Monads and enrichment in double categories

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Outline

1. Monoids in monoidal categories
2. Monads in double categories
3. Sweedler theory

Motivation: monoids

- A monoid M is a set equipped with an associative and unitary binary operation , namely $\mu\colon M \times M \to M$ and $\eta\colon 1 \to M$ such that

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] \arrow[dr, phantom, "(ab)c=a(bc)"] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times 1 \arrow[r, "1\times\eta"] \arrow[dr, "\cong"'] & M \times M \arrow[d, "\mu"] & 1 \times M \arrow[l, "\eta\times 1"'] \arrow[dl, "\cong"] \\
& M &
\end{tikzcd}$$

$\star$ Notion *internal* in sets and functions with cartesian product.

- A k-algebra A is a k-vector space with an associative and unitary bilinear operation , namely $\mu\colon A \otimes_k A \to A$ and $\eta\colon k \to A$ such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes k \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & k \otimes A \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

$\star$ Notion *internal* in k-vector spaces and linear maps with tensor product.

Monoidal categories

▶ A monoidal category $\mathcal{V}$ is equipped with functor $\otimes \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and object I , with $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$, $I \otimes X \cong X \cong X \otimes I$ +axioms.

Examples

- Sets and functions $(\mathrm{Set}, \times, 1)$
- Vector spaces and linear maps $(\mathrm{Vect}_k, \otimes_k, k)$
- Topological spaces and continuous functions $(\mathrm{Top}, \times, \{*\})$
- Bounded meet-semilattice $(L, \wedge, \top)$ as a (strict) monoidal category

$\star$ In fact, $(\mathcal{V}, \otimes, I)$ is a *pseudomonoid* in the cartesian monoidal 2-category Cat of categories, functors and natural transformations...

Models for multiplicative fragment of intuitionistic linear logic.

Monoids in monoidal categories

- A monoid in $(\mathcal{V},\otimes,I)$ is an object A together with maps $\mu\colon A\otimes A\to A$ and $\eta\colon I\to A$ which are associative and unital.

Examples

- Ordinary monoids, k-algebras, rings (for $\mathcal{V}$=Ab), topological monoids
- Monads on a category (for $\mathcal{V}$=End($\mathcal{C}$) with composition)
- Free monoid on set X is $\mathrm{List}(X)=\{x_1x_2\dots x_n\}$ with concatenation.

■ Together with maps $f\colon A\to B$ that preserve multiplications and units

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\ B\otimes B \arrow[r, "\mu"] & B \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, "\eta"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

we obtain a category Mon($\mathcal{V}$).

Comonoids in monoidal categories

- Dually, a comonoid is $(C, C \xrightarrow{\delta} C\otimes C, C \xrightarrow{\epsilon} I)$ such that

$$\begin{tikzcd}
C\otimes C\otimes C & C\otimes C \arrow[l, "\delta\otimes 1"'] \\
C\otimes C \arrow[u, "1\otimes\delta"] & C \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes C & C\otimes C \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & C\otimes I \\
 & C \arrow[ul] \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

Together with comonoid maps, they form a category $\mathrm{Comon}(\mathcal{V})$.

Examples

- In $(\mathrm{Set}, \times, 1)$, any set X with $\delta(x){=}(x,x)$ and $\epsilon(x){=}*$ ‘trivially’.
- In $(\mathrm{Vect}_k, \otimes_k, k)$, k-coalgebras: divided power coalgebra, tensor algebra, group-like coalgebra, trigonometric coalgebra...

■ When $\mathcal{V}$ is symmetric, i.e. $X\otimes Y \cong Y\otimes X$ naturally, (co)monoids become monoidal: $A\otimes B\otimes A\otimes B \xrightarrow{\cong} A\otimes A\otimes B\otimes B \xrightarrow{\mu\otimes\mu} A\otimes B$

Double categories

…as many-object generalizations of monoidal categories.

