Welcome to Math 010A!

Vectors

Idea: move from the real line to the plane and 3-dimensional space!

Cartesian coordinates for line, plane, space

- The real number line is $\mathbb{R}^1 = \mathbb{R}$ (1-dim)
- The set of all ordered pairs (x, y) of real numbers is $\mathbb{R}^2$ (2-dim)
- The set of all ordered triples (x, y, z) of real numbers is $\mathbb{R}^3$ (3-dim)

$\star$ In general, $\mathbb{R}^n$ is the n-dimensional Euclidean space.

A *vector*, denoted $\mathbf{a}$ or $\vec{a}$, is a directed line segment in space with a beginning (tail) and an end (head).

$\star$ To any point $(a_1, a_2, a_3) \in \mathbb{R}^3$ associate the vector with tail=origin and head=(a_1, a_2, a_3): vectors thought of as arrows emanating from the origin!

- Two vectors are *equal* if and only if all their components are equal.

Vector operations

Idea: $\mathbb{R}^3$ inherits various standard operations from $\mathbb{R}$!

Vector addition and scalar multiplication

The *sum* of two vectors $\vec{a}$ and $\vec{b}$ is a vector

$$\vec{a} + \vec{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

The *scalar multiple* of a real number κ and a vector $\vec{a}$ is a vector

$$\kappa\vec{a} = \kappa(a_1, a_2, a_3) := (\kappa a_1, \kappa a_2, \kappa a_3)$$

- The vector $\vec{0} = (0, 0, 0)$ is the *zero* of $\mathbb{R}^3$; the vector $-\vec{a} = (-a_1, -a_2, -a_2)$ is the *additive inverse* of $\vec{a}$

⋆ These have geometric interpretations: addition is placing vectors 'head to tail', scalar multiplication is 'stretching' (and possibly reversing).

Two key characteristics of vectors is their length and their direction.

Standard basis vectors

Define $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$. Any vector in $\mathbb{R}^3$ can be represented uniquely as a linear combination of these *standard basis vectors*

$$\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

Work out the following: suppose $\vec{a} = (3,-1,-2)$ and $\vec{b} = (0,1,1)$.

1. Compute the vector $3\vec{a} - 2\vec{b}$. $(9,-5,-8)$
2. Express $\vec{b}$ as a linear combination of the standard basis. $\vec{b} = \vec{j} + \vec{k}$

Vector Joining two Points

If $P = (x,y,z)$ and $Q = (u,v,w)$ are two points in $\mathbb{R}^3$, there is a vector $\overrightarrow{PQ} = (u-x, v-y, w-z)$ from the tip of P to the tip of Q.

$\star$ Geometric interpretation of vector subtraction: 'join the two heads'.

Line equations using vectors

Forms of lines — $t \in \mathbb{R}$ is the parameter

1. *Point-Direction*: a parametric equation of the line passing through the head of some $\vec{a}$ and parallel to some $\vec{v}$ is

$$\vec{\ell}(t) = \vec{a} + t\vec{v}, \text{ with coordinates } \begin{cases} x = a_1 + v_1 t \\ y = a_2 + v_2 t \\ z = a_3 + v_3 t \end{cases}$$

2. *Point-Point*: an equation of the line passing through some $P = (a_1, a_2, a_3)$ and $Q = (b_1, b_2, b_3)$ is

$$\vec{\ell}(t) = \begin{cases} x = a_1 + (b_1 - a_1)t \\ y = a_2 + (b_2 - a_2)t \\ z = a_3 + (b_3 - a_3)t \end{cases}$$

$\star$ In $\mathbb{R}^3$, two lines may NOT be parallel yet still NOT intersecting!

Work out the following: suppose
$P = (-2, -1), Q = (-3, -3), R = (-1, -4)$ in $\mathbb{R}^2$.

1. $\overrightarrow{PQ}$? $\overrightarrow{QR}$? $\overrightarrow{RP}$? $(-1, -2), (2, -1), (-1, 3)$
2. Write an equation for the line that passes through P and R.
   $\vec{\ell}(t) = (-2 - t, -1 + 3t)$

Inner Product of vectors

The *inner* (or *dot*) *product* of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is

$$\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle := a_1 b_1 + a_2 b_2 + a_3 b_3$$

The *norm* of a vector $\vec{a} = (a_1, a_2, a_3)$ is its length, given by

$$\|\vec{a}\| := \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

$\star$ Operations: sum $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{+} \mathbb{R}^3$, scalar multiplication $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, inner product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\langle\, ,\, \rangle} \mathbb{R}$, norm $\mathbb{R}^3 \xrightarrow{\|\,\|} \mathbb{R}_+$!

1. $\langle \vec{a}, \vec{a} \rangle \geq 0$
2. $\kappa \langle \vec{a}, \vec{b} \rangle = \langle \kappa \vec{a}, \vec{b} \rangle$
3. $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$
4. $\langle \vec{a}, \vec{a} \rangle = 0 \Leftrightarrow \vec{a} = \vec{0}$
5. $\kappa \langle \vec{a}, \vec{b} \rangle = \langle \vec{a}, \kappa \vec{b} \rangle$
6. $\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle$

- A *unit vector* has norm one unit, $\|\vec{a}\| = 1$; e.g. $\vec{i}, \vec{j}, \vec{k}$.

