xz} \colon \Sigma_y A_{x,y} \otimes A_{y,z} \to A_{x,z}, \quad j_x \colon I \to A_{x,x}$$

⋆ a $\bullet$-comonoid is a **Comon**$(\mathcal{V})$-graph with objects X

$$d_{xy} \colon A_{x,y} \to A_{x,y} \otimes A_{x,y}, \quad e_{xy} \colon A_{x,y} \to I$$

⋆ a bimonoid is a **Comon**$(\mathcal{V})$-category, a.k.a. *semi-Hopf category*

⋆ a Hopf monoid is a *Hopf category* [Batista-Caenapeel-Vercruysse]
i.e. $s_{xy} \colon A_{x,y} \to A_{y,x}$ satisfying 'many-object' antipode conditions

Hopf monads in $\mathcal{V}$-**Mat** are Hopf $\mathcal{V}$-categories. (X nat. Frobenius)

Double category of enriched matrices

⋆ $\mathcal{V}$-**Mat** is the *horizontal bicategory* of a double category $\mathcal{V}$-$\mathbb{M}$**at**.

A (pseudo) double category $\mathbb{D}$ consists of

- a category of objects $\mathbb{D}_0$ (0-cells & vertical 1-cells)
- a category of arrows $\mathbb{D}_1$ (horizontal 1-cells & 2-morphisms)
- stucture functors $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$
- natural isomorphisms $(M \circ N) \circ P \cong M \circ (N \circ P)$, $1_{\mathfrak{s}(M)} \circ M \cong M$, $M \circ 1_{\mathfrak{t}(M)} \cong M$ subject to axioms.

0-cells, horizontal 1-cells and globular 2-morphisms makes $\mathcal{H}(\mathbb{D})$.

Here $\mathcal{V}$-$\mathbb{M}\mathbf{at}_0$=**Set**, $\mathcal{V}$-$\mathbb{M}\mathbf{at}_1$ has objects $\mathcal{V}$-matrices and arrows

$$\begin{array}{ccc} X & \overset{S}{\longrightarrow} & Y \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ Z & \underset{T}{\longrightarrow} & W \end{array} \quad \text{are } \alpha_{xy} \colon S_{x,y} \to T_{fx,gy}.$$

$\star$ $\mathcal{V}$-$\mathbb{M}$**at** is *fibrant* double: every function $f\colon X \to Y$ has a *companion* $\hat{f}\colon X \to\!\!\!\!\mapsto Y$ and a *conjoint* $\check{f}\colon Y \to\!\!\!\!\mapsto X$ by

$$\hat{f}_{x,y} = \check{f}_{y,x} = \begin{cases} I, & \text{if } fx = y \\ 0, & \text{otherwise} \end{cases} \qquad (\hat{f} \dashv \check{f} \text{ in } \mathcal{V}\text{-}\mathbf{Mat})$$

inducing a fibration structure on $\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \xrightarrow{(\mathfrak{s},\mathfrak{t})} \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_0 \times \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_0$.

$\star$ $\mathcal{V}$-$\mathbb{M}$**at** is *monoidal* double: $(\mathbf{Set}, \times, \{*\})$ & $(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1, \otimes, 1_{\{*\}})$

$$\left(\begin{array}{ccc} X & \xrightarrow{S} & Y \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ X' & \xrightarrow[S']{} & Y' \end{array} , \begin{array}{ccc} Z & \xrightarrow{T} & W \\ h\downarrow & \Downarrow\beta & \downarrow k \\ Z' & \xrightarrow[T']{} & W' \end{array}\right) \longmapsto \begin{array}{ccc} X\times Z & \xrightarrow{S\otimes T} & Y\times W \\ f\times h\downarrow & \Downarrow\alpha\otimes\beta & \downarrow g\times k \\ X'\times Z' & \xrightarrow[S'\otimes T']{} & Y'\times W' \end{array}$$

and $(S \otimes T) \circ (S' \otimes T') \cong (S \circ T) \otimes (S' \circ T')$, $1_{X\times Y} \cong 1_X \otimes 1_Y$.

$\star$ Every 0-cell X is a comonoid in the cartesian $\mathcal{V}$-$\mathbb{M}\mathbf{at}_0$, via the canonical $\Delta\colon X \to X\times X$, $!\colon X \to \{*\}$.

