CATEGORY THEORY EXAMPLES 3

1. Show that a monoidal category with one object is given by the same data as a commutative monoid. What can you say about braidings and symmetries on such a category?

2. (i) Show that if $\mathcal{C}$ is a category with finite products, then $\mathcal{C}$ is a monoidal category with the binary product as the monoidal product and the terminal object as the monoidal unit.

   (ii) Deduce that if $\mathcal{C}$ is a category with finite coproducts, then $\mathcal{C}$ is a monoidal category with the binary coproduct as the monoidal product and the initial object as the monoidal unit.

   (iii) Explain, by the above observations, why the category of commutative k-algebras is a monoidal category with monoidal product the k-tensor product and monoidal unit k.

   (iv) We call a monoidal category of type (i) a *cartesian* monoidal category, and of type (ii) a *cocartesian* monoidal category. What can you say about braidings and symmetries on such monoidal categories?

3. Show that in any braided monoidal category $(\mathcal{V}, \otimes, I)$, the following diagram commutes:

$$\begin{array}{ccc} X \otimes I & \xrightarrow{\gamma_{X,I}} & I \otimes X \\ & {\scriptstyle r_X}\searrow \quad \swarrow{\scriptstyle l_X} & \\ & X. & \end{array}$$

4. Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories. If $F \dashv G$ with $G\colon \mathcal{D} \to \mathcal{C}$ monoidal, then the induced opmonoidal structure on F is strong in and only if F is monoidal an the unit and counit are monoidal natural transformations.

5. Show that taking the dual vector space induces a functor $(-)^*\colon \mathbf{Mod}_A^{\mathrm{op}} \to {}_A\mathbf{Mod}$ between right and left A-modules, for any k-algebra A.

6. Let A be a k-algebra A and $U\colon \mathbf{Mod}_A \to \mathbf{Vect}_k$ the forgetful functor.

   (a) Show that there exists a bijection between the sets
$$\mathbf{Nat}(U \circ (- \otimes A), \mathrm{id}_{\mathbf{Vect}_k}) \cong \mathrm{Hom}_A(A, A^*).$$

   (b) Show that A is Frobenius if and only if U has a left and right adjoint that are identical.

7. Describe 0- and 1-dimensional TQFT's.

8. Show that a Frobenius algebra in a monoidal category $\mathcal{C}$ is an algebra and a coalgebra, i.e. μ and Δ are associative and coassociative respectively.

9. Let A be a k-algebra, equipped with a functional $\epsilon \in A^*$. Show that any two Frobenius structures for A are necessarily identical. (Hint: use the description in terms of Casimir elements).

10. Show that $A = \mathbb{C}[x]/(x^2)$ is a Frobenius algebra, by writing explicitly the linear maps $\mu, \eta, \Delta, \epsilon$. Then show that the corresponding TQFT maps the sphere to the linear map $0\colon \mathbb{C} \to \mathbb{C}$ and the torus to 2.