⋆ Introduced by Ehresmann in '60s - "categories internal in categories"

- A double category $\mathbb{D}$ consists of
  - objects & vertical 1-cells which form a category $\mathbb{D}_0$
  - horizontal 1-cells & 2-maps which form a category $\mathbb{D}_1$
  - functors $s, t \colon \mathbb{D}_1 \to \mathbb{D}_0$ providing source and target

$$\begin{tikzcd} X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "B"', "\shortmid"{marking}] & W \end{tikzcd}$$

  - functor $1 \colon \mathbb{D}_0 \to \mathbb{D}_1$ providing units
  - functor $\circ \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ providing horizontal composition

together with natural coherent constraints, e.g. $(A \circ B) \circ C \cong A \circ (B \circ C)$.

E.g. the two kinds of compositions of 2-maps obey interchange law

$$\begin{array}{ccccc} & \relbar\mapstochar\rightarrow & & \relbar\mapstochar\rightarrow & \\ \downarrow & \Downarrow\alpha & \downarrow & \Downarrow\beta & \downarrow \\ & \relbar\mapstochar\rightarrow & & \relbar\mapstochar\rightarrow & \\ \downarrow & \Downarrow\gamma & \downarrow & \Downarrow\delta & \downarrow \\ & \relbar\mapstochar\rightarrow & & \relbar\mapstochar\rightarrow & \end{array} \qquad (\delta \circ \gamma) \cdot (\beta \circ \alpha) = (\delta \cdot \beta) \circ (\gamma \cdot \alpha)$$

$\star$ Traditional 2-dimensional category theory employs *2-categories* or *bicategories*. Double categories offer alternative – in fact more general – approach: for objects of interest, two different kinds of morphisms (with strict vs pseudo associative composition) encompassed in single structure.

A double category with a single object $*$ and only identity vertical 1-cell (i.e. $\mathbb{D}_0 = \mathbf{1}$) ‘is’ a monoidal category $(\mathbb{D}_1, \circ, 1_*)$

Examples of double categories

- $\mathbb{R}\mathrm{el}$ with sets as objects, functions as vertical 1-cells ($\mathbb{R}\mathrm{el}_0$=Set) , relations $A \subseteq X \times Y$ as horizontal 1-cells $A\colon X \nrightarrow Y$, maps of relations ($xAy \Rightarrow f(x)Bg(y)$) as 2-maps.

- $\mathbb{S}\mathrm{pan}$ with $\mathbb{S}\mathrm{pan}_0$=Set , horizontal 1-cells spans $\begin{tikzcd}[column sep=small] & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ and 2-maps

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[ddd, "k"] & \\ X \arrow[d, "f"'] \arrow[r, phantom, "\circlearrowleft"] & {} & Y \arrow[d, "g"] \arrow[l, phantom, "\circlearrowleft"] \\ Z & & W \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Horizontal composition given by taking pullbacks of spans.

- $\mathbb{B}\mathrm{im}$ with $\mathbb{B}\mathrm{im}_0 = \mathrm{Rng}$, the category of rings and homomorphisms , horizontal 1-cells $R \xrightarrow{M} S$ are (R, S)-bimodules. Horizontal composition $R \xrightarrow{M} S \xrightarrow{N} T$ is tensor product $M \otimes_S N$.

- $\mathbb{M}\mathrm{at}$ with $\mathbb{M}\mathrm{at}_0 = \mathrm{Set}$, horizontal 1-cells $X \xrightarrow{A} Y$ are sets $\{A(y,x)\}_{y,x}$. Composition is 'matrix multiplication' $(B \circ A)(z,x) = \sum_y B(z,y) \times A(y,x)$.

Double categories and logic

- Modular model of MLL: objects proof-nets, horizontal 1-cells modules, vertical 1-cells side-effects, 2-maps cut-elimination.
- Double-categorical logic: objects categories w. structure, horizontal 1-cells theory models, vertical 1-cells interpretations.
- Type theories as internal language of *virtual* double categories.