Abstract monoidal fibrant double context

[Shulman] Let $\mathbb{D}$ be monoidal fibrant double category. Its horizontal bicategory $\mathcal{H}(\mathbb{D})$ is monoidal.

Def. A *(Hopf) bimonad* in $\mathbb{D}$ is a (Hopf) bimonad in $\mathcal{H}(\mathbb{D})$.

Where might opmap monoidales in $\mathcal{H}(\mathbb{D})$ come from?

Proposition. There is a monoidal pseudofunctor $\widehat{(-)} \colon \mathbb{D}_0 \to \mathcal{H}(\mathbb{D})$.

- [Day-Street] (Co)monoids are mapped to pseudo(co)monoids.
- [Fibrant structure] For any $f \in \mathbb{D}_0$, $\hat{f} \dashv \check{f}$ in $\mathcal{H}(\mathbb{D})$.

Monoids (A, μ, η) in $(\mathbb{D}_0, \otimes_0, I)$ are mapped to map monoidales $(A, \hat{\mu}, \hat{\eta}, \check{\mu}, \check{\eta})$ in $\mathcal{H}(\mathbb{D})$; comonoids are mapped to opmap monoidales.

Bimonads and Hopf monads on vertical comonoids

A bimonad in $\mathbb{D}$ on a $\mathbb{D}_0$-comonoid (X, δ, ϵ) is a monad (A, m, j) with

$$
\begin{array}{ccc}
X & \xrightarrow{\;A\;} & X \\
{\scriptstyle\delta}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle\delta} \\
X\otimes_0 X & \xrightarrow[A\otimes_1 A]{} & X\otimes_0 X
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{\;A\;} & X \\
{\scriptstyle\epsilon}\downarrow & \Downarrow\phi_0 & \downarrow{\scriptstyle\epsilon} \\
I & \xrightarrow[1_I]{} & I
\end{array}
$$

satisfying axioms. The *Galois map* for N horizontal A-comodule is

$$
\begin{array}{ccccc}
X & \xrightarrow{\;N\;} & X & \xrightarrow{\;A\;} & X \\
{\scriptstyle\delta}\downarrow & \Downarrow\psi & \downarrow{\scriptstyle\delta} & \Downarrow & \downarrow{\scriptstyle\delta} \\
X\otimes_0 X & \xrightarrow{A\otimes_1 N} & X\otimes_0 X & \xrightarrow{A\otimes_1 J} & X\otimes_0 X \\
= & & \cong & & = \\
X\otimes_0 X & & \xrightarrow[(A\circ A)\otimes_1(N\circ J)]{} & & X\otimes_0 X \\
\Big\| & & \Downarrow m\otimes_1 1 & & \Big\| \\
X\otimes_0 X & & \xrightarrow[A\otimes_1(N\circ J)]{} & & X\otimes_0 X
\end{array}
$$

Hopf monad $\overset{(cond)}{\iff}$ double Galois corresponds to iso

- ⋆ Double approach captures the ‘right’ notion of morphisms
- ⋆ Most interesting examples naturally arise in this framework

Examples in double context

$\mathcal{V}$-$\mathbb{M}$**at** Canonical structure $(X, \Delta, !)$ – bimonad on any X is semi-Hopf $\mathcal{V}$-category, Hopf monad is a Hopf $\mathcal{V}$-category

$\mathbb{S}$**pan** Canonical structure $(X, \Delta, !)$ – bimonad on any X is a category, Hopf monad is a groupoid

$\mathbb{B}$**Mod** Monoid (A, μ, η) in $\mathbf{Alg}_R \equiv$ commutative algebra – bicomonad is A-bialgebroid, Hopf comonad is Hopf A-algebroid

$\mathbb{P}$**rof** (Pseudo)monoid $\mathcal{V}$ in **Cat** $\equiv$ monoidal category – (Hopf) bicomonad $\widehat{T} = \mathcal{V}(-, T-) \Leftrightarrow$ ordinary (Hopf) bicomonad T on (autonomous) monoidal $\mathcal{V}$

Vertical (co)monoids in a double category determine new examples of Hopf bi(co)monads in its horizontal bicategory.

Thank you for your attention!