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ANALYSIS AND APPROXIMATIONS OF THE EVOLUTIONARY STOKES EQUATIONS WITH INHOMOGENEOUS BOUNDARY AND DIVERGENCE DATA USING A PARABOLIC SADDLE POINT FORMULATION

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Abstract. This work concerns the analysis and finite element approximations of the evolutionary Stokes equations, with inhomogeneous boundary and divergence data. The proposed weak formulation can be viewed as an attempt to develop the parabolic analog of the well known saddle point theory for elliptic problems. Several results concerning the analysis and finite element approximations are presented. The key feature of the weak formulation under consideration is the treatment of Dirichlet boundary conditions within the Lagrange multiplier framework.

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1. Introduction

This work concerns the analysis and finite element approximations of the evolutionary Stokes equations with inhomogeneous boundary and/or divergence data. In particular, we are interested in developing and analyzing an appropriate weak formulation for the following problem: Given data $\boldsymbol{\phi}, \psi$ and initial velocity $\mathbf{u}_0$ we seek a pair $(\tilde{\mathbf{u}}, \tilde{p})$ such that

$$\begin{cases} \tilde{\mathbf{u}}_t - \nu\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} & = 0 & \text{in } \Omega \times (0,T] \\ \text{div } \tilde{\mathbf{u}} & = \psi & \text{in } \Omega \times (0,T] \\ \tilde{\mathbf{u}} & = \boldsymbol{\phi} & \text{on } \Gamma \times (0,T] \\ \tilde{\mathbf{u}}(0) & = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \in \mathbb{R}^d$, $d = 2, 3$, denotes a bounded polygonal (polyhedral when $d = 3$), and convex domain or a bounded domain with regular (enough) boundary Γ. Recall that the divergence Theorem implies the following compatibility condition,

$$\int_\Omega \psi(.,t) = \int_\Gamma \boldsymbol{\phi}(.,t) \cdot \mathbf{n}(.) \qquad \text{for a.e. } t \in (0,T].$$

It is worth noting that the analysis and finite element approximations of such problems are very important from the engineering view-point since they are closely related to boundary control problems with Dirichlet boundary control data, as well as to feedback control problems (see e.g. [19, 28]). In addition, another motive

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for this work, is the analysis and finite element approximations of a weak formulation suitable for handling essential inhomogeneous Dirichlet boundary data for the evolutionary Stokes problem. Our main goal is to develop the parabolic analog of the well known "saddle point" theory for elliptic problems and its finite element approximation within the context of mixed finite element methods. To our best knowledge there are no results regarding finite element approximations of such problems.

1.1. **The parabolic saddle point framework**

A weak formulation that resembles the classical saddle point formulation of the stationary Stokes equations will be developed. In particular, we examine weak problems of the following form: Given data $\mathbf{g}$, $\mathbf{u}_0$, find a solution pair $(\mathbf{u}, p)$ such that, for a.e. $t \in (0, T]$,

$$\begin{cases} \langle \mathbf{u}_t(t), \mathbf{v} \rangle_{(X^*,X)} + \nu A(\mathbf{u}(t), \mathbf{v}) + B(\mathbf{v}, p(t)) & = 0 \qquad \forall \mathbf{v} \in X \\ B(\mathbf{u}(t), q) & = \langle \mathbf{g}(t), q \rangle_{(M^*,M)} \quad \forall q \in M \\ (\mathbf{u}(0), \mathbf{z}) & = (\mathbf{u}_0, \mathbf{z}) \qquad \forall \mathbf{z} \in H, \end{cases} \tag{1.2}$$

where X, M, H are suitable Banach spaces, X^*, M^*, H their duals, $\nu > 0$ a positive constant, and $A(\cdot, \cdot)$, $B(\cdot, \cdot)$ are continuous bilinear forms defined on $X \times X$ and $X \times M$ respectively. The precise functional analytic framework is given in Section 2. A key feature of our analysis is that the bilinear forms $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ are defined in way to handle evolutionary problems with essential inhomogeneous boundary data, in particular within the framework of Lagrange multipliers. For instance, in the case of the evolutionary Stokes equations (1.1), we define the bilinear form

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \qquad \forall\, \mathbf{u}, \mathbf{v} \in X = H^1(\Omega).$$

All other terms, involving pressure and / or boundary terms resulting from integration by parts in space, are included into the bilinear form $B(\cdot, \cdot)$. The precise functional analytic formulation and its relation to Lagrange multipliers is presented in Section 4.

1.2. **Related results and comments**

Evolutionary Navier-Stokes problems with inhomogeneous Dirichlet boundary data have been studied in the works of [13, 14, 30]. Several results regarding the analysis of Dirichlet boundary value problems, as well as several applications to optimal boundary control problems were studied in [11, 12].

The evolutionary Stokes and Navier-Stokes equations with inhomogeneous divergence condition have also their own independent importance. To this end, we point out the work of [30], where the Stokes and the Oseen's equations with inhomogeneous divergence condition were analyzed. The analysis of [30] is also applicable within the context of feedback control. Saddle-point formulations suitable for space-time approximations, are studied in the recent work of [18] for the Stokes and Navier-Stokes equations with Navier slip boundary conditions. The main target of the work of [18] is the development of suitable weak formulations for space-time approximations with wavelet basis.

A key feature of the analysis presented here, is to impose regularity assumptions on the data to guarantee the existence of a suffieciently regular solution of (1.2) that allows the use of standard finite element approximations within the context of mixed finite elements.

Our work differs from the previously developed analysis of [30], since our main emphasis is to avoid the use of "divergence-free" spaces for the regularity of the time-derivative of the velocity $\mathbf{u}_t$, or very weak formulations resulting the validity of the pressure term in a distributional sense. Even though the various concepts of very weak solutions based on transposition techniques as presented in [30], guarantee existence and uniqueness under very low regularity assumptions on the data, they are not directly applicable within the framework of finite element analysis. This is due to the fact that the finite element discretization of weak solutions based on transposition techniques typically require nonstandard finite element spaces. To the contrary, the parabolic saddle point formulation of (1.2) allows us to define finite element approximations in a more standard (but not classical) way and to obtain error estimates for the semi-discrete (in space) approximations for the velocity and the Lagrange multiplier. Special care is exercised in order to obtain estimates which resemble the "symmetric" structure of the ones of the classical saddle point theory of elliptic problems. In addition, we prove error estimates when essential inhomogeneous data are being used in the definition of the discrete analog of the weak formulation (1.2). These estimates can be used in many physical applications, including optimal boundary control problems. A particular choice of subspaces allowing the decoupling of the computation of the velocity and pressure from the computation of the and Lagrange multiplier is analyzed in [7]. For results related to the analysis and finite element approximations of parabolic problems with inhomogeneous Dirichlet boundary data, we also refer the reader to [3, 6]. The Lagrange multiplier framework for the numerical treatment of essential inhomogeneous Dirichlet boundary data for elliptic problems, and for stationary Navier-Stokes equations has been considered in [1, 2, 20, 27, 34].

Saddle point problems are usually related to elliptic partial differential equations and result from certain minimization principles. The main concepts originate from solid and fluid mechanics since many problems in these areas can be viewed as saddle point problems. One of the main advantages of this approach, is the relation of saddle point problems to finite element methods of mixed type. Finite element spaces of mixed type were studied extensively in previous works (see e.g [4, 5]). For a comprehensive treatment of many important algorithms such as penalized, iterated penalized algorithms, augmented Lagrangian and Uzawa type, one may consult the classical works of [5, 16, 33]. Even though parabolic problems of saddle point type are not related to an optimization principle, this particular type of formulation can be very useful for the analysis and finite element approximations of time dependent problems such as (1.1).

This paper is organized as follows: In section 2 we present the notation and the main result concerning saddle point problems associated to elliptic partial differential equations. Furthermore, we state the main result concerning the existence and uniqueness of the solution of problem (1.2). In subsequent section 3 we establish the proof of the main theorem. In section 4, we present applications of the main theorem to the existence and uniqueness of weak solution for evolutionary problems with inhomogeneous boundary and divergence data. Finally, in section 5, we derive the main error estimates for the finite element approximations. Note that we also treat inhomogeneous essential boundary data.

2. Preliminaries and main results

2.1. Notation

Let Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$ which can be either convex and polygonal (convex and polyhedral, in $d = 3$) or with regular (enough) boundary Γ. We will denote all vector valued functions using the boldface notation $\mathbf{u}, \mathbf{v}$ etc. We use the standard notation $H^m(\Omega)$, $H^s(\Gamma)$ for Hilbert spaces of order $m, s \in \mathbb{R}$, defined on Ω and Γ repsectively, and their norms. Furthermore we denote by $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ and H^{-1} its dual. Abusing the notation we will not use different notation for their vector valued counterparts. For any Hilbert space U defined as above, the standard notation is being used for their corresponding time-space spaces $L^p(0, T; U)$ and their norms, i.e.,

$$\|v\|_{L^p(0,T;U)} = \Big(\int_0^T \|v\|_U^p dt\Big)^{\frac{1}{p}}, \qquad \|v\|_{L^\infty(0,T;U)} = \operatorname{esssup}_{t\in[0,T]} \|v\|_U.$$

We also employ the standard notation for the $L^2(\Omega)$ inner product $(\cdot,\cdot)_{L^2(\Omega)} = (\cdot,\cdot)$. In addition, we denote by X any vector valued version of the above spaces and by H a (vector valued version) of the above Hilbert spaces such that $X \subset H \subset X^*$ form an "evolution" triple, i.e., $X \subset H$ with compact embedding (for details see [35, Proposition 23.23]), satisfying $\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_H^2 = \langle \mathbf{u}_t(t), \mathbf{u}(t)\rangle_{(X^*,X)}$. In practice, H is always the vector valued version of $L^2(\Omega)$ and X a vector valued version of $H^1(\Omega)$, $H_0^1(\Omega)$ and/or their divergence free counterparts (see e.g. [15,33]). Abusing the notation, we will denote by $(.,.)$ the inner product of H, and by $\langle .,. \rangle \equiv \langle .,. \rangle_{(X^*,X)}$. Similarly, we denote by $L^2(0,T;X), L^\infty(0,T;X)$ the vector valued time dependent spaces with their norms defined as above. Finally, we will frequently use the space $H^1(0,T;X)$, endowed with norn $\|\mathbf{u}\|^2_{H^1(0,T;X)} = \|\mathbf{u}\|^2_{L^2(0,T;X)} + \|\mathbf{u}_t\|^2_{L^2(0,T;X)}$. For the pressure terms, we also use the space

$$L^2_0(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p dx = 0\},$$

endowed with norm $\|.\|_{L^2(\Omega)}$.

2.2. The elliptic saddle point problem

The classical theory of elliptic saddle point problems can be described as follows: Find $(\mathbf{u}, p) \in X \times M$ such that,

$$\begin{cases} \nu A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) & = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in X \\ B(\mathbf{u},q) & = \langle \mathbf{g}, q \rangle_{(M^*,M)} \qquad \forall q \in M, \end{cases} \tag{2.1}$$

where X, M are given spaces, $\mathbf{f} \in X^*$, and $\mathbf{g} \in M^*$ are given data. We also assume that $A(\cdot,\cdot)$ is a continuous bilinear form on $X \times X$, and $B(\cdot,\cdot)$ is a continuous bilinear form on $X \times M$. Moreover, we define the auxiliary subspaces (see e.g. [15])

$$Z(\mathbf{g}) := \{\mathbf{u} \in X : B(\mathbf{u},q) = \langle \mathbf{g}, q \rangle_{(M^*,M)} \quad \forall q \in M\}, \qquad Z \equiv Z(0).$$

In addition we require that the bilinear forms satisfy the standard coercivity assumptions:

$$A(\mathbf{z},\mathbf{z}) \geq \alpha \|\mathbf{z}\|^2_X \quad \forall \mathbf{z} \in Z, \tag{2.2}$$

$$\inf_{0\neq q \in M} \sup_{0 \neq \mathbf{u} \in X} \frac{B(\mathbf{u},q)}{\|\mathbf{u}\|_X \|q\|_M} \geq \beta > 0. \tag{2.3}$$

The last inequality is usually called inf-sup condition (see e.g., [1], [5], [15], [23], [26] and references within). The main result concerning the existence and uniqueness of a solution pair $(\mathbf{u}, p) \in X \times M$ is presented in the following theorem (see e.g. [15]).

Theorem 2.1. Let $A(\mathbf{u},\mathbf{v})$, $B(\mathbf{v},q)$ be bounded bilinear operators satisfying coercivity conditions (2.2)-(2.3). Then, for any given $\mathbf{f} \in X^*, \mathbf{g} \in M^*$, there exists a unique pair $(\mathbf{u}, p) \in X \times M$ such that (2.1) holds.

2.3. The parabolic saddle point framework and main results

We close this section by stating the main result and some additional comments regarding the existence and uniqueness of parabolic saddle point problems.

Theorem 2.2. Assume that the continuous bilinear forms $A(\cdot,\cdot), B(\cdot,\cdot)$ satisfy the coercivity properties (2.2)-(2.3). Furthermore, suppose a semi-norm is defined by the bilinear form, $|\mathbf{u}|^2_X \equiv A(\mathbf{u},\mathbf{u}) \quad \forall \mathbf{u} \in X$ with

$$A(\mathbf{u},\mathbf{v}) \leq \frac{1}{2} A(\mathbf{u},\mathbf{u}) + \frac{1}{2} A(\mathbf{v},\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in X. \tag{2.4}$$

If $\mathbf{g} \in H^1(0,T;M^*)$, $\mathbf{u}_0 \in X$, and $B(\mathbf{u}(0),q) = \langle \mathbf{g}(0), q \rangle_{(M^*,M)}, \forall q \in M$ then there exists $\mathbf{u} \in L^\infty(0,T;X) \cap H^1(0,T;H)$, and $p \in L^2(0,T;M)$ such that for a.e. $t \in (0,T]$,

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v} \rangle + \nu A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) & = 0 \qquad \forall \mathbf{v} \in X \\ B(\mathbf{u},q) & = \langle \mathbf{g}, q \rangle_{(M^*,M)} \qquad \forall q \in M \\ (\mathbf{u}(0),\mathbf{z}) & = (\mathbf{u}_0, \mathbf{z}) \qquad \forall \mathbf{z} \in H. \end{cases} \tag{2.5}$$

In addition, if we decompose $\mathbf{u}(.) = \mathbf{w}(.) + \mathbf{z}(.)$, where $\mathbf{w}(.) \in Z^{\perp}$, $\mathbf{z}(.) \in Z$, for a.e. $t \in (0,T]$, then $\mathbf{w}_t \in L^2(0,T;X)$.

Remark 2.3. For the examples stated in the introduction, inequality (2.4), states that the bilinear form $A(\cdot,\cdot)$ contains only gradient terms. In addition, note also that (2.4) implies the following inequality

$$A(\mathbf{u},\mathbf{u}-\mathbf{v}) \geq \frac{1}{2}A(\mathbf{u},\mathbf{u}) - \frac{1}{2}A(\mathbf{v},\mathbf{v}) \equiv \frac{1}{2}|\mathbf{u}|_X^2 - \frac{1}{2}|\mathbf{v}|_X^2. \tag{2.6}$$

The inhomogeneous evolutionary Stokes equations (1.1), can be included in the above setting provided that the data $\mathbf{g}$ are understood as a pair $\mathbf{g} \equiv (\psi, \boldsymbol{\phi}) \in H^1(0,T;M^*)$ with $M \equiv M_1 \times M_2$. Here M_1, M_2 denote appropriate spaces for the inhomogeneous divergence and boundary data respectively. Hence, we seek velocity $\mathbf{u} \equiv \tilde{\mathbf{u}} \in L^2(0,T;X) \cap H^1(0,T;X^*) \cap L^\infty(0,T;H)$ and a pair $p \equiv (\tilde{p}, \tilde{\boldsymbol{\lambda}}) \in L^2(0,T;M_1) \times L^2(0,T;M_2)$ consisting of the pressure $\tilde{p}$ and the Lagrange multiplier $\tilde{\boldsymbol{\lambda}}$ terms respectively. Under our assumptions we prove the enhanced regularity $\mathbf{u} \in H^1(0,T;H) \cap L^\infty(0,T;X)$ which is crucial in the development of error estimates. We emphasize that the Lagrange multiplier term $\tilde{\boldsymbol{\lambda}}$ contains all related boundary terms, including terms resulting from various applications of Green's Theorem. The presence of the Lagrange multiplier $\tilde{\boldsymbol{\lambda}}$ is an essential feature of our work which distinguishes it from other approaches. Note also, that the classical evolutionary Stokes problem with inhomogeneous Dirichlet data can be fit into the above framework (for $\psi \equiv 0$).