Monads in double categories

- A monad in $\mathbb{D}$ is $A\colon X \to\!\!\!\!\!\!+\;\; X$ with ‘multiplication’ and ‘unit’ 2-maps

$$\begin{tikzcd}
X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, equal] & X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\
X \arrow[rr, "A"', "\shortmid"{marking}] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1_X", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[d, equal] \\
X \arrow[r, "A"', "\shortmid"{marking}] & X
\end{tikzcd}$$

satisfying usual associativity and unitality axioms. E.g.

$$\begin{tikzcd}
X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, equal] & X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_A"] & X \arrow[d, equal] \\
X \arrow[rr, "A"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\mu"] & & X \arrow[r, "A"', "\shortmid"{marking}] & X \arrow[d, equal] \\
X \arrow[rrr, "A"', "\shortmid"{marking}] & & & X
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_A"] & X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, equal] & X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[d, equal] \\
X \arrow[r, "A"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\mu"] & X \arrow[rr, "A"', "\shortmid"{marking}] & & X \arrow[d, equal] \\
X \arrow[rrr, "A"', "\shortmid"{marking}] & & & X
\end{tikzcd}$$

$\star$ Coincides with monoid in monoidal category $(\mathbb{D}(X,X),\circ,1_X)$ of endo-1-cells and maps with id vertical boundaries! Not true for arrows...

- A monad map from $X \overset{A}{\nrightarrow} X$ to $Y \overset{B}{\nrightarrow} Y$ is a 2-map $\begin{tikzcd} X \arrow[r, "A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \\ Y \arrow[r, "B"', "|" marking] & Y \end{tikzcd}$ s.t.

$$\begin{tikzcd}
X \arrow[r, "A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[r, "A", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] & & X \arrow[r, "A", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & X \arrow[r, "A", "|" marking] & X \arrow[d, equal] \\
Y \arrow[r, "B"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & Y \arrow[r, "B"', "|" marking] & Y \arrow[d, equal] \arrow[rr, phantom, "="] & & X \arrow[rr, "A"', "|" marking] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\alpha"] & & X \arrow[d, "f"] \\
Y \arrow[rr, "B"', "|" marking] & & Y & & Y \arrow[rr, "B"', "|" marking] & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "1_X", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & X \arrow[d, equal] & & X \arrow[r, "1_X", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_f"] & X \arrow[d, "f"] \\
X \arrow[r, "A"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "f"] \arrow[rr, phantom, "="] & & Y \arrow[r, "1_Y"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & Y \arrow[d, equal] \\
Y \arrow[r, "B"', "|" marking] & Y & & Y \arrow[r, "B"', "|" marking] & Y
\end{tikzcd}$$

■ Monads and monad maps form a category $\mathsf{Mnd}(\mathbb{D})$ for any double $\mathbb{D}$. Dually, comonads and comonad maps form a category $\mathsf{Cmd}(\mathbb{D})$.

In the single-object, single-vertical arrow double case reducing to a monoidal category, $\mathsf{Mnd}(\mathbb{D})$ 'is' the category of monoids.

Examples of categories of monads

- For $\mathbb{D}=\mathbb{S}\mathrm{pan}$, monad $X \xleftarrow{d} A \xrightarrow{c} X$ is category: consists of set X of objects, set A of arrows, η picks identities and $\mu\colon A \times_X A \to A$ is composition. A monad map is a functor, so $\mathrm{Mnd}(\mathbb{S}\mathrm{pan}) = \mathrm{Cat}$!

- For $\mathbb{R}\mathrm{el}$, $\mathrm{Mnd}(\mathbb{R}\mathrm{el})=\mathrm{Preord}$ of preorders and order-preserving maps.

- For $\mathbb{B}\mathrm{im}$, a monad $R \overset{A}{\to\!\!\!\!\!\!\!+} R$ is an R-algebra and a monad map
$$\begin{tikzcd} R \arrow[r, "A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & R \arrow[d, "f"] \\ S \arrow[r, "B"'] & S \end{tikzcd}$$
is R-algebra map $\alpha\colon A \to B$ with B an R-algebra via restriction of scalars. So $\mathrm{Mnd}(\mathbb{B}\mathrm{im})=\mathrm{Alg}$, a ‘global’ category of algebras over arbitrary rings.

- For $\mathbb{M}\mathrm{at}$, a monad $X \overset{A}{\to\!\!\!\!\!\!\!+} X$ is a collection of sets $\{A(x,x')\}_{x,x'}$ with
$$\Big(\sum\Big) A(x,x')\otimes A(x',x'')\to A(x,x''), \quad I \to A(x,x)$$
i.e. again a category! Then $\mathrm{Mnd}(\mathbb{M}\mathrm{at})=\mathrm{Cat}$.

