BIT Numerical Mathematics (2007)
DOI: 10.1007/s10543-007-0141-0

APPROXIMATIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS USING WAVELET-GALERKIN COMPRESSION TECHNIQUES*

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Abstract.

Error estimates for Galerkin discretizations of parabolic integro-differential equations are presented under minimal regularity assumptions. The analysis is applicable in case that the full Galerkin matrix $\mathbf{A}$ associated to the integral operator is replaced by a compressed "sparse" matrix $\tilde{\mathbf{A}}$ using wavelet basis techniques. In particular, a semi-discrete (in space) scheme and a fully-discrete scheme which is discontinuous in time but conforming in space are analyzed.

AMS subject classification (2000): 65R20, 65M60.

Key words: parabolic integro-differential equations, error estimates, wavelet compression techniques, discontinuous Galerkin method, minimal regularity.

1 Introduction.

The purpose of this work is the analysis of numerical schemes for parabolic integro-differential equations based on wavelet-Galerkin methods. Approximations of the following problem are considered: Given data f, u_0 find u such that

$$\begin{cases} u_t = A(u) + f & \text{in } \Omega \times (0,T) \\ u(0,x) = u_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz boundary Γ, and f, u_0 denote the forcing term and the initial data respectively. The operator $A(u)$

* Received August 10, 2005. Accepted in revised form May 29, 2007. Communicated by Anders Szepessy.

** This work was initiated while the author was a Post-Doctoral associate at Institute of Numerical Simulations at University of Bonn, Germany. The author acknowledges the financial support provided through the European Community's Human Potential Programme under contract HPRN–CT–2002–00286 (Breaking Complexity).

consists of an integral operator, i.e.,

$$A(u)(x) = \int_{\Omega} k(x, x-y)u(y)dy, \quad x \in \Omega$$

but the analysis presented here is also applicable in case that $A(u) = A_D u(x)$ where $A_D u \equiv div(\mathbf{D}(x)\nabla u)$ is the standard diffusion operator.

The analysis of various classical finite element Galerkin schemes related to parabolic intregro-differential operators is quite extensive (see e.g. [4, 13, 21, 23]) where several discretization schemes are studied. For the construction and approximation properties of finite elements, as well as for finite element analysis of parabolic problems one may consult the classical works of [2, 20] respectively.

It is worth noting that since the integro-differential operator involved in these examples is typically elliptic but nonlocal, discretization based on finite element or wavelet-based Galerkin schemes lead to systems that do not have a banded structure. Therefore, the matrix $\mathbf{A}$ induced by the operator $A(.)$ by such schemes, lead to dense stiff matrices with $\mathcal{O}(N^2)$ nonzero elements (where N is the number of degrees of freedom for the standard Galerkin discretization of u).

In [17], an alternative approach based on wavelet Galerkin schemes is analyzed in case of a broad category of parabolic integro-differential equations, including equations such as the classical diffusion equations and the Kolmogorov forward equations for Lèvy process. The authors analyze θ-schemes and prove error estimates. The proposed discrete wavelet-Galerkin scheme is based on compression. The idea is to reduce the number for nonzero elements to $\mathcal{O}(N \log N)$ by compressing the wavelet-based stiffness matrix $\mathbf{A}$ to a new matrix $\tilde{\mathbf{A}}$ by setting elements of the original matrix equal to zero.

The main scope of this work is to utilize wavelet basis compression techniques similar to the ones proposed in [16, 17], in case of parabolic integro-differential equations, when data f, u_0 satisfy minimal regularity assumptions. Parabolic integro-differential equations typically exhibit low regularity, which significantly complicates the analysis of numerical schemes.

In this work, we prove semi-discrete in space error estimates of optimal order. In addition, we analyze a fully-discrete scheme that allows possible discontinuities in time in order to better handle the low regularity of solutions. To summarize, the following issues are being addressed:

- The basic estimate is derived under minimal regularity assumptions, i.e., we assume no more regularity than necessary for the existence and uniqueness of solutions.
- Our estimates are also valid when compressed stiffness matrices $\tilde{\mathbf{A}}$ (obtained by setting equal to zero elements of the original matrix $\mathbf{A}$ induced by the integral operator) are being used. This procedure reduces the complexity of the scheme.
- The fully-discrete scheme is discontinuous in time, which better accomodates the low regularity of solutions. The error estimates associated to the fully-discrete scheme are derived at partition points as well as at arbitrary

time points and are applicable for higher order approximations, provided that natural regularity assumptions hold.
- Different subspaces can be used in each time step to allow greater flexibility.