The above result can be extended for a nonzero forcing term $\mathbf{f}$, when it is combined with an analogous result for the homogeneous case $\mathbf{g} \equiv 0$.

Remark 2.4. Let the assumptions of Theorem 2.2 hold, and let the forcing term $\mathbf{f} \in L^2(0,T;H)$ Then there exists $\mathbf{u} \in L^2(0,T;X) \cap H^1(0,T;H)$, and $p \in L^2(0,T;M)$ such that for a.e. $t \in (0,T]$

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v}\rangle + \nu A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) &= \langle \mathbf{f},\mathbf{v}\rangle_{(X^*,X)} \qquad \forall\, \mathbf{v} \in X \\ B(\mathbf{u},q) &= \langle \mathbf{g}, q\rangle_{(M^*,M)} \qquad \forall\, q \in M \\ (\mathbf{u}(0),\mathbf{z}) &= (\mathbf{u}_0,\mathbf{z}) \qquad \forall\, \mathbf{z} \in H. \end{cases}$$

The regularity assumption for $\mathbf{f}$ and $\mathbf{u}_0$ is due to the coupling between p and $\mathbf{u}_t$ and our requirement for regularity $p \in L^2(0,T;M)$ within the above weak formulation. Recall that even in case of the evolutionary Stokes equation with homogeneous data, if we restrict the regularity assumptions to $\mathbf{f} \in L^2(0,T;Z^*)$, $\mathbf{u}_0 \in H$, where $Z = \{\mathbf{v} \in H_0^1(\Omega) : \mathrm{div}\mathbf{v} = 0\}$, and $H = \{\mathbf{v} \in L^2(\Omega) : \mathrm{div}\mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n} = 0\}$, then the existence of a pressure is only proved in a distributional sense (see e.g. [33]).

3. Inhomogeneous parabolic saddle point problem

In this section, we present the proof of the main Theorem 2.2. We will employ a semi-discretization (in time) approach, in order to fully utilize the inf-sup condition. In particular, we first obtain a-priori estimates for the semi-discrete solutions, and then we pass to the limit following the approach of [33, Chapter 3, Section 4].

3.1. **The semi-discrete (in time) approximation**

Let N be an integer, set $k = \frac{T}{N}$, and let $\{t^m\}_{m=0}^N$, denote the partition points of $[0,T]$, where $t^m = mk$, with $m = 0,...,N$. We recursively define a family of elements of X, M, denoted by $\{\mathbf{u}^m\}_{m=1}^N$, $\{p^m\}_{m=1}^N$ respectively where $\mathbf{u}^m, p^m$ are in some sense approximations of functions $\mathbf{u}(.), p(.)$ respectively, on the interval $t^{m-1} < t \leq t^m$, with $m = 1,...,N$. Here, we denote by $\mathbf{u}^0 \equiv \mathbf{u}(0)$. Taking into account that $\mathbf{g} \in H^1(0,T;M^*) \subset C([0,T];M^*)$ we may also define elements $\mathbf{g}^0, \mathbf{g}^2, ..., \mathbf{g}^N$ of M^* as:

$$\mathbf{g}^m = \mathbf{g}(t^m), \quad m = 0,...,N. \tag{3.1}$$

If $\{\mathbf{u}^i\}_{i=1}^{m-1}$, $\{p^i\}_{i=1}^{m-1}$, are known, we can define $\mathbf{u}^m, p^m$ as elements of X, M respectively satisfying:

$$\left\{\begin{array}{rll} (\frac{\mathbf{u}^m-\mathbf{u}^{m-1}}{k},\mathbf{v})+\nu A(\mathbf{u}^m,\mathbf{v})+B(\mathbf{v},p^m) & =0 & \forall \mathbf{v}\in X \\ B(\mathbf{u}^m,q) & =\langle \mathbf{g}^m,q\rangle_{(M^*,M)} & \forall q\in M \end{array}\right. \tag{3.2}$$

We will also impose the compatibility condition $B(\mathbf{u}(0),q)=\langle \mathbf{g}(0),q\rangle_{(M^*,M)}$, for all $q\in M$. The existence of the pair $(\mathbf{u}^m,p^m)\in X\times M$ can be easily justified by Theorem 2.1. Indeed, we can rewrite (3.2) as:

$$\left\{\begin{array}{rll} \frac{1}{k}(\mathbf{u}^m,\mathbf{v})+\nu A(\mathbf{u}^m,\mathbf{v})+B(\mathbf{v},p^m) & =(\frac{1}{k}\mathbf{u}^{m-1},\mathbf{v}) & \forall \mathbf{v}\in X \\ B(\mathbf{u}^m,q) & =\langle \mathbf{g}^m,q\rangle_{(M^*,M)} & \forall q\in M. \end{array}\right.$$

Note that $\frac{1}{k}\mathbf{u}^{m-1}+\mathbf{f}^m\in X^*$ are given data. Moreover, $\forall \mathbf{z}^m\in Z$

$$\frac{1}{k}(\mathbf{z}^m,\mathbf{z}^m)+\nu A(\mathbf{z}^m,\mathbf{z}^m)\geq \frac{1}{k}\|\mathbf{z}^m\|_H^2+\nu\alpha\|\mathbf{z}^m\|_X^2\geq \nu\alpha\|\mathbf{z}^m\|_X^2.$$

Since the coercivity inequality and the inf-sup condition on $B(\cdot,\cdot)$ hold, we may apply Theorem 2.1 to guarantee the existence and uniqueness of a pair $(\mathbf{u}^m,p^m)\in X\times M$. Moreover, it easy to check that the following inequality holds: There exists a positive constant C, depending only on α,β,ν,Ω such that

$$\|\mathbf{u}^m\|_X+\|p^m\|_M\leq C\Big(\frac{1}{k}\|\mathbf{u}^{m-1}\|_H+\|\mathbf{f}^m\|_{X^*}+\|\mathbf{g}^m\|_{M^*}\Big).$$

For $m=1,...,N$,we define the following auxiliary functions:

$$\begin{array}{l} \mathbf{u}_k:[0,T]\to X,\quad \mathbf{u}_k(t)=\mathbf{u}^m,\quad t\in(t^{m-1},t^m] \\ p_k:[0,T]\to M,\quad p_k(t)=p^m,\quad t\in(t^{m-1},t^m] \\ \bar{\mathbf{u}}_k:[0,T]\to H,\quad \bar{\mathbf{u}}_k \text{ is continuous, linear on each subinterval} \\ \qquad (t^{m-1},t^m],\ \text{ and }\quad \bar{\mathbf{u}}_k(t^m)=\mathbf{u}^m. \end{array}$$

We also note that due to the inf-sup condition we may decompose $\mathbf{u}^m\in X$ as $\mathbf{u}^m=\mathbf{w}^m+\mathbf{z}^m$, where $\mathbf{w}^m\in Z^\perp$ and $\mathbf{z}^m\in Z$, and for all $m=1,..,N$. We also define functions $\bar{\mathbf{w}}_k:(0,T]\to Z^\perp$, in a similar fashion. The next lemma relates various quantities of the semi-discrete (in time) values $\mathbf{g}^m$ in terms of regularity properties on data $\mathbf{g}$.

Lemma 3.1. Let $\mathbf{g}^m$ be defined as in (3.1) and $\mathbf{g}\in H^1(0,T;M^*)$. Then,

$$\|\mathbf{g}^m\|_{M^*}\leq C<\infty,\qquad k\sum_{m=1}^N\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k}\|_{M^*}^2\leq C<\infty. \tag{3.3}$$

Proof. The first estimate is obvious. For the second one, using standard calculations, Hölder's inequality and the fact that $\mathbf{g}_t\in L^2(0,T;M^*)$, we deduce that

$$\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k}\|_{M^*}\leq \frac{1}{k}\int_{t^{m-1}}^{t^m}\|\mathbf{g}_t(t)\|_{M^*}dt\leq \frac{1}{k^{1/2}}\Big(\int_{t^{m-1}}^{t^m}\|\mathbf{g}_t(t)\|_{M^*}^2dt\Big)^{\frac{1}{2}} \tag{3.4}$$

Hence (3.4) implies,

$$k\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k}\|_{M^*}^2\quad\leq\quad\int_{t^{m-1}}^{t^m}\|\mathbf{g}_t\|_{M^*}^2dt. \tag{3.5}$$

Adding inequalities (3.5), we conclude

$$k\sum_{m=1}^{N}\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k}\|_{M^*}^2 \leq \sum_{m=1}^{N}\int_{t^{m-1}}^{t^m}\|\mathbf{g}_t\|_{M^*}^2 dt \leq \int_0^T \|\mathbf{g}_t(t)\|_{M^*}^2 dt < \infty.$$

□

We now derive estimates for the approximation pair $(\mathbf{u}^m, p^m)$.

3.2. A priori estimates

First we derive a priori estimates for $\mathbf{w}^m \in Z^\perp$ using Lemma 3.1 and the inf-sup condition. Subsequently, we establish a priori estimates for the $\mathbf{z}^m \in Z$ terms based on estimates on $\mathbf{w}^m$.

Lemma 3.2. Assume that the bilinear forms are continuous and satisfy (2.2)-(2.3). Suppose that $\mathbf{g} \in H^1(0,T;M^*)$, $\mathbf{u}^0 \equiv \mathbf{u}(0) \in X$ are given data, $\mathbf{g}^m$, $m = 0,...N$ are defined as in (3.1) with $B(\mathbf{u}^0, q) = \langle \mathbf{g}(0), q\rangle_{(M^*.M)}$, for all $q \in M$. Let $(\mathbf{u}^m, p^m) \in X \times M$, $m = 1, ..., N$ satisfy (3.2). Then,

$$k\sum_{m=1}^{N}\|\frac{\mathbf{w}^m-\mathbf{w}^{m-1}}{k}\|_X^2 \leq C < \infty, \text{ and } \quad k\sum_{m=1}^{N}\|\mathbf{w}^m\|_X^2 \leq C < \infty, \tag{3.6}$$

where $C > 0$ depends only upon Ω, β.

Proof. Note that $B(\mathbf{u}^m - \mathbf{u}^{m-1}, q) = \langle \mathbf{g}^m - \mathbf{g}^{m-1}, q\rangle_{(M^*,M)} \quad \forall q \in M$, for all $m = 1, ..., N$, so using the inf-sup condition, the fact that $\mathbf{z}^m - \mathbf{z}^{m-1} \in Z$, and (3.2)

$$\begin{aligned}
\|\frac{\mathbf{w}^m-\mathbf{w}^{m-1}}{k^{1/2}}\|_X &\leq C\sup_{q\in M}\frac{B(\frac{\mathbf{w}^m-\mathbf{w}^{m-1}}{k^{1/2}}, q)}{\|q\|_M} \\
&\leq C\sup_{q\in M}\frac{|B(\frac{\mathbf{u}^m-\mathbf{u}^{m-1}}{k^{1/2}}, q)| + |B(\frac{\mathbf{z}^m-\mathbf{z}^{m-1}}{k^{1/2}}, q)|}{\|q\|_M} \\
&\leq C\sup_{q\in M}\frac{|\langle\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k^{1/2}}, q\rangle|}{\|q\|_M}.
\end{aligned}$$

Note that (3.4) implies $\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k^{1/2}} \in M^*$, since

$$\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k^{1/2}}\|_{M^*} \leq \Big(\int_{t^{m-1}}^{t^m}\|\mathbf{g}_t(t)\|_{M^*}^2 dt\Big)^{\frac{1}{2}} < \infty.$$

Therefore, we deduce,

$$\|\frac{\mathbf{w}^m-\mathbf{w}^{m-1}}{k^{1/2}}\|_X \leq C\sup_{q\in M}\frac{|\langle\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k^{1/2}}, q\rangle|}{\|q\|_M} \leq C\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k^{1/2}}\|_{M^*},$$

or equivalently, squaring both sides,

$$\frac{1}{k}\|\mathbf{w}^m-\mathbf{w}^{m-1}\|_X^2 \leq \frac{C}{k}\|\mathbf{g}^m-\mathbf{g}^{m-1}\|_{M^*}^2.$$

Hence, the above inequality together with (3.3),

$$k\sum_{m=1}^{N}\|\frac{\mathbf{w}^m-\mathbf{w}^{m-1}}{k}\|_X^2 \le Ck\sum_{m=1}^{N}\|\frac{\mathbf{g}^m-\mathbf{g}^{m-1}}{k}\|_{M^*}^2 \le C\|\mathbf{g}_t\|_{L^2(0,T;M^*)}^2 < \infty.$$

The other estimate is an immediate consequence of the inf-sup condition applied to $B(\mathbf{u}^m,q)=\langle \mathbf{g}^m,q\rangle$ which states that $\|\mathbf{w}^m\|_X \le C\|\mathbf{g}^m\|_{M^*} \le C < \infty$ by (3.3), and hence, $k\sum_{m=1}^N\|\mathbf{w}^m\|_X^2 \le Ck\sum_{m=1}^N\|\mathbf{g}^m\|_{M^*}^2 \le C < \infty$. □

It remains to estimate several quantities related to $\{\mathbf{z}^m\}_{m=1}^N$.

Lemma 3.3. Suppose that the assumptions of Lemma 3.2 hold and the bilinear forms are continuous and satisfy (2.2)-(2.3)-(2.4). Then,

$$\|\mathbf{z}^m\|_H \le C < \infty, \quad k\sum_{m=1}^{N}\|\mathbf{z}^m\|_X^2 \le C < \infty, \quad \sum_{m=1}^{N}\|\mathbf{z}^m-\mathbf{z}^{m-1}\|_H^2 \le C < \infty, \tag{3.7}$$

where C denotes constants depending only upon Ω,α, β, and ν.

Proof. We start from (3.2) and we substitute $\mathbf{u}^m, \mathbf{u}^{m-1}$ by their decomposition i.e., $\mathbf{u}^m=\mathbf{w}^m+\mathbf{z}^m$, $\mathbf{u}^{m-1}=\mathbf{w}^{m-1}+\mathbf{z}^{m-1}$,

$$(\mathbf{z}^m-\mathbf{z}^{m-1},\mathbf{v})+\nu kA(\mathbf{z}^m,\mathbf{v})+kB(\mathbf{v},p^m) = -(\mathbf{w}^m-\mathbf{w}^{m-1},\mathbf{v})-\nu kA(\mathbf{w}^m,\mathbf{v}). \tag{3.8}$$

Set $\mathbf{v}=2\mathbf{z}^m\in Z$ and note that $B(\mathbf{z}^m,p^m)=0$. Therefore,

$$\begin{aligned}&\|\mathbf{z}^m\|_H^2-\|\mathbf{z}^{m-1}\|_H^2+\|\mathbf{z}^m-\mathbf{z}^{m-1}\|_H^2+2k\nu\alpha\|\mathbf{z}^m\|_X^2\\ &\quad\le C\|\mathbf{w}^m-\mathbf{w}^{m-1}\|_H\|\mathbf{z}^m\|_X+Ck\nu\|\mathbf{w}^m\|_X\|\mathbf{z}^m\|_X\\ &\quad\le k\nu\alpha\|\mathbf{z}^m\|_X^2+C\Big(\frac{1}{\nu k}\|\mathbf{w}^m-\mathbf{w}^{m-1}\|_H^2+\nu k\|\mathbf{w}^m\|_X^2\Big).\end{aligned}$$

Here, C is a constant depending on α and on the domain. Hence,

$$\|\mathbf{z}^m\|_H^2-\|\mathbf{z}^{m-1}\|_H^2+\|\mathbf{z}^m-\mathbf{z}^{m-1}\|_H^2+k\nu\alpha\|\mathbf{z}^m\|_X^2 \le \frac{C}{\nu k}\|\mathbf{w}^m-\mathbf{w}^{m-1}\|_H^2+C\nu k\|\mathbf{w}^m\|_X^2.$$

Using the above relation recursively, we obtain

$$\begin{aligned}&\|\mathbf{z}^N\|_H^2+\sum_{m=1}^{N}\|\mathbf{z}^m-\mathbf{z}^{m-1}\|_H^2+k\nu\alpha\sum_{m=1}^{N}\|\mathbf{z}^m\|_X^2\\ &\le C\Big(\|\mathbf{z}^0\|_H^2+\frac{1}{\nu k}\sum_{m=1}^{N}\|\mathbf{w}^m-\mathbf{w}^{m-1}\|_H^2+\nu k\sum_{m=1}^{N}\|\mathbf{w}^m\|_X^2\Big).\end{aligned}$$

Equations (3.6) of Lemma 3.2 guarantee that the last two sums are finite. □

Lemma 3.4. Under the assumptions of Lemma 3.3, the following estimates hold:

$$\frac{1}{k}\sum_{m=1}^{N}\|\mathbf{z}^m-\mathbf{z}^{m-1}\|_H^2 \le C < \infty, \quad \text{and } |\mathbf{u}^m|_X \le C < \infty \text{ for all } m=1,...,N,$$

where C denotes constants depending only upon Ω,α, β, and ν.

