Monoids and Modules in Double Categories

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Double Categories

- A (pseudo) double category $\mathbb{D}$ consists of
  - a category of objects $\mathbb{D}_0$ (0-cells & vertical 1-cells);
  - a category of arrows $\mathbb{D}_1$ (horizontal 1-cells & 2-morphisms);
  - stucture functors $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \overset{\mathfrak{s}}{\underset{\mathfrak{t}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1$;
  - natural isomorphisms $\alpha : (M \odot N) \odot P \cong M \odot (N \odot P)$, $\lambda : 1_{\mathfrak{s}(M)} \odot M \cong M$, $\rho : M \odot 1_{\mathfrak{t}(M)} \cong M$ (+ conditions).

Each $\mathbb{D}$ induces a *horizontal bicategory* $\mathcal{H}(\mathbb{D})$ with 0-cells, horizontal 1-cells and globular 2-cells (**Mod**, **Prof**, **Span**$_{\mathcal{E}}$).

- A (pseudo) double functor $F : \mathbb{D} \to \mathbb{E}$ consists of $F_0 : \mathbb{D}_0 \to \mathbb{E}_0$, $F_1 : \mathbb{D}_1 \to \mathbb{E}_1$ and globular isos $F_1 M \odot F_1 N \cong F_1(M \odot N)$, $1_{F_0 A} \cong F_1(1_A)$ (+conditions). E.g. tensor product $\mathbb{D} \times \mathbb{D} \xrightarrow{\otimes} \mathbb{D}$.

Fibrant Double Categories

Idea: a way of turning vertical 1-cells into horizontal 1-cells.

- A *companion* of $f\downarrow\begin{smallmatrix}A\\ \\ B\end{smallmatrix}$ is $\hat{f}: A \to\!\!\!\!\bullet\!\!\to B$ with 2-morphisms

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}} & B \\ {\scriptstyle f}\downarrow & \Downarrow p_1 & \| \\ B & \xrightarrow[1_B]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \| & \Downarrow p_2 & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\hat{f}]{} & B \end{array}$$

s.t. $p_1 p_2 = 1_f$ and $p_1 \odot p_2 \cong 1_{\hat{f}}$. Duallly, a *conjoint* of f is $\check{f} : B \to\!\!\!\!\bullet\!\!\to A$ with q_1, q_2.

- If every vertical 1-cell has a companion and a conjoint, we have a *fibrant double category* ($\mathbb{M}$**od**, $\mathbb{P}$**rof**, $\mathbb{S}$**pan**$_{\mathcal{E}}$).

Theorem (Shulman)

The horizontal bicategory of a fibrant monoidal double category inherits a monoidal structure.

Monoids and Comonoids

- Category $\mathbb{D}_1^\bullet \subset \mathbb{D}_1$ of horizontal endo-1-cells ${}_AM_A$ & 2-morphisms

$$\begin{array}{ccc} A & \overset{M}{\to} & A \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle f} \\ B & \underset{N}{\to} & B \end{array}$$

. (Idea: base case for monoids and comonoids)

- **Mon**$(\mathbb{D}) \subset \mathbb{D}_1^\bullet$ of *monoids* $M : A \to A$ with

$$\begin{array}{ccccc} A & \overset{M}{\to} & A & \overset{M}{\to} & A \\ \| & & \Downarrow m & & \| \\ A & & \underset{M}{\longrightarrow} & & A, \end{array} \qquad \begin{array}{ccc} A & \overset{1_A}{\to} & A \\ \| & \Downarrow\eta & \| \\ A & \underset{M}{\to} & A \end{array}$$

satisfying usual associativity and unit laws, and *monoid homomorphisms* ${}_f\alpha_f : {}_AM_A \Rightarrow {}_BN_B$ respecting multiplication and unit. Dually, **Comon**$(\mathbb{D}) \subset \mathbb{D}_1^\bullet$ with $(C : A \to A, \Delta, \epsilon)$.

A monoid/comonoid in a double category $\mathbb{D}$ coincides with a monad/comonad in the horizontal bicategory $\mathcal{H}(\mathbb{D})$, i.e. a monoid/comonoid in the monoidal $(\mathcal{H}(\mathbb{D})(*,*), \odot, 1_*)$.

Fibrational Structure

If $\mathbb{D}$ is a fibrant double category, the forgetful functor $\mathbb{D}_1^\bullet \to \mathbb{D}_0$ is a bifibration.