Sweedler theory

⋆ For $C \in \mathrm{Coalg}_k$, $A \in \mathrm{Alg_k}$, $\mathrm{Hom}_k(C, A)$ obtains convolution k-algebra structure via $(f * g)(c) = \sum_{(c)} f(c_1)g(c_2)$.

[Sweedler, 1969] For any three vector spaces A, B and C,

$$\mathrm{Hom}_k(C \otimes_k B, A) \cong \mathrm{Hom}_k(B, \mathrm{Hom}_k(C, A)).$$

If C coalgebra, A, B algebras, when is it an *algebra* map $B \to \mathrm{Hom}(C, A)$?

Answer (low-level): when $f \colon C \otimes_k B \to A$ *measures*, i.e. satisfies

$$f(c \otimes aa') = \sum f(c_{(1)} \otimes a)f(c_{(2)} \otimes a')$$
$$f(c \otimes 1) = \epsilon(c)1$$

There exists a *universal measuring* coalgebra P, namely for any other measuring coalgebra C, we get a unique coalgebra map $C \to P$.

Answer (high-level): $\mathrm{Hom}_k(\text{-}, A)\colon \mathsf{Coalg}_k^{\mathrm{op}} \to \mathsf{Alg}_k$ has an adjoint $P(A, \text{-})$!

Special case of general result …for *monoidal closed* (existence of a hom-functor with $\text{-} \otimes X \dashv [X, \text{-}]$) and *locally presentable* (set of pres. objects that generate the rest under filtered colimits) categories.

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. There is an adjunction between

$$
\begin{aligned}
[\text{-}, \text{-}]\colon \mathsf{Comon}^{\mathrm{op}}(\mathcal{V}) \times \mathsf{Mon}(\mathcal{V}) &\to \mathsf{Mon}(\mathcal{V}) \quad \text{convolution} \\
P(\text{-}, \text{-})\colon \mathsf{Mon}^{\mathrm{op}}(\mathcal{V}) \times \mathsf{Mon}(\mathcal{V}) &\to \mathsf{Comon}(\mathcal{V}) \quad \text{universal measuring}
\end{aligned}
$$

⋆ 'Generalized algebra maps': k-coalgebra $P(A, B)$ contains all k-algebra maps as the group-like elements $\delta(f) = f \otimes f$.

[Wraith, 1970's] k-algebras are enriched in k-coalgebras. . .

A $(\mathcal{V},\otimes,I)$-enriched category has objects $\mathrm{ob}\mathcal{C}$ and hom-objects $\mathcal{C}(x,y)\in\mathcal{V}$ with composition and identity rules as maps in $\mathcal{V}$

$$\mathcal{C}(x,y)\otimes\mathcal{C}(y,z)\to\mathcal{C}(x,z),\quad I\to\mathcal{C}(x,x)$$

$\star$ Linear=Vect_k-enriched; additive$\subseteq$ Ab-enriched; 2-cats=Cat-enriched.

Suppose $\mathcal{V}$ is symmetric monoidal closed and locally presentable. The category $\mathrm{Mon}(\mathcal{V})$ is enriched in the symmetric monoidal $\mathrm{Comon}(\mathcal{V})$.

Hom-objects of enrichment are Sweedler's universal measuring coalgebras!
Proof uses theory of actions of monoidal on ordinary categories.

Sweedler theory for double categories

⋆ Generalizing from monoidal categories to double categories, obtain ‘many-object’ version of Sweedler's universal measuring coalgebras.

For $\mathbb{D}$ symmetric monoidal closed, loc. presentable double category, category of monads $\mathsf{Mnd}(\mathbb{D})$ is enriched in category of comonads $\mathsf{Cmd}(\mathbb{D})$.

Proof points

- New concepts of monoidal closed and locally presentable double categories, incorporating ‘fibrancy’ and ‘parallel limits’
- Running examples of $\mathbb{S}$pan and $\mathbb{M}$at satisfy conditions
- Obtain enrichment of categories in ‘cocategories’, along with supplementary results of (co)monadicity and loc. presentability

Thank you for your attention!

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$