Proof. We start from equation (3.2), and we substitute $\mathbf{u}^m, \mathbf{u}^{m-1}$ by their decomposition, i.e., $\mathbf{u}^m = \mathbf{w}^m + \mathbf{z}^m$, $\mathbf{u}^{m-1} = \mathbf{w}^{m-1} + \mathbf{z}^{m-1}$, where $\mathbf{z}^m, \mathbf{z}^{m-1} \in Z$ and $\mathbf{w}^m, \mathbf{w}^{m-1} \in Z^\perp$.

$$\begin{cases} \frac{1}{k}(\mathbf{z}^m - \mathbf{z}^{m-1}, \mathbf{v}) + \nu A(\mathbf{u}^m, \mathbf{v}) + B(\mathbf{v}, p^m) = -\frac{1}{k}(\mathbf{w}^m - \mathbf{w}^{m-1}, \mathbf{v}) \\ B(\mathbf{u}^m, q) = \langle \mathbf{g}^m, q\rangle_{(M^*,M)} \qquad \forall\, q \in M. \end{cases} \tag{3.9}$$

Set $\mathbf{v} = \mathbf{z}^m - \mathbf{z}^{m-1}$ into (3.9). Therefore, after noting that $B(\mathbf{z}^m - \mathbf{z}^{m-1}, p^m) = 0$,

$$\begin{aligned} \frac{1}{k}\|\mathbf{z}^m - \mathbf{z}^{m-1}\|_H^2 + \nu A(\mathbf{u}^m, \mathbf{z}^m - \mathbf{z}^{m-1}) &\le \|\frac{\mathbf{w}^m - \mathbf{w}^{m-1}}{k}\|_H \|\mathbf{z}^m - \mathbf{z}^{m-1}\|_H \\ &\le \frac{1}{4k}\|\mathbf{z}^m - \mathbf{z}^{m-1}\|_H^2 + k\|\frac{\mathbf{w}^m - \mathbf{w}^{m-1}}{k}\|_H^2. \end{aligned}$$

Using the decomposition once more together with (2.6), we rewrite the bilinear term as follows

$$\begin{aligned} A(\mathbf{u}^m, \mathbf{z}^m - \mathbf{z}^{m-1}) &= A(\mathbf{u}^m, \mathbf{u}^m - \mathbf{u}^{m-1}) - A(\mathbf{u}^m, \mathbf{w}^m - \mathbf{w}^{m-1}) \\ &\ge \frac{1}{2}|\mathbf{u}^m|_X^2 - \frac{1}{2}|\mathbf{u}^{m-1}|_X^2 - A(\mathbf{u}^m, \mathbf{w}^m - \mathbf{w}^{m-1}). \end{aligned}$$

Combining the last two inequalities,

$$\begin{aligned} &\frac{3}{4k}\|\mathbf{z}^m - \mathbf{z}^{m-1}\|_H^2 + \frac{\nu}{2}|\mathbf{u}^m|_X^2 - \frac{\nu}{2}|\mathbf{u}^{m-1}|_X^2 \\ &\quad \le k\|\frac{\mathbf{w}^m - \mathbf{w}^{m-1}}{k}\|_H^2 + \nu A(\mathbf{u}^m, \mathbf{w}^m - \mathbf{w}^{m-1}) \\ &\quad \le k\|\frac{\mathbf{w}^m - \mathbf{w}^{m-1}}{k}\|_H^2 + \nu\|\mathbf{u}^m\|_X \|\mathbf{w}^m - \mathbf{w}^{m-1}\|_X \\ &\quad \le k\|\frac{\mathbf{w}^m - \mathbf{w}^{m-1}}{k}\|_H^2 + \frac{\nu}{2k}\|\mathbf{w}^m - \mathbf{w}^{m-1}\|_X^2 + \frac{\nu k}{2}\|\mathbf{u}^m\|_X^2. \end{aligned} \tag{3.10}$$

Using the above relation recursively from $m = 1$ to $m = N$, we obtain

$$\begin{aligned} &\frac{1}{k}\sum_{m=1}^{N}\|\mathbf{z}^m - \mathbf{z}^{m-1}\|_H^2 + \nu|\mathbf{u}^N|_X^2 \\ &\le C\Big(|\mathbf{u}^0|_X^2 + \frac{1}{k}\sum_{m=1}^{N}\|\mathbf{w}^m - \mathbf{w}^{m-1}\|_H^2 + \frac{\nu}{k}\sum_{m=1}^{N}\|\mathbf{w}^m - \mathbf{w}^{m-1}\|_X^2 + \nu k\sum_{m=1}^{N}\|\mathbf{u}^m\|_X^2\Big). \end{aligned}$$

Lemmas 3.2-3.3 guarantee that the above sums are finite. Returning to (3.10), and summing from 1 to m, we easily obtain $|\mathbf{u}^m|_X^2 \le C < \infty$. □

Collecting the estimates of Lemmas 3.2-3.4, we obtain the main stability estimates.

Theorem 3.5. Assume that the bilinear forms $A(.,.), B(.,.)$ are continuous, satisfy (2.2)-(2.3)-(2.4), and let $\mathbf{g} \in H^1(0,T;M^*)$. Let $\mathbf{u}^0 = \mathbf{u}(0) \in X$ be given data, $\mathbf{g}^m$, $m = 0,...N$ be defined as in (3.1) with $B(\mathbf{u}(0), q) = \langle \mathbf{g}(0), q\rangle_{(M^*.M)} \equiv \langle \mathbf{g}^0, q\rangle_{(M^*,M)}$, for all $q \in M$. Let $(\mathbf{u}^m, p^m) \in X \times M$, $m = 1,...,N$ satisfy (3.2). Then, the following quantities are bounded by constants $C < \infty$ depending only upon $\Omega, \alpha, \beta, \nu$:

$$\begin{gathered} k\sum_{m=1}^{N}\|\mathbf{u}^m\|_X^2, \quad \sum_{m=1}^{N}\|\mathbf{u}^m - \mathbf{u}^{m-1}\|_H^2, \quad \frac{1}{k}\sum_{m=1}^{N}\|\mathbf{u}^m - \mathbf{u}^{m-1}\|_H^2, \\ k\sum_{m=1}^{N}\|p^m\|_M^2, \quad \frac{1}{k}\sum_{m=1}^{N}\|\mathbf{u}^m - \mathbf{u}^{m-1}\|_{X^*}^2, \quad \|\mathbf{u}^m\|_X \text{ for all } m = 1,...,N. \end{gathered}$$

Proof. Note that Lemmas 3.2-3.4 and the triangle inequality imply the first three estimates for the sums of $\mathbf{u}^m$ terms. From the inf-sup condition, it is clear that

$$\|p^m\|_M \le C\Big(\|\frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k}\|_H + \|\mathbf{u}^m\|_X + \|\mathbf{g}^m\|_{M^*}\Big)$$

or equivalently,

$$k\sum_{m=1}^{N}\|p^m\|_M^2 \le C\Big(\sum_{m=1}^{N} k\|\frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k}\|_H^2 + k\sum_{m=1}^{N}\|\mathbf{u}^m\|_X^2 + k\sum_{m=1}^{N}\|\mathbf{g}^m\|_{M^*}^2\Big).$$

It is now obvious that the desired estimate for the pressure holds, due to the first three estimates. Taking the supremum over $\mathbf{v} \in X$ into the first equation of (3.2), we obtain $k\sum_{m=1}^{N}\|\frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k}\|_{X^*}^2 < \infty$. In order to estimate $\|\mathbf{u}^m\|_X$ we simply need to estimate $\|\mathbf{u}^m\|_H \le C < \infty$, since Lemma 3.4 states that $|\mathbf{u}^m|_X \le C < \infty$, for all $m = 1, ..., N$. For this purpose, we return to (3.2) and set $\mathbf{v} = \mathbf{u}^m$ and $q = p^m$. Thus, we deduce,

$$(\mathbf{u}^m - \mathbf{u}^{m-1}, \mathbf{u}^m) + k\nu A(\mathbf{u}^m, \mathbf{u}^m) = -k\langle \mathbf{g}^m, p^m\rangle_{(M^*,M)} \le (k/2)\|\mathbf{g}^m\|_{M^*}^2 + (k/2)\|p^m\|_M^2.$$

The proof now follows upon summing the above inequalities from 1 to m, and using the previous bounds on $k\sum_{m=1}^{N}\|\mathbf{g}^m\|_{M^*}^2$ and $k\sum_{i=1}^{N}\|p^m\|_M^2$. □

Now we are ready to prove the main Theorem 2.2 by using the above a-priori bounds of the auxiliary functions on the semi-discretized (in time) approach.

3.3. **Proof of Theorem 2.2:**

The proof is similar to the one of [33, Chapter III, Section 4]. The functions $\mathbf{u}_k, \bar{\mathbf{u}}_k, p_k$ defined as above, together with Theorem 3.5 remain bounded in $L^2(0,T;X) \cap L^\infty(0,T;H)$, and $L^2(0,T;M)$ respectively. Also note that $\bar{\mathbf{u}}_{kt}$ remains bounded in $L^2(0,T;H)$. Indeed, these are simply the interpretations of the stability estimates of Theorem 3.5. Moreover, [33, Lemma 4.8, pp 328] implies that $\mathbf{u}_k - \bar{\mathbf{u}}_k \to 0$ in $L^2(0,T;H)$ as $k \to \infty$. Therefore, we can extract subsequences, still denoted by $\mathbf{u}_k, \bar{\mathbf{u}}_k, p_k$, such that

$$\begin{aligned}
&\mathbf{u}_k \to \mathbf{u} \quad \text{weakly in } L^2(0,T;X), \quad \mathbf{u}_k \to \mathbf{u} \quad \text{weakly-* in } L^\infty(0,T;H)\\
&p_k \to p \quad \text{weakly in } L^2(0,T;M), \quad \bar{\mathbf{u}}_k \to \mathbf{u}_* \quad \text{weakly in } L^2(0,T;X)\\
&\bar{\mathbf{u}}_k \to \mathbf{u}_* \quad \text{weakly in } L^\infty(0,T;H), \quad \frac{d\bar{\mathbf{u}}_k}{dt} \to \frac{d\mathbf{u}_*}{dt} \quad \text{weakly in } L^2(0,T;X^*).
\end{aligned}$$

But [33, Lemma 4.8, pp 328], also implies that $\mathbf{u} = \mathbf{u}_*$. Note also that the classical Aubin-Lions compactness Lemma (see [33, Chapter 3, Section 3]) implies that $\bar{\mathbf{u}}_k \to \mathbf{u}$ strongly in $L^2(0,T;H)$, since $X \subset H \subset X^*$ form an evolution triple and $X \subset H$ with compact embedding. It is evident that the limit $(\mathbf{u}, p)$ is the solution of (2.5). Indeed, using the definitions of the auxiliary functions, we can rewrite the equations (3.2) as:

$$\begin{cases} (\frac{d\bar{\mathbf{u}}_k(t)}{dt}, \mathbf{v}) + \nu A(\mathbf{u}_k, \mathbf{v}) + B(\mathbf{v}, p_k) = 0 \qquad \forall \mathbf{v} \in X \\ B(\mathbf{u}_k, q) = \langle \mathbf{g}_k, q\rangle_{(M^*,M)} \quad \forall q \in M^*, \end{cases} \tag{3.11}$$

where $\mathbf{g}_k$ is defined by:

$$\mathbf{g}_k(t) = \mathbf{g}^m, t \in (t^{m-1}, t^m].$$

Working identically to [33, Lemma 4.9, pp 429] we obtain that

$$\mathbf{g}_k \to \mathbf{g} \text{ weakly in } L^2(0,T;M^*).$$

Hence, using the convergence results together with the continuity properties of the bilinear forms we pass the limit into (3.11) to obtain (2.5). The improved regularity on $\mathbf{u} \in L^\infty(0,T;X)$ is evident by the estimate of Theorem 3.5. The regularity on $\mathbf{w}_t$ is due to the estimate of Lemma 3.2.

4. Applications to evolutionary problems with inhomogeneous data

We apply the main Theorem 2.2 in order to prove the existence and uniqueness of a solution pair $(\mathbf{u}, p)$ of problems with inhomogeneous boundary and/or divergence data. First, we begin by treating the evolutionary Stokes problem with inhomogeneous divergence data, but with zero boundary condition. In particular, given $\mathbf{u}_0$ and g, we seek velocity $\mathbf{u}$ and pressure p such that,

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= 0 \qquad \text{in } \Omega\times(0,T] \\ \text{div } \mathbf{u} &= g \qquad \text{in } \Omega\times(0,T] \\ \mathbf{u} &= 0 \qquad \text{on } \Gamma\times(0,T] \\ \mathbf{u}(0) &= \mathbf{u}_0 \qquad \text{in } \Omega. \end{cases} \tag{4.1}$$

First, we recast our problem into the parabolic saddle point framework. Assume that $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, $M = L_0^2(\Omega)$ and define the standard bilinear forms,

$$A(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}, \qquad B(\mathbf{v},q) = -\int_\Omega \text{div}\mathbf{v}\ \text{q}$$

for all $\mathbf{u},\mathbf{v} \in H_0^1(\Omega), q \in L_0^2(\Omega)$. Here we denote $(\nabla\mathbf{u}) : (\nabla\mathbf{v}) = \sum_{i,j=1}^d \mathbf{u}_{i,j}\mathbf{v}_{i,j}$, with the second index denoting the derivative with respect to x_j.

Theorem 4.1. Suppose that $\mathbf{u}_0 \in H_0^1(\Omega)$, and $g \in H^1(0,T;L_0^2(\Omega))$ with $g(0) = \text{div } \mathbf{u}(0)$. Then, there exists a unique weak solution pair $(\mathbf{u}, p)$ of (4.1) in the sense of (1.2), satisfying:

$$\mathbf{u} \in L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \qquad p \in L^2(0,T;L_0^2(\Omega)).$$

For a.e. $t \in (0,T]$ let $Z = \{\mathbf{u} \in H_0^1(\Omega) : B(\mathbf{u},q) = 0, \quad \forall q \in L_0^2(\Omega)\}$. Then, if $\mathbf{u}(.)$ is decomposed to $\mathbf{u}(.) = \mathbf{z}(.) + \mathbf{w}(.)$, for a.e $t \in (0,T]$ with $\mathbf{z}(.) \in Z$, $\mathbf{w}(.) \in Z^\perp$, we obtain, $\mathbf{w}_t \in L^2(0,T;H_0^1(\Omega))$.

