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NUMERICAL ANALYSIS OF HIGH ORDER TIME STEPPING SCHEMES FOR A PREDATOR-PREY SYSTEM

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This paper is dedicated to Prof. Max Gunzburger on the occasion of his 75th birthday.

Abstract. Finite element discretisations of the modified predator-prey system are examined. In particular, fully-discrete schemes based on the discontinuous Galerkin time stepping approach for the temporal discretisation combined with standard finite elements for the spatial discretisation are considered. Stability estimates are derived for schemes of arbitrary order and error estimates that maintain a symmetric structure are proved.

Key words. Predator-Prey systems, discontinuous Galerkin schemes, stability properties, error estimates.

1. Introduction

The scope of this work is the stability and error analysis of fully-discrete schemes for the predator-prey system. The predator-prey system under consideration, consists of two coupled parabolic pdes, i.e.,

$$\begin{cases} u_t - d_1\Delta u - u(1-|u|) + vh(au) &= 0 \qquad \text{in } (0,T)\times\Omega \\ \dfrac{\partial u}{\partial n} &= 0 \qquad \text{on } (0,T)\times\Gamma \\ u(0,x) &= u_0 \qquad \text{in } \Omega, \end{cases}$$

$$\begin{cases} v_t - d_2\Delta v - bvh(au) + cv &= 0 \qquad \text{in } (0,T)\times\Omega \\ \dfrac{\partial v}{\partial n} &= 0 \qquad \text{on } (0,T)\times\Gamma \\ v(0,x) &= v_0 \qquad \text{in } \Omega. \end{cases}$$

Here, $d_1, d_2 > 0$ denote diffusion constants, with $d_1 \neq d_2$, $b, c, a > 0$ are positive parameters and $\Omega \subset \mathbb{R}^3$ is a bounded domain with suitably smooth boundary Γ. The initial data are denoted by u_0, v_0 respectively. Our analysis covers two of the most commonly used functional responses $h(.)$, the Holling type II and type III functionals, defined by:

$$h(au) = \frac{au}{1+a|u|} \qquad \text{or} \qquad h(au) = \frac{au^2}{1+au^2},$$

respectively, and involves the nonlinear reaction function $u(1-|u|)$. These type of functional responses were proposed in [30, 31]. For an overview of the role of such functional responses in these models we refer the reader to [32]. The above system is often called the "modified predator-prey system". Our goal is to establish stability and error estimates for fully-discrete schemes of arbitrary order. The schemes under consideration are based on a discontinuous Galerkin -in time- approach combined with standard conforming finite elements in space. Such schemes are known to maintain the structural properties of the underlying pde model, in the sense, that it

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possible to prove stability estimates under minimal regularity assumptions. Indeed, given initial data $u_0, v_0 \in L^2(\Omega)$, in Section 3, we prove the following estimate:

$$\|u_h\|_{W(0,T)} + \|v_h\|_{W(0,T)} \leq C \left(\|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)}\right),$$

where u_h, v_h denote the fully-discrete approximations of weak solutions u, v, and $\|.\|_{W(0,T)} := \|.\|_{L^\infty[0,T;L^2(\Omega)]} + \|.\|_{L^2[0,T;H^1(\Omega)]}$ denotes the natural energy norm associated to the discontinuous Galerkin approximation in time. The key difficulty involves the derivation of estimates for higher order schemes at the $\|.\|_{L^\infty[0,T;L^2(\Omega)]}$ norm in presence of the nonlinear coupling. We note that the above stability estimate under minimal regularity assumptions, is the key step in order to develop a-priori error estimates. In addition, for $u, v \in W(0,T) \cap L^\infty[0,T;L^\infty(\Omega)]$ we establish the fully-discrete analog of the classical Céa Lemma which in this context, is an estimate of the form,

$$\|u - u_h\|_{W(0,T)} + \|v - v_h\|_{W(0,T)} \leq C \left(\|u - P_h^{loc} u\|_{W(0,T)} + \|v - P_h^{loc}\|_{W(0,T)}\right).$$

Here, P_h^{loc} denotes the standard projection associated to discontinuous Galerkin schemes that exhibits best approximation properties in terms of the available regularity of the solution.

We emphasise that this estimate is also derived under minimal regularity assumptions on data and it is applicable when high order schemes are employed. Such estimate demonstrates that the error of the fully-discrete scheme will convergence at the maximal rate that the chosen approximation spaces and the regularity of the solution will allow. The estimate is valid for a suitable choice of the temporal discretisation parameter τ in terms of the parameters a, b, c, d_1, d_2, but it can be chosen independent of the size of the spatial discretisation parameter h. Our work uses ideas and techniques of [9, 8], developed for proving estimates at arbitrary time points, combined with a suitable "boot-strap" argument that decouples the two involved pdes without imposing additional regularity and / or very stringent conditions between the discretisation parameters and the physical parameters of our system. In addition the estimate is derived without making assumptions regarding point-wise space-time stability on u_h, v_h. To our best knowledge these estimates are new.

Various issues related to numerical analysis and computational efficiency of discretisation schemes for systems of reaction-diffusion pdes that resemble the predator-prey system have been considered before (see e.g. [5, 6, 7, 14, 23, 24, 25, 27, 28, 33, 34, 35, 36, 40]). In particular, we point out [27] where a-priori estimates are established for the fully-discrete approximation of the predator-prey system using semi-implicit Euler scheme in time combined with conforming finite elements in space and [14] where the analysis of first-order in time implicit-symplectic method is considered.

Stability analysis and a-priori error estimates involving the Brusselator nonlinear coupling structure are presented in the work of [7]. A finite volume scheme for the Brusselator model with cross diffusion is considered in [35], while an alternative direction (ADI) extrapolated Crank-Nicolson orthogonal collocation algorithm is analysed in [23]. Both papers include various informative computational results and are applicable in other nonlinear reaction diffusion systems. In [40], implicit-explicit schemes for various reaction diffusion systems arising in pattern formation are considered, while analytical and computational aspects of moving grid time-stepping schemes are studied in [36]. In [24], optimal error bounds of a fully-discrete scheme based on the implicit-explicit Euler method combined with a lumped surface finite element method for the spatial discretisation of reaction-diffusion equations on closed compact surfaces are proved. Other works that are related to somewhat different but still relevant couplings, such as the forced-Fisher equation, the FitzHugh-Nagumo system, and other parameter dependent systems can be found in [28, 5, 6, 13, 25, 33]. In [6, 7] various computational examples related to high order discontinuous Galerkin schemes are presented. In [34] an error analysis of exponential time differencing schemes for the epitaxial growth model is presented. An overview of various properties of exponential time differencing schemes can be found in [15].

There is an abundant literature concerning numerical analysis of general semilinear parabolic pdes: we refer the reader to [42] and the references therein. The discontinuous (in time) Galerkin technique is analyzed in the works [3, 12, 16, 18, 19, 20, 21, 37, 39] for linear and semilinear problems. Other approaches including implicit-explicit multistep methods and linear implicit schemes can be found in [1, 2] respectively. An otherview of results regarding a posteriori error estimation of reaction-diffusion systems are presented in [17] (see also references therein).

2. Preliminaries

We use standard notation for Hilbert spaces $L^2(\Omega)$, $H^s(\Omega)$, $0 < s \in \mathbb{R}$, $H^1_0(\Omega) := \{w \in H^1(\Omega) : w|_\Gamma = 0\}$, related norms and inner products (see e.g. [22, Chapter 5]). We denote by $\langle .,. \rangle$ the duality pairing between $H^1(\Omega)$ and its dual $H^1(\Omega)^*$. For any Banach space X, we denote by $L^p[0,T;X], L^\infty[0,T;X]$ the time-space spaces, endowed with norms,

$$\|w\|_{L^p[0,T;X]} = \Big(\int_0^T \|w\|_X^p dt \Big)^{\frac{1}{p}}, \quad \|w\|_{L^\infty[0,T;X]} = \text{esssup}_{t\in[0,T]} \|w\|_X.$$

The set of all continuous functions w:$[0,T]\to X$, is denoted by $C[0,T;X]$, with norm $\|w\|_{C[0,T;X]} = \max_{t\in[0,T]} \|w(t)\|_X$. We refer the reader to [38, 22] for the definition of spaces $H^s[0,T;X]$. In particular, we will use the space $H^1[0,T;X]$ with norm,

$$\|w\|_{H^1[0,T;X]} = \Big(\int_0^T \|w\|_X^2 dt \Big)^{\frac{1}{2}} + \Big(\int_0^T \|w_t\|_X^2 dt \Big)^{\frac{1}{2}}.$$

We define the (weak) solution space by $W(0,T) = L^2[0,T;H^1(\Omega)] \cap L^\infty[0,T;L^2(\Omega)]$ with norm

$$\|w\|^2_{W(0,T)} = \|w\|^2_{L^2[0,T;H^1(\Omega)]} + \|w\|^2_{L^\infty[0,T;W(\Omega)]}.$$

