Welcome to Math 007B!

Approximating areas

Idea: want to compute an area on the plane, by dividing it in small rectangular regions and summing their areas.

- Tool for summation? *Sigma notation*!

For any real numbers $a_1, a_2, \ldots, a_n$, we write

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

Properties of Sums

1. $\sum_{k=1}^{n} 1 = n$
2. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
3. $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$
4. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
   sum of first n integers
5. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

A *partition* P of an interval $[a, b]$ is a collection of subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$$

with $a < x_1 < x_2 < \ldots < x_{n-1} < b$.

The length of k-th subinterval $[x_{k-1}, x_k]$ is written Δx_k; if all equal, Δx.

- We can use any point of the subinterval (e.g. left endpoint, midpoint, right endpoint etc.) to form the rectangles.
- The finer the partition, i.e. the larger the number n of subintervals - the shorter their length, the better the area approximation.

Definite Integral

Let $P = [a, x_1, ..., x_{n-1}, b]$ partitions of $[a, b]$, and $c_k \in [x_{k-1}, x_k]$. The *definite integral* of a function f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \overbrace{f(c_k)}^{\text{height}} \overbrace{\Delta x}^{\text{base}}$$

If the limit exists, f is (Riemann) *integrable* on $[a, b]$; e.g. all continuous f.

Integrals are signed areas

⋆ Geometrically, integrals are *signed* areas between curves and the x-axis: above or below changes the sign!

[POSITIVE] If f is integrable on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)dx = \text{area between } f\text{-graph and } x\text{-axis, from } a \text{ to } b$$

[ANY] If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$

Properties of the Riemann Integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

⋆ All the proofs follow from properties of limits and Sigma notation!

Work out the following: (answers are now posted in green):

1. Compute $\int_{-1}^{2} (x-1)dx$ by finding the respective area. $-\frac{3}{2}$

The Fundamental Theorem of Calculus

Idea: computing areas [integrals] is connected to tangents [derivatives]!

- If f continuous, then it is integrable: $\int_a^x f(u)du$ exists, for arbitrary x.

Fundamental Theorem of Calculus (FTC), pt.1

Suppose f is continuous on $[a, b]$. Then the function defined as

$$F(x) = \int_a^x f(u)du, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\left(\frac{d}{dx}F(x)=\right) \boxed{\text{F'(x)=f(x)}}$$

$\star$ Integration and differentiation are 'inverses' to one another.

What if the upper and/or lower limits of the integral are more complicated functions of x?

Leibniz's Rule

If $g(x)$ and $h(x)$ are differentiable, $f(u)$ continuous for $g(x) \leq u \leq h(x)$,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

Work out the following:

1. Compute $\frac{d}{dx}\int_{-2x}^{x^2} \frac{t}{2}dt$. $x^3 - 2x$

Indefinite Integrals

Idea: FTC says $F'(x) = f(x)$ where $F(x) = \int\limits_a^x f(u)du$, for arbitrary $a \in \mathbb{R}$!

- An *antiderivative* of $f(x)$ is any $F(x)$ for which $F'(x) = f(x)$.

They are infinitely many, and differ from each other by a constant C.

Indefinite Integral

The general antiderivative of a function is denoted by

$$\int f(x)dx = C + \int_a^x f(u)du$$

and is called an *indefinite integral*.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
2. $\int e^x dx = e^x + C$
3. $\int \sin(x)dx = -\cos(x) + C$
4. $\int \frac{1}{x^2+1} dx = \arctan(x) + C$
5. $\int \frac{1}{x} dx = \ln|x| + C$
6. $\int a^x dx = \frac{a^x}{\ln a} + C$
7. $\int \cos(x)dx = \sin(x) + C$
8. $\int \sec^2(x)dx = \tan(x) + C$

What if we want to compute a definite integral, i.e. a specific area?

Fundamental Theorem of Calculus (FTC), pt.2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

for $F(x)$ any antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

$\star$ Any antiderivative will do (and they all differ by C), so go for simplest.