Arises as the Grothendieck category for the pseudofunctors

$$
\begin{array}{ccc}
\mathscr{M} : \mathbb{D}_0^{\mathrm{op}} & \longrightarrow & \mathbf{Cat}, \\
A & \longmapsto & \mathcal{H}(\mathbb{D})(A,A) \\
{\scriptstyle f}\downarrow & & \uparrow {\scriptstyle (\check{f} \odot \text{-} \odot \hat{f})} \\
B & \longmapsto & \mathcal{H}(\mathbb{D})(B,B)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{F} : \mathbb{D}_0 & \longrightarrow & \mathbf{Cat} \\
A & \longmapsto & \mathcal{H}(\mathbb{D})(A,A) \\
{\scriptstyle f}\downarrow & & \downarrow {\scriptstyle (\hat{f} \odot \text{-} \odot \check{f})} \\
B & \longmapsto & \mathcal{H}(\mathbb{D})(B,B).
\end{array}
$$

For a fibrant $\mathbb{D}$, $\mathbf{Mon}(\mathbb{D})$ is fibred over $\mathbb{D}_0$ and $\mathbf{Comon}(\mathbb{D})$ is opfibred over $\mathbb{D}_0$.

The functors $\mathscr{M}f$ and $\mathscr{L}f$ restrict to $\mathbf{Mon}(\mathcal{H}(\mathbb{D})(*,*))$ and $\mathbf{Comon}(\mathcal{H}(\mathbb{D})(*,*))$ respectively.

Modules and Comodules

- A *M-module* for a monoid $M : A \relbar\!\!\bullet\!\!\rightarrow A$ is a horizontal 1-cell $\Psi : Z \relbar\!\!\bullet\!\!\rightarrow A$ with

$$\begin{array}{ccccc} Z & \overset{\Psi}{\relbar\!\!\bullet\!\!\rightarrow} & A & \overset{M}{\relbar\!\!\bullet\!\!\rightarrow} & A \\ \| & & \Downarrow \mu & & \| \\ Z & & \underset{\Psi}{\relbar\!\!\relbar\!\!\bullet\!\!\relbar\!\!\rightarrow} & & A \end{array}$$

compatible with multiplication and unit. A *module homomorphism* $\Psi \to \Xi$ is a monoid map ${}_f\alpha_f : {}_AM_A \Rightarrow {}_BN_B$ &

$$\begin{array}{ccc} Z & \overset{\Psi}{\relbar\!\!\bullet\!\!\rightarrow} & A \\ k\downarrow & \Downarrow \beta & \downarrow f \\ W & \underset{\Xi}{\relbar\!\!\bullet\!\!\rightarrow} & B \end{array}$$

compatible with actions. Dually, a *$\mathcal{C}$-comodule* $(\Phi : X \relbar\!\!\bullet\!\!\rightarrow A, \delta)$.

$\star$ We obtain global categories of modules and comodules $\mathbf{Mod}(\mathbb{D})$ and $\mathbf{Comod}(\mathbb{D})$ with subcategories ${}^Z\mathbf{Mod}(\mathbb{D})/{}^X\mathbf{Comod}(\mathbb{D})$, ${}_M\mathbf{Mod}(\mathbb{D})/{}_C\mathbf{Comod}(\mathbb{D})$, and ${}^Z_M\mathbf{Mod}(\mathbb{D})/{}^X_C\mathbf{Comod}(\mathbb{D})$.

If $\mathbb{D}$ is a fibrant double category, the forgetful $\mathbf{Mod}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$ is a fibration and $\mathbf{Comod}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$ is an op-fibration.

The corresponding pseudofunctors are

$$\mathscr{H} : \mathbf{Mon}(\mathbb{D})^{\mathrm{op}} \longrightarrow \mathbf{Cat}, \qquad \mathscr{S} : \mathbf{Comon}(\mathbb{D}) \longrightarrow \mathbf{Cat}$$

$$\begin{array}{ccc} M & \longmapsto & {}_M\mathbf{Mod}(\mathbb{D}) \\ {\scriptstyle \alpha_f}\downarrow & & \uparrow{\scriptstyle (\check{f}\odot\text{-})} \\ N & \longmapsto & {}_N\mathbf{Mod}(\mathbb{D}) \end{array} \qquad \begin{array}{ccc} C & \longmapsto & {}_C\mathbf{Comod}(\mathbb{D}) \\ {\scriptstyle \alpha_f}\downarrow & & \downarrow{\scriptstyle (\hat{f}\odot\text{-})} \\ D & \longmapsto & {}_D\mathbf{Comod}(\mathbb{D}). \end{array}$$