We use the standard bilinear form

$$a(y,w) = \int_\Omega \nabla y \nabla w dx \qquad \forall\, y, w \in H^1(\Omega),$$

which satisfies the classical coercivity and continuity properties,

$$a(y,y) = \|\nabla y\|^2_{L^2(\Omega)} \qquad a(y,w) \leq \|y\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}, \qquad \forall y, v \in H^1(\Omega).$$

We will also use the following classical inequalities:
Gagliardo-Nirenberg-Ladyzhenskaya inequalities (GNL): For all $w \in H^1(\Omega)$,

$$\|w\|_{L_4(\Omega)} \leq C \|w\|^{1/2}_{L_2(\Omega)} \|w\|^{1/2}_{H^1(\Omega)}, \text{ for } d = 2,$$
$$\|w\|_{L_4(\Omega)} \leq C \|w\|^{1/4}_{L_2(\Omega)} \|w\|^{3/4}_{H^1(\Omega)}, \text{ for } d = 3,$$

where $C > 0$ depends only on Ω and it is independent of w.
Young's inequality: For any $\epsilon > 0$, $a, b \geq 0$, $p, q > 1$ it holds

$$ab \leq \frac{\epsilon^p}{p} a^p + \frac{1}{q\epsilon^q} b^q, \quad \text{where} \quad 1/p + 1/q = 1.$$

Throughout this work we will denote $C > 0$ positive constants that might change at each occurance but depend only on the domain.

The weak formulation is stated as follows: Given data $u_0, v_0 \in L^2(\Omega)$, we seek $u, v \in W(0,T)$ such that for every $w \in L^2[0,T;H^1(\Omega)] \cap H^1[0,T;H^1(\Omega)^*]$,

$$(1) \qquad \begin{cases} (u(T), w(T)) + \int_0^T (-\langle u, w_s\rangle + d_1 a(u,w) - \langle u(1-|u|), w\rangle)\, ds \\ \quad + \int_0^T \langle vh(au), w\rangle ds = (u(0), w(0)), \end{cases}$$

$$(2) \qquad \begin{cases} (v(T), w(T)) + \int_0^T (-\langle v, w_s\rangle + d_2 a(v,w) - b\langle vh(au), w\rangle + c\langle v, w\rangle)\, ds \\ \quad = (v(0), w(0)). \end{cases}$$

where we denote by $w_s := \frac{d}{ds} w$ the derivative with respect to time.

Existence and uniqueness of (1)-(2) is well known and can be derived by standard techniques. Indeed, given initial data $u_0, v_0 \in L^2(\Omega)$, there exists a unique solution $u, v \in W(0,T)$ of (1) and (2). For the derivation of the symmetric error estimate we will assume that $u, v \in W(0,T) \cap L^\infty[0,T;L^\infty(\Omega)]$ (see for instance [26, 41] for a related results). Note that we will work solely with the modified model, that uses the term $\int_{t^{n-1}}^{t^n} \langle u(1-|u|), w\rangle ds$. Otherwise, it is well known that the solution of the predator-prey system, as well as its discrete approximations, may blow up in finite time and further restrictions on the size of the data will be necessary in order to guarantee stability estimates in the prescribed time interval.

3. The fully-discrete scheme and its stability properties

Approximations of (1) are constructed on a partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0,T]$ and we denote the length of each subinterval by $\tau_n = t^n - t^{n-1}$, $n = 1, ..., N$. On each interval of the form $(t^{n-1}, t^n]$ a subspace X_h of $H^1(\Omega)$ is specified. We seek approximate solutions who belong to the space

$$\mathcal{X}_h = \{w_h \in L^2[0,T;H^1(\Omega)] : w_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k[t^{n-1}, t^n; X_h]\}.$$

Here $\mathcal{P}_k[t^{n-1}, t^n; X_h]$ denotes the space of polynomials of degree k or less having values in X_h. Notice that, by convention, the functions of $\mathcal{X}_h$ are left continuous with right limits and hence will subsequently write w_-^n for $w_h(t^n)$, and w_+^n for $\lim_{s\to 0^+} w_h(t^n + s) := w_h(t_+^n)$. The above notation will be also used with functions u, v, w as well as for the corresponding errors $e_u := u_h - u, e_v := v_h - v$. Due to a well known embedding result, it is assumed that the exact solution, u, v, is in $C[0,T;L^2(\Omega)]$ so that the jump in the error at t^n, denoted by $[e_u^n]$, $[e_v^n]$ is given by $[e_u^n] = [u^n] = u_+^n - u^n$ and $[e_v^n] = v_+^n - v^n$. The finite element space X_h satisfies classical approximation theory results: If $l \geq 1$ denote the degree of polynomials (in space), and $w \in H^{l+1}(\Omega)$, then,

$$(3) \quad \inf_{w_h \in X_h} \|w - w_h\|_{H^s(\Omega)} \leq C h^{l+1-s} \|w\|_{H^{l+1}(\Omega)}, \qquad 0 \leq l \leq \ell, \qquad s = -1, 0, 1.$$

We also assume that X_h satisfies the inverse estimate: $\|w_h\|_{H^1(\Omega)} \leq \frac{C}{h} \|w_h\|_{L^2(\Omega)}$, $\forall\, w_h \in X_h$. Throughout this work we employ a quasi-uniform partition in time, i.e., there exists $\theta \in (0,1]$ such that $\theta\tau \leq \min_{n=1,...N} \tau_n$, where $\tau = \max_{n=1,...,N} \tau_n$.

We are ready to define the fully-discrete formulation of (1)-(2). We seek $u_h, v_h \in \mathcal{X}_h$ such that, for all $n = 1, .., N$ and $w_h \in \mathcal{X}_h$,

$$\text{(4)} \qquad \langle u_{h-}^n, w_{h-}^n \rangle + \int_{t^{n-1}}^{t^n} \Big(- \langle u_h, w_{hs} \rangle + d_1 a(u_h, w_h) + \langle v_h h(au_h), w_h \rangle \Big) ds$$
$$= \langle u_{h-}^{n-1}, w_{h+}^{n-1} \rangle + \int_{t^{n-1}}^{t^n} \langle u_h(1 - |u_h|), w_h \rangle ds,$$

$$\text{(5)} \qquad \langle v_{h-}^n, w_{h-}^n \rangle + \int_{t^{n-1}}^{t^n} \Big(- \langle v_h, w_{hs} \rangle + d_2 a(v_h, w_h) + c\langle v_h, w_h \rangle \Big) ds$$
$$= \langle v_{h-}^{n-1}, w_{h+}^{n-1} \rangle + b \int_{t^{n-1}}^{t^n} \langle v_h h(au_h), w_h \rangle ds.$$

Integrating by parts in time, we obtain the following equivalent formulation:

$$\text{(6)} \qquad \int_{t^{n-1}}^{t^n} \Big(\langle u_{hs}, w_h \rangle + d_1 a(u_h, w_h) + \langle v_h h(au_h), w_h \rangle \Big) ds$$
$$+ \langle [u_h^{n-1}], w_{h+}^{n-1} \rangle = \int_{t^{n-1}}^{t^n} \langle u_h(1 - |u_h|), w_h \rangle ds,$$

$$\text{(7)} \qquad \int_{t^{n-1}}^{t^n} \Big(\langle v_{hs}, w_h \rangle + d_2 a(v_h, w_h) + c\langle v_h, w_h \rangle \Big) ds$$
$$+ \langle [v_h^{n-1}], w_{h+}^{n-1} \rangle = b \int_{t^{n-1}}^{t^n} \langle v_h h(au_h), w_h \rangle ds.$$

We will frequently alternate between these two formulations. We are ready to proceed with the main stability estimate, which is also the key ingredient in the development of the error estimate. The stability estimate is derived under minimal regularity assumptions on data. Apart from the nonlinear coupling and the presence of different diffusion constants, another key technical challenge, when dealing with high order schemes, is the derivation of the $\|.\|_{L_\infty(0,T;L^2(\Omega))}$ stability estimates. Here, the structure of the response functional h plays important role.