Work out the following:

1. Calculate the following antiderivative $\int x(1 - x^2)dx$. $\frac{x^2}{2} - \frac{x^4}{4} + C$
2. Evaluate $\int_0^\pi (\sin(x) - e^x + \cos(x))dx$. $3 - e^\pi$

Substitution Rule, indefinite integrals

Idea: substitution rule is for integration what chain rule is for derivation!

Recall: if $f(x)$ and $u(x)$ are differentiable, then

$$(f \circ u)'(x) = f'(u(x)) \cdot u'(x) \text{ or } \frac{d}{dx}\Big(f(u(x))\Big) = \frac{df}{du} \cdot \frac{du}{dx}$$

Substitution Rule for Indefinite Integrals

If $u = u(x)$, then $$\int f(u(x))\,u'(x)dx = \int f(u)du.$$

Work out the following: evaluate, using the substitution rule,

1. $\int x^2 \sin(x^3) dx$
   $-\frac{1}{3}\cos(x^3) + C$
2. $\int x e^{1-3x^2} dx$
   $-\frac{1}{6} e^{1-3x^2} + C$
3. $\int \frac{3x}{x+4} dx$
   $3(x+4) - 12 \ln|x+4| + C$

Substitution Rule, definite integrals

Idea: for definite integrals, also substitute limits of integration!

Substitution Rule for Definite Integrals

$$\int_a^b f(u(x)) \cdot u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Work out the following: evaluate, using the substitution rule,

1. $\int_1^2 \frac{3x^2+1}{x^3+x}dx$ $\ln(5)$
2. $\int_0^{\pi/6} \cos(x)e^{\sin(x)}dx$ $\sqrt{e}-1$
3. $\int_0^1 \frac{x^3}{x^2+1}dx$ $\frac{1-\ln 2}{2}$

Computing areas via integration

Idea: want to compute areas between two arbitrary curves on the plane!

If f and g are continuous on $[a, b]$, with $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is

$$A = \int_a^b (f(x) - g(x))dx$$

$\star$ Always a positive number: the 'real' area between the graphs.

Always draw the corresponding graphs: finding intersecting points, or splitting areas in parts bounded only by two curves, may be necessary!

⋆ Sometimes convenient to compute areas, "form rectangles", from y-axis!

For a region bounded by $x = f(y)$ and $x = g(y)$, $g(y) \leq f(y)$ in $[c, d]$,

$$A = \int_c^d (f(y) - g(y))dx$$

Now $f(y)$ is to the *right* of $g(y)$.

- Next, compute the *net* or *cumulative change* of a quantity.

FTC,pt.2 $\int_a^b F'(x)dx = F(b) - F(a)$ says that
the integral (sums) of instantaneous rate of change=net change

Average values and the Mean Value Theorem

Idea: use the 'summing rectangles' method to find average values.

Average value of a function

If $f(x)$ is continuous on $[a, b]$, its average value on that interval is

$$f_{\mathrm{avg}} = \frac{1}{b-a}\int_a^b f(x)dx$$

$\star$ In fact, the value f_{avg} is the output for some input of f!

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then there exists some $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)dx$$

Work out the following:

1. Average value of $f(x) = x^2 - 2 + e^{3x}$ over the interval $[0, 2]$? $\frac{e^6-5}{6}$

The Volume of a Solid

Idea: split a solid into partitioned cylinder 'slices' and sum volumes up!
When their number $n \to \infty$, circular cylinders becomes disks...

- Cross-section: intersection of solid and plane perpendicular to x-axis.

The *volume of a solid* of cross-sectional area $A(x)$ from a to b is

$$\int_a^b A(x)dx$$

For any solid of *revolution*, obtained by rotating some curve $y = f(x)$ about the x-axis, compute its volume by the *disk method*:

$$V = \int_a^b \pi (f(x))^2 dx.$$

If cross-sectional area is between two curves, also called *washer method*.

Integration by Parts

Idea: integration by parts is what product rule is for derivation!

