Monads and modules in double categories

Christina Vasilakopoulou

National Technical University of Athens, Greece

10th Greek Algebra and Number Theory Conference
University of Crete

24 May 2025

GSRT
GENERAL SECRETARIAT FOR RESEARCH AND TECHNOLOGY
H.F.R.I.
Hellenic Foundation for Research & Innovation

Outline

Double categories

$\star$ Introduced by Ehresmann in the '60s - strict version "category internal in the category of categories"

- A double category $\mathbb{D}$ consists of
  - 0-cells & vertical 1-cells which form a category $\mathbb{D}_0$
  - horizontal 1-cells & 2-maps which form a category $\mathbb{D}_1$
  - functor $1\colon \mathbb{D}_0 \to \mathbb{D}_1$ providing units
  - functors $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$ providing source and target

$$\begin{tikzcd} X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "B"', "\shortmid"{marking}] & W \end{tikzcd}$$

  - functor $\circ\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ providing horizontal composition

together with natural $(A \circ B) \circ C \cong A \circ (B \circ C)$, $A \circ 1_X \cong A \cong 1_Y \circ A$ with identity vertical boundaries, satisfying coherence axioms.

E.g. the two kinds of compositions of 2-maps obey interchange law

$$\begin{array}{ccccc} & \relbar\joinrel\mapstochar\joinrel\rightarrow & & \relbar\joinrel\mapstochar\joinrel\rightarrow & \\ \downarrow & \Downarrow\alpha & \downarrow & \Downarrow\beta & \downarrow \\ & \relbar\joinrel\mapstochar\joinrel\rightarrow & & \relbar\joinrel\mapstochar\joinrel\rightarrow & \\ \downarrow & \Downarrow\gamma & \downarrow & \Downarrow\delta & \downarrow \\ & \relbar\joinrel\mapstochar\joinrel\rightarrow & & \relbar\joinrel\mapstochar\joinrel\rightarrow & \end{array} \qquad (\delta \circ \gamma) \cdot (\beta \circ \alpha) = (\delta \cdot \beta) \circ (\gamma \cdot \alpha)$$

0-cells, horizontal 1-cells, *globular* 2-maps
make horizontal bicategory $\mathcal{H}(\mathbb{D})$.

$\star$ Alternative approach to 2-dimensional category theory, often more rich: for objects (0-cells) of interest, two different kinds of morphisms (with strict vs pseudo associative composition) encompassed in single structure.

Examples of double categories

- $\mathbb{R}$el with sets as 0-cells, functions as vertical 1-cells ($\mathbb{R}\mathrm{el}_0$=Set), relations $A \subseteq X \times Y$ as horizontal 1-cells $A\colon X \nrightarrow Y$, maps of relations $(xAy \Rightarrow f(x)Bg(y))$ as 2-maps.

Works in any regular category $\mathcal{C} \rightsquigarrow$ double category $\mathbb{R}\mathrm{el}(\mathcal{C})$.

- $\mathbb{S}$pan with $\mathbb{S}\mathrm{pan}_0$=Set, horizontal 1-cells spans $\begin{tikzcd}[column sep=small] & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ and 2-maps

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[ddd, "k"] & \\ X \arrow[d, "f"'] & & Y \arrow[d, "g"] \\ Z & & W \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Horizontal composition given by taking pullbacks of spans.

Works in any $\mathcal{C}$ with pullbacks$\rightsquigarrow$ double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$.

- $\mathbb{B}\text{im}$ with $\mathbb{B}\text{im}_0 = \text{Rng}$, the category of rings and ring homomorphisms, horizontall 1-cells $R \overset{M}{\nrightarrow} S$ are (R,S)-bimodules and 2-maps
$$\begin{tikzcd} R \arrow[r, "M", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & S \arrow[d, "g"] \\ R' \arrow[r, "M'"', "|"{marking}] & S' \end{tikzcd}$$
homomorphisms $\phi\colon M \to M'$ s.t. $\phi(rm) = f(r)\phi(m), \phi(ms) = \phi(m)g(s)$. Horizontal composition $R \overset{M}{\nrightarrow} S \overset{N}{\nrightarrow} T$ is tensor product $M \otimes_S N$.

