Enrichment of Categories of Algebras and Modules

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1. Motivation
2. Universal Measuring Comonoid and enrichment of monoids in comonoids
3. Global Categories of Comodules and Modules
4. Universal Measuring Comodule and enrichment of modules in comodules

Measuring Coalgebras

• (*Sweedler, 1969*) A,B algebras, C coalgebra. When is the linear map $\rho : A \to \mathrm{Hom}(C, B)$, corresponding to $\sigma : A \otimes C \to B$, an algebra map? $\Longrightarrow$ Measuring coalgebras (σ, C)
Universal measuring coalgebra (terminal object) $P(A, B)$

$$\mathbf{Alg}_k(A, \mathrm{Hom}_k(C, B)) \cong \mathbf{Coalg}_k(C, P(A, B)).$$

• (*Wraith, 1970s*) $P(A, B)$ provides an enrichment of algebras in coalgebras...

• (*Batchelor, 1990s*) Measuring coalgebras as sets of generalized maps between algebras, applications (non-commutative geometry).

Measuring Comodules and Enriched Fibration

• (*Batchelor, 1998*) Definition of measuring comodules, terminal object *universal measuring comodule* $Q(M,N)$, applications (loop algebras, bundles, representations)

$$\mathbf{Comod}_C(X, Q(M,N)) \cong \mathbf{Mod}_A(M, \mathrm{Hom}(X,N))$$

- Underlying idea: *There is no evident notion of fibration in the enriched context!*

Well-known fibration **Mod** over $\mathbf{Alg}_R$ + opfibration **Comod** over $\mathbf{Coalg}_R$ + enrichment

$$\begin{array}{ccc} \mathbf{Mod} & \dashrightarrow & \mathbf{Comod} \\ \downarrow & & \downarrow \\ \mathbf{Alg}_R & \dashrightarrow & \mathbf{Coalg}_R \end{array}$$

Aim: generalization of existence of Sweedler's measuring coalgebra

$$\mathbf{Mon}(\mathcal{V})(A,[C,B]) \cong \mathbf{Comon}(\mathcal{V})(C,P(A,B))$$

for a monoidal category $\mathcal{V}$. (properties?)

Internal hom functor in a symmetric monoidal closed category

$$[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

is a lax monoidal functor, so it induces

$$\mathbf{Mon}[-,-] = H : \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})$$

Concretely: $[C,B]$ obtains the structure of a monoid, for C a comonoid and B a monoid (e.g. convolution structure).

- Existence of a right adjoint $P(-,B)$ for the functor

$$H(-,B)^{\mathrm{op}} : \mathbf{Comon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

Theorem (*Kelly*)

If the cocomplete $\mathcal{C}$ has a small dense subcategory, then every cocontinuous $K : \mathcal{C} \to \mathcal{D}$ has a right adjoint.

- (*Porst, 2008*) If $\mathcal{V}$ is a symmetric monoidal closed category which is locally presentable, then $\mathbf{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$ and locally presentable itself (small dense subcategory).
- The functor $H(-,B)^{\mathrm{op}}$ is cocontinuous:

$$\begin{array}{ccc} \mathbf{Comon}(\mathcal{V}) & \xrightarrow{H(-,B)^{\mathrm{op}}} & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V^{\mathrm{op}}} \\ \mathcal{V} & \xrightarrow{[-,VB]^{\mathrm{op}}} & \mathcal{V}^{\mathrm{op}} \end{array}$$

so there exists $P(-,B)$ for all B, and H^{op} has a parametrised adjoint

$$P : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Comon}(\mathcal{V})$$

and $P(A,B)$ is the *universal measuring comonoid*.

An *action* of a monoidal category $\mathcal{V}$ on $\mathcal{A}$ is given by a functor $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ with coherent isomorphisms

$$\alpha_{XYA} : (X \otimes Y) * A \xrightarrow{\sim} X * (Y * A), \quad \lambda_A : I * A \xrightarrow{\sim} A.$$

Theorem (*Janelidze, Kelly*)

If each $- * A$ has a right adjoint $F(A, -)$ with

$$\mathcal{A}(X * A, B) \cong \mathcal{V}(X, F(A, B)),$$

then we can enrich $\mathcal{A}$ in $\mathcal{V}$, with hom-object functor F.