Theorem 3.1. Suppose that $u_0, v_0 \in L^2(\Omega)$ and let a, b, c, d_1, d_2 positive constants. In addition, suppose that τ satisfy $\tau \leq \begin{cases} \min\{\frac{1}{C_k(b+d_2)}, \frac{1}{C_k(b+c)}\} & \text{when } \frac{c}{2} < b + d_2, \\ \frac{1}{C_k(b+c)} & \text{when } \frac{c}{2} \geq b + d_2, \end{cases}$ where C_k is a constant depending only on the polynomial degree k of the temporal discretisation. Then, the following estimates hold:

$$\|v_h^N\|^2_{L^2(\Omega)} + \sum_{i=1}^N \|[v_h^{i-1}]\|^2_{L^2(\Omega)} + 2d_2\int_0^T \|v_h\|^2_{H^1(\Omega)}ds + c\int_0^T \|v_h\|^2_{L^2(\Omega)}ds \tag{8}$$

$$\leq C_{st,1} =: \begin{cases} \|v_0\|^2_{L^2(\Omega)} & \text{when } \frac{c}{2} \geq b + d_2, \\ Ce^{C_k(b+d_2)T}\|v_h^0\|^2_{L^2(\Omega)} & \text{when } \frac{c}{2} < b + d_2, \end{cases}$$

$$\|v_h(t)\|^2_{L^\infty[t^{n-1},t^n,L^2(\Omega)]} \leq C_k C_{st,1}\|v_h^0\|^2_{L^2(\Omega)}, \tag{9}$$

$$\|u_h^N\|^2_{L^2(\Omega)} + \sum_{i=1}^N \|[u_h^{i-1}]\|^2_{L^2(\Omega)} + d_1\int_0^T \|\nabla u_h\|^2_{L^2(\Omega)}ds + \int_0^T \|u_h\|^3_{L^3(\Omega)}ds \tag{10}$$

$$\leq C\left(\|u_h^0\|^2_{L^2(\Omega)} + C_{d_1}T + \frac{C_{st,1}}{c}\right) := C_{st,2},$$

$$\|u_h(t)\|^2_{L^\infty[0,T,L^2(\Omega)]} \leq C_k C_{st,2}. \tag{11}$$

where C is an algebraic constant and $C_{d_1} := \frac{1}{4}\left(2 + \frac{4}{3}d_1\right)^3$.

Proof. We select $w_h = v_h$ into (7), and since $|h(au_h)| \leq 1$, we easily deduce,

$$\frac{1}{2}\|v_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[v_h^{n-1}]\|^2_{L^2(\Omega)} + d_2\int_{t^{n-1}}^{t^n} \|v_h\|^2_{H^1(\Omega)}ds \tag{12}$$

$$+ c\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)}ds \leq \frac{1}{2}\|v_h^{n-1}\|^2_{L^2(\Omega)} + (b + d_2)\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)}ds,$$

where we have added $d_2\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)}ds$ term at both sides. Let us point out that if $\frac{c}{2} \geq (b + d_2)$ then summing the above inequalities, we obtain that

$$\|v_h^N\|^2_{L^2(\Omega)}+\sum_{i=1}^N \|[v_h^{i-1}]\|^2_{L^2(\Omega)}+2d_2\int_0^T \|v_h\|^2_{H^1(\Omega)}ds+c\int_0^T \|v_h\|^2_{L^2(\Omega)}ds \leq \|v_h^0\|^2_{L^2(\Omega)}.$$

We focus our attention to the case where $\frac{c}{2} < b + d_2$. We need to bound the last term of (12). We will use a technique proposed in [8]. First, we fix $z_h \in X_h$ and $t \in (t^{n-1}, t^n)$. Then, set $w_h(s) = \phi(s)z_h$, where $\phi \in \mathcal{P}_k(t^{n-1}, t^n)$ satisfies,

$$\phi(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n} \phi\psi = \int_{t^{n-1}}^{t} \psi, \qquad \psi \in \mathcal{P}_{k-1}(t^{n-1}, t^n).$$

Note that [8, Lemma 3.2], implies that $\|\phi\|_{L^\infty(t^{n-1},t^n)} \leq C_k$, where constant C_k is independent of the fixed t and depends on the polynomial degree of the temporal discretisation. Therefore, the above construction with the particular choice of w_h, allow us to integrate,

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} (v_{hs}, w_h)ds + (v_{h+}^{n-1} - v_{h-}^{n-1}, w_{h+}^{n-1}) \\
&= \int_{t^{n-1}}^{t} (v_{hs}, z_h)ds + (v_{h+}^{n-1} - v_{h-}^{n-1}, \phi(t^{n-1})z_h) \\
&= (v_h(t) - v_{h+}^{n-1}, z_h) + (v_{h+}^{n-1} - v_{h-}^{n-1}, z_h) = (v_h(t) - v_{h-}^{n-1}, z_h).
\end{aligned}$$

Therefore, selecting $w_h := \phi(s)z_h$ into (7) and substituting the above equality, (7) yields,

$$(v_h(t) - v_{h-}^{n-1}, z_h) = -\int_{t^{n-1}}^{t^n} (d_2 a(v_h, z_h\phi(s)) + c(v_h, z_h\phi(s)))\, ds$$
$$+ b\int_{t^{n-1}}^{t^n} (v_h h(au_h), z_h\phi(s))ds.$$

Hence, using the fact the $\|\phi\|_{L^\infty(t^{n-1},t^n)} \leq C_k$, and since $z_h \in X_h$ is independent of the integration variable, we deduce,

$$\Big|(v_h(t) - v_{h-}^{n-1}, z_h)\Big| \leq C_k\Big(d_2\|z_h\|_{H^1(\Omega)}\int_{t^{n-1}}^{t^n}\|v_h\|_{H^1(\Omega)}ds$$
$$+ (b+c)\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|v_h\|_{L^2(\Omega)}ds\Big),$$

where we have used Hölder's inequality, and the fact that $|h(au_h)| \leq 1$. Thus, using Hölder's inequality with respect to time to bound $\int_{t^{n-1}}^{t^n}\|v_h(t)\|_{L^2(\Omega)}dt \leq \tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ and $\int_{t^{n-1}}^{t^n}\|v_h(t)\|_{H^1(\Omega)}dt \leq \tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$, we obtain,

$$(v_h(t) - v_{h-}^{n-1}, z_h) \leq C_k\Big(d_2\|z_h\|_{H^1(\Omega)}\tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$$
$$+ (b+c)\|z_h\|_{L^2(\Omega)}\tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big).$$

For the fixed t, set $z_h = v_h(t)$ and integrate the resulting inequality with respect to time,

$$\int_{t^{n-1}}^{t^n}\|v_h(t)\|^2_{L^2(\Omega)}dt \leq \|v_{h-}^{n-1}\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|v_h(t)\|_{L^2(\Omega)}dt$$
$$+ C_k d_2\left(\int_{t^{n-1}}^{t^n}\|v_h(t)\|_{H^1(\Omega)}dt\right)\tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$$
$$+ C_k(b+c)\left(\int_{t^{n-1}}^{t^n}\|v_h(t)\|_{L^2(\Omega)}dt\right)\tau_n^{1/2}\|v_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}.$$

Using Hölder's inequality once more, Young's inequality to hide the $\|v_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ on the left, and for all τ_n such that $C_k(b+c)\tau_n \leq 1/4$, we obtain,

$$\int_{t^{n-1}}^{t^n}\|v_h(t)\|^2_{L^2(\Omega)}dt \leq C_k\tau_n\left(\|v_{h-}^{n-1}\|^2_{L^2(\Omega)} + d_2\int_{t^{n-1}}^{t^n}\|v_h\|^2_{H^1(\Omega)}dt\right). \tag{13}$$

Substituting (13) into (12), it yields,

$$\frac{1}{2}\|v_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[v_h^{n-1}]\|^2_{L^2(\Omega)} + d_2\int_{t^{n-1}}^{t^n}\|v_h\|^2_{H^1(\Omega)}ds + c\int_{t^{n-1}}^{t^n}\|v_h\|^2_{L^2(\Omega)}ds \tag{14}$$
$$\leq \left(\frac{1}{2} + (b+d_2)C_k\tau_n\right)\|v_h^{n-1}\|^2_{L^2(\Omega)} + (b+d_2)C_k\tau_n d_2\int_{t^{n-1}}^{t^n}\|v_h\|^2_{H^1(\Omega)}dt.$$

