The commuting tensor product of multicategories

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Context and motivation

	$\mathbb{B}$Mod $\mathbb{B}$im(Ab)	$\mathbb{P}$rof $\mathbb{B}$im($\mathbb{M}$at)	$\mathbb{S}$MultProf $\mathbb{B}$im($\mathbb{S}$ym)
objects	rings	categories	sym.multicategories
vertical 1-cells	homomorphisms	functors	sym.multifunctors
commuting tensor	$R \otimes S$	$\mathcal{A} \times \mathcal{B}$	$\mathcal{A} \otimes_{\mathrm{BV}} \mathcal{B}$
horizontal 1-cells	bimodules $M \colon R \to\!\!\!\!\!\mapsto S$	profunctors $F \colon \mathcal{A} \to\!\!\!\!\!\mapsto \mathcal{B}$	bimodules $\mathcal{A} \to\!\!\!\!\!\mapsto \mathcal{B}$
commuting tensor	$R \otimes R' \overset{M \otimes M'}{\to\!\!\!\!\!\mapsto} S \otimes S'$	$\mathcal{A} \times \mathcal{A}' \overset{F \times G}{\to\!\!\!\!\!\mapsto} \mathcal{B} \times \mathcal{B}'$	?

$\star$ Goal: uniform account for double categories $\mathbb{B}\mathrm{im}(\mathbb{C})$ of monads, monad maps (i.e. $\mathrm{Mnd}(\mathbb{C})$ vertically), bimodules and bimodule maps in $\mathbb{C}$.

Fix an oplax monoidal closed and fibrant double category $(\mathbb{C}, \boxtimes, I)$ $(+\dots)$

$$\begin{tikzcd}
A_1 \boxtimes A_2 \arrow[rr, "(y_1 \circ x_1) \boxtimes (y_2 \circ x_2)", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\xi"] & & C_1 \boxtimes C_2 \arrow[d, equal] \\
A_1 \boxtimes A_2 \arrow[r, "x_1 \boxtimes x_2", "|" marking] & B_1 \boxtimes B_2 \arrow[r, "y_1 \boxtimes y_2", "|" marking] & C_1 \boxtimes C_2
\end{tikzcd}$$

Theorem: The double category $\mathbb{B}\mathrm{im}(\mathbb{C})$ is oplax monoidal closed.

- Monads with *multimaps* form a representable closed multicategory.

$\rightsquigarrow$ $\mathrm{Mnd}(\mathbb{C})$ with $a \,\square\, b = a \boxtimes 1 + 1 \boxtimes b$ is a monoidal closed category.

- Monads with *commuting* multimaps form a rep. closed multicategory.

$\rightsquigarrow$ $\mathrm{Mnd}(\mathbb{C})$ with $a \otimes b$, coequalizer of $a \boxtimes b \rightrightarrows a \,\square\, b$, is monoidal closed.

- Define $m \otimes n$ on bimodules of monads, using $\boxtimes$ on free ones and extending via reflexive coequalizers.