We also emphasize that our estimates are also applicable in case of finite element basis, when effects of numerical approximate integration of the stiffness matrix are also taken into account.

For other wavelet based methods for the solution of general operator equations, and their relation to adaptive and multiscale strategies, we refer the reader to the works of [3, 7, 9] (see also references within). In particular, several results regarding the complexity analysis of adaptive schemes constructed by utilizing wavelet-Galerkin techniques are given in [3] for abstract operator equations. For some relevant results in case of integro-differential equations one may consult [16], while adaptive wavelet algorithms for elliptic control problems were studied in [8].

The paper is organized as follows: In Section 2, we present the weak formulation of (1.1) and we state the basic results with respect to existence and uniqueness. In Section 3, we treat the semi-discrete (in space) case and we prove error estimates using a perturbed weak bilinear form. In Section 4, we analyze a fully-discrete scheme which is conforming in space but discontinuous in time. Finally, Section 5 is devoted to the discussion of complexity issues due to the wavelet compression of the stiffness matrix $\mathbf{A}$ while Section 6 contains some concluding remarks.

2 Preliminaries.

We shall use standard notation for Sobolev spaces $L^2(\Omega), H^1(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$, and we denote by H^{-1} the dual space of $H_0^1(\Omega)$. For the definition of Sobolev spaces of fractional order see [1]. For $s \geq 0$, we denote by

$$\bar{H}^s(\Omega) = \big\{u|_\Omega : u \in H^s(\mathbb{R}^d), u|_{\mathbb{R}^d/\Omega} = 0\big\}.$$

Note that if $s + \frac{1}{2}$ is not an integer, then $\bar{H}^s(\Omega) \equiv H_0^s(\Omega)$ (see e.g. [1]). We will use the standard evolution triple (see e.g. [22]) $V \subset L^2(\Omega) \equiv H \subset V^*$, where V^* denotes the dual of a Hilbert space V, with $V = \bar{H}^{(\rho/2)}(\Omega)$, and $0 \leq \rho \leq 2$ denotes the order of the operator $A(.)$.

EXAMPLE 1. For the pure diffusion problem, $\rho = 2$, and the operator A takes the form $A = div(\mathbf{D}(x)\nabla)$. Then $V = H_0^1(\Omega), V^* = H^{-1}(\Omega)$ and $a(u, v) = \int_\Omega \nabla v \cdot (\mathbf{D}(x)\nabla u)dx$. Standard assumptions are imposed on $\mathbf{D}$ in order to guarantee continuity and coercivity for the bilinear form $a(.\,,.)$.

EXAMPLE 2. The classical pseudo-differential operators of order $0 \leq \rho < 2$, defined on an open, bounded and Lipschitz domain Ω of $\mathbb{R}^d$. Here, $V = \bar{H}^{(\rho/2)}$, $A : V \to V^*$, while A has a representation in terms of a kernel $k(x, x - y) \in \mathcal{D}'(\Omega \times \Omega)$, in distributional sense, due to the Schwartz kernel theorem. The bilinear form is then defined as $a(u, v) = \langle Au, v\rangle_{V^*,V} = \langle k(x, x-y), v(x) \times v(y)\rangle$.

We also employ the standard notation for inner products, norms, and duality pairings. To simplify the notation the duality pairing $\langle .\,,.\rangle_{V^*,V}$ is occasionally denoted by $\langle .\,,.\rangle$. If V is a Hilbert space, we denote by $L^p(0,T;V), H^1(0,T;V)$ the time-space function spaces such that

$$\|v\|^p_{L^p(0,T,V)} \equiv \int_0^T \|v(t)\|^p_V < \infty \quad \forall v \in L^p(0,T;V),\ 1 \leq p < \infty$$

and

$$\|v\|^2_{H^1(0,T;V)} \equiv \int_0^T \left(\|v(t)\|^2_V + \|v_t(t)\|^2_V\right) < \infty \quad \forall v \in H^1(0,T;V),$$

together with the standard modification for $L^\infty(0,T;V)$. We associate to the operator A a bilinear form $a(.\,,.) : V \times V \to \mathbb{R}$, via

$$a(u,v) := \langle Au, v\rangle \quad \forall u, v \in V,$$

and we assume that standard coercivity and continuity assumptions hold. In particular, there exist positive constants α, β depending only on the domain, such that

$$a(u,v) \leq \alpha \|u\|_V \|v\|_V, \quad \forall\, u, v \in V, \tag{2.1}$$

$$a(u,u) \geq \beta \|u\|^2_V, \quad \forall\, v \in V. \tag{2.2}$$

In most cases associated to pseudo-differential operators, the weaker Gårding's inequality is valid instead of (2.2), i.e., we assume that there exists $\beta_1, \beta_2 > 0$ such that

$$a(u,u) + \beta_1 \|u\|^2_{L^2(\Omega)} \geq \beta_2 \|u\|^2_V, \quad \forall\, u \in V. \tag{2.3}$$