Proof. It is an immediate consequence of the main Theorem 2.2. Indeed, note that $Z \equiv \{\mathbf{u} \in H_0^1(\Omega) : \text{div}\mathbf{u}{=}0\}$, and hence the continuity and coercivity conditions (2.2)-(2.3)-(2.4) can be easily proven (see e.g. [15]), as in the elliptic case. The second result, is an immediate consequence of the inf-sup condition (see also regularity estimate of Lemma 3.2). □

The second application of Theorem 2.2 is the Lagrange multiplier method for a weak solution of the evolutionary Stokes, with inhomogeneous Dirichlet boundary data, i.e., the problem

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= 0 \qquad \text{in } \Omega\times(0,T) \\ \text{div } \mathbf{u} &= 0 \qquad \text{in } \Omega\times(0,T) \\ \mathbf{u} &= \boldsymbol{\phi} \qquad \text{on } \Gamma\times(0,T) \\ \mathbf{u}(0) &= \mathbf{u}_0 \qquad \text{in } \Omega, \end{cases} \tag{4.2}$$

together with the compatibility condition $\int_\Gamma \boldsymbol{\phi}(.,t)\cdot\mathbf{n}(.) = 0$ for a.e. $t \in (0,T]$. In this problem, we enforce the boundary condition weakly which implies that we need to introduce an additional variable, the Lagrange multiplier $\boldsymbol{\lambda}$ corresponding to the boundary stress. Our preferred weak formulation, now can be defined as follows: We seek $\mathbf{u} \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*)$, $p \in L^2(0,T;L_0^2(\Omega))$ and $\boldsymbol{\lambda} \in L^2(0,T;H^{-1/2}(\Gamma))$

such that for a.e. $t \in (0,T]$, and for all $\mathbf{v} \in H^1(\Omega)$, $q \in L^2_0(\Omega)$, $\mathbf{s} \in H^{-1/2}(\Gamma)$,

$$\begin{cases} \langle \mathbf{u}_t(t), \mathbf{v}\rangle + \nu(\nabla \mathbf{u}(t), \nabla \mathbf{v}) - (p(t),\ \mathrm{div}\mathbf{v}) & -\langle \boldsymbol{\lambda}(t), \mathbf{v}\rangle_{(H^{-1/2}(\Gamma),H^{1/2}(\Gamma))} = 0 \\ (\ \mathrm{div}\ \mathbf{u}(t), q) & = 0 \\ \langle \mathbf{u}(t), \mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} & = \langle \boldsymbol{\phi}(t), \mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} \\ (\mathbf{u}(0), \mathbf{v}) & = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{4.3}$$

It is evident that if $\int_\Gamma \boldsymbol{\phi} \cdot \mathbf{n} = 0$, and $\mathbf{u}, p, \boldsymbol{\lambda}$ sufficiently smooth (see e.g. [32]), then the formulation (4.3) is equivalent to (4.2). Next we put (4.3) into our parabolic saddle point framework. For this purpose, we define $X = H^1(\Omega)$, $M = L^2_0(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ and we denote by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, \qquad b(\mathbf{v}, q) = -\int_\Omega \mathrm{div}\mathbf{v}\ \mathrm{q},$$

the standard bilinear forms associated to the evolutionary Stokes problem. Note that $a(\mathbf{u}, \mathbf{u}) \equiv |\mathbf{u}|^2_X$ denotes a semi-norm, and satisfies (2.4). Now, it is clear that we can recast (4.3) as a "parabolic" saddle point problem by simply defining the bilinear forms, for a.e. $t \in (0,T]$, for all $\mathbf{u}(.), \mathbf{v} \in H^1(\Omega)$, $q \in L^2_0(\Omega)$, $\mathbf{s} \in H^{-1/2}(\Gamma)$,

$$A(\mathbf{u}(.), \mathbf{v}) \equiv a(\mathbf{u}(.), \mathbf{v}),$$

$$B(\mathbf{v}(.), (q, \mathbf{s})) \equiv b(\mathbf{v}(.), q) - \langle \mathbf{v}, \mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}.$$

Then, problem (4.3) can be written as a parabolic saddle point problem (1.2), as follows: For a.e. $t \in (0,T]$, and for all $\mathbf{v} \in H^1(\Omega)$ and $(q, \mathbf{s}) \in L^2_0(\Omega) \times H^{-1/2}(\Gamma)$,

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v}\rangle + \nu A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, (p, \boldsymbol{\lambda})) & = 0 \\ B(\mathbf{u}, (q, \mathbf{s})) & = -\langle \boldsymbol{\phi}, \mathbf{s}\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} \\ (\mathbf{u}(0), \mathbf{v}) & = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{4.4}$$

It remains to define the space $Z = \{\mathbf{u} \in H^1(\Omega) : B(\mathbf{u}, (q, \mathbf{s})) = 0\,\forall\,(q, \mathbf{s}) \in L^2_0(\Omega) \times H^{-1/2}(\Gamma)\}$, upon which the coercivity condition on $A(.,.)$ should be verified. We are ready to prove our main result.

Theorem 4.2. Suppose that $\boldsymbol{\phi} \in H^1(0,T; H^{\frac{1}{2}}(\Gamma))$ with $\langle \boldsymbol{\phi}(0), s\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} = \langle \mathbf{u}(0), s\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}$, for all $s \in H^{-1}(\Gamma)$, and $\mathbf{u}_0 \in H^1(\Omega)$, with div$\mathbf{u}_0$=0, then there exists a unique solution

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)),\ p \in L^2(0,T; L^2_0(\Omega)),\ \boldsymbol{\lambda} \in L^2(0,T; H^{-\frac{1}{2}}(\Gamma))$$

satisfying system (4.4).

Proof. It is easy to prove the continuity and coercivity assumption on Z for the bilinear form $A(\cdot,\cdot)$, since $Z \subset H^1_0(\Omega)$. The continuity of the bilinear form $B(\cdot,\cdot)$ is also evident. Then the proof follows directly from Theorem 2.2, since the inf-sup condition is proved in [20, Proposition 3]. □

Remark 4.3. We note that more spacial regularity can be recovered, under additional assumptions. Indeed, the fact that $\mathbf{u}_t \in L^2(0,T; L^2(\Omega))$ may be used to improve the spacial regularity of $\mathbf{u}$ in a standard fashion and hence to recover a strong solution, by exploring techniques of parabolic regularity and classical boot-strap arguments provided that some additional compatibility conditions, and smoothness on the boundary are assumed (for instance $\Gamma \in C^{1,1}$). For evolutionary Stokes equations, with inhomogeneous Dirichlet boundary data, if $\boldsymbol{\phi} \in L^2(0,T; H^{3/2}(\Gamma)) \cap H^{3/4}(0,T; L^2(\Gamma))$ then we can recover $L^2(0,T; H^2(\Omega))$ regularity for the strong solution (see for instance [32]). We note that the case of convex and polygonal domains requires further attention (see for instance [17]) since the polygonal structure of the domain acts as a barrier for higher regularity. However, for our analysis including the error estimates of the semi-discrete scheme, $L^2(0,T; H^2(\Omega))$ regularity for the velocity will not be necessary.

Remark 4.4. Even though the regularity on $\mathbf{g}, \mathbf{u}_0$ is not optimal, compared to the notion of very weak solutions of [30], the above formulation clearly represents the parabolic analog of saddle point theory.

Combining the above results, we may obtain the existence of a weak solution of the evolutionary Stokes equations, with inhomogeneous divergence and Dirichlet boundary data. We will treat the inhomogeneous Dirichlet boundary data for the evolutionary Stokes problem, as a parabolic saddle point problem by using a Lagrange multiplier principle similar to the elliptic case (see e.g. [4]). As before, we denote by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}, \qquad b(\mathbf{v},q) = -\int_\Omega \mathrm{div}\mathbf{v}\ \mathrm{q}.$$

Introducing the Lagrange multiplier, the weak formulation is given as follows: Seek $(\mathbf{u},p) \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*) \times L^2(0,T;L^2(\Omega))$, and a Lagrange multiplier $\boldsymbol{\lambda} \in L^2(0,T;H^{-1/2}(\Gamma))$ such that, for all $\mathbf{v} \in H^1(\Omega)$, $q \in L^2(\Omega)$ and $\mathbf{s} \in H^{-1/2}(\Gamma)$, and for a.e. $t \in (0,T]$,

$$\left\{\begin{array}{rl} \langle \mathbf{u}_t,\mathbf{v}\rangle + \nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - \langle \boldsymbol{\lambda},\mathbf{v}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} & = 0 \\ b(\mathbf{u},q) & = (-\psi,q) \\ \langle \mathbf{u},\mathbf{s}\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)} & = \langle \boldsymbol{\phi},\mathbf{s}\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)} \\ (\mathbf{u}(0),\mathbf{v}) & = (\mathbf{u}_0,\mathbf{v}). \end{array}\right. \tag{4.5}$$

In order to recast problem (4.5) as a parabolic saddle point problem, we define by

$$\begin{array}{rcl} A(\mathbf{u},\mathbf{v}) & = & a(\mathbf{u},\mathbf{v}) \qquad \forall\, \mathbf{u},\mathbf{v} \in H^1(\Omega) \\ B(\mathbf{u},(q,\mathbf{s})) & = & b(\mathbf{u},q) - \langle \mathbf{u},\mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} \qquad \forall\,(q,\mathbf{s}) \in L^2_0(\Omega)\times H^{-1/2}(\Gamma), \end{array}$$

and

$$Z = \{\mathbf{u} \in H^1(\Omega) \quad : \quad B(\mathbf{u},(q,\mathbf{s})) = 0, \qquad \forall\,(q,\mathbf{s}) \in L^2_0(\Omega)\times H^{-1/2}(\Omega)\}.$$

Then, problem (4.5) can be rewritten as follows: For a.e. $t \in (0,T]$, for all $\mathbf{v} \in H^1(\Omega)$, $q \in L^2_0(\Omega)$ and $\mathbf{s} \in H^{-1/2}(\Gamma)$,

$$\left\{\begin{array}{rl} \langle \mathbf{u}_t,\mathbf{v}\rangle + \nu A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},(\boldsymbol{\lambda},p)) & = 0 \\ B(\mathbf{u},(\mathbf{s},q)) & = -(\psi,q) - \langle \boldsymbol{\phi},\mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}. \end{array}\right. \tag{4.6}$$

Theorem 4.5. Given initial and boundary data satisfying

$$\mathbf{u}_0 \in H^1(\Omega), \quad \boldsymbol{\phi} \in H^1(0,T;H^{1/2}(\Gamma)), \quad \psi \in H^1(0,T;L^2(\Omega)),$$

and the compatibility conditions $(\mathrm{div}\mathbf{u}_0,q) = (\psi(0),q)$ for all $q \in L^2_0(\Omega)$ and $\langle \mathbf{u}(0),\mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} = \langle \boldsymbol{\phi}(0),\mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}$ for all $s \in H^{-1/2}(\Gamma)$, there exists a unique weak solution

$$(\mathbf{u},(p,\boldsymbol{\lambda})) \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \times L^2(0,T;L^2_0(\Omega)) \times L^2(0,T;H^{-1/2}(\Gamma))$$

of the weak problem (4.6). Let $\mathbf{u}$ be decomposed to $\mathbf{u}(.) = \mathbf{z}(.) + \mathbf{w}(.)$, with $\mathbf{z}(.) \in Z$ and $\mathbf{w}(.) \in Z^\perp$ for a.e. $t \in (0,T]$. Then $\mathbf{w}_t \in L^2(0,T;H^1(\Omega))$.

Proof. Note that the continuity and coercivity assumption on bilinear form $A(.,.)$ can be easily verified, since $Z \subset H^1_0(\Omega)$. It remains to prove the inf-sup condition, which can be verified identically to Theorem 4.2, since the bilinear form $B(\cdot,\cdot)$ is defined as in Theorem 4.2. □

5. Finite element approximations of parabolic saddle point problems

We now turn our attention to the error analysis of finite element approximations of such parabolic saddle point problems. The main goal is to derive "best approximation" type of estimates for semi-discrete (in space) approximations.

5.1. **Preliminaries and assumptions**

Let $V^h \subset X$ and $M^h \subset M$ be standard finite element spaces, associated to the approximation of elliptic saddle point problems (see e.g. [5, 15]) satisfying the classical approximation theory properties: There exists an integer k, and a constant C, independent of h such that, $\forall\, \mathbf{v} \in H^{m+1}(\Omega) \cap X, \quad 0 \le m \le k$ and $\forall\, q \in H^m(\Omega) \cap M, \quad 0 \le m \le k$, the following inequalities hold:

$$\begin{cases} \inf_{\mathbf{v}^h \in V^h} \|\mathbf{v} - \mathbf{v}^h\|_X \le Ch^m \|\mathbf{v}\|_{H^{m+1}(\Omega)\cap X} \\ \inf_{\mathbf{v}^h \in V^h} \|\mathbf{v} - \mathbf{v}^h\|_H \le Ch^{m+1} \|\mathbf{v}\|_{H^{m+1}(\Omega)\cap X} \\ \inf_{q^h \in M^h} \|q - q^h\|_M \le Ch^m \|q\|_{H^m(\Omega)\cap M}. \end{cases} \tag{5.1}$$

In addition, we assume that the discrete analog of the inf-sup condition holds for our choice of subspaces V^h and M^h:

$$\inf_{0\neq q^h\in M^h} \sup_{0\neq \mathbf{u}^h \in V^h} \frac{B(\mathbf{u}^h, q^h)}{\|\mathbf{u}^h\|_X \|q^h\|_M} \ge \beta, \tag{5.2}$$

with $\beta > 0$ and independent of the discretization parameter h. First we note that it is possible to construct finite element spaces satisfying (5.1) and (5.2).

Indeed, for the model problem (4.1), we may consider standard finite element spaces $V_0^h \subset X \equiv H_0^1(\Omega)$, $M^h \subset M \equiv L_0^2(\Omega)$, satisfying standard approximation properties (5.1) and the classical discrete inf-sup condition (5.2).

For model problem (4.2), we use $X = H^1(\Omega)$, $H = L^2(\Omega)$ for the velocity and $M = M_1 \times M_2$, with $M_1 = L_0^2(\Omega)$ for the pressure, and $M_2 = H^{-1/2}(\Gamma)$ for the Lagrange multiplier term respectively. Therefore, we consider $V^{h_1} \subset H^1(\Omega)$, for the velocity and $M^h = M_1^{h_1} \times M_2^{h_2} \subset L_0^2(\Omega) \times H^{1/2}(\Gamma)$ for the pressure and the boundary data respectively. We also assume that V^{h_1} and $M_1^{h_1}$ satisfy the standard approximation properties (5.1). The approximation properties of $M_2^{h_2}$, in terms of the given regularity assumptions on data as well as on the boundary regularity are more complicated (see for instance [20]), since in (4.2) the computation of the velocity and pressure is coupled to that of the boundary (stress) terms. Hence, in order to satisfy the discrete inf-sup condition (5.2), the choice of $M_2^{h_2}$ should be related to that of V^{h_1} and $M_1^{h_1}$.

To this end, we first treat the case of convex and polygonal (polyhedral in $\mathbb{R}^3$) domains. We choose $V^{h_1} \subset H^1(\Omega)$ and $M_1^{h_1} \subset L^2(\Omega)$ such that the spaces $V_0^{h_1} = V^{h_1} \cap H_0^1(\Omega)$, and $M_1^h = M_1^{h_1} \cap L_0^2(\Omega)$ satisfy (5.1) and the discrete inf-sup condition (5.2). Then, we choose $M_2^{h_2} \subset H^{1/2}(\Gamma)$ (note that h_2 might be different from h_1) such that the following approximation and inverse estimates hold (see e.g. [8] and [20]): There exists a constant $C > 0$ and an integer k, $0 \le m \le k$, such that,

$$\begin{cases} \inf_{\boldsymbol{\phi}^h \in M_2^h} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{M_2} \le Ch_2^m \|\boldsymbol{\phi}\|_{H^{m-\frac{1}{2}}(\Gamma)\cap M_2}, \qquad \forall \boldsymbol{\phi} \in H^{m-\frac{1}{2}}(\Gamma), 0 \le m \le 1, \\ \inf_{\boldsymbol{\phi}^h \in M_2^h} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{M_2} \le Ch_2^m \inf_{\hat{\mathbf{u}} \in H^m(\Omega), \hat{\mathbf{u}}|_\Gamma = \phi} \|\hat{\mathbf{u}}\|_{H^m(\Omega)}, \qquad \forall \boldsymbol{\phi} \in H^m(\Omega)|_\Gamma, 1 \le m \le k, \end{cases} \tag{5.3}$$

and

$$\|\boldsymbol{\phi}^h\|_{H^s(\Gamma)} \le Ch_2^{t-s} \|\boldsymbol{\phi}^h\|_{H^t(\Gamma)}, \quad \forall \boldsymbol{\phi}^h \in M_2^{h_2}, \quad -(1/2) \le t \le s \le (1/2).$$

Then, under the above assumptions we finally, set $h = \max\{h_1, h_2\}$ and $V^h \equiv V^{h_1}$, $M^h = M_1^{h_1} \times M_2^{h_2}$.

Remark 5.1. (1) Despite the fact that the choice of $M_2^{h_2}$ is independent of the pair (V^{h_1}, M^{h_1}) the dimension of $M_2^{h_2}$ can not exceed the one of $V^{h_1}|_\Gamma$.

(2) The verification of the discrete inf-sup condition of the above pair V^h and M^h typically requires the existence of a suitably large constant C, such that $h_2 \geq Ch_1$ (see for example [20, Proposition 5]).