Select τ_n such that $(b+d_2)C_k\tau_n d_2 \leq d_2/2$ and $(b+d_2)C_k\tau_n < 1$ to deduce (8) upon using the standard discrete Gronwall Lemma. The estimate at partition point $\|v_h^m\|^2_{L_2(\Omega)}$ follows by summing from 1 to m where $m \in \{1, .., N\}$. Returning back to (13), using an inverse estimate in time, and the estimate in $\|.\|_{L^2[0,T;H^1(\Omega)]}$

and the estimate at partition points, we obtain the estimate (9). We proceed with estimate on u_h. Now, we set $w_h = u_h$ in (6),

$$(15)\quad \frac{1}{2}\|u_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[u_h^{n-1}]\|^2_{L^2(\Omega)} + d_1 \int_{t^{n-1}}^{t^n} \|u_h\|^2_{H^1(\Omega)} ds + \int_{t^{n-1}}^{t^n} \|u_h\|^3_{L^3(\Omega)} ds$$
$$\leq \frac{1}{2}\|u_h^{n-1}\|^2_{L^2(\Omega)} + (1+d_1)\int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)} ds + \int_{t^{n-1}}^{t^n} (v_h h(au_h), u_h) ds$$
$$\leq \frac{1}{2}\|u_h^{n-1}\|^2_{L^2(\Omega)} + (\frac{3}{2}+d_1)\int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)} ds + \frac{1}{2}\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)} ds.$$
$$\leq \frac{1}{2}\|u_h^{n-1}\|^2_{L^2(\Omega)} + (\frac{3}{2}+d_1)\left(\frac{\delta^3}{3}\int_{t^{n-1}}^{t^n} 1 ds + \frac{2}{3\delta^{3/2}}\int_{t^{n-1}}^{t^n} \|u_h\|^3_{L^3(\Omega)} ds\right)$$
$$+ \frac{1}{2}\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)} ds.$$

where at the last two steps we have used Young's inequality. Choosing $\delta > 0$ such that $(\frac{3}{2}+d_1)\frac{2}{3\delta^{3/2}} = \frac{1}{2}$, (15) yields,

$$(16)\quad \frac{1}{2}\|u_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[u_h^{n-1}]\|^2_{L^2(\Omega)} + d_1 \int_{t^{n-1}}^{t^n} \|u_h\|^2_{H^1(\Omega)} ds + \frac{1}{2}\int_{t^{n-1}}^{t^n} \|u_h\|^3_{L^3(\Omega)} ds$$
$$\leq \frac{1}{2}\|u_h^{n-1}\|^2_{L^2(\Omega)} + C_{d_1}\int_{t^{n-1}}^{t^n} 1 ds + \frac{1}{2}\int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)} ds,$$

with $C_{d_1} = \frac{1}{4}\left(2 + \frac{4d_1}{3}\right)^3$ which implies estimate (10) upon summation and substitution of $\|v_h\|^2_{L^2[0,t^n;L^2(\Omega)]}$ from (8). It remains to prove (11), for which we will use the same technique to bound $\int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)} ds$ as in (9). Fix $z_h \in X_h$, $t \in (t^{n-1}, t^n)$ and set $w_h(s) = \phi(s) z_h$ into (6) to get

$$(u_h(t) - u_{h-}^{n-1}, z_h) = -\int_{t^{n-1}}^{t^n} \Big(d_1 a(u_h, z_h\phi(s)) + (u_h(1-|u_h|), z_h\phi(s))\Big) ds$$
$$+ \int_{t^{n-1}}^{t^n} (v_h h(au_h), z_h\phi(s)) ds.$$

Working identically as for the estimate of u_h and using Hölder's inequality to bound $\int_{t^{n-1}}^{t^n}\int_\Omega |u_h|^2|z_h| dx ds \leq \int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^3(\Omega)}\|z_h\|_{L^3(\Omega)} ds$ we obtain,

$$\Big|(u_h(t) - v_{h-}^{n-1}, z_h)\Big| \leq C_k\Big(d_1\|z_h\|_{H^1(\Omega)}\int_{t^{n-1}}^{t^n} \|u_h\|_{H^1(\Omega)} ds$$
$$+ \|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n} (\|v_h\|_{L^2(\Omega)} + \|u_h\|_{L^2(\Omega)}) ds + \|z_h\|_{L^3(\Omega)}\int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^3(\Omega)} ds\Big).$$

Using the Hölder inequality with respect to time,

$$\Big|(u_h(t) - u_{h-}^{n-1}, z_h)\Big| \leq C_k\Big(d_1\|z_h\|_{H^1(\Omega)}\tau_n^{1/2}\|u_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$$
$$+ \|z_h\|_{L^2(\Omega)}\tau_n^{1/2}(\|u_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|v_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]})$$
$$+ \|z_h\|_{L^3(\Omega)}\tau_n^{1/3}\|u_h\|^2_{L^3[t^{n-1},t^n;L^3(\Omega)]}\Big).$$

Set $z_h = u_h(t)$ and integrate with respect to time the resulting inequality to deduce,

(17)
$$\int_{t^{n-1}}^{t^n} \|u_h(t)\|^2_{L^2(\Omega)} dt \leq \tau_n C_k \Big(\|u^{n-1}_{h-}\|^2 + d_1 \int_{t^{n-1}}^{t^n} \|u_h\|^2_{H^1(\Omega)} dt + \int_{t^{n-1}}^{t^n} \|v_h\|^2_{L^2(\Omega)} dt + \int_{t^{n-1}}^{t^n} \|u_h\|^3_{L^3(\Omega)} dt \Big).$$

where at the last step have used Hölder's inequality to bound $\int_{t^{n-1}}^{t^n} \|u_h(t)\|_{L^3(\Omega)} dt \leq \tau_n^{2/3} \|u_h\|_{L^3[t^{n-1},t^n;L^3(\Omega)]}$. The estimate now follows using an inverse estimate and substituting the estimates (8) and (10). □

4. Error estimates

4.1. Projections and orthogonality relations. The following projections related to discontinuous Galerkin time-stepping schemes will be used.

Definition 4.1. (1) $P_h : L^2(\Omega) \to X_h$ is the orthogonal L^2 projection operator onto X_h, i.e., $(P_h w, w_h) = (w, w_h)$, $\forall w_h \in X_h, w \in L^2(\Omega)$.

(2) The projection $P^{loc}_n : C[t^{n-1}, t^n; L^2(\Omega)] \to \mathcal{P}_k[t^{n-1}, t^n; X_h]$ satisfies $(P^{loc}_n w)^n = P_h w(t^n)$, and

$$\int_{t^{n-1}}^{t^n} (w - P^{loc}_n w, w_h) ds = 0, \qquad \forall\, w_h \in \mathcal{P}_{k-1}[t^{n-1}, t^n; X_h].$$

In the above definition, we have used the convention $(P^{loc}_n w)^n \equiv (P^{loc}_n w)(t^n)$.

(3) The projection $P^{loc}_h : C[0, T; L^2(\Omega)] \to \mathcal{X}_h$ satisfies

$$P^{loc}_h w \in \mathcal{X}_h \text{ and } (P^{loc}_h w)|_{(t^{n-1},t^n]} = P^{loc}_n (w|_{[t^{n-1},t^n]}).$$

In the following Lemma, we collect several results regarding rates of convergence for the above projection (see e.g. [8]).

Lemma 4.2. Let $X_h \subset H^1(\Omega)$ satisfy (3) and let P^{loc}_h denote the projection of Definition 4.1. Then, for all $w \in L^2[0, T; H^{l+1}(\Omega)] \cap H^{k+1}[0, T; L^2(\Omega)]$ there exists constant $C > 0$ independent of h, τ such that

$$\|w - P^{loc}_h w\|_{L^2[0,T;L^2(\Omega)]} \leq C\big(h^{l+1} \|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]}\big),$$
$$\|w - P^{loc}_h w\|_{L^2[0,T;H^1(\Omega)]} \leq C\big(h^{l} \|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + (\tau^{k+1}/h) \|w^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]}\big),$$
$$\|w - P^{loc}_h w\|_{L^\infty[0,T;L^2(\Omega)]} \leq C\big(h^{l+1} \|w\|_{L^\infty[0,T;H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^\infty[0,T;L^2(\Omega)]}\big).$$

If $w \in L^2[0, T; H^{l+1}(\Omega)] \cap H^{k+1}[0, T; H^1(\Omega)]$, then there exists constant $C > 0$ independent of τ, h such that

$$\|w - P^{loc}_h w\|_{L^2[0,T;H^1(\Omega)]} \leq C\big(h^{l} \|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]}\big).$$

Let $k = 0, l = 1$, and $w \in L^2[0, T; H^2(\Omega)] \cap H^1[0, T; L^2(\Omega)]$. Then, there exists constant $C > 0$ independent of h, τ such that,

$$\|w - P^{loc}_h w\|_{L^\infty[0,T;L^2(\Omega)]} + \|w - P^{loc}_h w\|_{L^2[0,T;H^1(\Omega)]} \leq C\big(h \|w\|_{L^2[0,T;H^2(\Omega)]} + \tau^{1/2} (\|w_t\|_{L^2[0,T;L^2(\Omega)]} + \|w\|_{L^2[0,T;H^2(\Omega)]})\big).$$