- $\mathcal{V}$-$\mathbb{M}\text{at}$ for $(\mathcal{V}, \otimes, I)$ monoidal category+assumptions. $\mathcal{V}$-$\mathbb{M}\text{at}_0 = \text{Set}$, $X \overset{A}{\nrightarrow} Y$ are $\mathcal{V}$-matrices $Y \times X \xrightarrow{A} \mathcal{V}$ i.e. $\{A(y,x)\}_{y,x}$ in $\mathcal{V}$, 2-maps are
$$\begin{tikzcd} Y \times X \arrow[rr, bend left, "A"] \arrow[dr, "g\times f"'] & \Downarrow\alpha & \mathcal{V} \\ & W \times Z \arrow[ur, "B"'] & \end{tikzcd} \qquad \alpha_{x,y}\colon A(y,x) \to B(gy, fx) \in \mathcal{V}$$

Composition is 'matrix multiplication' $(B \circ A)(z,x) = \sum_y B(z,y) \otimes A(y,x)$.

Monads in double categories

- A monad in $\mathbb{D}$ is $A\colon X \to\!\!\!\!\!\!\mid\;\, X$ with 'multiplication' and 'unit' 2-maps

$$\begin{tikzcd} X \arrow[r, "A", "|"{marking}] \arrow[d, equal] & X \arrow[r, "A", "|"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\ X \arrow[rr, "A"', "|"{marking}] & & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[d, equal] \\ X \arrow[r, "A"', "|"{marking}] & X \end{tikzcd}$$

satisfying usual associativity and unitality axioms. E.g.

$$\begin{tikzcd} X \arrow[r, "A", "|"{marking}] \arrow[d, equal] & X \arrow[r, "A", "|"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[r, "A", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_A"] & X \arrow[d, equal] \\ X \arrow[rr, "A"', "|"{marking}] \arrow[d, equal] & & X \arrow[r, "A"', "|"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\ X \arrow[rrr, "A"', "|"{marking}] & & & X \end{tikzcd} = \begin{tikzcd} X \arrow[r, "A", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_A"] & X \arrow[r, "A", "|"{marking}] \arrow[d, equal] & X \arrow[r, "A", "|"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\ X \arrow[r, "A"', "|"{marking}] \arrow[d, equal] & X \arrow[rr, "A"', "|"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & & X \arrow[d, equal] \\ X \arrow[rrr, "A"', "|"{marking}] & & & X \end{tikzcd}$$

$\star$ Since all 2-maps are globular, coincide with monads in *bicategories*. However, maps of monads are different!

▶ A monad map from $X \overset{A}{\nrightarrow} X$ to $Y \overset{B}{\nrightarrow} Y$ is a 2-map $\begin{tikzcd} X \arrow[r, "A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \\ Y \arrow[r, "B"', "|" marking] & Y \end{tikzcd}$ s.t.

$$
\begin{tikzcd}
X \arrow[r, "A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[r, "A", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \\
Y \arrow[r, "B"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & Y \arrow[r, "B"', "|" marking] & Y \arrow[d, equal] \\
Y \arrow[rr, "B"', "|" marking] & & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "A", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & X \arrow[r, "A", "|" marking] & X \arrow[d, equal] \\
X \arrow[rr, "A"', "|" marking] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\alpha"] & & X \arrow[d, "f"] \\
Y \arrow[rr, "B"', "|" marking] & & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "1_X", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[d, equal] \\
X \arrow[r, "A"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \\
Y \arrow[r, "B"', "|" marking] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "1_X", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_f"] & X \arrow[d, "f"] \\
Y \arrow[r, "1_Y"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & Y \arrow[d, equal] \\
Y \arrow[r, "B"', "|" marking] & Y
\end{tikzcd}
$$

■ Monads and monad maps form a category $\mathrm{Mnd}(\mathbb{D})$ for any double $\mathbb{D}$.

$\star$ When $\mathbb{D}$ has single 0-cell and vertical 1-cell, becomes a monoidal category $\mathcal{V}$ ($\circ = \otimes$). Then $\mathrm{Mnd}(\mathbb{D})$ is the category of monoids in $\mathcal{V}$!

Examples of categories of monads

- For $\mathbb{D}$=$\mathbb{S}\mathrm{pan}(\mathcal{C})$, monad $\begin{tikzcd}[column sep=small] & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\ X & & X \end{tikzcd}$ is category *internal* in $\mathcal{C}$: consists of $\mathcal{C}$-object X of objects, $\mathcal{C}$-object A of arrows, η picks identities and μ

$$\begin{tikzcd}[column sep=small] & & A \times_X A \arrow[dl] \arrow[dr] \arrow[dddd, dashed, bend left] & & \\ & A \arrow[dl, "d"'] \arrow[dr, "c"] & & A \arrow[dl, "d"'] \arrow[dr, "c"] & \\ X & & X & & X \\ & & & & \\ & & A \arrow[uull, "d"] \arrow[uurr, "c"'] & & \end{tikzcd}$$

is composition rule.