(3) For the choice $h = h_1 = h_2$, and $M_2^h = V^h|_\Gamma$, we note that $M_2^h \subset C(\bar{\Gamma})$, and hence $M_2^h \subset H^1(\Gamma)$. It is clear that the first inequality of (5.3) is valid due to the approximation properties of V^h, (see [20, Lemma 13 and Proposition 14]) at least when $\boldsymbol{\phi} \in H^{m-\frac{1}{2}}(\Gamma), 0 \leq m \leq 1$. For the second inequality of (5.3), we note that for convex polyhedral domains, in general, it is not possible to define $H^s(\Gamma)$ when $s > 1$. Despite this fact, if $\mathbf{v} \in H^{s+\frac{1}{2}}(\Omega)$, with $s > 1$, it is still expected that its trace has approximation properties compatible with its regularity on Ω. We refer the reader to [20, Section 3] for a detailed discussion.

The above approximation properties easily result to approximation properties in time-space spaces. For example (see also [22, Section 2]), there exists an integer k and a constant C (independent of h) such that $\forall \mathbf{v} \in L^2(0,T;H^{m+1}(\Omega) \cap X)$, $\quad 0 \leq m \leq k$ and $\forall q \in L^2(0,T;H^m(\Omega) \cap M_1)$, $\quad 0 \leq m \leq k$, the following inequalities hold:

$$\begin{cases} \inf_{\mathbf{v}^h \in L^2(0,T;V^h)} \|\mathbf{v} - \mathbf{v}^h\|_{L^2(0,T;X)} \leq Ch_1^m \|\mathbf{v}\|_{L^2(0,T;H^{m+1}(\Omega)\cap X)} \\ \inf_{\mathbf{v}^h \in L^2(0,T;V^h)} \|\mathbf{v} - \mathbf{v}^h\|_{L^2(0,T;H)} \leq Ch_1^{m+1} \|\mathbf{v}\|_{L^2(0,T;H^{m+1}(\Omega)\cap X)} \\ \inf_{q^h \in L^2(0,T;M_1^h)} \|q - q^h\|_{L^2(0,T;M_1)} \leq Ch_1^m \|q\|_{L^2(0,T;H^m(\Omega))\cap M_1)}. \end{cases}$$

Similarly, there exists a constant $C > 0$ and an integer k, $0 \leq m \leq k$, such that,

$$\begin{cases} \inf_{\boldsymbol{\phi}^h \in L^2(0,T;M_2^h)} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{L^2(0,T;M_2)} \leq Ch_2^m \|\boldsymbol{\phi}\|_{L^2(0,T;H^{m-\frac{1}{2}}(\Gamma)\cap M_2)}, \quad \forall \boldsymbol{\phi} \in L^2(0,T;H^{m-\frac{1}{2}}(\Gamma)),\ 0 \leq m \leq 1, \\ \inf_{\boldsymbol{\phi}^h \in L^2(0,T;M_2^h)} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{L^2(0,T;M_2)} \leq Ch_2^m \inf_{\hat{\mathbf{u}} \in L^2(0,T;H^{m+1}(\Omega)), \hat{\mathbf{u}}|_\Gamma = \boldsymbol{\phi}} \|\hat{\mathbf{u}}\|_{L^2(0,T;H^m(\Omega)\cap X)}, \\ \qquad \forall \boldsymbol{\phi} \in L^2(0,T;H^m(\Omega)|_\Gamma), \qquad 1 \leq m \leq k. \end{cases}$$

We will frequently combine the approximation properties of M_1 and M_2, using the space $M = M_1 \times M_2$, by denoting $\tilde{q} \equiv (q, \boldsymbol{\phi}) \in M = M_1 \times M_2$. In this case, recall that $h = \max\{h_1, h_2\}$, $V^h = V^{h_1}$, $M^h = M_1^{h_1} \times M_2^{h_2}$. Then, the approximation property is stated as follows: $\forall \tilde{q} = (q, \boldsymbol{\phi})$ such that $q \in L^2(0,T;H^m(\Omega) \cap M_1)$, $\quad 0 \leq m \leq k$ and $\boldsymbol{\phi} \in L^2(0,T;H^{m-\frac{1}{2}}(\Gamma) \cap M_2)$, there exists a constant $C > 0$ such that,

$$\inf_{\tilde{q}^h \in L^2(0,T;M^h)} \|\tilde{q} - \tilde{q}^h\|_{L^2(0,T;M)} \leq Ch^m.$$

As before, we will abuse the notation to denote $\tilde{q}^h = q^h$, $\tilde{p}^h = p^h$, etc. To formulate the discrete analog of (2.5) we define the discretely "divergence" and/ or "divergence-free" analogs of the above finite element spaces by

$$Z^h(g) \equiv \{\mathbf{x}^h(.) \in V^h \quad \text{with } B(\mathbf{x}^h(.), q^h) = \langle \mathbf{g}(.), q^h \rangle \quad \forall q^h \in M^h \text{ for a.e } t \in (0,T]\}.$$

Note that $Z^h(0) \equiv Z^h$, where

$$Z^h = \{\mathbf{v}^h \in V^h : B(\mathbf{v}^h, q^h) = 0 \qquad \forall q^h \in M^h\}.$$

Here, the bilinear form $B(.,.)$ is defined in a similar spirit as in Section 4, i.e., it contains all boundary terms resulting from integration by parts, and related pressure terms. The semi-discrete (in space) finite element approximations of parabolic saddle point problem, can be defined as follows: Given $\mathbf{u}_0^h \in V^h$, and $\mathbf{g} \in H^1(0,T;M^*)$ we seek a discrete solution pair

$$(\mathbf{u}^h, p^h) \in H^1(0,T;V^h) \times L^2(0,T;M^h)$$

satisfying, for a.e. $t \in (0,T]$

$$\begin{cases} \langle \mathbf{u}_t^h(t), \mathbf{v}^h \rangle + \nu A(\mathbf{u}^h(t), \mathbf{v}^h) + B(\mathbf{v}^h, p^h(t)) &= 0 \qquad \forall \mathbf{v}^h \in V^h \\ B(\mathbf{u}^h(t), q^h) &= \langle \mathbf{g}(t), q^h \rangle \qquad \forall q^h \in M^h \\ (\mathbf{u}^h(0) - \mathbf{u}_0^h, v^h) &= 0 \qquad \forall \mathbf{v}^h \in V^h. \end{cases} \tag{5.4}$$

Throughout the remaining of this work, we assume that the discrete initial data $\mathbf{u}_0^h$ are chosen in a way to satisfy the standard approximation property, $\|\mathbf{u}_0 - \mathbf{u}_0^h\|_H \le Ch^m \|\mathbf{u}_0\|_{H^m(\Omega)}$.

Now, we turn our attention to the case of smooth domains (for simplicity in $\mathbb{R}^2$). In this case, it is assumed that the domain can be approximated appropriately by the corresponding finite element domain in the sense of [20, Section 3.4] or [34], and an approximation $\boldsymbol{\phi}^h$ of the boundary data $\boldsymbol{\phi}$ is actually computed. As a consequence, we may construct our subspaces on the approximated polygonal domain, as above, while the second equation of the discrete formulation (5.4) is now modified to

$$B(\mathbf{u}^h(.), q^h) = \langle \mathbf{g}^h(.), q^h \rangle \qquad \forall q^h \in M^h, \quad \text{and for a.e } t \in (0,T].$$

This case is also very important within the context of optimal control problems, where the control is applied on the Dirichlet part of the boundary, and it is an actual unknown. We view this case as the *essential data* case. Typical choices for the approximation of $\mathbf{g}$ are the L^2 projections. For instance, recall that for the boundary data $\boldsymbol{\phi}$, one may choose the $L^2(\Gamma)$ projection operator from $L^2(\Gamma)$ to $M_2^{h_2}$.

The rest of this Section is organized as follows: First, in Section 5.2, we consider (5.4), with $\mathbf{g}$ fixed (which covers the case of convex and polygonal or polyhedral domains), while the case of smooth domains and the case of essential boundary data where $\mathbf{g}$ is approximated by an element $\mathbf{g}^h$ will be treated subsequently in Section 5.3. In both cases, the role of the discrete inf-sup condition is carefully analyzed.

5.2. Preliminary best approximation estimates

The key difference between the inhomogeneous divergence and boundary data case, and the homogeneous one concerns the treatment of the inhomogeneous divergence data constraint equation. In addition, we note that the coupling between $\mathbf{u}_t$ and p creates additional difficulties, within the context of numerical approximations.

In order to obtain estimates for the differences $\mathbf{u}_t - \mathbf{u}_t^h$, and $p - p^h$, we will need to define various projections that satisfy the best approximation properties. We emphasize that we are interested in estimates at the natural energy norms, $\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;X^*)}$ and $\|p - p^h\|_{L^2(0,T;M)}$ respectively. We note that even for the homogeneous evolutionary Stokes equations the estimates on the pressure and the time-derivative are both suboptimal, due to the coupling between the time-derivative and the pressure through the incompressibility constraint.

For this purpose, we will follow the techniques of [22]. We denote by P^h the H projection $P^h : H \to V^h$ such that

$$(P^h \mathbf{v}, \mathbf{w}^h) = (\mathbf{v}, \mathbf{w}^h) \qquad \forall\, \mathbf{w}^h \in V^h$$

and by P_Z^h the "discretely divergence-free analog", $P_Z^h : H \to Z^h$ which satisfies,

$$(P_Z^h \mathbf{v}, \mathbf{z}^h) = (\mathbf{v}, \mathbf{z}^h) \qquad \forall\, \mathbf{z}^h \in Z^h.$$

We also assume that P^h satisfies stability properties in $\|.\|_X$ and $\|.\|_H$ norms, while P_Z^h satisfy the standard stability property in $\|.\|_H$. In particular, $\forall\, \mathbf{v} \in X$,

$$\|P^h \mathbf{v}\|_X \le C\|\mathbf{v}\|_X, \quad \|P_Z^h \mathbf{v}\|_H \le C\|\mathbf{v}\|_H. \tag{5.5}$$

In addition, the following inverse estimate $\|P_h \mathbf{v}\|_X \le C/h \|P_h \mathbf{v}\|_H$ will be frequently used. We also note that $\|P_Z^h \mathbf{v}\|_X \le C\|\mathbf{v}\|_X$ for all $\mathbf{v} \in X \cap Z$. Then, the following properties hold (see e.g. [22, Section 2]) for projections P^h, P_Z^h in $L^2(0,T;X)$: There exists a constant $C > 0$ independent of h, such that

$$\begin{cases} \|\mathbf{v} - P^h \mathbf{v}\|_{L^2(0,T;X)} \to 0 \text{ as } h \to 0, & \forall \mathbf{v} \in L^2(0,T;X), \\ \|\mathbf{v} - P_Z^h \mathbf{v}\|_{L^2(0,T;X)} \le C\|\mathbf{v}\|_{L^2(0,T;X)}, & \forall \mathbf{v} \in L^2(0,T; X \cap Z). \end{cases}$$

Finally, there exists constant C and an integer k such that for $0 \leq m \leq k$, the following error estimates for the projections P^h, P^h_Z hold respectively:

$$\begin{cases} \|\mathbf{v} - P^h\mathbf{v}\|_{L^2(0,T;X)} \leq Ch^m\|\mathbf{v}\|_{L^2(0,T;H^{m+1}(\Omega))} & \forall\, \mathbf{v} \in L^2(0,T;H^{m+1}(\Omega)\cap X), \\ \|\mathbf{v} - P^h\mathbf{v}\|_{L^2(0,T;H)} \leq Ch^{m+1}\|\mathbf{v}\|_{L^2(0,T;H^{m+1}(\Omega))} & \forall\, \mathbf{v} \in L^2(0,T;H^{m+1}(\Omega)\cap X), \\ \|\mathbf{v} - P^h_Z\mathbf{v}\|_{L^2(0,T;X)} \leq Ch^m\|\mathbf{v}\|_{L^2(0,T;H^{m+1}(\Omega))} & \forall\, \mathbf{v} \in L^2(0,T;H^{m+1}(\Omega)\cap X\cap Z). \end{cases}$$

In the subsequent proposition, we obtain the basic estimate, which relates the error $\mathbf{u} - \mathbf{u}^h$ to the best approximation error $\mathbf{u} - \mathbf{x}^h$, where $\mathbf{x}^h \in L^2(0,T;V^h \cap Z^h(g)) \cap H^1(0,T;V^h)$ and $\mathbf{x}^h_t \in L^2(0,T;Z^h(g_t))$. Note that if $\mathbf{x}^h \in L^2(0,T;V^h \cap Z^h(g))$ and $\mathbf{x}^h \in H^1(0,T;V^h)$ then $\mathbf{x}^h_t \in L^2(0,T;Z^h(g_t))$, since we assume that the bilinear form $B(.,.)$ does not contain time-dependent coefficients. We are now ready to obtain the preliminary "best approximation" estimate in $Z^h(g)$ for the velocity, while for the pressure the discrete inf-sup condition is needed, similar to the elliptic case (see [15] for the stationary Stokes case).

Theorem 5.2. Let $\mathbf{g}, \mathbf{u}_0$ satisfy the regularity assumptions of Theorem 2.2 and that the continuous bilinear forms $A(.,.), B(.,.)$ satisfy the coercivity conditions (2.2)-(2.3)-(2.4). Assume that $\mathbf{u}, \mathbf{u}^h$ are the solutions of the parabolic saddle point problem (1.2) and of the discrete parabolic saddle point problem (5.4) respectively. Moreover, let $A(\mathbf{z}^h, \mathbf{z}^h) \geq C\|\mathbf{z}^h\|^2_X, \quad \forall \mathbf{z}^h \in Z^h$. Suppose also that $\mathbf{u}^h_0 \in V^h$. Then, for any arbitrary $\mathbf{x}^h \in H^1(0,T;V^h) \cap L^2(0,T;Z^h(g))$, $q^h \in L^2(0,T;M^h)$ the following estimate holds:

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}^h\|^2_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} \\ &\leq C\Big(\|\mathbf{u}_0 - \mathbf{u}^h_0\|_H + \inf_{q^h \in L^2(0,T;M^h)} \|p - q^h\|_{L^2(0,T;M)} \\ &\quad + \inf_{\mathbf{x}^h \in H^1(0,T;V^h)\cap L^2(0,T;Z^h(g))} \left(\|\mathbf{u} - \mathbf{x}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{x}^h_t\|_{L^2(0,T;X^*)}\right)\Big). \end{aligned} \tag{5.6}$$

Proof. The orthogonality condition states that for almost every $t \in (0,T]$

$$\begin{cases} \langle \mathbf{u}^h_t - \mathbf{u}_t, \mathbf{v}^h\rangle + \nu A(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + B(\mathbf{v}^h, p^h - p) &= 0 \qquad \forall\, \mathbf{v}^h \in V^h \\ B(\mathbf{u}^h - \mathbf{u}, q^h) &= 0 \qquad \forall\, q^h \in M^h. \end{cases} \tag{5.7}$$

Let $\mathbf{x}^h \in L^2(0,T;V^h \cap Z^h(g)), q^h \in L^2(0,T;M^h)$ be arbitrary elements. Then, adding and subtracting $\mathbf{x}^h$ in (5.7), we obtain,

$$\begin{aligned} &\langle \mathbf{u}^h_t - \mathbf{x}^h_t, \mathbf{v}^h\rangle + \nu A(\mathbf{u}^h - \mathbf{x}^h, \mathbf{v}^h) + B(\mathbf{v}^h, p^h - p) \\ &\quad = -\langle \mathbf{x}^h_t - \mathbf{u}_t, \mathbf{v}^h\rangle - A(\mathbf{x}^h - \mathbf{u}, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in X^h. \end{aligned} \tag{5.8}$$

Note that $\mathbf{u}^h - \mathbf{x}^h \in Z^h$, and hence $B(\mathbf{u}^h - \mathbf{x}^h, p^h - p) = B(\mathbf{u}^h - \mathbf{x}^h, q^h - p)$ for any $q^h \in L^2(0,T;M^h)$. Setting $\mathbf{v}^h = \mathbf{u}^h - \mathbf{x}^h$ in (5.8) and using the coercivity inequality on Z^h we obtain,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^h - \mathbf{x}^h\|^2_H + C\|\mathbf{u}^h - \mathbf{x}^h\|^2_X \leq C\left(\|\mathbf{x}^h_t - \mathbf{u}_t\|^2_{X^*} + \|\mathbf{x}^h - \mathbf{u}\|^2_X + \|q^h - p\|^2_M\right).$$

The last inequality clearly implies estimate (5.6) by standard Grönwall Lemma. □

Proposition 5.3. Suppose that the assumptions of Theorem 5.2 and the discrete inf-sup condition (5.2) for the choice of V^h, M^h hold. Then, for any arbitrary $\mathbf{x}^h \in H^1(0,T;V^h) \cap L^2(0,T;Z^h(g))$, $q^h \in L^2(0,T;M^h)$ the following estimates hold:

$$\begin{aligned}
&\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;X^*)} + \|p - p^h\|_{L^2(0,T;M)} \\
&\le \frac{C}{h}\Big(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_H + \inf_{q^h \in L^2(0,T;M^h)} \|p - q^h\|_{L^2(0,T;M)} \\
&\quad + \inf_{\mathbf{x}^h \in H^1(0,T;V^h)\cap L^2(0,T;Z^h(g))} \big(\|\mathbf{u} - \mathbf{x}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{x}_t^h\|_{L^2(0,T;X^*)}\big)\Big), \\
&\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;Z^*)} \le C\Big(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_H + \inf_{q^h \in L^2(0,T;M^h)} \|p - q^h\|_{L^2(0,T;M)} \\
&\quad + \inf_{\mathbf{x}^h \in H^1(0,T;V^h)\cap L^2(0,T;Z^h(g))} \big(\|\mathbf{u} - \mathbf{x}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{x}_t^h\|_{L^2(0,T;X^*)}\big)\Big).
\end{aligned}$$

Here $C > 0$ denotes a constant depending only on Ω, ν, α, β.