Any double functor $F : \mathbb{D} \to \mathbb{E}$ induces ordinary functors

$$\mathbf{Mon}(\mathbb{D}) \xrightarrow{\mathbf{Mon}F} \mathbf{Mon}(\mathbb{E}),\ \mathbf{Comon}(\mathbb{D}) \xrightarrow{\mathbf{Comon}F} \mathbf{Comon}(\mathbb{E})$$

$$\mathbf{Mod}(\mathbb{D}) \xrightarrow{\mathbf{Mod}F} \mathbf{Mod}(\mathbb{E}),\ \mathbf{Comod}(\mathbb{D}) \xrightarrow{\mathbf{Comod}F} \mathbf{Comod}(\mathbb{E}).$$

E.g. monoidal structure is inherited.

The double category of $\mathcal{V}$-Matrices

$\mathcal{V}$ is a cocomplete monoidal category s.t. $\otimes$ preserves colimits.
Double category $\mathcal{V}$-$\mathbb{M}\mathbf{at}$:

- $\mathcal{V}$-$\mathbb{M}\mathbf{at}_0 = \mathbf{Set}$;
- $\mathcal{V}$-$\mathbb{M}\mathbf{at}_1$ with objects $S : X \looparrowright Y$ given by $\{S(y,x)\}_{Y\times X}$, morphisms ${}_f\alpha_g : S \Rightarrow T$ by $S(y,x) \xrightarrow{\alpha_{y,x}} T(gy,fx)$ in $\mathcal{V}$;
- evident source and target, $\odot$ given by

$$(S \odot T)(z,x) = \sum_{y\in Y} T(z,y) \otimes S(y,x).$$

Fibrant: each function $f : X \to Y$ determines $X \overset{f_*}{\looparrowright} Y$, $\quad Y \overset{f^*}{\looparrowright} X$

with $f_*(y,x) = f^*(x,y) = \begin{cases} I & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases}$.

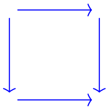
Monoidal: $\otimes_0 = \times$, $(S \otimes_1 T)((y,w),(x,z)) = S(y,x) \otimes_{\mathcal{V}} T(w,z)$.

- $\mathcal{H}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Mat}$ bicategory of $\mathcal{V}$-matrices $\Rightarrow$ monoidal.
- $\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1^\bullet = \mathcal{V}\text{-}\mathbf{Grph}$ of $\mathcal{V}$-graphs $\Rightarrow$ $\mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$ bifibration.
- $\mathbf{Mon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Cat}$ of $\mathcal{V}$-categories (vs monads in $\mathcal{V}\text{-}\mathbf{Mat}$), $\mathbf{Comon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Cocat}$ of $\mathcal{V}$-*cocategories* $\Rightarrow$ $\mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set}$ fibration, $\mathcal{V}\text{-}\mathbf{Cocat} \to \mathbf{Set}$ opfibration.
- ${}^{\{*\}}\mathbf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Mod}$ of one-sided $\mathcal{V}$-enriched modules, ${}^{\{*\}}\mathbf{Comod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Comod}$ $\Rightarrow$ $\mathcal{V}\text{-}\mathbf{Mod} \to \mathcal{V}\text{-}\mathbf{Cat}$ fibration, $\mathcal{V}\text{-}\mathbf{Comod} \to \mathcal{V}\text{-}\mathbf{Cocat}$ opfibration.
- All categories inherit a monoidal structure.

Motivation: look for enrichment relations between (op)fibrations

$$
\begin{array}{ccccc}
{}^{I}\mathbf{Mod}(\mathbb{D}) & \mathcal{V}\text{-}\mathbf{Mod} & \overset{\text{enriched}}{\dashrightarrow} & \mathcal{V}\text{-}\mathbf{Comod} & {}^{I}\mathbf{Comod}(\mathbb{D}) \\
 & \downarrow & & \downarrow & \\
\mathbf{Mon}(\mathbb{D}) & \mathcal{V}\text{-}\mathbf{Cat} & \overset{\text{enriched}}{\dashrightarrow} & \mathcal{V}\text{-}\mathbf{Cocat} & \mathbf{Comon}(\mathbb{D}) \\
 & \downarrow & & \downarrow & \\
\mathbb{D}_0 & \mathbf{Set} & \overset{\text{enriched}}{\dashrightarrow} & \mathbf{Set} & \mathbb{D}_0
\end{array}
$$

Thank you for your attention!



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