To normalize a non-zero vector $\vec{a}$ amounts to keeping the same direction but making its length 1:

$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}(a_1, a_2, a_3)$$

⋆ Geometrically, inner product relates to angle between vectors!

Inner product and angle between vectors

If $\vec{a}, \vec{b} \in \mathbb{R}^3$ and $0 \leq \theta \leq \pi$ the angle between them,

$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \, ||\vec{b}|| \cos(\theta) \qquad \boxed{\vec{a} \perp \vec{b} \Leftrightarrow \langle \vec{a}, \vec{b} \rangle = 0}$$

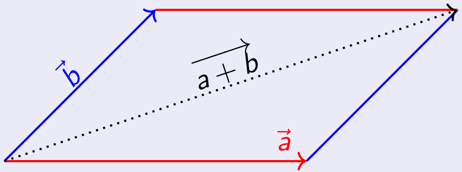
Given two vectors $\vec{a}$ and $\vec{b} \neq \vec{0}$, the *orthogonal projection* of $\vec{a}$ along $\vec{b}$ is

$$\vec{p} = \frac{\langle \vec{a}, \vec{b} \rangle}{\langle \vec{b}, \vec{b} \rangle} \vec{b}$$

Triangle Inequality

For any vectors $\vec{a}$ and $\vec{b}$,

$$\left\|\vec{a}+\vec{b}\right\| \leq \|\vec{a}\| + ||\vec{b}||$$



Work out the following:

1. Normalize the vector $(0, 3, -4)$. $(0, \frac{3}{5}, -\frac{4}{5})$
2. What is the angle between the vectors $\vec{i} - 2\vec{k}$ and $2\vec{i} + 5\vec{j} + \vec{k}$? $\theta = \frac{\pi}{2}$, orthogonal

Matrices and the Determinant

Matrix

An $m \times n$ *matrix* consists of m rows and n columns of real numbers; write

$$A = (a_{ij}) \quad \text{where } a_{ij} \text{ is the component in the position } (i,j)$$

$\star$ If the matrix is $n \times n$, we can find its determinant.

$\boxed{2 \times 2}$ For a matrix A with two rows and two columns,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22} - a_{12}a_{21}$$

$\boxed{3 \times 3}$ For a matrix A with three rows and three columns,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Properties of determinants *also for* 3×3 *matrices*

- Swapping two lines or two columns changes the sign of det

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = - \begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix}$$

- Scalars can be factored out a single row or column

$$\kappa \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \kappa a_{11} & \kappa a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \kappa a_{11} & a_{12} \\ \kappa a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ \kappa a_{21} & \kappa a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & \kappa a_{12} \\ a_{21} & \kappa a_{22} \end{vmatrix}$$

- Adding a row/column to an existing row/column does not change det

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ a_{21} & a_{22} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} & a_{22} \end{vmatrix}$$

Geometry of determinants

Idea: geometrically, det corresponds to area (2×2) or volume (3×3)

2×2 The area of the parallelogram with adjacent sides the vectors

$\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is $\left| \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right|$

3×3 The volume of the parallelepiped with adjacent sides $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

⋆ Absolute value due to physical meaning: area&volume always positive!

Work out the following: find the determinant of $\begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 4 \\ 5 & 2 & -3 \end{pmatrix}$ 39

The Cross Product

Idea: new vector operation, $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ (only like $+$).

Cross Product

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, their *cross product* is the vector

$$\vec{a} \times \vec{b} := \begin{vmatrix} \text{``}\vec{i} & \vec{j} & \vec{k}\text{''} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$\star$ The properties of this operation follow from those of the determinant.

Properties of cross product

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\kappa(\vec{a} \times \vec{b}) = (\kappa\vec{a}) \times \vec{b} = \vec{a} \times (\kappa\vec{b})$
- $(\vec{b} + \vec{c}) \times \vec{d} = \vec{b} \times \vec{d} + \vec{c} \times \vec{d}$

Geometry of cross product

Direction and norm of cross product

- $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ (right-hand rule)
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then the norm of their cross product

$$\left\|\vec{a} \times \vec{b}\right\| = \|\vec{a}\| \, ||\vec{b}|| \sin(\theta) \quad \boxed{\text{area of parallelogram spanned by } \vec{a}, \vec{b}}$$

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}||\vec{b}$ (parallel)

⋆ In $\mathbb{R}^2$, the area $\left\|\vec{a} \times \vec{b}\right\|$ reduces to the absolute value of det (as earlier!)