Recall that a standard substitution $w = \exp(-\beta_1 t)u$ into the original equation, leads to an equation $w' + (A + \beta_1 I)w = \tilde{f}$, which satisfies (2.2).

A weak solution for given data f, u_0 satisfying minimal regularity assumptions, is defined as follows: Given

$$f \in L^2(0,T;V^*), \quad u_0 \in H$$

we seek

$$u \in L^2(0,T;V) \cap H^1(0,T;V^*)$$

such that for all $v \in L^2(0,T;V) \cap H^1(0,T;V^*)$,

$$(u(T), v(T)) - (u_0, v(0)) + \int_0^T (-\langle u, v_t\rangle + a(u(t), v)) = \int_0^T \langle f, v\rangle \tag{2.4}$$

where the continuous embedding $L^2(0,T;V) \cap H^1(0,T;V^*) \subset C(0,T;L^2(\Omega))$ (see e.g. [10, 15, 22]) justifies the above definition. For existence and uniqueness of a weak solution under minimal regularity assumptions one may consult [14, 22], or [10, Chapter 7] for $\rho = 2$. We close this preliminary section by noting that we may define an equivalent norm on V, the standard energy norm, by

$$\|v\|_a \equiv (a(u,u))^{1/2} \approx \|v\|_V.$$

3 Discretization in space.

3.1 The semi-discrete (in space) formulation.

We discretize (2.4) based on a Galerkin scheme. For that purpose, we assume that a family of approximation spaces $V^h \subset V$ is given, satisfying the "standard approximation properties" (see e.g. [2, 20]). V^h consists of piecewise polynomials of degree $p \geq 0$, constructed on a quasi-uniform triangulation. In order to properly state some approximation properties for functions in V^h having zero boundary values we also define the spaces

$$\mathcal{H}^s(\Omega) = \begin{cases} \tilde{H}^{\rho/2} & \text{if } s = \rho/2, \\ \tilde{H}^{\rho/2} \cap H^s(\Omega) & \text{if } s \geq \rho/2. \end{cases}$$

Then, we assume that V^h has the following approximation properties: $\forall v \in \mathcal{H}^t(\Omega)$, $t \geq (\rho/2)$,

$$\inf_{v^h \in V^h} \|v - v^h\|_{\tilde{H}^s(\Omega)} \leq C h^{t-s} \|v\|_{\mathcal{H}^t(\Omega)}, \quad 0 \leq s \leq \rho/2,\ \rho/2 \leq t \leq p+1. \tag{3.1}$$

In addition, standard inverse inequalities are required:

$$\|v^h\|_{\tilde{H}^s(\Omega)} \leq C h^{-s} \|v^h\|_{L^2(\Omega)}, \quad 0 \leq s \leq \rho/2. \tag{3.2}$$

Then, the semi-discrete approximation of (2.4) is defined as follows: We seek $u^h \in H^1(0,T;V^h)$ such that

$$(u^h(T), v^h(T)) - \big(u_0^h, v^h\big) + \int_0^T \big(-\langle u^h, v_t^h \rangle + a(u^h, v^h)\big) = \int_0^T \langle f, v^h \rangle \tag{3.3}$$

for all $v^h \in H^1(0,T;V^h)$, where the initial approximation to the initial data u_0^h is chosen as the standard L_2 projection of the initial data into V^h, i.e., $u_0^h = P^h u_0$.

3.2 A class of perturbed problems.

In this section we relax the assumption that the bilinear form $a(.,.) : V^h \times V^h \to \mathbb{R}$ can be evaluated exactly. Throughout the rest of the paper, we assume that only a perturbed bilinear form can be computed, i.e., we introduce a bilinear form $\tilde{a}(.,.) : V^h \times V^h \to \mathbb{R}$. Note that the subsequent analysis covers the case where the original stiffness matrix $\mathbf{A}$ is compressed, resulting a perturbed matrix $\tilde{\mathbf{A}}$, as well as the standard case of errors due to numerical integration, or domain approximation by isoparametric elements. Our particular interest is to derive error estimates suitable for wavelet compression techniques and we pay extra attention on the regularity assumptions on the given data.