Recall: if $u = u(x)$ and $v = v(x)$ are differentiable, then

$$(uv)' = u'v + uv'$$

Integration by parts rule

For $u(x)$ and $v(x)$ differentiable functions of x,

$$\int v du = uv - \int u dv$$

$\star$ Product: choose $v(x)$ to be the factor whose derivative is 'easier'!

- Same way for definite integrals: evaluate uv at the limits of integration.

Work out the following: evaluate, using integration by parts,

1. $\int x \cos(x) dx$ $x \sin x + \cos x + C$
2. $\int_1^e \ln(x^3) dx$ 3

Integration by parts, techniques

$$\int v du = uv - \int u dv$$

- Multiply by 1 ($du = 1$ so $u = x$), if derivative of integrand is easier
- Sometimes, need to apply IBP repeatedly: at every step, integral becomes easier
- Choice matters; if u and v do not work, check the other option
- Very often, first substitute and then integrate by parts

Work out the following: evaluate, using integration by parts,

1. $\int e^x \sin(x) dx$ $\frac{e^x(sin(x) - cos(x))}{2} + C$
2. $\int_0^1 e^{\sqrt{x}} dx$ (hint: first substitution!) 2

Rational Functions

Idea: break up rational functions $f(x) = \dfrac{p(x)}{d(x)}$ into simpler fractions!

- Polynomial division algorithm $\boxed{p(x) = d(x) \cdot q(x) + r(x)}$

Method: to decompose $p(x)/d(x)$

1. if $\deg(p(x)) \geq \deg(d(x))$, long division; then (2)-(4) for $r(x)/d(x)$
2. if $d(x)$ is of higher degree, factor it into linear factors $(ax+b)^n$ and/or irreducible quadratic factors (ax^2+bx+c) [no real zeros]
3. to each linear factor $(ax+b)^n$, assign the sum of partial fractions

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \ldots + \frac{C}{(ax+b)^n}$$

4. to each quadratic factor $(ax^2+bx+c)^m$, assign the sum

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \ldots + \frac{Zx+W}{(ax^2+bx+c)^n}$$

- Then clear equation from fractions; use in values of x that easily give $A, B, \ldots$

Integral parts will either be $\int \frac{1}{x+b}\, dx = \ln|x+b|$ or some form of $\int \frac{1}{x^2+1}\, dx = \arctan(x)$ after algebraic transformation of quadratics.

Work out the following: evaluate, using partial fraction decomposition,

1. $\int \frac{2x^2+5x-1}{x+2} dx$ $x^2 + x - 3\ln|x+2| + C$
2. $\int \frac{1}{x^3+x^2}$ $-\frac{1}{x} + \ln\left|\frac{x+1}{x}\right| + C$

Improper integrals: Unbounded intervals

Idea: use limits to compute integrals whose endpoints are $\pm\infty$.

If $f(x)$ is continuous, we define the following *improper integrals*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$
$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

⋆ The area of an unbounded region may, or may not, approach a number!

If $f(x)$ is continuous on $[a,\infty)$ (resp. $(-\infty,a]$), we say the improper

$$\int_a^\infty f(x)dx \ \ (\text{resp.} \ \int_{-\infty}^a f(x)dx)$$

converges when the limit exists and has a finite value. Otherwise the improper integral *diverges*.

What happens when both endpoints are infinite?

If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

for any real number a, when both right-hand side improper integrals converge; if any of them diverges, the left-hand side diverges as well.

- Notice that $\displaystyle\int_{-\infty}^{\infty} f(x)dx \neq \lim_{b\to\infty} \int_{-b}^{b} f(x)dx$!

Work out the following: do these converge or diverge? If C, evaluate:

1. $\displaystyle\int_{-\infty}^{0} \frac{1}{(x-1)^2}\, dx$ Converges, 1
2. $\displaystyle\int_{-\infty}^{\infty} 3e^{6x}\, dx$ Diverges

Improper integrals: Unbounded Integrand

What about integrands being undefined on some part of interval?