Plane equations

An equation of the plane passing through the point $P = (x_0, y_0, z_0)$ perpendicular to a vector $\vec{n} = (A, B, C)$ is

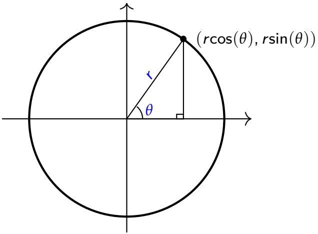
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

$\star$ Notice that for any given plane $Ax + By + Cz + D = 0$, the vector $\vec{n} = (A, B, C)$ is orthogonal/normal to the plane!

Work out the following: find a unit vector which is orthogonal to both $-2\vec{i} + 3\vec{k}$ and $\vec{j} - 5\vec{k}$. $\left(-\frac{3}{\sqrt{113}}, -\frac{10}{\sqrt{113}}, -\frac{2}{\sqrt{113}}\right)$

Polar and Cylindrical Coordinates

Polar coordinates (r, θ) express point in plane by positioning it on a circle of radius r and its angle $0 \leq \theta \leq 2\pi$ from x-axis



Cylindrical coordinates

The *cylindrical coordinates* (r, θ, z) of a point (x, y, z) are

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Conversely, the cartesian coordinates (x, y, z) of a point (r, θ, z) are

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)(+\pi \text{ or } 2\pi\ldots), \quad z = z$$

Spherical Coordinates

What if we view a 3-D point as inhabiting the surface of a sphere?

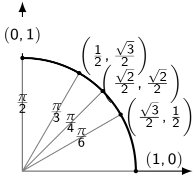
Spherical coordinates

The *spherical coordinates* (ρ, θ, ϕ) of a point (x, y, z) are

$$x = \rho \sin(\phi)\cos(\theta), \quad y = \rho \sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$$

where $\rho \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$. Conversely,

$$\rho = \sqrt{x^2 + y^2 + z^2}, \ \theta = \arctan\left(\frac{y}{x}\right)(+\ldots), \ \phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$



Work out the following:

1. What are the cartesian coordinates (x, y, z) for $(4, \frac{\pi}{4}, \frac{\pi}{3})$? $(\sqrt{6}, \sqrt{6}, 2)$
2. What are its cylindrical coordinates(r, θ, z)? $(2\sqrt{3}, \frac{\pi}{4}, 2)$

Vectors in n-dim space

Idea: earlier operations in $\mathbb{R}^2$ or $\mathbb{R}^3$ generalize in higher dimensions!

For vectors $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$,

- their *sum* is the n-vector

$$\vec{a} + \vec{b} := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

- for $\kappa \in \mathbb{R}$, *scalar multiplication* gives the n-vector

$$\kappa\vec{a} := (\kappa a_1, \kappa a_2, \ldots, \kappa a_n)$$

- their *inner* or *dot* product is the real number

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

- the *norm* of any n-vector is its length, given by the real number

$$\|\vec{a}\| := \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Operations of general matrices

For general $m \times n$ matrices $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$ we can add 'componentwise' $A + B = (a_{ij} + b_{ij})$, or scalarly multiply $\kappa A = (\kappa a_{ij})$.

Matrix multiplication

If $A = (a_{ij})$ is an $m \times n-$matrix and $B = (b_{ij})$ is an $n \times p-$matrix, their product is defined to be an $m \times p-$matrix $AB = C$ with

$$c_{ij} := a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{np} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$\star$ ...the inner product of the i-th row of A and the j-th column of B!

- $AB \neq BA$, i.e. matrix mutliplication is NOT commutative.

Invertible matrices

- The $n \times n-$matrix $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ is called the *identity* matrix:

  $IA = A, BI = B$ whenever the multiplication is defined.

Invertible matrices

An $n \times n$-matrix A is *invertible* if there exists some $n \times n$-matrix B such that

$$AB = BA = I_n.$$

$$\boxed{A \text{ is invertible} \Leftrightarrow \det(A) \neq 0}$$

Multivariable Functions

Idea: how to draw graphs of functions of multiple variables?

A function $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, namely $f(x_1, \ldots, x_n) \in \mathbb{R}^m$, is called *of several variables*. If $m = 1$, real-valued function; if $m > 1$, vector-valued.

- A *level set* for a real-valued function f of n variables is the collection

$$L_c = \{(x_1, \ldots, x_n) \in U \subseteq \mathbb{R}^n | f(x_1, \ldots, x_n) = c\}$$

for some constant c. If $n = 2$, level curve; if $n = 3$, level surface.

Graphs of multivariable functions

The graph of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is the set of all $(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ for any $(x_1, \ldots, x_n)$ in the domain of f; to draw it, compute the level sets and then 'raise' them to the appropriate level!

Limits of single-variable functions

Recall: $\lim_{x\to c} f(x) = L \Leftrightarrow$ if $|x-c| \to 0$, $|f(x)-L| \to 0$.

One-sided limits: $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$

We call f **continuous** at c if $\lim_{x\to c} f(x) = f(c)$.

⋆ The limit of a function at some input c, and the value/output $f(c)$, are in principle unrelated. They coincide? Continuity!

Suppose $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = K$ for functions f, g and $c, L, K \in \mathbb{R}$.