A monad map is a functor internal in $\mathcal{C}$, so $\mathsf{Mnd}(\mathbb{S}\mathrm{pan}(\mathcal{C})) = \mathsf{Cat}(\mathcal{C})$!

- For $\mathbb{R}\mathrm{el}(\mathcal{C}) \subseteq \mathbb{S}\mathrm{pan}(\mathcal{C})$, category of monads $\mathsf{Mnd}(\mathbb{R}\mathrm{el}(\mathcal{C}))$ is $\mathsf{Preord}(\mathcal{C})$, category of internal preorders and order-preserving maps in $\mathcal{C}$.

- For $\mathbb{B}\mathrm{im}$, a monad $R \overset{A}{\twoheadrightarrow} R$ is an R-algebra and a monad map
$$\begin{tikzcd} R \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & R \arrow[d, "f"] \\ S \arrow[r, "B"', "\shortmid"{marking}] & S \end{tikzcd}$$
is R-algebra map $\alpha\colon A \to B$ with B an R-algebra via restriction of scalars. So Mnd($\mathbb{B}\mathrm{im}$)=Alg, a 'global' category of algebras over arbitrary rings.

- For $\mathcal{V}$-$\mathbb{M}\mathrm{at}$, a monad $X \overset{A}{\twoheadrightarrow} X$ is $\{A(x,x')\}_{x,x'}$ in $\mathcal{V}$ with

$$\Big(\sum\Big) A(x,x') \otimes A(x',x'') \to A(x,x''), \quad I \to A(x,x)$$

$+$ axioms , i.e. a $\mathcal{V}$-category! Moreover, a monad map is a $\mathcal{V}$-functor between $\mathcal{V}$-categories, thus Mnd($\mathcal{V}$-$\mathbb{M}\mathrm{at}$)=$\mathcal{V}$-Cat.

$\star$ Both internal and enriched categories can be studied in this context!

Modules in double categories

- A left A-module for monad $X \overset{A}{\nrightarrow} X$ in $\mathbb{D}$ is $U \overset{M}{\nrightarrow} X$ with 'action'

$$\begin{tikzcd}
U \arrow[r, "M"] \arrow[d, equal] & X \arrow[r, "A"] \arrow[d, phantom, "\Downarrow\lambda"] & X \arrow[d, equal] \\
U \arrow[rr, "M"'] & & A
\end{tikzcd}$$

compatible with μ, η. A module morphism from A-module $U \overset{M}{\nrightarrow} X$ to B-module $Z \overset{N}{\nrightarrow} Y$ is a monad map $A \to B$ and a 2-map

$$\begin{tikzcd}
U \arrow[r, "M"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\zeta"] & X \arrow[d, "f"] \\
Z \arrow[r, "N"'] & Y
\end{tikzcd}$$ with f the boundary of monad map

compatible with the actions.

■ Module and module maps form a category $\mathrm{Mod}(\mathbb{D})$ for any double $\mathbb{D}$.

Monoidal structure

▶ A double category $\mathbb{D}$ is monoidal when $\mathbb{D}_0$ and $\mathbb{D}_1$ are monoidal, $s, t\colon \mathbb{D}_1 \to \mathbb{D}_0$ are strict monoidal functors, and

$$(M \otimes N) \circ (M' \otimes N') \cong (M \circ M') \otimes (N \circ N'), \quad 1_{X \otimes Y} \cong 1_X \otimes 1_Y$$

subject to coherence conditions.