Subtracting (1) from (4) and (2) from (5), we obtain the fully-discrete orthogonality condition, which reads as: For every $w_h \in \mathcal{X}_h$ and for $n = 1, ..., N$,

$$(18)\quad (e^n_{u-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} (-\langle e_u, w_{hs}\rangle + d_1 a(e_u, w_h) + (v_h h(au_h) - vh(au), w_h))\, ds$$
$$= \int_{t^{n-1}}^{t^n} (u^h(1 - |u_h|) - u(1 - |u|, w_h) ds + (e^{n-1}_-, w^{n-1}_{h+}),$$

and

$$(19)\quad (e^n_{v-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} (-\langle e_v, w_{hs}\rangle + d_2 a(e_v, w_h) - b(v_h h(au_h) - vh(au), w_h))\, ds$$
$$+c \int_{t^{n-1}}^{t^n} (e_v, w_h) ds = (e^{n-1}_-, w^{n-1}_{h+}),$$

where $e_u = u_h - u$, $e_v = v_h - v$ denote the error for the u, v components respectively. We will split the error as $e_u = (u_h - u_p) + (u_p - u) := e_{uh} + e_{up}$, and $e_v = (v_h - v_p) + (v_p - v) := e_{vh} + e_{vp}$ respectively, where u_p is the discontinuous Galerkin solution of a linear parabolic pde with right hand side $u_t - d_1 \Delta u$, and initial data $u_{p0} = P_h u_0$, and v_p is the discontinuous Galerkin solution of a linear parabolic pde with right hand side $v_t - d_2 \Delta v$ and initial data $v_{p0} = P_h v_0$. Hence, for every $w_h \in \mathcal{X}_h$ and for $n = 1, ..., N$, we define $u_p, v_p \in \mathcal{X}_h$ as follows,

$$(20)\quad (u^n_{p-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \Big(- \langle u_p, w_{hs}\rangle + d_1 a(u_p, w_h)\Big) ds$$
$$= (u^{n-1}_{p-}, w^{n-1}_+) + \int_{t^{n-1}}^{t^n} \langle u_t - d_1 \Delta u, w_h\rangle ds,$$

$$(21)\quad (v^n_{p-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \Big(- \langle v_p, w_{hs}\rangle + a(v_p, w_h)\Big) ds$$
$$= (v^{n-1}_{p-}, w^{n-1}_+) + \int_{t^{n-1}}^{t^n} \langle v_t - d_2 \Delta v, w_h\rangle ds.$$

Integrating by parts the last term of the right-hand side and noting that $u, v \in C[0, T; L^2(\Omega)]$, we obtain the equivalent form of the orthogality condition: For $n = 1, ..., N$, and $w_h \in \mathcal{X}_h$

$$(22)\quad (e^n_{p-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \Big(- \langle e_p, w_{hs}\rangle + d_1 a(e_p, w_h)\Big) ds = (e^{n-1}_{p-}, w^{n-1}_+),$$

$$(23)\quad (e^n_{p-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \Big(- \langle e_p, w_{hs}\rangle + d_2 a(e_p, w_h)\Big) ds = (e^{n-1}_{p-}, w^{n-1}_+).$$

We can estimate e_{up}, e_{vp} in terms of the standard dG projection of Definition 4.1 which is a straightforward application of [9, Theorem 2.2 and Theorem 2.3]):

$$(24)\quad \|e_{up}\|_{L^\infty[0,T;L^2(\Omega)]} + d_1^{1/2} \|e_{up}\|_{L^2[0,T;H^1(\Omega)]} \le C_k \Big(\|P_h u(0) - u(0)\|_{L^2(\Omega)}$$
$$+ \|u - P^{loc}_h u\|_{L^\infty[0,T;L^2(\Omega)]} + d_1^{1/2} \|u - P^{loc}_h u\|_{L^2[0,T;H^1(\Omega)]}\Big),$$

$$(25)\quad \|e_{vp}\|_{L^\infty[0,T;L^2(\Omega)]} + d_2^{1/2} \|e_{vp}\|_{L^2[0,T;H^1(\Omega)]} \le C_k \Big(\|P_h v(0) - v(0)\|_{L^2(\Omega)}$$
$$+ \|v - P^{loc}_h v\|_{L^\infty[0,T;L^2(\Omega)]} + d_2^{1/2} \|v - P^{loc}_h v\|_{L^2[0,T;H^1(\Omega)]}\Big).$$

Here C_k is a constant depending upon Ω and on the polynomial degree of the temporal discretisation. Therefore, using (22) and (23) into (18) and (19) respectively we deduce the of following relations for e_{uh} and e_{vh}: For all $n = 1, .., N$ and for all $w_h \in \mathcal{X}_h$,

(26)
$$(e^n_{uh-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} (-\langle e_{uh}, w_{hs}\rangle + d_1 a(e_{uh}, w_h) + (v_h h(au_h) - vh(au), w_h))\, ds$$
$$= \int_{t^{n-1}}^{t^n} (u^h(1 - |u_h|) - u(1 - |u|), w_h) ds + (e^{n-1}_{uh-}, w^{n-1}_{h+}),$$
(27)
$$(e^n_{vh-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} (-\langle e_{vh}, w_{hs}\rangle + d_2 a(e_{vh}, w_h) - b(v_h h(au_h) - vh(au), w_h))\, ds$$
$$+ c \int_{t^{n-1}}^{t^n} (e_{vh} + e_{vp}, w_h) ds = (e^{n-1}_{vh-}, w^{n-1}_{h+}).$$

Our goal now is to bound e_{uh} and e_{vh} in terms of e_{up} and e_{vp}.

4.2. The main error estimate. We proceed by deriving the initial estimate that bounds norms at partition point t^n and the energy norms $\|e_{uh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}$ and $\|e_{vh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}$, in terms of the similar norms of the projection error e_{up}, e_{vp} plus $\|e_{uh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ and $\|e_{vh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ terms.

Proposition 4.3. Let u, v satisfy (1),(2) respectively and u_h, v_h denote their fully-discrete approximations defined by (4), (5) respectively. Suppose that the assumptions of Theorem 3.1 hold, and let u_p, v_p defined by (20), (21) respectively. Denote by $e_{uh} := u_h - u_p$, $e_{up} := u_p - u$, and by $e_{vh} := v_h - v_p$, $e_{vp} := v_p - v$. Then, the following estimate holds: For all $n = 1, .., N$

$$\frac{1}{2}\|e^n_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{uh}]^{n-1}\|^2_{L^2(\Omega)} + d_1 \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds$$
$$+ \frac{1}{2}\|e^n_{vh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{vh}]^{n-1}\|^2_{L^2(\Omega)} + d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{H^1(\Omega)} ds + \frac{c}{2} \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds$$
$$\leq \frac{1}{2}\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|e^{n-1}_{vh-}\|^2_{L^2(\Omega)}$$
$$+ \tilde{\mathbf{C}} \int_{t^{n-1}}^{t^n} \Big(\|e_{vh}\|^2_{L^2(\Omega)} + \|e_{uh}\|^2_{L^2(\Omega)} + \|e_{vp}\|^2_{L^2(\Omega)} + \|e_{up}\|^2_{L^2(\Omega)} + \|e_{vp}\|^2_{H^1(\Omega)} \Big) ds,$$

where $\tilde{\mathbf{C}} := C \max\{C_{\infty,a,b}, c, d_1, d_2, C^2_{st,2}/d_1^3, C_{st,2}/d_1\}$. Here $C_{st,2}$ denotes the stability constant (11) of Theorem 3.1, and $C_{\infty,a,b} = \max\{C_{v,a,b}, C_u\}$ where $C_{v,a,b} := C \max\{2, (1 + 3a)\|v\|_{L^\infty[0,T;L^\infty(\Omega)]}\} \max\{1, b\}$ and $C_u := 1 + \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$ respectively.

Proof. We present the proof in case of the Howling type II functional response. The of case of Holling type III functional can be treated in a similar way (see also Remark 4.5). We focus on the three dimensional case. The two dimensional case can be treated similarly and more easily. We need to treat the nonlinear terms of (26) and (27). We begin first with the two nonlinear terms of (26). For the first one, note that standard algebraic manipulations lead to,

$$
\begin{aligned}
(28)\qquad &\Big|v_h h(au_h) - v h(au)\Big| = \Big|\frac{a v_h u_h}{1+a|u_h|} - \frac{avu}{1+a|u|}\Big| \\
&\le \frac{a\big|v_h u_h - vu\big|}{(1+a|u_h|)(1+a|u|)} + \frac{a^2\Big|v_h u_h|u| - vu|u_h|\Big|}{(1+a|u_h|)(1+a|u|)} \\
&\le a\frac{\big|v_h u_h - vu_h + vu_h - vu\big|}{(1+a|u_h|)(1+a|u|)} + a^2\frac{\Big|v_h u_h|u| - vu_h|u| + vu_h|u| - vu|u_h|\Big|}{(1+a|u_h|)(1+a|u|)} \\
&\le a\frac{|u_h||v_h - v|}{(1+a|u_h|)(1+a|u|)} + a\frac{|v||u_h - u|}{(1+a|u_h|)(1+a|u|)} \\
&\quad + a^2\frac{|u_h||u||v_h - v|}{(1+a|u_h|)(1+a|u|)} + a^2|v|\frac{\big|u_h|u| - u|u_h|\big|}{(1+a|u_h|)(1+a|u|)} \\
&\le 2|v_h - v| + (1+3a)|v||u - u_h|.
\end{aligned}
$$