1. $\lim_{x\to c} (f(x) \pm g(x)) = L \pm K$
2. $\lim_{x\to c} (\kappa f(x)) = \kappa \cdot L$
3. $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot K$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ for $K \neq 0$

- Polynomials, rational functions, exponentials/logarithms and trigonometric functions are continuous at their domain.

Limits of multivariable functions

Idea: distance in $\mathbb{R}^n$ is now measured as $\left\|\vec{b}-\vec{a}\right\| = \sqrt{\sum_{i=1}^{n}(b_i-a_i)^2}$

Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$. We say a vector $\vec{b} \in \mathbb{R}^m$ is the limit as $\vec{x}$ approaches $\vec{c} \in U$ when, for any $\vec{x} \neq \vec{c}$,

$$\text{if } \|\vec{x}-\vec{c}\| \to 0 \text{ then } \left\|f(\vec{x})-\vec{b}\right\| \to 0.$$

In that case, denote $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = \vec{b}$. Otherwise, the limit does not exist.

⋆ 'One-sided' limits are now endless - from all possible directions! If any two particular ones disagree, DNE.

Properties of multivariable limits

1. (Uniqueness) If $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = \vec{b_1}$ and $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = \vec{b_2}$, then $\vec{b_1} = \vec{b_2}$.
2. $\lim_{\vec{x}\to\vec{c}} \left(\kappa f(\vec{x})\right) = \kappa \lim_{\vec{x}\to\vec{c}} f(\vec{x})$ for $\kappa \in \mathbb{R}$ when the limit exists
3. $\lim_{\vec{x}\to\vec{c}} (f(\vec{x}) \pm g(\vec{x})) = \lim_{\vec{x}\to\vec{c}} f(\vec{x}) + \lim_{\vec{x}\to\vec{c}} g(\vec{x})$ if both limits exist
4. $\lim_{\vec{x}\to\vec{c}} \left(f(\vec{x})g(\vec{x})\right) = \lim_{\vec{x}\to\vec{c}} f(\vec{x}) \cdot \lim_{\vec{x}\to\vec{c}} g(\vec{x})$ if both limits exist & are reals
5. $\lim_{\vec{x}\to\vec{c}} \left(\frac{f(\vec{x})}{g(\vec{x})}\right) = \frac{\lim_{\vec{x}\to\vec{c}} f(\vec{x})}{\lim_{\vec{x}\to\vec{c}} g(\vec{x})}$ if both limits exist (bottom$\neq 0$) & are reals

Work out the following: do the following exist? If so, evaluate.

1. $\lim_{(x,y)\to(0,0)} \frac{y}{y+x}$ DNE

Multivariable Continuity

A function $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\vec{c} \in U$ if

- $\lim_{\vec{x}\to\vec{c}} f(\vec{x})$ exists;
- $f(\vec{c})$ exists;
- $\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = f(\vec{c})$.

⋆ Multivariable polynomial, rational, trigonometric, exponential and logarithmic (real-valued) functions are continuous, at their domain.

Every vector-valued function with range $\mathbb{R}^m$ can be written as $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$. Its limit is

$$\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m) \text{ where } b_i = \lim_{\vec{x}\to\vec{c}} f_i(\vec{x})$$

If each $f_i(\vec{x})$ is continuous, then so is $f(\vec{x})$.

Limit of composition

If $\lim_{\vec{x}\to\vec{c}} u(\vec{x})=b$ and $\lim_{\vec{x}\to\vec{b}} f(\vec{x})=\vec{d}$, the limit of the composite $(f \circ u)(\vec{x})$ is

$$\lim_{\vec{x}\to\vec{c}} f(u(\vec{x})) = \vec{d}$$

Work out the following:

1. $\lim_{(x,y)\to(1,1)} (x^2+y^3)e^{x-y}$ 2
2. What value should we assign to $\dfrac{e^{xy}-e}{xy-1}$ to make it continuous at $(1,1)$? e

Single Variable Differentiation

Recall: $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim_{t \to 0} \dfrac{f(x+t) - f(x)}{t}$

Derivation Rules for single variable functions

If $f(x), g(x), h(x)$ are differentiable functions with $h(x) \neq 0$ and $\kappa \in \mathbb{R}$

- $(\kappa f(x))' = \kappa f'(x)$ — *scalar multiple*
- $(f(x) + g(x))' = f'(x) + g'(x)$ — *sum*
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ — *product*
- $\left(\dfrac{f(x)}{h(x)}\right)' = \dfrac{f'(x)h(x) - f(x)h'(x)}{h^2(x)}$ — *quotient*
- $\Big(f(u(x))\Big)' = f'(u(x))u'(x)$ — *chain rule*

⋆ In the multivariable setting, consider all but one variables as constants and compute single-variable derivative!

Partial Derivatives

If $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a real-valued function, its *partial derivatives* are

$$\frac{\partial f}{\partial x_j}(x_1, \ldots x_n) := \lim_{t\to 0} \frac{f(\vec{x} + t\vec{e_j}) - f(\vec{x})}{t} = \lim_{t\to 0} \frac{f(x_1, \ldots, x_j + t, \ldots, x_n)}{t}$$

where $\vec{e_j} = (0, \ldots, \underbrace{1}_{jth}, 0, \ldots, 0)$ are the standard basis vectors in $\mathbb{R}^n$.