Proof. We begin by estimating the time-derivative. First, we note that if $\mathbf{x}^h \in L^2(0,T;Z^h(g))$ then for a.e $t \in (0,T]$, we obtain $\mathbf{u}^h(.) - \mathbf{x}^h(.) \in Z^h$, and $\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.) \in Z^h$. Recall, $\|\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.)\|_{X^*} = \sup_{\mathbf{v}\in X} \frac{\langle \mathbf{u}_t^h(.) - \mathbf{x}_t^h(.), \mathbf{v}\rangle}{\|\mathbf{v}\|_X}$. Adding and subtracting $P_Z^h\mathbf{v}$, we obtain,

$$\|\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.)\|_{X^*} = \sup_{\mathbf{v}\in X} \frac{\langle \mathbf{u}_t^h(.) - \mathbf{x}_t^h(.), \mathbf{v} - P_Z^h\mathbf{v}\rangle + \langle \mathbf{u}_t^h(.) - \mathbf{x}_t^h(.), P_Z^h\mathbf{v}\rangle}{\|\mathbf{v}\|_X}.$$

Note that since $\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.) \in Z^h$ the definition of the projection P_Z^h implies that $\langle \mathbf{u}_t^h(.) - \mathbf{x}_t^h(.), \mathbf{v} - P_Z^h\mathbf{v}\rangle = 0$. For the remaining term, from the orthogonality condition (5.8), we obtain

$$\begin{aligned}
\|\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.)\|_{X^*} \quad \le \quad & \sup_{\mathbf{v}\in X}\Big(\frac{\nu|A(\mathbf{u}^h(.) - \mathbf{x}^h(.), P_Z^h\mathbf{v})|}{\|\mathbf{v}\|_X} + \frac{|B(P_Z^h\mathbf{v}, p^h - p)|}{\|\mathbf{v}\|_X} \\
& + \frac{|\langle \mathbf{u}_t(.) - \mathbf{x}_t^h(.), P_Z^h\mathbf{v}\rangle|}{\|\mathbf{v}\|_X} + \frac{|A(\mathbf{x}^h - \mathbf{u}, P_Z^h\mathbf{v})|}{\|\mathbf{v}\|_X}\Big).
\end{aligned} \tag{5.9}$$

Observe that $B(P_Z^h\mathbf{v}, p^h - p) = B(P_Z^h\mathbf{v}, q^h - p)$. Then, using the inverse estimate $\|P_Z^h\mathbf{v}\|_X \le \frac{C}{h}\|P_Z^h\mathbf{v}\|_H$, squaring both sides, and integrating with respect to time we obtain the desired estimate. Once, we have shown an estimate on the time derivative on $\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;X^*)}$ the estimate on the $p - p^h$ term follows directly from the discrete inf-sup condition (5.2). Indeed, note that

$$\begin{aligned}
B(\mathbf{v}^h, p^h - q^h) &= B(\mathbf{v}^h, p^h - p) - B(\mathbf{v}^h, p - q^h) \\
&= -\langle \mathbf{u}_t^h - \mathbf{u}_t, \mathbf{v}^h\rangle - \nu A(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) - B(\mathbf{v}^h, p - q^h) \\
&\le C(\|\mathbf{u}_t^h - \mathbf{u}_t\|_{X^*}\|\mathbf{v}^h\|_X + \|\mathbf{u}^h - \mathbf{u}\|_X\|\mathbf{v}^h\|_X + \|\mathbf{v}^h\|_X\|p - q^h\|_M).
\end{aligned}$$

Hence, dividing by $\|\mathbf{v}^h\|_X$, taking the supremum over V^h, using the discrete inf-sup (5.2) and standard algebra we derive the estimate on $p - p^h$ term. At the second equality, we have used the orthogonality condition. For the last estimate, we note that $\|\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.)\|_{Z^*} = \sup_{\mathbf{v}\in Z} \frac{\langle \mathbf{u}_t^h(.) - \mathbf{x}_t^h(.), \mathbf{v}\rangle}{\|\mathbf{v}\|_X}$ and hence working identically as above we obtain the analogue of (5.9),

$$\begin{aligned}
\|\mathbf{u}_t^h(.) - \mathbf{x}_t^h(.)\|_{Z^*} \quad \le \quad & \sup_{\mathbf{v}\in Z}\Big(\frac{\nu|A(\mathbf{u}^h(.) - \mathbf{x}^h(.), P_Z^h\mathbf{v})|}{\|\mathbf{v}\|_X} + \frac{|B(P_Z^h\mathbf{v}, p^h - p)|}{\|\mathbf{v}\|_X} \\
& + \frac{|\langle \mathbf{u}_t(.) - \mathbf{x}_t^h(.), P_Z^h\mathbf{v}\rangle}{\|\mathbf{v}\|_X} + \frac{|A(\mathbf{x}^h - \mathbf{u}, P_Z^h\mathbf{v})|}{\|\mathbf{v}\|_X}\Big).
\end{aligned} \tag{5.10}$$

The estimate now follows using similar arguments and the stability estimate $\|P_Z \mathbf{v}\|_X \le C\|\mathbf{v}\|_X$ for all $\mathbf{v} \in Z \cap X$. □

Remark 5.4. The structure of the estimate (5.10) on $\|\mathbf{u}_t^h - \mathbf{x}_t^h\|_{L^2(0,T;Z^*)}$ is similar to the estimate of the velocity in $L^2(0,T;X)$ and hence it leads to similar rates. In particular, we have shown the following best-approximation and almost symmetric error estimate:

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;Z^*)} \\
&\le C\Big(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_H + \inf_{q^h \in L^2(0,T;M^h)} \|p - q^h\|_{L^2(0,T;M)} \\
&\quad + \inf_{\mathbf{x}^h \in H^1(0,T;V^h) \cap L^2(0,T;Z^h(g))} \big(\|\mathbf{u} - \mathbf{x}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{x}_t^h\|_{L^2(0,T;X^*)}\big)\Big).
\end{aligned}$$

However, the above estimate is not useful since we cannot apply the inf-sup condition to recover an estimate for the pressure and the Lagrange multiplier term. Indeed, despite the fact that the estimate in $L^2(0,T;Z^*)$ is of the same order to the velocity one, it seems unlikely to obtain a better rate in $L^2(0,T;X^*)$ norm simply because $B(\mathbf{v}, q) = 0, \forall \mathbf{v} \in Z, q \in M$. The reduced rate for the estimate on the time derivative and the pressure is even present for the homogeneous evolutionary Stokes equations, and it is due to the coupling between the time-derivative and the pressure. The rate reduction was caused because we have used a suboptimal bound for the time-derivative in $L^2(0,T;X^*)$ norm by applying an inverse estimate. The inverse estimate was necessary since we cannot assume the stability property $\|P_Z \mathbf{v}\|_X \le C\|\mathbf{v}\|_X$ for any $\mathbf{v} \in X$, but only if $\mathbf{v} \in X \cap Z$. On the other hand, the definition of the projection, implies the stability in the H norm.

Next, we will relate the approximation properties on $H^1(0,T;V^h) \cap L^2(0,T;Z^h(g))$ to standard best approximation properties on $H^1(0,T;V^h)$. This is necessary in order to quantify the error estimate. The discrete inf-sup condition will be used similar to the elliptic case (see [15, Theorem 1.1, pp 114]). We note that we will use the enhanced regularity $\mathbf{w}_t \in H^1(0,T;X)$, in order to obtain estimate on the time derivative via the inf-sup condition. Recall, that we have shown that $\mathbf{u}_t \in L^2(0,T;H)$ and if the decomposition $\mathbf{u}(.) = \mathbf{w}(.) + \mathbf{z}(.)$, with $\mathbf{w}(.) \in Z^\perp$, $\mathbf{z}(.) \in Z$ holds for a.e. $t \in (0,T]$ then $\mathbf{w}_t \in L^2(0,T;X)$.

Lemma 5.5. Let the assumptions of Theorem 5.2 hold. In addition, suppose that the finite element subspaces V^h, M^h satisfy the discrete inf-sup condition (5.2). Then, for any $\mathbf{v}^h \in H^1(0,T;V^h)$, $q_h \in L^2(0,T;M^h)$ the following estimates hold:

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} + \\
&\quad + h\left(\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;X^*)} + \|p - p^h\|_{L^2(0,T;M)}\right) \\
&\le C\Big(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_H + \inf_{\mathbf{v}^h \in H^1(0,T;V^h)} (\|\mathbf{u} - \mathbf{v}^h\|_{L^2(0,T;X)} + \|\mathbf{u}_t - \mathbf{v}_t^h\|_{L^2(0,T;X^*)}) \\
&\quad + \inf_{\mathbf{v}^h \in H^1(0,T;V^h)} \|\mathbf{w}_t - \mathbf{v}_t^h\|_{L^2(0,T;X)} + \|p - q^h\|_{L^2(0,T;M)}\Big).
\end{aligned}$$

Here we denote by $\mathbf{u}(.) = \mathbf{w}(.) + \mathbf{z}(.)$, where $\mathbf{z}(.) \in Z$, $\mathbf{w}(.) \in Z^\perp$ for a.e. $t \in (0,T]$. If in addition, $\mathbf{u} \in H^1(0,T;H^{m+1}(\Omega) \cap X)$, and $p \in L^2(0,T;H^m(\Omega) \cap M)$ then there exists a constant C such that

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} \\
&\quad + h\left(\|\mathbf{u}_t - \mathbf{u}_t^h\|_{L^2(0,T;X^*)} + \|p - p^h\|_{L^2(0,T;M)}\right) \le Ch^m.
\end{aligned}$$

Proof. If the finite element subspaces V^h, M^h satisfy the discrete inf-sup condition (5.2) then using the Banach-Babuška-Nečas Lemma (see e.g. [5, 15, 26]) for a.e $t \in (0,T]$ there exists $\mathbf{w}^h(.) \in (Z^h)^\perp$ (depending on q) such that

$$B(\mathbf{w}^h(.), q^h) = B(\mathbf{u}(.) - \mathbf{v}^h(.), q^h), \quad \forall \mathbf{v}^h \in L^2(0,T;V^h) \quad \text{and} \quad \forall q^h \in M^h. \tag{5.11}$$

In addition, the discrete inf-sup implies $\|\mathbf{w}^h(.)\|_X \le C\|\mathbf{u}(.) - \mathbf{v}^h(.)\|_X$. Set $\mathbf{x}^h(.) = \mathbf{w}^h(.) + \mathbf{v}^h(.)$ and note that $\mathbf{x}^h(.) \in L^2(0,T;V^h)$, and $\mathbf{x}^h(.) \in Z^h(g)$, since

$$B(\mathbf{x}^h(.), q^h) = B(\mathbf{w}^h(.) + \mathbf{v}^h(.), q^h) = B(\mathbf{u}(.), q^h) = \langle \mathbf{g}(.), q^h \rangle,$$

due to (5.11). Therefore, $\|\mathbf{u}(.) - \mathbf{x}^h(.)\|_X \le \|\mathbf{u}(.) - \mathbf{v}^h(.)\|_X + \|\mathbf{w}^h(.)\|_X \le C\|\mathbf{u}(.) - \mathbf{v}^h(.)\|_X$. The last inequality implies that $\|\mathbf{u} - \mathbf{x}^h\|_{L^2(0,T;X)} \le C\|\mathbf{u} - \mathbf{v}^h\|_{L^2(0,T;X)}$, for any arbitrary $\mathbf{v}^h \in L^2(0,T;V^h)$. Suppose now that $\mathbf{v}^h \in H^1(0,T;V^h)$. The unique decomposition of $\mathbf{u}(.) = \mathbf{w}(.) + \mathbf{z}(.)$, with $\mathbf{w}(.) \in Z^\perp$, $\mathbf{z}(.) \in Z$ implies that $B(\mathbf{w}^h(.), q^h) = B(\mathbf{u}(.) - \mathbf{v}^h(.), q^h) = B(\mathbf{w}(.) - \mathbf{v}^h(.), q^h)$. Hence, since $\mathbf{w}_t, \mathbf{v}^h_t \in L^2(0,T;X)$, differentiating with respect to time, we deduce for a.e. $t \in (0,T]$,

$$B(\mathbf{w}^h_t(.), q^h) = B(\mathbf{w}_t(.) - \mathbf{v}^h_t(.), q^h).$$

The discrete inf-sup condition, and the Banach-Babuška-Nečas Lemma (see e.g. [5, 15, 26]) imply that there exists $\tilde{\mathbf{w}}^h \in (Z^h)^\perp$ such that $B(\tilde{\mathbf{w}}^h(.), q^h) = B(\mathbf{w}_t(.) - \mathbf{v}^h_t(.), q^h)$. Therefore, we obtain

$$B(\tilde{\mathbf{w}}^h(.), q^h) = B(\mathbf{w}_t(.) - \mathbf{v}^h_t(.), q^h) = B(\mathbf{w}^h_t(.), q^h). \tag{5.12}$$

The discrete inf-sup implies $\|\tilde{\mathbf{w}}^h(.)\|_X \le C\|\mathbf{w}_t(.) - \mathbf{v}^h_t(.)\|_X$. Note that since $\tilde{\mathbf{w}}^h \in L^2(0,T;(Z^h)^\perp)$ we also deduce from (5.12) that $\mathbf{w}^h_t \in L^2(0,T;(Z^h)^\perp)$, and $\|\mathbf{w}^h_t\|_{L^2(0,T;X)} \le C\|\mathbf{w}_t(.) - \mathbf{v}^h_t(.)\|_X$. Hence using triangle inequality and the previous estimates we obtain the following estimate,

$$\begin{aligned}&\|\mathbf{u}_t - \mathbf{x}^h_t\|_{L^2(0,T;X^*)} \le C(\|\mathbf{u}_t - \mathbf{v}^h_t\|_{L^2(0,T;X^*)} + \|\mathbf{w}^h_t\|_{L^2(0,T;X^*)})\\ &\le C\big(\|\mathbf{u}_t - \mathbf{v}^h_t\|_{L^2(0,T;X^*)} + \|\mathbf{w}_t - \mathbf{v}^h_t\|_{L^2(0,T;X)}\big),\end{aligned}$$

since $\|\mathbf{w}^h_t\|_{L^2(0,T;X^*)}$ is estimated by $\|\mathbf{w}^h_t\|_{L^2(0,T;X)} \le C\|\mathbf{w}_t - \mathbf{v}^h_t\|_{L^2(0,T;X)}$. The desired estimate easily follows by substituting $\mathbf{x}^h$ into the estimate of Theorem 5.2 and Proposition 5.3 (see also Remark 5.4). The last estimate of the Theorem now follows by the approximation properties. □

We are ready to state the main best approximation type error estimates for the semi-discrete approximations of problems (4.1) and (4.2). First, we simply point out that we may recast problem (4.1) into the discrete parabolic saddle point framework of (5.4), for $V^h \subset H^1_0(\Omega)$, $M^h \subset L^2_0(\Omega)$ and by defining the bilinear forms similar to Section 4. Then it is evident that the assumptions of Theorem 5.1, and Proposition 5.2 hold, and hence the estimates of Theorem 5.1, Proposition 5.2, and Lemma 5.3 hold. In particular, we have the following result:

Corollary 5.6. Suppose that $\mathbf{u}_0 \in H^1_0(\Omega)$, and $g \in H^1(0,T;L^2_0(\Omega))$ with $g(0) = \text{div}\,\mathbf{u}(0)$. Let V^h, M^h satisfy the approximation properties of Section 5.1, $\mathbf{u}^h_0 \in V^h$ an approximation of $\mathbf{u}_0$. Then, there exists a constant $C > 0$ independent of h, such that,

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} \to 0.$$

If in addition, $\mathbf{u} \in H^1(0,T;H^{m+1}(\Omega) \cap X)$, and $p \in L^2(0,T;H^m(\Omega) \cap M)$ then there exists a constant C such that

$$\begin{aligned}&\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)}\\ &\quad + h\left(\|\mathbf{u}_t - \mathbf{u}^h_t\|_{L^2(0,T;X^*)} + \|p - p^h\|_{L^2(0,T;M)}\right) \le Ch^m.\end{aligned}$$

Now, for the model problem (4.2), we choose the spaces $V^h = V^{h_1}$ for the velocity and $M^h = M^{h_1}_1 \times M^{h_2}_2$ for the pressure and the boundary data term, satisfying the assumptions of Section 5.1. Then, denoting by

$h=\max\{h_1,h_2\}$, the discrete analog of (4.2), is to seek $\mathbf{u}^h\in H^1(0,T;V^{h_1})$, $p^h\in L^2(0,T;M^{h_1})$, and a Lagrange multiplier $\boldsymbol{\lambda}^h\in L^2(0,T;M_2^{h_2})$ such that for a.e. $t\in(0,T]$, and for all $\mathbf{v}\in V^{h_1}$ and $(q^h,\mathbf{s}^h)\in M_1^{h_1}\times M_2^{h_2}$,

$$\begin{cases} \langle \mathbf{u}_t^h(t),\mathbf{v}^h\rangle+\nu a(\mathbf{u}^h(t),\mathbf{v}^h)-(p^h(t),\ \mathrm{div}\mathbf{v}^h) & -\langle \boldsymbol{\lambda}^h(t),\mathbf{v}^h\rangle_{(H^{-1/2}(\Gamma),H^{1/2}(\Gamma))}=0\\ (\ \mathrm{div}\ \mathbf{u}^h(t),q^h) & =0\\ \langle \mathbf{u}^h(t),\mathbf{s}^h\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))} & =\langle \boldsymbol{\phi}(t),\mathbf{s}^h\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}\\ (\mathbf{u}^h(0),\mathbf{v}^h) & =(\mathbf{u}_0^h,\mathbf{v}^h). \end{cases}$$

which is written equivalently,

$$\begin{cases} \langle \mathbf{u}_t^h,\mathbf{v}^h\rangle+\nu A(\mathbf{u}^h,\mathbf{v}^h)+B(\mathbf{v}^h,(p^h,\boldsymbol{\lambda}^h)) & =0\\ B(\mathbf{u}^h,(q^h,\mathbf{s}^h)) & =-\langle \boldsymbol{\phi},\mathbf{s}^h\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}\\ (\mathbf{u}^h(0),\mathbf{v}^h) & =(\mathbf{u}_0^h,\mathbf{v}^h). \end{cases}$$

Here, similar to Section 4, we denote by $A(\mathbf{u},\mathbf{v})=a(.,.)$, $\forall\,\mathbf{u},\mathbf{v}\in H^1(\Omega)$ and by $B(\mathbf{u},(q,\mathbf{s}))=b(\mathbf{u},q)-\langle \mathbf{u},\mathbf{s}\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}$, $\forall\,(q,\mathbf{s})\in L_0^2(\Omega)\times H^{-1/2}(\Gamma)$. Then, a direct application of Theorem 5.2, Proposition 5.3, and Lemma 5.5, imply the following estimates:

Corollary 5.7. Suppose that $\mathbf{u}_0\in H^1(\Omega)$, $\phi\in H^1(0,T;H^{1/2}(\Gamma))$ given data with $\langle \boldsymbol{\phi}(0),s\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}=\langle \mathbf{u}(0),s\rangle_{(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))}$, for all $s\in H^{-1}(\Gamma)$, and $\mathbf{u}_0\in H^1(\Omega)$, with div$\mathbf{u}_0$=0. Let V^h,M^h satisfy the approximation properties of Section 5.1, and $\mathbf{u}_0^h\in V^h$ an approximation of $\mathbf{u}_0$. Then, there exists a constant $C>0$ independent of h, such that,

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}^h\|_{L^\infty(0,T;L^2(\Omega)))}+\|\mathbf{u}-\mathbf{u}^h\|_{L^2(0,T;H^1(\Omega))}\\ &\quad +h(\|p-p^h\|_{L^2(0,T;L_0^2(\Omega))}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{L^2(0,T;H^{-1/2}(\Gamma))})\to 0. \end{aligned}$$

If in addition, $\mathbf{u}\in H^1(0,T;H^{m+1}(\Omega)\cap X)$, and $p\in L^2(0,T;H^m(\Omega)\cap L_0^2(\Omega))\times L^2(0,T;H^{m-\frac{1}{2}}(\Gamma)\cap M_2)$ then there exists a constant C such that

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}^h\|_{L^\infty(0,T;H)}+\|\mathbf{u}-\mathbf{u}^h\|_{L^2(0,T;X)}\\ &\quad +h\left(\|\mathbf{u}_t-\mathbf{u}_t^h\|_{L^2(0,T;X^*)}+\|(p,\lambda)-(p^h,\boldsymbol{\lambda}^h)\|_{L^2(0,T;M)}\right)\le Ch^m. \end{aligned}$$

Remark 5.8. The choice of finite element spaces $(V^{h_1},M_1^{h_1})$, and $M_2^{h_2}$ is related through the verification of the coercivity of bilinear form $A(.,.)$ and the inf-sup condition (5.2) (see Remark 5.1). For a detailed discussion we refer the reader to [20] (see also references within).

5.3. **Treating essential inhomogeneous data**

Our next goal is to derive estimates in terms of projections when $\mathbf{g}$ is approximated by a function $\mathbf{g}^h$. This case arises when we consider essential inhomogeneous Dirichlet boundary data and/or a curved boundary.

The discrete parabolic saddle point problem is defined as follows: Given $\mathbf{u}_0^h\in V^h$, and $\mathbf{g}^h\in H^1(0,T;M^h)$ we seek a discrete solution pair

$$(\mathbf{u}^h,p^h)\in H^1(0,T;V^h)\times L^2(0,T;M^h),$$

satisfying, for almost every $t\in(0,T]$

$$\begin{cases} \langle \mathbf{u}_t^h(t),\mathbf{v}^h\rangle+\nu A(\mathbf{u}^h(t),\mathbf{v}^h)+B(\mathbf{v}^h,p^h(t)) & =0 & \forall\,\mathbf{v}^h\in V^h\\ B(\mathbf{u}^h(t),q^h) & =\langle \mathbf{g}^h(t),q^h\rangle & \forall\,q^h\in M^h\\ (\mathbf{u}^h(0)-\mathbf{u}_0^h,v^h) & =0 & \forall\,\mathbf{v}^h\in V^h, \end{cases} \tag{5.13}$$

where $\mathbf{g}^h$ is a suitable approximation of $\mathbf{g}$.

Theorem 5.9. Let $\mathbf{g}, \mathbf{u}_0$ satisfy the regularity assumptions of Theorem 2.2 and that the continuous bilinear forms $A(.,.), B(.,.)$ satisfy the coercivity conditions (2.2)-(2.3)-(2.4). Moreover, let $V^h \subset X$, $M^h \subset M$ be finite element subspaces satisfying the discrete inf-sup condition (5.2) and standard approximation properties. Suppose also that $\mathbf{u}_0^h \in V^h$, $\mathbf{g}^h \in H^1(0,T;M^h)$, and $\mathbf{u}_t \in L^2(0,T;X)$ Then, for any arbitrary $\mathbf{x}^h \in H^1(0,T;V^h), q^h \in L^2(0,T;M^h)$, the following estimate holds:

$$\|\mathbf{u}-\mathbf{u}^h\|_{L^2(0,T;X)} \leq C\Big(\|\mathbf{u}_0-\mathbf{u}_0^h\|_H + \|\mathbf{u}-P^h\mathbf{u}\|_{H^1(0,T;X)} + \|\mathbf{g}-\mathbf{g}^h\|_{H^1(0,T;M^*)} + \|p-q^h\|_{L^2(0,T;M)}\Big). \quad (5.14)$$

If in addition, the approximation $\mathbf{g}^h$ of $\mathbf{g}$ satisfies the estimate $\|\mathbf{g}-\mathbf{g}^h\|_{H^1(0,T;M^*)} \leq Ch^m$ and $\mathbf{u} \in H^1(0,T;H^{m+1}(\Omega)\cap X)$, $p \in L^2(0,T;H^m(\Omega)\cap M)$ then, $\|\mathbf{u}-\mathbf{u}^h\|_{L^2(0,T;X)} \leq Ch^m$.

Proof. The orthogonality condition reads as follows: For almost every $t \in (0,T]$

$$\begin{cases} \langle \mathbf{u}_t^h - \mathbf{u}_t, \mathbf{v}^h\rangle + \nu A(\mathbf{u}^h-\mathbf{u},\mathbf{v}^h) + B(\mathbf{v}^h, p^h-p) &= 0 \qquad \forall\, \mathbf{v}^h \in V^h \\ B(\mathbf{u}^h-\mathbf{u}, q^h) &= \langle \mathbf{g}^h-\mathbf{g}, q^h\rangle \qquad \forall\, q^h \in M^h. \end{cases} \quad (5.15)$$

We split the error as follows: $\mathbf{u}^h - \mathbf{u} = (\mathbf{u}^h - P^h\mathbf{u}) + (P^h\mathbf{u}-\mathbf{u})$, where $P^h\mathbf{u} : H \to V^h$, denotes the standard orthogonal projection. First we prove an estimate for $\mathbf{u}^h - P^h\mathbf{u}$ term. For that purpose, we note that for a.e. $t \in (0,T]$, $\mathbf{u}^h(t) - P^h\mathbf{u}(t) \in V^h$, and we use the decomposition $\mathbf{u}^h(t) - P^h\mathbf{u}(t) = \mathbf{w}^h(t) + \mathbf{z}^h(t)$ where $\mathbf{w}^h(t) \in {Z^h}^{\perp}, \mathbf{z}^h(t) \in Z^h$. We can bound the $\mathbf{w}^h(t)$ term using the discrete inf-sup condition (5.2) and the orthogonality condition (5.15), (dropping the t notation),

$$\begin{aligned} \|\mathbf{w}^h\|_X &\leq C \sup_{q^h\in M^h} \frac{|B(\mathbf{w}^h,q^h)|}{\|q^h\|_M} \leq C \sup_{q^h\in M^h} \frac{|B(\mathbf{u}^h - P^h\mathbf{u} - \mathbf{z}^h, q^h)|}{\|q^h\|_M} \\ &\leq C \sup_{q^h\in M^h} \frac{|B(\mathbf{u}^h - P^h\mathbf{u}, q^h)|}{\|q^h\|_M} \\ &\leq C \sup_{q^h\in M^h} \frac{|B(\mathbf{u}^h - \mathbf{u}, q^h) + B(\mathbf{u} - P^h\mathbf{u}, q^h)|}{\|q^h\|_M} \\ &\leq C \left(\sup_{q^h\in M^h} \frac{|\langle \mathbf{g}^h - \mathbf{g}, q^h\rangle|}{\|q^h\|_M} + \sup_{q^h\in M^h} \frac{|B(\mathbf{u} - P^h\mathbf{u}, q^h)|}{\|q^h\|_M} \right) \\ &\leq C\left(\|\mathbf{g}-\mathbf{g}^h\|_{M^*} + \|\mathbf{u}-P^h\mathbf{u}\|_X\right). \end{aligned} \quad (5.16)$$

Therefore, after integrating (5.16),

$$\|\mathbf{w}^h\|^2_{L^2(0,T;X)} \leq C\left(\|\mathbf{g}-\mathbf{g}^h\|^2_{L^2(0,T;M^*)} + \|\mathbf{u}-P^h\mathbf{u}\|^2_{L^2(0,T;X)}\right). \quad (5.17)$$

Note also that $\mathbf{u}_t^h, \mathbf{w}_t^h \in L^2(0,T;V^h)$ and that $\mathbf{u}_t \in L^2(0,T;X)$. Hence, the orthogonality condition implies after differentiation with respect to time,

$$\frac{d}{dt}B(\mathbf{u}^h(t) - \mathbf{u}(t), q^h) = \langle \mathbf{g}_t^h(t) - \mathbf{g}_t(t), q^h\rangle \quad \forall\, q^h \in M^h.$$

Therefore, following exactly the same steps as above the inf-sup condition implies for a.e. $t \in (0,T]$, (dropping the t notation)

$$\begin{aligned}
\|\mathbf{w}_t^h\|_X &\leq C \sup_{q^h \in M^h} \frac{|B(\mathbf{w}_t^h, q^h)|}{\|q^h\|_M} \leq C \sup_{q^h \in M^h} = C \sup_{q^h \in M^h} \frac{|\frac{d}{dt} B(\mathbf{w}^h, q^h)|}{\|q^h\|_M} \\
&\leq C \sup_{q^h \in M^h} \frac{|\frac{d}{dt} B(\mathbf{u}^h - P^h\mathbf{u} - \mathbf{z}^h, q^h)|}{\|q^h\|_M} \\
&\leq C \sup_{q^h \in M^h} \frac{|\frac{d}{dt} B(\mathbf{u}^h - P^h\mathbf{u}, q^h)|}{\|q^h\|_M} \\
&\leq C \sup_{q^h \in M^h} \frac{|\frac{d}{dt} (B(\mathbf{u}^h - \mathbf{u}, q^h) + B(\mathbf{u} - P^h\mathbf{u}, q^h))|}{\|q^h\|_M} \\
&\leq C\Big(\|\mathbf{g}_t^h - \mathbf{g}_t\|_{M^*} + \|\mathbf{u}_t - P^h\mathbf{u}_t\|_X\Big)
\end{aligned}$$

which clearly implies,

$$\|\mathbf{w}_t^h\|^2_{L^2(0,T;X)} \leq C\Big(\|\mathbf{g}_t^h - \mathbf{g}_t\|^2_{L^2(0,T;M^*)} + \|\mathbf{u}_t - P^h\mathbf{u}_t\|^2_{L^2(0,T;X)}\Big).$$

It remains to bound the $\mathbf{z}^h$ term. Note that the regularity assumptions on $\mathbf{g}$, $\mathbf{u}_0$ imply that $\mathbf{u}_t \in L^2(0,T;H)$ (see Theorem 2.2). Adding and subtracting appropriate terms at the orthogonality condition (5.15) we obtain,

$$\begin{aligned}
&\langle \mathbf{u}_t^h - P^h\mathbf{u}_t, \mathbf{v}^h\rangle + \nu A(\mathbf{u}^h - P^h\mathbf{u}, \mathbf{v}^h) + B(\mathbf{v}^h, p^h - p) \\
&\quad = -\langle P^h\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}^h\rangle - \nu A(P^h\mathbf{u} - \mathbf{u}, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in V^h.
\end{aligned} \tag{5.18}$$

Note that by the definition of projection P^h we obtain that $\langle P^h\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}^h\rangle = 0$. Using once more the decomposition $\mathbf{u}^h - P^h\mathbf{u} = \mathbf{w}^h + \mathbf{z}^h$, we obtain from (5.18) for all $\mathbf{v}^h \in V^h$ and for a.e. $t \in (0,T]$,

$$\begin{aligned}
&\langle \mathbf{z}_t^h, \mathbf{v}^h\rangle + \nu A(\mathbf{z}^h, \mathbf{v}^h) + B(\mathbf{v}^h, p^h - p) \\
&\quad = -\nu A(P^h\mathbf{u} - \mathbf{u}, \mathbf{v}^h) - \langle \mathbf{w}_t^h, \mathbf{v}^h\rangle - \nu A(\mathbf{w}^h, \mathbf{v}^h).
\end{aligned} \tag{5.19}$$