- (H and) $H^{\mathrm{op}} : \mathbf{Comon}(\mathcal{V}) \times \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ is an action of the monoidal category $\mathbf{Comon}(\mathcal{V})$ on $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$.
- Each $H(-, B)^{\mathrm{op}}$ has a right adjoint, $P(-, B)$.

($\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ and so) $\mathbf{Mon}(\mathcal{V})$ is enriched in $\mathbf{Comon}(\mathcal{V})$, with hom-objects $P(A, B)$.

Each comonoid arrow $C \xrightarrow{f} D$ induces the *corestriction of scalars*

$$\begin{aligned} f_* : \mathbf{Comod}_{\mathcal{V}}(C) &\longrightarrow \mathbf{Comod}_{\mathcal{V}}(D) \\ (X, \delta) &\longmapsto (X, (1 \otimes f) \circ \delta) \end{aligned}$$

The *Global category of comodules* **Comod** has

→ objects X_C, where C is a comonoid and X a C-comodule

→ arrows $X_C \xrightarrow{(k,f)} Y_D$ where $\begin{cases} f_*X \xrightarrow{k} Y & \text{in } \mathbf{Comod}_{\mathcal{V}}(D) \\ C \xrightarrow{f} D & \text{in } \mathbf{Comon}(\mathcal{V}) \end{cases}$

→ appropriate composition and identities

Comod is the Grothendieck category for the functor which sends each comonoid C to the category of its comodules $\mathbf{Comod}_{\mathcal{V}}(C)$

- **Comod** is comonadic over $\mathcal{V} \times \mathbf{Comon}(\mathcal{V})$.

Mod is the Grothendieck category for the functor which sends each monoid A to the category of its modules $\mathbf{Mod}_{\mathcal{V}}(A)$

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The functor $H = \mathbf{Mon}[-,-]$ induces

$$\begin{aligned} \bar{H} : \mathbf{Comod}_{\mathcal{V}}(C)^{\mathrm{op}} \times \mathbf{Mod}_{\mathcal{V}}(B) &\longrightarrow \mathbf{Mod}_{\mathcal{V}}([C,B]) \\ (\,X\,,\,N\,) &\longmapsto [X,N] \end{aligned}$$

Furthemore, between the global categories

$$\begin{aligned} \mathrm{Hom} : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} &\longrightarrow \mathbf{Mod} \\ (\,X_C\,,\,N_B\,) &\longmapsto [X,N]_{[C,B]} \end{aligned}$$

Theorem

Suppose $\mathcal{V}$ is a locally presentable, symmetric monoidal closed category. Then the functor $\mathrm{Hom}(-,N_B)^{\mathrm{op}}$ has a right adjoint $Q(-,N_B)$, with a natural isomorphism

$$\mathbf{Mod}(M_A,[X,N]_{[C,B]}) \cong \mathbf{Comod}(X_C, Q(M,N)_{P(A,B)}).$$

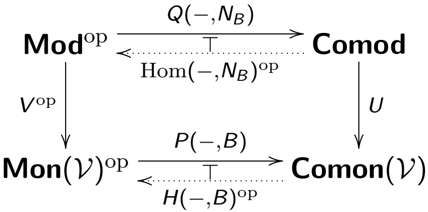
The object $Q(M,N)$ is called the *universal measuring comodule*.

- The functor (Hom and) $\mathrm{Hom}^{\mathrm{op}} : \mathbf{Comod} \times \mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mod}^{\mathrm{op}}$ is an action of the monoidal category **Comod** on $\mathbf{Mod}^{\mathrm{op}}$.
- $\mathrm{Hom}(-, N_B)^{\mathrm{op}} \dashv Q(-, N_B)$ for each $N_B \in \mathbf{Mod}$.

($\mathbf{Mod}^{\mathrm{op}}$ and so) **Mod** is enriched in **Comod**, with hom-objects $Q(M, N)_{P(A,B)}$.

Enriched Fibration?

- The (op) forgetful $V^{\mathrm{op}} : \mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ is an opfibration.
- The forgetful $U : \mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$ is an opfibration.
- $Q(-, N_B)$ and $P(-, B)$ are the hom-functors of the enriched categories.



Thank you for your attention!