Using the perturbed form $\tilde{a}(.,.)$ we define the perturbed semi-discrete (in space) approximations as follows: We seek $\tilde{u} \in H^1(0,T;V^h)$ such that

$$(\tilde{u}^h(T), v^h(T)) - \big(u_0^h, v^h\big) + \int_0^T \big(-\langle \tilde{u}^h, v_t^h \rangle + \tilde{a}(\tilde{u}^h, v^h)\big) = \int_0^T \langle f, v^h \rangle \tag{3.4}$$

for all $v^h \in H^1(0,T;V^h)$. The following consistency assumptions are made on the perturbed bilinear form $\tilde{a}(.,.)$.

ASSUMPTION 3.1. *There exists $\delta < 1$ independent of h such that*

$$|a(u^h, v^h) - \tilde{a}(u^h, v^h)| \leq \delta \|u^h\|_a \|v^h\|_a \quad \forall\, u^h, v^h \in V^h. \tag{3.5}$$

There exists $\nu > 0$ depending on the choice of bilinear form $\tilde{a}(.,.)$, and a projector $I^h : V \to V^h$ such that for every $v^h \in V^h$ and $u \in \mathcal{H}^{p+1}(\Omega)$

$$|a(I^h u, v^h) - \tilde{a}(I^h u, v^h)| \leq C h^{p+1-\rho/2} |\log h|^{\nu} \|u\|_{\mathcal{H}^{p+1}(\Omega)} \|v_h\|_{\bar{H}^{\rho/2}(\Omega)} \tag{3.6}$$

where $C > 0$ depends only on the domain.

Then, it is easy to see that the following coercivity and continuity properties of $\tilde{a}(.,.)$ also hold.

LEMMA 3.2. *Suppose that Assumption 3.1 is valid. Then, there exist positive constants $\tilde{\alpha}, \tilde{\beta}$ such that*

$$|\tilde{a}(u^h, v^h)| \leq \tilde{\alpha} \|u^h\|_a \|v^h\|_a \quad \forall\, u^h, v^h \in V^h$$

and

$$|\tilde{a}(u^h, u^h)| \geq \tilde{\beta} \|u^h\|_a^2.$$

PROOF. See [17, Proposition 3.2] □

As we will state in the last section (for proofs see e.g. [17] and references within) the perturbed bilinear forms $\tilde{a}(.,.)$ obtained by wavelet basis compression techniques satisfy the consistency assumptions, as well as the basic approximation theory assumptions (3.1). In our proofs we will also employ L^2 projections similar [5]. Recall that the standard L^2 projection $P^h : L^2(\Omega) \to V^h$ is defined by $(v - P^h v, v^h) = 0,\ \forall\, v^h \in V^h$.

3.3 Main estimate.

We are now ready to prove the main estimate. This estimate can be viewed as a generalization of [5, Theorem 3.2].

THEOREM 3.3. *Suppose that V^h is a finite dimensional subspace of V satisfying the standard approximation properties* (3.1)*. In addition, suppose that the perturbed bilinear form $\tilde{a}(.,.)$ satisfies Assumption 3.1. Given $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;V^*)$, let $u, \tilde{u}$ be the solutions of* (2.4)–(3.4)*. Then the following estimate holds:*

(3.7)

$$\begin{aligned}
&\|u(T) - \tilde{u}^h(T)\|^2_{L^2(\Omega)} + \|u - \tilde{u}^h\|^2_{L^2(0,T;V)} \\
&\quad \leq C\Big(\|u_0 - \tilde{u}_0^h\|^2_{L^2(\Omega)} + \|u - P^h u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u - P^h u\|^2_{L^2(0,T;V)} \\
&\qquad + \int_0^T |a(I^h u, P^h u - \tilde{u}^h) - \tilde{a}(I^h u, P^h u - \tilde{u}^h)| \Big).
\end{aligned}$$

In addition, assuming $u \in L^2(0,T;\mathcal{H}^{p+1}(\Omega)) \cap H^1(0,T;\mathcal{H}^{p-1}(\Omega))$ then,

$$\|u - \tilde{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \|u - \tilde{u}^h\|_{L^2(0,T;V)} \leq Ch^{p+1-\rho/2}(|\log h|^\nu + 1)$$

where $\nu > 0$ is the constant of Assumption 3.1 and depends on the compressed bilinear form $\tilde{a}(,.,)$. The constant C depends on $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, \delta$ and on the domain.