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$, its *matrix of partial derivatives* is the $m \times n$ matrix

$$Df := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \ldots & & \ldots \\ \frac{\partial f_m}{\partial x_1} & \ldots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

If f is real-valued, this $1 \times n$ matrix a.k.a. n-vector is its *gradient*

$$\nabla f := Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_1} & \ldots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Multivariable Differentiability

A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$ is *differentiable* at $\vec{x_0}$ if all partial derivatives exist at $\vec{x_0}$ and

$$\lim_{\vec{x}\to\vec{x_0}} \frac{\left\| f(\vec{x}) - f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) \right\|}{\|\vec{x} - \vec{x_0}\|} = 0$$

- If all $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at $\vec{x_0}$, then f is differentiable at $\vec{x_0}$.

For $f\colon \mathbb{R}^2 \to \mathbb{R}$ differentiable at (x_0, y_0), its **tangent plane** in $\mathbb{R}^3$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) =: T_{(x_0,y_0)}(x, y)$$

⋆ Differentiability requires tangent plane to be a good approximation of output $z = f(a, b)$ for any $(a, b) \to (x_0, y_0)$: $f(a, b) \approx T_{(x_0,y_0)}(a, b)$!

Work out the following: find ∇f for $f(x, y) = \cos(xy) + x\cos(3y)$.

$(-y\sin(xy) + \cos(3y), -x\sin(xy) - 3x\sin(y))$

Iterated Partial Derivatives

⋆ If partial derivatives exist and are continuous, we say $\boxed{f \text{ is of class } C^1}$.
If the *second* partial derivatives exist and are continuous, f is of class C^2!

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) \text{ etc.}$$

Equality of Mixed Partials

If f is of class C^2, namely twice continuously differentiable, then its mixed partial derivatives are equal: e.g. $f_{xy} = f_{yx}$, or generally

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Work out the following:

1. If $f(x, y) = \cos(3x)\sin^2(y)$, find f_{xy}. $-6\sin(3x)\sin(y)\cos(y)$
2. Approximate $(0.98)^2 - (0.01)^3$. 0.96

Paths and Curves

- A **path** in $\mathbb{R}^n$ is a function $\vec{c}\colon [a,b] \to \mathbb{R}^n$. For $n=2$, it is called path in plane and for $n=3$ path in space.
- The collection of points $C = \{\vec{c}(t) | t \in [a,b]\} \subseteq \mathbb{R}^n$ is called the **curve traced out by** $\vec{c}$, with endpoints $\vec{c}(a)$ and $\vec{c}(b)$.
- For a path in space, write $\vec{c}(t) = (x(t), y(t), z(t))$ for its component functions $x(t), y(t), z(t)$.

$\star$ We say that $\vec{c}(t)$ *traces* or *parameterizes* the curve C.

If a path $\vec{c}$ in $\mathbb{R}^n$ is diff., its *velocity* or *tangent vector* at time t is

$$\vec{c}'(t) = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h} = (x_1'(t), x_2'(t), \ldots, x_n'(t))$$

Its *speed* at each t is given by $\left\|\vec{c}(t)\right\|$, the length of its tangent vector.

$\star$ If $\vec{c}'(t_0) \neq 0$, draw this vector with tail $\vec{c}(t_0)$ tangent to the curve.

- The *tangent line* of a curve C traced by a path $\vec{c}(t)$ at time t_0 is $\boxed{\vec{\ell}(t) = \vec{c}(t_0) + \vec{c}'(t_0)(t - t_0)}$ with direction vector $\vec{c}'(t_0)$.

$\star$ Using $t - t_0$ rather than just t in the line equation ensures that $\vec{\ell}(t_0) = \vec{c}(t_0)$, meaning the line goes through that point at time t_0.

Work out the following: suppose the position of a particle is given by

$$\vec{c}(t) = (t, t^2, \sqrt{t}).$$

1. What is the particle's speed at time $t = 1$? $\sqrt{6}$
2. If the particle flies off at its tangent at $t = 1$, what is its position at $t = 2$? $(2, 3, 2)$

Derivation Rules for multivariable functions

Idea: rules for multivariable derivation are very similar to single variable, but now things are expressed using partial derivative matrices!

