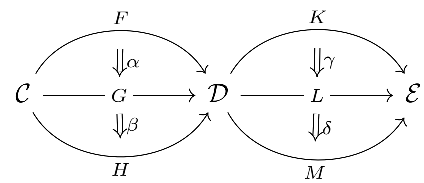
CATEGORY THEORY EXAMPLES 1

1. Let $\mathcal{C}$ be a category and $I \in \mathcal{C}_0$ an object. Show that $\mathcal{C}/I$ as defined in Example 1.2.5 indeed forms a category, the *category of arrows over I* or *slice category*.

2. Prove the interchange law for categories, functors and natural transformations:

   for  $(\delta * \beta) \circ (\gamma * \alpha) = (\delta \circ \gamma) * (\beta \circ \alpha).$

3. A morphism $e\colon A \to A$ in a category $\mathcal{C}$ is called *idempotent* if $e \circ e = e$. Denote by $\mathrm{dom} e = \mathrm{cod} e$ the domain and codomain of e.

   (i) Suppose $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ are functors, and $\alpha\colon 1_{\mathcal{C}} \Rightarrow G \circ F$, $\beta\colon F \circ G \Rightarrow 1_{\mathcal{D}}$ are natural transformations such that $G\beta \circ \alpha_G\colon G \Rightarrow GFG \Rightarrow G$ is the identity. Show that $\beta_F \circ F\alpha\colon F \Rightarrow F$ is an idempotent in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, for $\mathcal{C}$ small.

   (ii) If $\mathcal{E}$ is a class of idempotents in a category $\mathcal{C}$, show that there exists a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f\colon \mathrm{dom} e \to \mathrm{cod} d$ in $\mathcal{C}$ for which $d \circ f \circ e = f$, and whose composition coincides with composition in $\mathcal{C}$ (hint: the identity is not the same as in $\mathcal{C}$!)

4. Use Yoneda to prove that $\alpha\colon F \Rightarrow G$ is a monomorphism in $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ if and only if its components $\alpha_A\colon FA \to GA$ are injective functions.

5. (i) Viewing a group G as an one-object category, show that natural transformations $1_G \Rightarrow 1_G$ correspond to elements in the centre of the group.

   (ii) Deduce Cayley's embedding theorem using the Yoneda embedding theorem.

6. Show that there exist functors $\mathrm{ob}, \mathrm{mor}\colon \mathbf{Cat} \to \mathbf{Set}$ picking the set of objects and morphisms of categories. Are they full? Are they faithful?

7. Prove the following:

   (i) any retraction is an epimorphism, and faithful functors reflect them;

   (ii) an isomorphism is a mono and an epi, and the converse is not always true;

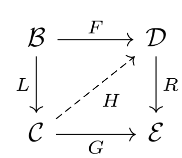
   (iii) (*two-out-of-three property*) for $A \xrightarrow{f} B \xrightarrow{g} C$, if two out of $f, g, g \circ f$ are isos then so is the third;

   (iv) all functors preserve isos and fully faithful functors reflect them.

8. Show that any functor $F\colon \mathcal{C} \to \mathcal{D}$ can be factorized as

   $$\mathcal{C} \xrightarrow{L} \mathcal{E} \xrightarrow{R} \mathcal{D}$$

   where L is bijective-on-objects and R is fully faithful. Also, show that for any commutative square

   where L is b.o.b. and R is ff, there exists a unique functor $H\colon \mathcal{C} \to \mathcal{D}$ such that $H \circ L = F$ and $R \circ H = G$.

9. By an *automorphism* of a small category $\mathcal{C}$ we mean an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ which has a (2-sided) inverse. We say an automorphism is *inner* if it is naturally isomorphic to the identity functor.

   (i) Show that inner $\mathcal{C}$-automorphisms form a normal subgroup of all $\mathcal{C}$-automorphisms, viewed as a group with composition as multiplication.

   (ii) If F is a $\mathcal{C}$-automorphism and $\mathbf{1}$ is a terminal object in $\mathcal{C}$, show that $F(\mathbf{1})$ is also a terminal object in $\mathcal{C}$ (hence isomorphic to $\mathbf{1}$).

10. (i) Express the universal property of a coproduct of a family of objects $(FZ)_{Z\in\mathcal{Z}}$, for a functor $F\colon \mathcal{Z} \to \mathcal{C}$ from a discrete category $\mathcal{Z}$.

    (ii) (Exercise 3.2.3) Consider a poset $(P, \leq)$. Let $(x_i)_{i\in I}$ be a family of elements in P, what is the product and coproduct of $(x_i)_{i\in I}$ considered as a family of objects in the poset category?

11. (Proposition 3.2.10) Consider the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_B} & B \\ {\scriptstyle p_A}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

    Then if g is a monomorphism (respectively, isomorphism), then p_A is a monomorphism (respectively, isomorphism) as well.

12. Consider the following commutative squares:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow[h]{} & U & \xrightarrow[k]{} & W \end{array}$$

    Prove the following statements:

    (i) if both small rectangles are pullbacks, then so is the large one;

    (ii) if the large rectangle and the small right one are pullbacks, then so is the left one.

13. (Theorem 3.3.5) For a category $\mathcal{C}$, the following are equivalent:

    (i) $\mathcal{C}$ is finitely complete;

    (ii) $\mathcal{C}$ has a terminal object, binary products and equalizers;

    (iii) $\mathcal{C}$ has a terminal object and pullbacks.

14. We say that a functor $G\colon \mathcal{C} \to \mathcal{D}$ *creates limits* of shape $\mathcal{Z}$ if, given $F\colon \mathcal{Z} \to \mathcal{C}$ and a limit (M, μ_Z) for $G \circ F$, there exist a cone (L, λ_Z) over F in $\mathcal{C}$ whose image is isomorphic to (M, μ_Z); and any such cone is a limit in $\mathcal{C}$.

    (i) If $\mathcal{D}$ has and G creates limits of shape $\mathcal{Z}$, then $\mathcal{C}$ has and G preserves them.

    (ii) If G creates limits of shape $\mathcal{Z}$, then G reflects them.