• $\mathbb{R}\mathrm{el}(\mathcal{C})$ and $\mathbb{S}\mathrm{pan}(\mathcal{C})$ are (cartesian) monoidal, $\mathbb{B}\mathrm{im}$ is monoidal with $R \otimes S$ and $M \otimes_{\mathbb{Z}} N$, $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ is monoidal with $(X \otimes Y) = X \times Y$ and

$$(A \otimes B)((x, y), (z, w)) = A(x, z) \otimes B(y, w).$$

■ If $\mathbb{D}$ is monoidal double, $\mathsf{Mnd}(\mathbb{D})$ and $\mathsf{Mod}(\mathbb{D})$ are monoidal categories. If A and B are monads, $A \otimes B$ becomes a monad via

$$(A \otimes B) \circ (A \otimes B) \xrightarrow{\cong} (A \circ A) \otimes (B \circ B) \xrightarrow{\mu \otimes \mu} A \otimes B$$

Fibrant structure

- $\mathbb{D}$ is fibrant when the functor $(s,t)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a fibration.

$F\colon \mathcal{C} \to \mathcal{X}$ is a fibration when for every $f\colon X \to F(B)$ in $\mathcal{X}$ there exists unique lifting $f^*(B) \to B$ of f in $\mathcal{C}$ with factorization property.

Gives canonical way to turn vertical 1-cells $X \xrightarrow{f} Y$ to two horizontal ones, the *companion* $\hat{f}\colon X \nrightarrow Y$ & the *conjoint* $\check{f}\colon Y \nrightarrow X$.

- In $\mathbb{S}\mathrm{pan}$, function $X \xrightarrow{f} Y$ gives spans

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\ Y & & X \end{tikzcd}$$

- In $\mathcal{V}$-$\mathbb{M}\mathrm{at}$, $X \xrightarrow{f} Y$ gives matrices $\hat{f}(x,y) = \check{f}(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

■ $\mathrm{Mnd}(\mathbb{D}) \to \mathbb{D}_0$ is a fibration, $\mathrm{Mod}(\mathbb{D}) \to \mathrm{Mnd}(\mathbb{D})$ is a fibration.

For a vertical $X \xrightarrow{f} Y$ and monad $Y \overset{A}{\nrightarrow} Y$, $f^*(A)\colon X \overset{\hat{f}}{\nrightarrow} Y \overset{A}{\nrightarrow} Y \overset{\check{f}}{\nrightarrow} X$.

Parallel limits and colimits

$\star$ Double categorical (co)limits exist and have been studied (Paré et al). Here, more specific notion is more relevant.

▶ $\mathbb{D}$ has parallel (co)limits if $\mathbb{D}_0$, $\mathbb{D}_1$ have (co)limits, s, t preserve them.

• $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has all parallel limits that $\mathcal{C}$ has. $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ has parallel coproducts, and is parallel cocomplete when $\mathcal{V}$ is and $\otimes$ preserves colimits.

■ When $\mathbb{D}$ is fibrant and has parallel limits, $\mathsf{Mnd}(\mathbb{D})$ has limits, $\mathsf{Mod}(\mathbb{D})$ has limits and $\mathsf{Mod}(\mathbb{D}) \to \mathsf{Mnd}(\mathbb{D})$ preserves them.

■ When $\mathbb{D}$ is fibrant and has parallel colimits, preserved by $- \circ -$, then

- $\mathsf{Mnd}(\mathbb{D})$ is monadic over horizontal endo-1-cells (no μ or η);
- $\mathsf{Mnd}(\mathbb{D})$ is cocomplete.

Sweedler theory

⋆ (1960s) Sweedler defines 'universal measuring k-coalgebra' $P(A, B)$ for any two k-algebras. For $B = k$, 'finite dual' coalgebra A^o for which

$$\{\text{algebra maps } A \to C^*\} \cong \{\text{coalgebra maps } C \to A^o\}$$

⋆ (later) P provides an enrichment of k-algebras Alg_k in k-coalgebras Coalg_k. Can replace Vect_k by monoidal category $(\mathcal{V}, \otimes, I)$+assumptions

⋆ (many-object version) $\mathsf{Mon}(\mathcal{V})$ of monoids 'is' $\mathsf{Mnd}(\mathbb{D})$ for double $\mathbb{D}$ with $\mathbb{D}_0 = \{*\}$, also exist dual concept of comonads and comodules...

For $\mathbb{D}$ a braided monoidal closed and locally presentable double category

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{D}) \arrow[r, dotted, "\text{enriched}"] \arrow[d, "\text{fibered}"'] & \mathsf{Comod}(\mathbb{D}) \arrow[d, "\text{opfibered}"] \\
\mathsf{Mnd}(\mathbb{D}) \arrow[r, dotted, "\text{enriched}"'] & \mathsf{Cmd}(\mathbb{D})
\end{tikzcd}$$

Thank you for your attention!

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$