[Left Endpoint] If $f(x)$ is continuous on $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$, define

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

[Right Endpoint] If $f(x)$ is continuous on $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$,

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

If the limit exists, the imporoper integral *converges*; otherwise it *diverges*.

Work out the following: does this converge or diverge? If C, evaluate:

1. $\int_0^9 \frac{1}{\sqrt{9-x}}dx$ Converges, 6

Improper integrals: Comparison Theorems

$\star$ If $f(x)$ is discontinuous on some $p \in [a, b]$, break up integral as

$$\int_a^b f(x)dx = \int_a^{p} f(x)dx + \int_{p}^b f(x)dx$$

Convergence and Divergence of improper integrals via Comparison

- [Converge] If $0 \leq f(x) \leq g(x)$ for any $x \in [a, \infty)$,

$$0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

implies that if $\int_a^\infty g(x)dx$ converges, so does $\int_a^\infty f(x)dx$.

- [Diverge] If $0 \leq h(x) \leq f(x)$ for any $x \in [a, \infty)$,

$$0 \leq \int_a^\infty h(x)dx \leq \int_a^\infty f(x)dx$$

implies that if $\int_a^\infty h(x)dx$ diverges, so does $\int_a^\infty f(x)dx$.

For $f(x)$, find 'easier' function bounding it either from above or below and compute!

Trigonometric Integrals

Idea: use $\sin^2(x) + \cos^2(x) = 1$ to antiderive trigonometric combinations.

Integrals of the form $\int \sin^m(x)\cos^n(x)dx$

1. if m is odd, then $m = 2k+1$ for some integer k; rewrite

$$\sin^m(x) = \sin^{2k+1}(x) = \overbrace{\sin^{2k}(x)}^{(\sin^2(x))^k} \cdot \sin(x) = (1-\cos^2(x))^k \sin(x)$$

   and substitute $u = \cos(x)$, $du = -\sin(x)dx$.
2. if n is odd, similarly rewrite $\cos^n(x) = (1-\sin^2(x))^k\cos(x)$ and substitute $u = \sin(x)$, $du = \cos(x)dx$.
3. if both m and n are even, use the ‘power-reducing’ identities

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Work out the following: evaluate $\int_0^{\pi/2} \sin^2(x)\cos^3(x)dx$ $\frac{2}{15}$

Trigonometric Integrals, pt.2

Integrals of the form $\int \tan^m(x)\sec^n(x)dx$

1. if $n = 2k$ is even, rewrite

$$\sec^n(x) = \sec^{2k}(x) = \sec^2(x)\cdot\sec^{2k-2}(x) = \sec^2(x)\Big(1+\tan^2(x)\Big)^{k-1}$$

and substitute $u = \tan(x)$, with $du = \sec^2(x)dx$.

2. if $n = 0$, rewrite (and any method until all integrals are evaluated)

$$\tan^m(x) = \tan^{m-2}(x)\tan^2(x) = \tan^{m-2}(x)(\sec^2(x)-1)$$

3. if $m = 2k+1$ is odd, rewrite

$$\tan^{2k+1}(x)\sec^n(x) = (\sec^2(x)-1)^k\sec^{n-1}(x)\Big(\tan(x)\sec(x)\Big)$$

and substitute $u = \sec(x)$, with $du = \tan(x)\sec(x)dx$.

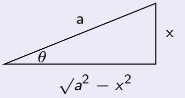
Work out the following: $\int \tan^3(x)dx = \tan^2(x)/2 + \ln|\cos(x)| + C$

Trigonometric Substitution

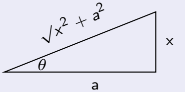
Idea: substitute x by some trig function of an angle θ
to simplify square roots & other integrands.

Trigonometric Substitution

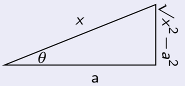
- whenever $\sqrt{a^2 - x^2}$, substitute $x = a\sin(\theta)$; by $(a\sin(\theta))^2+(a\cos(\theta))^2=a^2$, $\sqrt{a^2-x^2}=a\cos(\theta)$.