At the last step we have used the inequalities $\frac{a|u_h|}{1+a|u_h|} \le 1$ and $\frac{1}{1+a|u|} \le 1$ for the first term, $\frac{1}{(1+a|u_h|)(1+a|u|)} \le 1$ for the second term, $\frac{a^2|u_h||u|}{(1+a|u_h|)(1+a|u|)} \le 1$ for the third term, while for the fourth term (adding and subtracting $u|u|$ first) the bounds $\frac{a|u|}{1+a|u|} \le 1$, and $\frac{1}{1+|u_h|} \le 1$. It remains to bound the second nonlinear term of (26). For this purpose, we simply observe that standard algebraic manipulations lead to,

$$
(29)\qquad \Big|u(1-|u|) - u_h(1-|u_h|)\Big| \le (1+|u|+|u_h|)|u-u_h|.
$$

Setting $w_h = e_{uh}$ and $w_h = e_{vh}$ into (26) and (27) respectively, using (28), (29) and standard algebra we deduce,

$$
\begin{aligned}
(30)\qquad &\frac{1}{2}\|e^n_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{uh}]^{n-1}\|^2_{L^2(\Omega)} + d_1\int_{t^{n-1}}^{t^n}\|\nabla e_{uh}\|^2_{L^2(\Omega)}ds \\
&\le \frac{1}{2}\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n}\int_\Omega \left(2|v-v_h| + (1+3a)|v||u-u_h|\right)|e_{uh}|dxds \\
&\quad + \int_{t^{n-1}}^{t^n}\int_\Omega (1+|u|+|u_h|)|u-u_h||e_{uh}|dxds,
\end{aligned}
$$

and

$$
\begin{aligned}
(31)\qquad &\frac{1}{2}\|e^n_{vh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{vh}]^{n-1}\|^2_{L^2(\Omega)} + d_2\int_{t^{n-1}}^{t^n}\|\nabla e_{vh}\|^2_{L^2(\Omega)}ds \\
&\quad + \frac{c}{2}\int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{L^2(\Omega)}ds \le \frac{1}{2}\|e^{n-1}_{vh-}\|^2_{L^2(\Omega)} + \frac{c}{2}\int_{t^{n-1}}^{t^n}\|e_{vp}\|^2_{L^2(\Omega)}ds \\
&\quad + b\int_{t^{n-1}}^{t^n}\int_\Omega \left(2|v-v_h| + (1+3a)|v||u-u_h|\right)|e_{vh}|dxds.
\end{aligned}
$$

Adding $d_1 \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{L^2(\Omega)} ds$ on both sides of (30) and $d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds$ on both sides of (31) and adding the resulting inequalities we finally arrive at,

$$
\begin{aligned}
&\frac{1}{2}\|e^n_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{uh}]^{n-1}\|^2_{L^2(\Omega)} + d_1 \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds \\
&+ \frac{1}{2}\|e^n_{vh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{vh}]^{n-1}\|^2_{L^2(\Omega)} + d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{H^1(\Omega)} ds \\
&+ \frac{c}{2}\int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds \\
&\le \frac{1}{2}\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|e^{n-1}_{vh-}\|^2_{L^2(\Omega)} + \frac{c}{2}\int_{t^{n-1}}^{t^n} \|e_{vp}\|^2_{L^2(\Omega)} ds \\
&+ d_1 \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{L^2(\Omega)} ds + d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds \\
&+ \int_{t^{n-1}}^{t^n} \int_\Omega (2|e_{vh} + e_{vp}| + (1+3a)|v||e_{uh} + e_{up}|)\,(|e_{uh}| + b|e_{vh}|)\,dxds \\
&+ \int_{t^{n-1}}^{t^n} \int_\Omega (1 + |u| + |u_h|)|e_{uh} + e_{up}||e_{uh}| dxds. \\
&=: \frac{1}{2}\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|e^{n-1}_{vh-}\|^2_{L^2(\Omega)} + \frac{c}{2}\int_{t^{n-1}}^{t^n} \|e_{vp}\|^2_{L^2(\Omega)} ds \\
&+ d_1 \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{L^2(\Omega)} ds + d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds + \mathbf{I}_1 + \mathbf{I}_2.
\end{aligned} \tag{32}
$$

We need to estimate the integrals $\mathbf{I}_i$, for $i = 1, 2$. For the first term, we easily observe that

$$
\mathbf{I}_1 \le C_{v,a,b} \int_{t^{n-1}}^{t^n} \left(\|e_{vh}\|^2_{L^2(\Omega)} + \|e_{vp}\|^2_{L^2(\Omega)} + \|e_{uh}\|^2_{L^2(\Omega)} + \|e_{up}\|^2_{L^2(\Omega)} \right) ds
$$

where $C_{v,a,b} = C \max\{2, (1+3a)\|v\|_{L^\infty[0,T;L^\infty(\Omega)]}\} \max\{1, b\}$, For the second term using Young's inequality,

$$
\begin{aligned}
\mathbf{I}_2 \le C_u \int_{t^{n-1}}^{t^n} &\left(\|e_{uh}\|^2_{L^2(\Omega)} + \|e_{up}\|^2_{L^2(\Omega)} \right) ds \\
&+ \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)} \left(\|e_{uh}\|^2_{L^4(\Omega)} + \|e_{uh}\|_{L^4(\Omega)} \|e_{up}\|_{L^4(\Omega)} \right) ds,
\end{aligned}
$$

where $C_u = 1 + \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$. Therefore, combing the last two bounds,

$$
\begin{aligned}
\mathbf{I}_1 + \mathbf{I}_2 \le C_{\infty,a,b} \int_{t^{n-1}}^{t^n} &\left(\|e_{vh}\|^2_{L^2(\Omega)} + \|e_{vp}\|^2_{L^2(\Omega)} + \|e_{uh}\|^2_{L^2(\Omega)} + \|e_{vh}\|^2_{L^2(\Omega)} \right) ds \\
&+ \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)} \left(\|e_{uh}\|^2_{L^4(\Omega)} + \|e_{up}\|_{L^4(\Omega)} \|e_{uh}\|_{L^4(\Omega)} \right) ds
\end{aligned}
$$

where $C_{\infty,a,b} := \max\{C_{v,a,b}, C_u\}$. It remains to bound the last integral. The *GNL* interpolation inequality and Young's inequality with $p_1 = 4$, $p_2 = 4/3$ and $\epsilon_1 > 0$ (to be determined) for the first integral and the embedding $H^1(\Omega) \subset L^4(\Omega)$ and Young's inequality with $p_1 = p_2 = 2$ and $\epsilon_2 > 0$ (to be determined) for the second one imply that

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)} \Big(\|e_{uh}\|^2_{L^4(\Omega)} + \|e_{up}\|_{L^4(\Omega)} \|e_{uh}\|_{L^4(\Omega)} \Big) ds \\
&\leq C \|u_h\|_{L^\infty[0,T;L^2(\Omega)]} \int_{t^{n-1}}^{t^n} \Big(\|e_{uh}\|^{1/2}_{L^2(\Omega)} \|e_{uh}\|^{3/2}_{H^1(\Omega)} dt + \|e_{uh}\|_{H^1(\Omega)} \|e_{up}\|_{H^1(\Omega)} \Big) ds \\
&\leq C \frac{\epsilon_1^4 C^2_{st,2}}{4} \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{L^2(\Omega)} ds + \frac{3}{4\epsilon_1^{4/3}} \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds \\
&\quad + C \frac{\epsilon_2 C_{st,2}}{2} \int_{t^{n-1}}^{t^n} \|e_{up}\|^2_{H^1(\Omega)} ds + \frac{1}{2\epsilon_2} \int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds.
\end{aligned}$$

where $C_{st,2}$ denotes the stability constant of Theorem 3.1 and C is an algebraic constant depending on the domain. We select ϵ_1 such that $\frac{3}{4\epsilon_1^{4/3}} = \frac{d_1}{4}$ and $\frac{1}{2\epsilon_2} = \frac{d_1}{4}$, and substituting the resulting bound into $\mathbf{I}_1 + \mathbf{I}_2$ and then $\mathbf{I}_1 + \mathbf{I}_2$ into (32) we derive the desired estimate. □

For low order schemes, i.e., when $k = 0$ or $k = 1$, standard Gronwall techniques imply the desired estimates. In order to include high order schemes in our analysis we need to provide a sharp local estimate for $\int_{t^{n-1}}^{t^n} \Big(\|e_{uh}\|^2_{L^2(\Omega)} + \|e_{vh}\|^2_{L^2(\Omega)} \Big) ds$. This is achieved in the subsequent proposition.