Note that $B(\mathbf{v}^h, p^h - p) = B(\mathbf{v}^h, p^h - q^h) + B(\mathbf{v}^h, q^h - p)$, so (5.19) can be rewritten as follows: For all $\mathbf{v}^h \in V^h$, and for a.e. $t \in (0,T]$,

$$\begin{aligned}
&\langle \mathbf{z}_t^h, \mathbf{v}^h\rangle + \nu A(\mathbf{z}^h, \mathbf{v}^h) + B(\mathbf{v}^h, p^h - q^h) \\
&\quad = -\nu A(P^h\mathbf{u} - \mathbf{u}, \mathbf{v}^h) - B(\mathbf{v}^h, q^h - p) - \langle \mathbf{w}_t^h, \mathbf{v}^h\rangle - A(\mathbf{w}^h, \mathbf{v}^h).
\end{aligned} \tag{5.20}$$

Set $\mathbf{v}^h = \mathbf{z}^h \in Z^h$ in (5.20) and note that $B(\mathbf{z}^h, p^h - q^h) = 0$. Therefore, using the coercivity condition of $A(\cdot,\cdot)$ on Z^h, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{z}^h\|_H^2 + \nu\alpha\|\mathbf{z}^h\|_X^2 \leq C\Big(\|P^h\mathbf{u} - \mathbf{u}\|_X\|\mathbf{z}^h\|_X + \|\mathbf{z}^h\|_X\|q^h - p\|_M + \big(\|\mathbf{w}^h\|_X + \|\mathbf{w}_t^h\|_{X^*}\big)\|\mathbf{z}^h\|_X\Big).$$

Using Cauchy's inequality with appropriate constants,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{z}^h\|_H^2 + \frac{\nu\alpha}{2}\|\mathbf{z}^h\|_X^2 \leq C\left(\|P^h\mathbf{u} - \mathbf{u}\|_X^2 + \|q^h - p\|_M^2 + \|\mathbf{w}^h\|_X^2 + \|\mathbf{w}_t^h\|_{X^*}^2\right).$$

Clearly, the above inequality together with previous estimate for $\mathbf{w}^h$, leads to the desired estimate for the $\mathbf{z}^h$ term, i.e,

$$\|\mathbf{z}^h\|_{L^2(0,T;X)} \leq C\left(\|P^h\mathbf{u}-\mathbf{u}\|_{H^1(0,T;X)} + \|q^h-p\|_{L^2(0,T;M)} + \|\mathbf{g}^h-\mathbf{g}\|_{H^1(0,T;M^*)}\right).$$

Here, we have used the bound $\|P^h\mathbf{u}-\mathbf{u}\|_{L^2(0,T;X^*)} \leq C\|P^h\mathbf{u}-\mathbf{u}\|_{L^2(0,T;X)}$ (since the later norm also appears in the estimate of $\mathbf{w}^h_t$). Combining estimates for $\mathbf{z}^h, \mathbf{w}^h$, we finally arrive at:

$$\|\mathbf{u}^h - P^h\mathbf{u}\|_{L^2(0,T;X)} \leq C\big(\|P^h\mathbf{u}-\mathbf{u}\|_{H^1(0,T;X)} + \|q^h-p\|_{L^2(0,TM)} + \|\mathbf{g}^h-\mathbf{g}\|_{H^1(0,T;M^*)}\big).$$

Using the triangle inequality we finally arrive at the desired estimate (5.14). □

Next we focus on the estimate on the time derivative term. We emphasize that we derive an estimate on the natural dual norm $L^2(0,T;X^*)$, instead of $L^2(0,T;Z^*)$. To achieve this, the enhanced regularity on $\mathbf{u}_t \in L^2(0,T;X)$ will be used. Theorem 2.2 implies that $\mathbf{u}_t \in L^2(0,T;H)$ and that $\mathbf{w}_t \in L^2(0,T;X)$ where $\mathbf{w}$ is defined through the decomposition $\mathbf{u} = \mathbf{w}+\mathbf{z}$, with $\mathbf{w} \in (Z^h)^\perp$ and $\mathbf{z} \in Z^h$. The enhanced regularity appears to be necessary in order to utilize the discrete inf-sup condition and control $\mathbf{w}_t$. In addition, we will employ the discrete divergence free projection P^h_Z, and will assume stability properties in the X norm.

Theorem 5.10. Suppose that the assumptions as of Theorem 5.9 hold. Then, there exists a constant $C>0$ independent of h such that the following estimate holds:

$$\|\mathbf{u}^h_t - \mathbf{u}_t\|_{L^2(0,T;H)} + \|p-p^h\|_{L^2(0,T;M)} \leq \frac{C}{h}\Big(\|\mathbf{u}^h-\mathbf{u}\|_{L^2(0,T;X)} + \|P^h\mathbf{u}-\mathbf{u}\|_{H^1(0,T;X)} + \|q^h-p\|_{L^2(0,T;M)} + \|\mathbf{g}^h-\mathbf{g}\|_{H^1(0,T;M^*)}\Big).$$

Proof. We focus on $\mathbf{u}^h_t - P^h\mathbf{u}_t$ term. Recall that $\mathbf{u}_t \in L^2(0,T;H)$ and hence $P^h\mathbf{u}_t(.)$ is well defined for a.e. $t\in(0,T]$. Then the estimate follows from triangle inequality. Note that we use the decomposition $\mathbf{u}^h(t) - P^h\mathbf{u}(t) = \mathbf{w}^h(t) + \mathbf{z}^h(t)$ as before. Working identically to Theorem 5.9 we may derive an estimate for $\mathbf{w}^h_t$, i.e

$$\|\mathbf{w}^h_t\|_{L^2(0,T;X)} \leq C\left(\|\mathbf{g}^h_t - \mathbf{g}_t\|_{L^2(0,T;M^*)} + \|\mathbf{u}_t - P^h\mathbf{u}_t\|_X\right).$$

Using (5.18), we obtain for $\mathbf{v}\in X$,

$$\begin{aligned}\langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}\rangle &= \langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}-\mathbf{v}^h\rangle + \langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}^h\rangle \\ &= \langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}-\mathbf{v}^h\rangle - \Big(\nu A(\mathbf{u}^h - P^h\mathbf{u}, \mathbf{v}^h) + B(\mathbf{v}^h, p^h-q^h) \\ &\qquad + \langle P^h\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}^h\rangle + \nu A(P^h\mathbf{u}-\mathbf{u}, \mathbf{v}^h) + B(\mathbf{v}^h, q^h-p)\Big). \end{aligned} \tag{5.21}$$

Setting $\mathbf{v}^h = P^h_Z\mathbf{v}$ into (5.21) and noting that $\langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}-\mathbf{v}^h\rangle = (\mathbf{w}^h_t + \mathbf{z}^h_t, \mathbf{v}-P^h_Z\mathbf{v})$ and $(\mathbf{z}^h_t, \mathbf{v}-P^h_Z\mathbf{v}) = 0$ due to the definition of P^h_Z, we obtain

$$\begin{aligned}\langle \mathbf{u}^h_t - P^h\mathbf{u}_t, \mathbf{v}\rangle \quad &\leq \quad C\Big(\|\mathbf{w}^h_t\|_H\|\mathbf{v}-P^h_Z\mathbf{v}\|_H + \|\mathbf{u}^h - P^h\mathbf{u}\|_X\|P^h_Z\mathbf{v}\|_X \\ &\qquad + (\|P^h\mathbf{u}_t - \mathbf{u}_t\|_{X^*} + \|P^h\mathbf{u}-\mathbf{u}\|_X + \|q^h-p\|_M)\|P^h_Z\mathbf{v}\|_X\Big).\end{aligned} \tag{5.22}$$

Using the inverse estimate $\|P^h_Z\mathbf{v}\|_X \leq \frac{C}{h}\|P^h_Z\mathbf{v}\|_H$ the stability of the projection P^h_Z in H and taking the supremum in (5.22) over all $\mathbf{v}\in H$ with $\|\mathbf{v}\|_H = 1$ and using the stability properties of the projection P^h_Z we arrive at:

$$\|\mathbf{u}^h_t - P^h\mathbf{u}_t\|_H \leq C\Big(\|\mathbf{w}^h_t\|_X + \frac{1}{h}\left(\|\mathbf{u}^h - P^h\mathbf{u}\|_X + \|q^h-p\|_M + \|P^h\mathbf{u}_t - \mathbf{u}_t\|_{X^*} + \|P^h\mathbf{u}-\mathbf{u}\|_X\right)\Big). \tag{5.23}$$

Taking the square, integrating with respect to time, using the estimate on $\|\mathbf{w}_t\|_{L^2(0,T;X}$, and estimates from Theorem 5.9, we obtain from (5.23) the desired estimate on the time derivative. For the pressure term note that

$$\begin{aligned}B(\mathbf{v}^h, p^h - q^h) &= B(\mathbf{v}^h, p^h - p) - B(\mathbf{v}^h, p - q^h)\\ &= -\langle \mathbf{u}_t^h - \mathbf{u}_t, \mathbf{v}^h\rangle - \nu A(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) - B(\mathbf{v}^h, p - q^h)\\ &\le C\left(\|\mathbf{u}_t^h - \mathbf{u}_t\|_{X^*}\|\mathbf{v}^h\|_X + \|\mathbf{u}^h - \mathbf{u}\|_X\|\mathbf{v}^h\|_X + \|\mathbf{v}^h\|_X\|p - q^h\|_M\right).\end{aligned}$$

The discrete inf-sup condition, together with Theorem 5.9 and using the estimate on $\|\mathbf{u}_t - \mathbf{u}_t^h\|_H$ to estimate $\|\mathbf{u}_t - \mathbf{u}_t^h\|_{X^*}$, we establish the estimate on the pressure term. □

Combining Theorems 5.9, 5.10, we obtain an estimate for smooth solutions.

Proposition 5.11. Let the assumption of Theorem 5.9 hold. If in addition, the approximation $\mathbf{g}^h$ of $\mathbf{g}$ satisfies the estimate $\|\mathbf{g} - \mathbf{g}^h\|_{H^1(0,T;M^*)} \le Ch^m$ and $\mathbf{u} \in H^1(0, T; H^{m+1}(\Omega) \cap X)$, $p \in L^2(0, T; H^m(\Omega) \cap M)$ then,

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} + h\left(\|\mathbf{u}_t - \mathbf{u}^h\|_{L^2(0,T;H)} + \|p - p^h\|_{L^2(0,T;M)}\right) \le Ch^m.$$

Proof. The proof follows directly from Theorems 5.9, 5.10, and the approximation properties of the related finite element spaces and of the projection P_Z^h. □

Remark 5.12. Similar to Section 5.2, the reduced rate for the pressure is justified since the coupling of the time-derivative and pressure requires the use of estimates on the time-derivative in $L^2(0, T; X^*)$ norm, and it is present even in case of homogeneous data, i.e., $\mathbf{g} = 0$. Theorems 5.9, and 5.10 provide estimates, which originate to various best approximation properties of suitable projections, under the additional regularity assumption on $\mathbf{u}_t \in L^2(0, T; X)$. Once again this is due to the coupling of the pressure, and the time-derivative constraint as well as the inhomogeneous divergence data. A careful inspection of the proof reveals that it is not possible to directly estimate $\mathbf{w}_t^h \in L^2(0, T; X^*)$ without invoking the discrete inf-sup condition, which actually estimates $\mathbf{w}_t^h$ in the bigger norm $L^2(0, T; X)$.

Remark 5.13. Compared to [21], our theory treats also inhomogeneous essential boundary problems.

Finally we apply the results of Theorems 5.9,5.10, and Proposition 5.11, to obtain error estimates for the semi-discrete approximations of problem (4.2). Recall that, we have chosen the spaces $V^h = V^{h_1}$ for the velocity and $M^h = M_1^{h_1} \times M_2^{h_2}$ for the pressure and the boundary data term, satisfying the assumptions of Section 5.1. Then, denoting by $h = \max\{h_1, h_2\}$, the semi-discrete analog of (4.2), is written now as follows: Given $\boldsymbol{\phi}^h \in H^1(0, T; M_2^h)$, we seek $\mathbf{u}^h \in H^1(0, T; V^{h_1})$, $p^h \in L^2(0, T; M^{h_1})$, and a Lagrange multiplier $\boldsymbol{\lambda}^h \in L^2(0, T; M_2^{h_2})$ such that for a.e. $t \in (0, T]$, and for all $\mathbf{v} \in V^{h_1}$ and $(q^h, \mathbf{s}^h) \in M_1^{h_1} \times M_2^{h_2}$,

$$\left\{\begin{aligned} \langle \mathbf{u}_t^h, \mathbf{v}^h\rangle + \nu A(\mathbf{u}^h, \mathbf{v}^h) + B(\mathbf{v}^h, (p^h, \boldsymbol{\lambda}^h)) &= 0\\ B(\mathbf{u}^h, (q^h, \mathbf{s}^h)) &= -\langle \boldsymbol{\phi}^h, \mathbf{s}^h\rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma))}\\ (\mathbf{u}^h(0), \mathbf{v}^h) &= (\mathbf{u}_0^h, \mathbf{v}^h).\end{aligned}\right. \tag{5.24}$$

Here, similar to Section 4, we denote by $A(\mathbf{u}, \mathbf{v}) = a(.,.)$, $\forall\, \mathbf{u}, \mathbf{v} \in H^1(\Omega)$ and by $B(\mathbf{u}, (q, \mathbf{s})) = b(\mathbf{u}, q) - \langle \mathbf{u}, \mathbf{s}\rangle_{(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))}$, $\forall\, (q, \mathbf{s}) \in L_0^2(\Omega) \times H^{-1/2}(\Gamma)$. For the approximation of data, we assume that for a.e. $t \in (0, T]$, $\boldsymbol{\phi}^h(t) = P_{L^2(\Gamma)}^h \boldsymbol{\phi}(t)$, where $P_{L^2(\Gamma)}^h$ denotes the standard $L^2(\Gamma)$ projection of the boundary data, i.e., $(P_{L^2(\Gamma)}^h \boldsymbol{\phi}, \mathbf{s}^h)_{L^2(\Gamma)} = (\boldsymbol{\phi}, \mathbf{s}^h)_{L^2(\Gamma)}$, $\forall\, \mathbf{s}^h \in M_2^h$. We note that due to [20, Proposition 14], Theorems 5.9, 5.10 and Proposition 5.11, we deduce $\|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{H^1(0,T;H^{1/2}(\Gamma))} \to 0$, and if more regularity is available, then $\|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_{H^1(0,T;H^{1/2}(\Gamma))} \le Ch^m \inf_{\hat{\mathbf{u}} \in H^1(0,T;H^{m+1}(\Omega)), \hat{\mathbf{u}}|_\Gamma = \boldsymbol{\phi}} \|\hat{\mathbf{u}}\|_{H^1(0,T;H^{m+1}(\Omega))}$. Hence, we arrive at the following estimate.

Corollary 5.14. Suppose that the assumptions of Theorem 5.8 hold, and let $\boldsymbol{\phi} \in H^1(0,T;H^{1/2}(\Gamma))$. Then, there exists a constant $C > 0$ independent of h, such that,

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;L^2(\Omega)))} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;H^1(\Omega))} \\ +h(\|p - p^h\|_{L^2(0,T;L^2_0(\Omega))} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{L^2(0,T;H^{-1/2}(\Gamma))}) \to 0.$$

If in addition, $\mathbf{u} \in H^1(0,T;H^{m+1}(\Omega) \cap X)$, and $p \in L^2(0,T;H^m(\Omega) \cap L^2_0(\Omega)) \times L^2(0,T;H^{m-\frac{1}{2}}(\Gamma) \cap M_2)$ then there exists a constant C such that

$$\|\mathbf{u} - \mathbf{u}^h\|_{L^\infty(0,T;H)} + \|\mathbf{u} - \mathbf{u}^h\|_{L^2(0,T;X)} \\ +h\left(\|\mathbf{u}_t - \mathbf{u}^h_t\|_{L^2(0,T;X^*)} + \|(p,\lambda) - (p^h,\boldsymbol{\lambda}^h)\|_{L^2(0,T;M)}\right) \leq Ch^m.$$

Remark 5.15. The discrete weak formaluation for the evolutionary Stokes equations with inhomogeneous Dirichlet boundary data (5.24) resembles the classical saddle point formulation for elliptic problems. However, the computation of the velocity and the pressure is coupled to the computation of the Lagrange multiplier. A more practical choice of finite element spaces is considered in [7], allowing the decoupling of the computation of the velocity and the pressure from the computation of the Lagrange multiplier.

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