CATEGORY THEORY EXAMPLES 2

1. Prove that there is an adjunction $- \times B \dashv (-)^B$ of endofunctors on **Set**.

2. Let $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ ($\perp$) be an adunction with unit η and counit ϵ. Show that the following conditions are equivalent:

   (i) $L\eta_A$ is an isomorphism, for all $A \in \mathrm{ob}\mathcal{C}$;
   (ii) ϵ_{LA} is an isomorphism, for all $A \in \mathrm{ob}\mathcal{C}$;
   (iii) $R\epsilon_{LA}$ is an isomorphism, for all $A \in \mathrm{ob}\mathcal{C}$;
   (iv) $RL\eta_A = \eta_{RLA}$ for all $A \in \mathrm{ob}\mathcal{C}$;
   (v) $RL\eta_{RB} = \eta_{RLRB}$ for all $B \in \mathrm{ob}\mathcal{D}$;
   (vi) – (x) duals of (i)–(v).

   Such an adjunction is called *idempotent*. (Hint: choose the cyclic order of implications as given!)

3. Find an equivalent definition of adjoint functors involving only a unit (and no counit). (Hint: each $\eta_C : C \to RLC$ is *universal* from C to R).

4. Re-prove the equivance between the unit-counit definition of adjoint functors and the Hom-bijection definition of adjoint functors by means of the Yoneda Lemma.

5. For any adjunction $L \dashv R$, we have a full subcategory $\mathrm{FIX}(RL) \subseteq \mathcal{C}$ of objects A such that η_A is an isomorphism and similarly $\mathrm{FIX}(LR) \subseteq \mathcal{D}$ of objects with invertible counit.

   (i) If $L \dashv R$ is idempotent, show that $\mathrm{FIX}(RL)$ is a *reflective* and $\mathrm{FIX}(LR)$ is a *coreflective* subcategory (i.e. the inclusions have a left and right adjoint respectively).
   (ii) If $L \dashv R$ is idempotent, show that L and R restrict to an equivalence between $\mathrm{FIX}(RL)$ and $\mathrm{FIX}(LR)$.
   (iii) Deduce that an adjunction is idempotent if and only if it can be factored as a reflection followed by a coreflection.

6. For any categories $\mathcal{Z}, \mathcal{D}$, there is a 'discrete diagram' functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ given by $C \mapsto \Delta_C$ which maps all objects of $\mathcal{Z}$ to C and all morphisms of $\mathcal{Z}$ to 1_C. Prove that it has a right (respectively left) adjoint if and only if $\mathcal{C}$ has limits (respectively colimits) of shape $\mathcal{Z}$.

7. Suppose $F : \mathcal{C} \simeq \mathcal{D} : G$ via natural isomorphisms $\alpha : \mathbf{1}_\mathcal{C} \xrightarrow{\sim} GF$ and $\beta : FG \xrightarrow{\sim} \mathbf{1}_\mathcal{D}$. Show that there exists an adjunction $F \dashv G$ with unit α and counit $\beta' := \beta \circ (F\alpha_G)^{-1} \circ \beta_{FG}^{-1}$.

8. Fix a field k. Let **Mat** be the category with objects natural numbers and hom-sets $\mathbf{Mat}(n, m) = \{n \times m \text{ matrices over } k\}$. After showing that this is indeed a category, prove that it is equivalent to $\mathbf{fdVect}_k$, the category of finite-dimensional vector spaces over k and linear maps between them.

9. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and a category $\mathcal{A}$, first define a functor $F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{A})$ defined on objects by $H \mapsto H \circ F$, and then show that any adjunction $F \dashv G$ gives rise to an adjunction $G^* \dashv F^*$ (hint: use the unit/counit formulation).

10. Let $\mathbf{n}$ denote an n-element ordered set, viewed as a category.

    (i) Describe the functor category $\mathrm{Fun}(\mathbf{n}, \mathbf{Set})$.
    (ii) Show that there exist functors $F_0, \ldots, F_{n+1} : \mathrm{Fun}(\mathbf{n}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{n+1}, \mathbf{Set})$ and $G_0, \ldots, G_n : \mathrm{Fun}(\mathbf{n+1}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{n}, \mathbf{Set})$ which form an adjoint string
    $$(F_0 \dashv G_0 \dashv F_1 \dashv \ldots \dashv G_n \dashv F_{n+1}).$$
    (iii) Show that this string is maximal, i.e. F_0 has no left adjoint and F_{n+1} has no right adjoint. (Hint: do they preserve the terminal and initial object respectively?)