PROOF. Subtracting (3.4) from (2.4), and denoting by $e = u - \tilde{u}^h$, we obtain:

$$(e(T), v^h(T)) - (e(0), v^h(0)) + \int_0^T \big(- \langle e, v_t^h \rangle + a(u, v^h) - \tilde{a}(\tilde{u}^h, v^h)\big) = 0 \tag{3.8}$$

for all $v^h \in H^1(0,T;V^h)$. We decompose the error into two parts: $e = (u - P^h u) + (P^h u - \tilde{u}^h) \equiv e_p + e_h$, and calculate:

(3.9)

$$\begin{aligned}
\big(e_h(T), v^h(T)\big) - \big(e_h(0), v^h(0)\big) &+ \int_0^T \big(- \langle e_h, v_t^h \rangle + a(e_h, v^h)\big) \\
&= (e(T), v^h(T)) - (e(0), v^h(0)) + \int_0^T \big(- \langle e, v_t^h \rangle + a(e, v^h)\big) \\
&\quad - \big(e_p(T), v^h(T)\big) + \big(e_p(0), v^h(0)\big) + \int_0^T \big(\langle e_p, v_t^h \rangle - a\big(e_p, v^h\big)\big) \\
&= (e(T), v^h(T)) - (e(0), v^h(0)) + \int_0^T \big(- \langle e, v_t^h \rangle + a(e, v^h)\big) \\
&\quad - \int_0^T a\big(e_p, v^h\big)
\end{aligned}$$

due to projection properties. Note that $a(e, v^h) = a(u, v^h) - \tilde{a}(\tilde{u}^h, v^h) + \tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h)$, so Equation (3.9) together with the orthogonality condition (3.8) imply,

$$\begin{aligned}
\big(e_h(T), v^h(T)\big) - \big(e_h(0), v^h(0)\big) &+ \int_0^T \big(- \langle e_h, v_t^h \rangle + a\big(e_h, v^h\big)\big) \\
&= \int_0^T (\tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h)) - \int_0^T a\big(e_p, v^h\big).
\end{aligned} \tag{3.10}$$

Adding and subtracting appropriate terms, we may rewrite

$$\tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h) = \tilde{a}(\tilde{u}^h - I^h u, v^h) - a(\tilde{u}^h - I^h u, v^h) + \mathrm{Con}_1$$

where $\mathrm{Con}_1 = \tilde{a}(I^h u, v^h) - a(I^h u, v^h)$ denotes the consistency term of Assumption (3.1). Note also that adding and subtracting $P^h u$ and using the definition of e_h we arrive to

$$\begin{aligned}
\tilde{a}(\tilde{u}^h - I^h u, v^h) - a(\tilde{u}^h - I^h u, v^h) = &-\tilde{a}\big(e_h, v^h\big) + \tilde{a}(P^h u - I^h u, v^h) \\
&+ a\big(e_h, v^h\big) - a(P^h u - I^h u, v^h).
\end{aligned}$$

Combining the last two relations and Equation (3.10) we obtain,

$$\begin{aligned}(3.11)\quad &\big(e_h(T), v^h(T)\big) - \big(e_h(0), v^h(0)\big) + \int_0^T \big(- \langle e_h, v_t^h\rangle + \tilde{a}(e_h, v^h)\big) \\ &= \int_0^T \big(\tilde{a}(P^h u - I^h u, v^h) - a(P^h u - I^h u, v^h) - a\big(e_p, v^h\big) + \mathrm{Con}_1\big).\end{aligned}$$

Setting $v^h = e_h$ into (3.11) and using the continuity and coercivity properties of bilinear forms, we may bound the first three terms on the right hand side by

$$\big|\tilde{a}\big(P^h u - I^h u, e_h\big) - a\big(P^h u - I^h u, e_h\big)\big| \leq \frac{\tilde{\beta}}{4}\|e_h\|_a^2 + C(\alpha, \tilde{\alpha}, \tilde{\beta})\|P^h u - I^h u\|_a^2$$

and

$$|a(e_p, e_h)| \leq \frac{\tilde{\beta}}{4}\|e_h\|_a^2 + C(\alpha, \tilde{\beta})\|e_p\|_a^2$$

which clearly lead to

$$\begin{aligned}\frac{1}{2}\big(\|e_h(T)\|_{L^2(\Omega)}^2 - \|e_h(0)\|_{L^2(\Omega)}^2\big) + \frac{\tilde{\beta}}{2}\int_0^T \|e_h\|_a^2 \\ \leq C\Big(\int_0^T \big(\|e_p\|_a^2 + \|P^h u - I^h u\|_a^2\big) \\ + \int_0^T |a(I^h u, P^h u - \tilde{u}^h) - \tilde{a}(I^h u, P^h u - \tilde{u}^h)|\Big).\end{aligned}$$