If $f, g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $h, k\colon \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{x}_0$ and $\kappa \in \mathbb{R}$

1. $D(\kappa f)(\vec{x}_0) = \kappa Df(\vec{x}_0)$ — *scalar multiple rule*
2. $D(f+g)(\vec{x}_0) = Df(\vec{x}_0) + Dg(\vec{x}_0)$ — *sum rule*
3. $D(hk)(\vec{x}_0) = Dh(\vec{x}_0)k(\vec{x}_0) + h(\vec{x}_0)Dk(\vec{x}_0)$ — *product rule*
4. $D\left(\dfrac{h}{k}\right)(\vec{x}_0) = \dfrac{Dh(\vec{x}_0)k(\vec{x}_0) - h(\vec{x}_0)Dk(\vec{x}_0)}{k^2(\vec{x}_0)}$ — *quotient rule* ($k \neq 0$)

The Chain Rule

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}^p$ are differentiable functions, then $g \circ f\colon \mathbb{R}^n \to \mathbb{R}^q$ is differentiable with

$$D(g \circ f)(\vec{x}_0) = Dg(f(\vec{x}_0)) \cdot Df(\vec{x}_0).$$

Work out: $f(x,y)=(x+1, y-1)$, $g(x,y)=3x-y^2$. $D(g\circ f)(9,1)$? $(3, 0)$

Chain rule $\boxed{D(g \circ f)(\vec{x}) = Dg(f(\vec{x})) \cdot Df(\vec{x})}$

Special Cases of the Chain Rule

For path $\vec{c}\colon \mathbb{R} \to \mathbb{R}^3$ and real-valued $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f \circ \vec{c}\colon \mathbb{R} \to \mathbb{R}$ has

$$(f \circ \vec{c})'(t) = \langle \nabla f(\vec{c}(t)), \vec{c}'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

For $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $g(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ and a real-valued $f\colon \mathbb{R}^3 \to \mathbb{R}$, $h{:=}(f \circ g)\colon \mathbb{R}^3 \to \mathbb{R}$ has

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$
$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$
$$\frac{\partial h}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$\star$ These all arise from the matrix multiplication of the chain rule!

Work out: $(f \circ \vec{c})'(1)$ for $f(x,y,z){=}xy + z$ and $\vec{c}(t){=}(t+1, t^2, 1-t)$? 4

Directional Derivatives

Idea: for an object moving on some line $\vec{\ell}(t) = \vec{x} + \vec{v}t$, how 'fast' are the values of some $f(x, y, z)$ changing at a specific point?

The **directional derivative** of $f : \mathbb{R}^3 \to \mathbb{R}$ at $\vec{x}$ along the (unit) vector $\vec{v}$ is

$$\frac{d}{dt} f(\vec{x} + t\vec{v})|_{t=0} = \langle \nabla f(\vec{x}), \vec{v} \rangle$$

namely $D(f \circ \vec{c})(0)$ for any path $\vec{c}$ with $\vec{c}(0)=\vec{x}$ & $\vec{c}'(0)=\vec{v}$, unit speed.

⋆ When is RoC $\langle \nabla f(\vec{x}), \vec{v} \rangle = \left\| \nabla f(\vec{x}) \right\| \cos(\theta)$ maximum?

$\boxed{-1 \leq \cos(\theta) \leq 1}$ so when $\theta = 0$!

Direction of fastest increase or decrease

If $\nabla f(\vec{x}) \neq 0$, the vector $\nabla f(\vec{x})$ points in the direction along which f increases the fastest. Similarly, f decreases the fastest along $-\nabla f(\vec{x})$.

Tangent Planes

If $f: \mathbb{R}^3 \to \mathbb{R}$ is diff. & $(x_0, y_0, z_0) \in L_c = \{(x, y, z) | f(x, y, z) = c\}$ level surface, then $\nabla f(x_0, y_0, z_0)$ is orthogonal to L_c at (x_0, y_0, z_0).

- Recall that $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ is the plane passing from (x_0, y_0, z_0) and is perpendicular to vector (A, B, C).

Tangent Plane on Level Surface

The tangent plane of surface L_c for $f(x, y, z)$ at (x_0, y_0, z_0) is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

⋆ Reduces also to tangent plane for graph of some $g(x, y)$!
Using level surface L_0 for $f(x, y, z) = g(x, y) - z$ ends up in previous

$$\boxed{z = T_{(x_0, y_0)}(x, y) = g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)}$$

Taylor's Theorem

Idea: earlier, used tangent plane $Ax+By+Cz=D$ to *linearly* approximate some $f(x_0, y_0)$. Now, quadratic or higher-order approximations!

Single-Variable Taylor Theorem

For a smooth function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\underbrace{f(x_0+h) = f(x_0) + f'(x_0)h}_{\text{linear approximation}} + \frac{f''(x_0)}{2}h^2 + \ldots + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0, h)$$

where $R_k(x_0, h)$ is the k-th order remainder (small error term). For $k=1$ **first-order Taylor formula**, for $k=2$ **second-order Taylor formula**.

⋆ Express either as above formula $f(x_0+h)$, or as approximation function

$$f(x) = f(x_0) + f'(x_0)\underbrace{(x-x_0)}_{h} + \frac{1}{2}f''(x_0)(x-x_0)^2 + \ldots$$

- First-order for two variables is tangent plane approximation from 2.3.