Proposition 4.4. Suppose that the assumptions of Theorem 3.1 and Proposition 4.3 hold. If in addition, τ satisfies $C_k \tilde{\mathbf{C}}(d_1 + C^{1/2}_{st,2})\tau < d_1/2$, $C_k \tilde{\mathbf{C}} \tau < 1/2$, $C_k(C_{\infty,a,b} + c)\tau < 1/4$, and $C_k C_{\infty,a} \tau < 1/2$ where $C_{st,2}$ is the stability constant (11) of Theorem 3.1, $\tilde{\mathbf{C}}$ denotes the constant of Proposition 4.3, and $C_{\infty,a} := C \max\Big\{ \max\{2, (1+3a)C_v\}, 1 + C_u \Big\}$, $C_{\infty,a,b} = bC_{\infty,a}$with $C_v = \|v\|_{L^\infty[0,T;L^\infty(\Omega)]}$ and $C_u := \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$, then the following estimate holds: For all $n = 1, ..., N$,

$$\begin{aligned}
&\|e^n_{uh-}\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|[e_{uh}]^{i-1}\|^2_{L^2(\Omega)} + d_1 \int_0^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds \\
&+ \|e^n_{vh-}\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|[e_{vh}]^{i-1}\|^2_{L^2(\Omega)} + d_2 \int_0^{t^n} \|e_{vh}\|^2_{H^1(\Omega)} ds + c \int_0^{t^n} \|e_{vh}\|^2_{L^2(\Omega)} ds \\
&\leq \tilde{\mathbf{C}} e^{C_k \tilde{\mathbf{C}} T} \Bigg\{ \int_0^{t^n} \Big(\|e_{vp}\|^2_{H^1(\Omega)} + \|e_{up}\|^2_{H^1(\Omega)} \Big) ds \Bigg\}.
\end{aligned}$$

Proof. We will closely follow the proof of Theorem 3.1. We fix $z_h \in X_h$ and $t \in (t^{n-1}, t^n)$ and we return back to (26) to set $w_h(s) = \phi(s) z_h$, where $\phi \in \mathcal{P}_k(t^{n-1}, t^n)$ satisfies,

$$\phi(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n} \phi\psi = \int_{t^{n-1}}^{t} \psi, \qquad \psi \in \mathcal{P}_{k-1}(t^{n-1}, t^n).$$

Note that due to [8, Lemma 3.2], $\|\phi\|_{L^\infty(t^{n-1},t^n)} \leq C_k$, where constant C_k is independent of the fixed t. Therefore, this particular choice of w_h, allow us to integrate,

$$\int_{t^{n-1}}^{t^n} (e_{uh,t}, w_h) ds + (e^{n-1}_{uh+} - e^{n-1}_{hh-}, w^{n-1}_{h+}) = (e_{uh}(t) - v^{n-1}_{h-}, z_h).$$

Hence, working identically as in Theorem 3.1, we obtain that

$$(33)\qquad (e_{uh}(t)-e_{uh-}^{n-1},z_h) = -d_1\int_{t^{n-1}}^{t^n} a(e_{uh},z_h\phi(s))ds - \int_{t^{n-1}}^{t^n}\left((v_h h(au_h)-vh(au),z_h\phi(s))\right)ds + \int_{t^{n-1}}^{t^n}(u^h(1-|u_h|)-u(1-|u|,z_h\phi(s))ds.$$

Therefore, since $\|\phi\|_{L^\infty(t^{n-1},t^n)}\le C_k$, and $z_h\in X_h$, and applying bounds (28) and (29) to (33), it yields,

$$\Big|(e_{uh}(t)-e_{uh-}^{n-1},z_h)\Big| \le C_k\Bigg\{d_1\int_{t^{n-1}}^{t^n}\int_\Omega |\nabla e_{uh}||\nabla z_h|dxds + \int_{t^{n-1}}^{t^n}\int_\Omega\left(2|v_h-v|+(1+3a)|v||u-u_h|\right)|z_h|dxds + \int_{t^{n-1}}^{t^n}\int_\Omega(1+|u|+|u_h|)|u-u_h||z_h|dxds\Bigg\}.$$

Using Hölder's inequalities and since $z_h\in X_h$ (and independent of s) the above inequality leads to,

$$\Big|(e_{uh}(t)-e_{uh-}^{n-1},z_h)\Big| \le C_k\Bigg\{d_1\|\nabla z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|\nabla e_{uh}\|_{L^2(\Omega)}ds + C_{\infty,a}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\left(\|e_{vh}\|_{L^2(\Omega)}+\|e_{vp}\|_{L^2(\Omega)}+\|e_{up}\|_{L^2(\Omega)}+\|e_{uh}\|_{L^2(\Omega)}\right)ds + \|z_h\|_{L^4(\Omega)}\int_{t^{n-1}}^{t^n}\|u_h\|_{L^2(\Omega)}\|e_{uh}+e_{up}\|_{L^4(\Omega)}ds\Bigg\},$$

where here we denote by $C_{\infty,a}:=C\max\Big\{\max\{2,(1+3a)C_v\},1+C_u\Big\}$ with $C_v=\|v\|_{L^\infty[0,T;L^\infty(\Omega)]}$ and $C_u:=\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$. Hence, using the embedding $H^1(\Omega)\subset L^4(\Omega)$ and working identically as in Theorem 3.1 we obtain,

$$(34)\qquad \int_{t^{n-1}}^{t^n}\|e_{uh}\|^2_{L^2(\Omega)} \le C_k\tau_n\Bigg\{\|e_{h-}^{n-1}\|^2_{L^2(\Omega}+(d_1+C_{st,2}^{1/2})\|e_{uh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} + C_{\infty,a}\left(\|e_{uh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\|e_{vh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\right) + C_{\infty,a}\left(\|e_{up}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\|e_{vp}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\right) + C_{st,2}^{1/2}\|e_{up}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}\Bigg\}.$$

We proceed in a similar way to derive the bound on $\int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{L^2(\Omega)}ds$. Indeed, once again we fix $z_h\in X_h$ and set $w_h(s)=\phi(s)z_h$, in (27) where $\phi\in\mathcal{P}_k(t^{n-1},t^n)$ satisfies,

$$\phi(t^{n-1})=1,\qquad \int_{t^{n-1}}^{t^n}\phi\psi=\int_{t^{n-1}}^{t}\psi,\qquad \psi\in\mathcal{P}_{k-1}(t^{n-1},t^n).$$

Hence, working in an identical way, we get

$$\Big|(e_{vh}(t)-e_{vh-}^{n-1}, z_h)\Big| \le C_k d_2 \int_{t^{n-1}}^{t^n}\int_\Omega |\nabla e_{vh}||\nabla z_h| + c|e_{vh}||z_h| dxds \\ + C_k b \int_{t^{n-1}}^{t^n}\int_\Omega (2|v_h - v| + (1+3a)|v||u-u_h|)\,|z_h| dxds.$$

Thus, we arrive at

$$\Big|(e_{vh}-e_{vh-}^{n-1}, z_h)\Big| \\ \le C_k \tau_n^{1/2}\Bigg\{ d_2\|\nabla z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|\nabla e_{vh}\|_{L^2(\Omega)}ds + c\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|e_{vh}\|_{L^2(\Omega)}ds \\ + C_{\infty,a,b}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\Big(\|e_{uh}\|_{L^2(\Omega)} + \|e_{vh}\|_{L^2(\Omega)} + \|e_{up}\|_{L^2(\Omega)} + \|e_{vp}\|_{L^2(\Omega)}\Big)ds\Bigg\},$$

where $C_{\infty,a,b} = bC_{\infty,a}$. Setting now $z_h = e_{vh}(t)$ and integrating with respect to time we easily deduce,

$$\text{(35)}\qquad \int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{L^2(\Omega)}dt \le C_k\tau_n\Bigg\{\|e_{vh-}^{n-1}\|^2_{L^2(\Omega)} + d_2\|\nabla e_{vh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} \\ + (C_{\infty,a,b}+c)\Big(\|e_{vh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|e_{uh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big) \\ + C_{\infty,a,b}\Big(\|e_{up}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|e_{vp}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big)\Bigg\}.$$

Adding (34) and (35), and for $C_k(C_{\infty,a,b}+c)\tau_n \le 1/4$, $C_kC_{\infty,a}\tau_n \le 1/2$ (note that C_k might be different in different occurrences), we hide the terms $\|e_{uh}\|^2_{L_2[t^{n-1},t^n;L^2(\Omega)]}$ and $\|e_{vh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ at the left hand side to obtain,

$$\text{(36)}\quad \int_{t^{n-1}}^{t^n}\Big(\|e_{uh}\|^2_{L^2(\Omega)} + \|e_{vh}\|^2_{L^2(\Omega)}\Big)dt \le CC_k\tau_n\Bigg\{\|e_{uh-}^{n-1}\|^2_{L^2(\Omega)} + \|e_{vh-}^{n-1}\|^2_{L^2(\Omega)} \\ + (d_1 + C_{st,2}^{1/2})\|e_{uh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} + d_2\|e_{vh}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} \\ + \max\{C_{\infty,a}, C_{\infty,a,b}\}\Big(\|e_{up}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|e_{vp}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big) \\ + C_{st,2}^{1/2}\|e_{up}\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}\Bigg\}.$$