- whenever $\sqrt{x^2 + a^2}$, substitute $x = a\tan(\theta)$; by $a^2+(a\tan(\theta))^2=(a\sec(\theta))^2$, $\sqrt{x^2+a^2}=a\sec(\theta)$.



- whenever $\sqrt{x^2 - a^2}$, substitute $x = a\sec(\theta)$; then $\sqrt{x^2 - a^2} = a\tan(\theta)$.



- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\int \sec(\theta)d\theta = \ln|\tan(\theta) + \sec(\theta)| + C$

Differential Equations

Idea: find a function $y(x)$ for which $\frac{dy}{dx}$ =some function of x and/or y.

Separable first-order differential equation

An equation including only first derivatives, of the general form

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Pure-Time Differential Equations are of the form $\frac{dy}{dx} = f(x)$. Then by the FTC, $y = \int f(x)\,dx$, an antiderivative of $f(x)$.

⋆ Use initial condition to determine the constant C for final answer.

Work out the following:

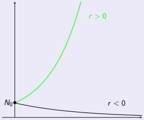
1. $\frac{dy}{dx} = \frac{1}{3-x}$, $y(0) = 0$ (simplify) $y = \ln(\frac{3}{|3-x|})$

Autonomous Differential Equations

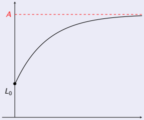
- $\frac{dy}{dx} = g(y)$: by separating variables, $\int \frac{1}{g(y)} dy = \int dx$.

Applications - Models

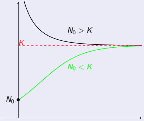
- *Exponential Population Growth* $\boxed{\frac{dN}{dt} = rN}$ where $N(t)$=population size at time t, $N(0) > 0$ and $r = \frac{1}{N}\frac{dN}{dt} \lessgtr 0$ per capita rate of growth.



- *Restricted Growth* $\boxed{\frac{dL}{dt} = k(A - L)}$ where $L(t)$=length of fish at age t and $L(0) < A$, the asymptotic length of the fish.



- *The Logistic Equation* $\boxed{\frac{dN}{dt} = rN(1 - \frac{N}{K})}$ $r, K>0$ where K=carrying capacity and per capita rate of growth decreases.



Work out the following: $N'(t) = 4N$, $N(0) = 10$ $N(t) = 10e^{4t}$

General separable equations of the form $\dfrac{dy}{dx} = g(y)f(x)$ are solved by separating variables: $\displaystyle\int \frac{1}{g(y)}dy = \int f(x)dx$.

Work out the following:

1. $y'(x) = y(y-1)$, $y(0) = \frac{1}{2}$
   $y(x) = \frac{1}{1+e^x}$
2. $\dfrac{dy}{dx} = 2\dfrac{y}{x}$, $y(1) = 1$ $y(x) = x^2$

Equilibria and their Stability

Idea: look for certain y's where the system possibly 'balances'.

For an autonomous differential equation $\frac{dy}{dx} = g(y)$, an *equilibrium* is some $\hat{y}$ such that $g(\hat{y}) = 0$, i.e. a solution of $\boxed{\frac{dy}{dx} = 0}$.

- If $y(0) = \hat{y}$, then $y(x) = \hat{y}$ for all $x > 0$; but in general, it is not guaranteed that the system will ever reach $\hat{y}$.

⋆ What about their stability, i.e. endurance after a small pertrubation?

If $\hat{y}$ is an equilibrium of $\frac{dy}{dx} = g(y)$, then

- $\hat{y}$ is *locally stable* if $g'(y) < 0$
- $\hat{y}$ is *unstable* if $g'(y) > 0$

Work out the following:

1. Equilibria and stability for $y'=2-3y$. $\hat{y}=2/3$, stable
2. Solve the differential equation, with $y(0) = 2/3$. $y(x) = 2/3$!!!
3. Equilibria and for $y' = y^2 - 2$. $\sqrt{2}$ unstable, $-\sqrt{2}$ stable