Multi-Variable Taylor Theorem $f:\mathbb{R}^n \to \mathbb{R}$

- **First-Order**: $f(\vec{x_0}+\vec{h}) = f(\vec{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x_0})h_i + R_1(\vec{x_0},\vec{h})$
- **Second-Order**:
$$f(\vec{x_0}+\vec{h}) = f(\vec{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x_0})h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x_0})h_i h_j + R_2(\vec{x_0},\vec{h})$$

  - Second-order for two variables $f(x,y)$ at point $P=(x_0,y_0)$ is

$$f(x_0+h_1, y_0+h_2) = f(x_0,y_0) + f_x(x_0,y_0)h_1 + f_y(x_0,y_0)h_2 + \frac{1}{2}\left(f_{xx}(x_0,y_0)h_1^2 + 2f_{xy}(x_0,y_0)h_1h_2 + f_{yy}(x_0,y_0)h_2^2\right) + R_2$$

Work out: second-order Taylor for $f(x,y)=e^x\cos(y)$ at $(0,0)$?
$f(0+h_1, 0+h_2) = \ldots 1 + h_1 + \frac{1}{2}h_1^2 - \frac{1}{2}h_2^2 + R_2((0,0),(h_1,h_2))$

Critical points and Extrema of Real-Valued Functions

Idea: similarly to single-variable case, derivatives relate to max/min values!

For a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$, a point $\vec{x}_0$ is

- a **critical point** if f is NOT differentiable at $\vec{x}_0$, or if $Df(\vec{x}_0) = \vec{0}$
- a **local minimum** if $f(\vec{x}) \geq f(\vec{x}_0)$ for all $\vec{x} \in V$, a neighborhood of $\vec{x}_0$
- a **local maximum** if $f(\vec{x}) \leq f(\vec{x}_0)$ for all $\vec{x} \in V$, a neighborhood of $\vec{x}_0$
- a **saddle** if it is a critical point, but not an extremum.

First Derivative Test

Every local extremum $\vec{x}_0$ (max or min) of $f\colon \mathbb{R}^n \to \mathbb{R}$ has $Df(\vec{x}_0) = \vec{0}$, in particular is a critical point. Equivalently,

$$\frac{\partial f_i}{\partial x_i}(\vec{x}_0) = 0 \text{ for all } i = 1, \ldots, n$$

The Hessian of a function

Idea: like partial derivative matrix Df,
but now including all second partial derivatives!

The **Hessian matrix** of a real-valued $f : \mathbb{R}^n \to \mathbb{R}$ is an $n \times n$ matrix

$$Hf = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \dots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \dots & f_{x_2x_n} \\ \vdots & \vdots & \dots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \dots & f_{x_nx_n} \end{pmatrix} = \begin{pmatrix} \nabla f_{x_1} \\ \nabla f_{x_2} \\ \vdots \\ \nabla f_{x_n} \end{pmatrix}$$

⋆ By the law of mixed partials $f_{x_ix_j} = f_{x_jx_i}$, this matrix is *symmetric*: changing rows by columns (i.e. taking the *transpose*) gives same matrix!

- $\frac{1}{2}(h_1, \dots, h_n) \cdot Hf(\vec{x}_0) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ is second-order Taylor for critical $\vec{x}_0$.

A symmetric $n \times n$ matrix H is **positive-definite** when all diagonal sub-matrices H_k (from top left) for $1 \leq k \leq n$ satisfy $\det(H_k) > 0$; it is **negative-definite** when $\det(H_1) < 0$ and the rest alternate signs.

Second Derivative Test

A critical point $\vec{x_0}$ of $f : \mathbb{R}^n \to \mathbb{R}$ is

- a local minimum, when $Hf(\vec{x_0})$ is positive-definite;
- a local maximum, when $Hf(\vec{x_0})$ is negative-definite;
- a saddle-type, when $Hf(\vec{x_0})$ is neither of the two: it is a saddle point, unless $\det(H) = 0$ when it is inconclusive.

Work out the following: consider $f(x, y) = x^2 + xy$.

1. Find its critical points. $(0, 0)$
2. Find its Hessian matrix. $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$
3. Classify the critical points. $(0, 0)$ saddle point

Global Extrema

For a function $f\colon A \to \mathbb{R}$ defined on $A \subset \mathbb{R}^2$ or $\mathbb{R}^3$, a point $\vec{x}_0 \in A$ is

- an **absolute maximum** if $f(\vec{x}) \leq f(\vec{x}_0)$ for all $\vec{x} \in A$
- an **absolute minimum** if $f(\vec{x}) \geq f(\vec{x}_0)$ for all $\vec{x} \in A$

⋆ [Single-var] A continuous f on *closed* interval has global max and min!

▶ A point $\vec{x}$ is called a *boundary point* of A if every neighborhood of $\vec{x}$ contains at least one point in A and at least one not in A.

A set A is **closed** if it contains all its boundary points. It is **bounded** if $\|\vec{x}\| < M$ for all $\vec{x} \in A$ for some number M.

Global existence theorem

If a continuous real-valued f is defined on a bounded and closed subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, it has an absolute maximum and minimum value.