Replacing (36) into Proposition 4.3, we finally obtain,

$$\frac{1}{2}\|e^n_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{uh}]^{n-1}\|^2_{L^2(\Omega)} + d_1\int_{t^{n-1}}^{t^n}\|e_{uh}\|^2_{H^1(\Omega)}ds$$
$$+\frac{1}{2}\|e^n_{vh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[e_{vh}]^{n-1}\|^2_{L^2(\Omega)} + d_2\int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{H^1(\Omega)}ds + \frac{c}{2}\int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{L^2(\Omega)}ds$$
$$\leq \left(\frac{1}{2} + C_k\tilde{\mathbf{C}}\tau_n\right)\left(\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \frac{1}{2}\|e^{n-1}_{vh-}\|^2_{L^2(\Omega)}\right)$$
$$+ CC_k\tilde{\mathbf{C}}(d_1 + C^{1/2}_{st,2})\tau_n\int_{t^{n-1}}^{t^n}\|e_{uh}\|^2_{H^1(\Omega)} + CC_k\tilde{\mathbf{C}}d_2\tau_n\int_{t^{n-1}}^{t^n}\|e_{vh}\|^2_{H^1(\Omega)}ds$$
$$+ C_k\tilde{\mathbf{C}}(1+\max\{C_{\infty,a,b}, C_{\infty,a}\}\tau_n)\int_{t^{n-1}}^{t^n}\left(\|e_{vp}\|^2_{L^2(\Omega)} + \|e_{up}\|^2_{L^2(\Omega)}\right)ds$$
$$+ C_k\tilde{\mathbf{C}}(1+C^{1/2}_{st,2}\tau_n)\int_{t^{n-1}}^{t^n}\left(\|e_{vp}\|^2_{H^1(\Omega)} + \|e_{up}\|^2_{H^1(\Omega)}\right)ds.$$

Choosing τ_n such that $CC_k\tilde{\mathbf{C}}(d_1+C^{1/2}_{st,2})\tau_n \leq \frac{d_1}{2}$, $CC_k\tilde{\mathbf{C}}d_2\tau_n \leq \frac{d_2}{2}$ and $C_k\tilde{\mathbf{C}}\tau_n < 1$, we obtain the desired estimate using standard algebra. □

Remark 4.5. Let us briefly consider the case of the functional response of Holling type III. Then, we only need to prove an analog of (28) for the case when $h(au) = \frac{au^2}{1+au^2}$. For this purpose, we simply observe that

$$\text{(37)}\quad \Big|v_h h(au_h) - vh(au)\Big| = \Big|\frac{av_hu_h^2}{(1+au_h^2)} - \frac{avu^2}{1+au^2}\Big|$$
$$\leq \frac{a\Big|v_hu_h^2 - vu^2\Big|}{(1+au_h^2)(1+au^2)} + \frac{a^2\Big|v_hu_h^2u^2 - vu^2u_h^2\Big|}{(1+au_h^2)(1+au^2)}$$
$$\leq a\frac{\big|v_hu_h^2 - vu_h^2 + vu_h^2 - vu^2\big|}{(1+au_h^2)(1+au^2)} + a^2\frac{|v_h - v|u_h^2u^2}{(1+au_h^2)(1+au^2)}$$
$$\leq a\frac{u_h^2|v_h - v|}{(1+au_h^2)(1+au^2)} + a\frac{|v||u_h - u|(|u_h| + |u|)}{(1+au_h^2)(1+au^2)} + a^2\frac{|v_h - v|u_h^2u^2}{(1+au_h^2)(1+au^2)}$$
$$\leq 2|v - v_h| + \frac{a^{1/2}|v|}{4}|u_h - u|.$$

For the first term we have used that $\frac{au_h^2}{(1+a_h^2)} \leq 1$ and $\frac{1}{(1+au^2)} \leq 1$, while for the second one $\frac{\sqrt{a}|u_h|}{(1+au^h_2)(1+au_h^2)} \leq \frac{1}{2}$, and $\frac{\sqrt{a}|u|}{1+au^2} \leq \frac{1}{2}$. Finally for the third term term we have that $\frac{(au_h^2)(au^2)}{(1+au_h^2)(1+au^2)} \leq 1$. Hence, the remaing of the proofs of Proposition 4.3 and Proposition 4.4 remain the same upon using different algebraic constants C.

4.3. The symmetric estimate and rates of convergence. We close this section by stating the main result.

Theorem 4.6. *Let the assumptions of Theorem 3.1 and Propositions 4.3 and 4.4 hold and let P_h^{loc} defined as in Defintion 4.1. Then, there exists a constant $\tilde{D}$ such that*

$$\|u-u_h\|_{W(0,T)} + \|v-v_h\|_{W(0,T)} \leq \tilde{D}\Big(\|u - P_h^{loc}u\|_{W(0,T)} + \|v - P_h^{loc}\|_{W(0,T)}$$
$$+ \|u_0 - P_hu_0\|_{L^2(\Omega)} + \|v_0 - P_hv_0\|_{L^2(\Omega)}\Big).$$

If in addition $u, v \in\in L^2[0,T; H^{l+1}(\Omega)] \cap H^{k+1}[0,T; L^2(\Omega)]$ *there exists constant* $\tilde{D} \geq 0$ *such that*

$$\|u - u_h\|_{W(0,T)} + \|v - v_h\|_{W(0,T)} \leq \tilde{D}\big(h^l + \frac{\tau^{k+1}}{h}\big).$$

If in addition $u, v \in\in L^2[0,T; H^{l+1}(\Omega)] \cap H^{k+1}[0,T; H^1(\Omega)]$ *then there exists constant* $\tilde{D}$ *such that*

$$\|u - u_h\|_{W(0,T)} + \|v - v_h\|_{W(0,T)} \leq \tilde{D}\big(h^l + \tau^{k+1}\big).$$

The constant $\tilde{D}$ *depends upon the constant* $(\frac{C_{st,2}^{1/2}}{d_1} + 1)\tilde{\mathbf{C}}e^{C_k T \tilde{\mathbf{C}}}$*, where* $\tilde{\mathbf{C}}$ *is defined in Proposition 4.4,* $C_{st,2}$ *is the stability constant* (11) *of Theorem 3.1 and it is independent of* τ, h*.*

Proof. First note that Proposition 4.4 implies an estimate at arbitrary time points. Indeed, from (36) and the classical inverse estimate - in time applied to e_{uh} and e_{vh} repsectively, we obtain,

$$\begin{aligned}
&\|e_{uh}\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)} + \|e_{vh}\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \\
&\leq \frac{C_k}{\tau_n}\Big(\|e_{uh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|e_{vh}\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big) \\
&\leq CC_k\Bigg\{\|e^{n-1}_{uh-}\|^2_{L^2(\Omega)} + \|e^{n-1}_{vh-}\|^2_{L^2(\Omega)} \\
&+ (C_{st,2}^{1/2} + d_1)\int_{t^{n-1}}^{t^n} \|e_{uh}\|^2_{H^1(\Omega)} ds + d_2 \int_{t^{n-1}}^{t^n} \|e_{vh}\|^2_{H^1(\Omega)} ds \\
&+ C_{st,2}^{1/2}\int_{t^{n-1}}^{t^n} \|e_{up}\|^2_{H^1(\Omega)} ds + \max\{C_{\infty,a}, C_{\infty,a,b}\}\int_{t^{n-1}}^{t^n} \|e_{up}\|^2_{L^2(\Omega)} + \|e_{vp}\|^2_{L^2(\Omega)} ds\Bigg\}.
\end{aligned}$$

Using now Proposition 4.4, the above estimate implies that

$$\begin{aligned}
(38)\qquad &\|e_{uh}\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} + \|e_{vh}\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \\
&\leq (\frac{C_{st,2}^{1/2}}{d_1} + 1)\tilde{\mathbf{C}}e^{C_k \tilde{\mathbf{C}}}\Bigg\{\Big(\|e_{vp}\|^2_{L^2[0,T;H^1(\Omega)]} + \|e_{up}\|^2_{L^2[0,T;H^1(\Omega)]}\Big)\Bigg\}.
\end{aligned}$$

The first estimate now follows from Proposition 4.4, inequality (38) and triangle inequality after splitting $e_u = e_{uh} + e_{up}$, $e_v = e_{vh} + e_{vp}$ and substituting the bounds of e_{up}, e_{vp} by (24). The second and the third estimate follow by Lemma 4.2. □

5. Conclusion

In this work we have studied high order schemes for the predator-prey system based on a discontinuous Galerkin approach in time combined with standard finite elements for the spatial discretization. After proving the main stability estimate in the natural energy norm, under minimal regularity assumptions on the given data, we were able to obtain an a-priori error estimate of almost symmetric type that can be viewed as the Céa Lemma's analogue. The estimate is applicable when high order discretizations both in space and time are used and it is derived without making any assumption regarding point-wise discrete stability estimates on the fully-discrete solutions. Future work will include extensive testing of high order schemes under various choices of the physical parameters involved.

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