Methodology for global extrema for $f(x, y)$

1. Find critical points in interior of A
2. Find critical points on boundary of A [reduce to single variable case]
3. Compute the values of f at all above points
4. Compare the values and select largest & smallest

⋆ A multivariable function, similarly to the single-variable case, does *not* need to have a global max or min in general; however, a function restricted to a bounded and closed set always does, by the existence theorem!

Lagrange Multipliers

Idea: when a function is defined on some curve, can find critical points from viewing it as a level set of a different function!

Suppose $f, g\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ are C^1, and $L_c = \{\vec{x} \in \mathbb{R}^n | g(\vec{x}) = c\}$ is a level set for g. If $\vec{x_0}$ is a local extremum of f restricted to L_c and $\nabla g(\vec{x_0}) \neq 0$, there exists some scalar λ, the **Lagrange multiplier**, with

$$\nabla f(\vec{x_0}) = \lambda \nabla g(\vec{x_0})$$

- For finding global extrema, we can locate critical points on the *boundary* of a region in step (2) using Lagrange Multipliers.

Work out the following: find the critical points for $f(x, y, z) = x - y + z$ under the condition that $\frac{1}{2}x^2 + y^2 + z^2 = 1$. $(1, -\frac{1}{2}, \frac{1}{2})$ and $(-1, \frac{1}{2}, -\frac{1}{2})$

Arc Length

Idea: what is the length of a path $\vec{c}\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}^3$?

The length of the path $\vec{c}(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ is

$$L(\vec{c}) = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

namely the integral of its speed $\|\vec{c}'(t)\|$.

Some useful identities

- Power-reducing: $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$, $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- Trig-sub: $\int\sqrt{x^2+a^2}dx = \frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\ln(x+\sqrt{x^2+a^2})\right) + C$

Work out the following: what is the arc length of the path $\vec{c}(t) = (3\cos(t), 3\sin(t))$ for $t \in [0, 2\pi]$? $6\pi = 2\pi * 3$, circle's circumference!

Vector Fields

A **vector field** in $\mathbb{R}^n$ is a function $\vec{F}\colon A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\vec{x}$ a vector $\vec{F}(\vec{x})$. If $n{=}2$, vector field in the plane; if $n{=}3$, in space.

- For any $f\colon \mathbb{R}^3 \to \mathbb{R}$, its gradient gives rise to the *gradient vector field*

$$\begin{aligned} \nabla f\colon \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)) \end{aligned}$$

A **flow line** for a vector field $\vec{F}$ is a (same-dimension) path $\vec{c}(t)$ such that

$$\vec{c}\,'(t) = \vec{F}(\vec{c}(t)) \qquad \text{for all } t$$

namely for curve traced out by $\vec{c}(t)$, all tangent vectors are values of $\vec{F}$.

Work out the following: find some function whose gradient vector field is $\vec{F}(x,y,z) = (3yz-1, 3xz, 3xy)$. $f(x,y,z) = 3xyz - x$

Divergence

[Single var calc] The *differentiation operator* $\frac{d}{dx}$ applies to f and gives f'.

- The **del** or **nabla operator** in the n-dimensional space is given by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

The **divergence** of a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is

$$\mathrm{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \langle \nabla, \vec{F} \rangle$$

⋆ Physically: if $\vec{F}$ is the flow of fluid, its div represents rate of expansion per unit volume in $\mathbb{R}^3$ or unit area in $\mathbb{R}^2$.

- $\mathrm{div}\vec{F} > 0$?expand
- $\mathrm{div}\vec{F} < 0$?compress
- $\mathrm{div}\vec{F} = 0$?same

Laplacian

Idea: for gradiant vector fields, their divergence involves second derivatives.

The **Laplacian** of a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is

$$\Delta f = \nabla^2 f = \mathrm{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- A function f is called *harmonic* if $\Delta f = 0$.

Work out the following: suppose $\vec{F} = (5xy, y^2 + 1, 3x - z)$.

1. Find its divergence. $7y - 1$
2. Is $\vec{c}(t) = (5t, t^2 + 1, \sqrt{t})$ a flow line for $\vec{F}$? No

Curl

⋆ Divergence=inner product of ∇ & vector field; curl=cross product!

The **curl** of a vector field $\vec{F} = (F_1, F_2, F_3)$ is

$$\mathrm{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \nabla \times \vec{F}$$

Gradients are curl free

For any $f : \mathbb{R}^3 \to \mathbb{R}$, its gradient vector field has zero curl: $\nabla \times \nabla f = \vec{0}$

Curls are divergence free

For any C^2-vector field $\vec{F}$, $\mathrm{div}\ \mathrm{curl}\ \vec{F} = \langle \nabla, \nabla \times \vec{F} \rangle = 0$.

Work out the following: consider the vector field
$\vec{F}(x, y, z) = (x^2 y, \cos(yz), e^{z+y})$.

1. Find the divergence. $\mathrm{div}\vec{F} = 2xy - z\sin(yz) + e^{z+y}$
2. Find the curl. $(e^{x+z} - y\sin(yz), 0, -x^2)$
3. Is $\vec{F}$ a gradient vector field? No: its curl is not 0!