The first part of the theorem follows using standard techniques. For the the second part we also use the approximation properties of the projections, the equivalence of $\|.\|_a, \|.\|_V$ norms and the consistency Assumption 3.1, which allows to hide e_h to the left hand side. ◻

REMARK 3.4. The structure of the estimate of Theorem 3.3 is as follows: The first term is the standard initial data approximation error, while the second and third terms contain the error due to standard approximation properties. Finally the last term contains the "consistency error". Note that we were able to uncouple the estimates on u from estimates on u_t, and our estimate does not contain any time-derivative term on the right hand side.

REMARK 3.5. Suppose that $\rho = 2$ and that the bilinear forms are induced by the standard diffusion operator. If we assume that $u \in L^2(0,T;\mathcal{H}^{p+1}(\Omega)) \cap H^1(0,T;\mathcal{H}^{p-1}(\Omega))$ then an embedding theorem, implies that $u \in C(0,T;\mathcal{H}^p(\Omega))$, which guarantees that $u_0 \in \mathcal{H}^p(\Omega)$ and $f \in L^2(0,T;\mathcal{H}^{p-1}(\Omega))$. However, if we suppose that $f \in L^2(0,T;\mathcal{H}^{p-1}(\Omega)), u_0 \in \mathcal{H}^p(\Omega)$ there is no guarantee that the solution $u \in L^2(0,T;\mathcal{H}^{p+1}(\Omega)) \cap H^1(0,T;\mathcal{H}^{p-1}(\Omega))$. Appropriate compatibility conditions (see e.g. [10, Section 7]) as well as additional regularity on f and its time-derivatives are needed to guarantee the existence of a solution that satisfies the desired regularity properties. Similar observations also hold for the more general case of integro-differential equations.

4 The discontinuous (in time) scheme.

4.1 The fully-discrete formulation.

The main theme of this section concerns the discretization of (1.1) in space and time. We have chosen a discontinuous (in time), conforming in space scheme which allows us flexibility in the choice of the discrete subspaces and accomodates the low regularity of the weak solutions. Similar to the semi-discrete case, the weak formulation of (1.1) under minimal regularity assumptions is to find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$(u(T), v(T)) + \int_0^T (-\langle u, v_t\rangle + a(u,v)) = (u(0), v(0)) + \int_0^T \langle f, v\rangle. \tag{4.1}$$

The approximate solution is introduced on a partition $0 = t^0 < t^1 < \cdots < t^N = T$, and on each partition we construct a closed subspace $V_h^n \subset V$. We denote by $\tau^n = t^n - t^{n-1}, n = 1,\dots,N$ and by $\tau = \max_{n=1,\dots,N} \tau^n$. The space of approximate solutions is denoted by

$$\mathcal{U}_h = \left\{u^h \in L^2(0,T;V) : u^h|_{(t^{n-1},t^n]} \in \mathcal{P}_k\big(t^{n-1}, t^n, V_h^n\big)\right\}$$

where $\mathcal{P}_k(t^{n-1}, t^n, V_h^n)$ denotes the polynomials of degree k or less with values on V_h^n. Therefore, using the notation of Section 3 for the perturbed bilinear form, the discontinuous Galerkin method constructs an approximate solution $\tilde{u}^h|_{(t^{n-1},t^n]} \in \mathcal{P}_k(t^{n-1}, t^n, V_h^n)$ such that

$$\begin{aligned}(\tilde{u}^n, v^n) &+ \int_{t^{n-1}}^{t^n} \big(- \langle \tilde{u}^h, v_t^h\rangle + \tilde{a}(\tilde{u}^h, v^h)\big) \\ &= (\tilde{u}^{n-1}, v_+^{n-1}) + \int_{t^{n-1}}^{t^n} \langle f, v^h\rangle \quad \forall v^h \in \mathcal{P}_k\big(t^{n-1}, t^n; V_h^n\big).\end{aligned} \tag{4.2}$$

In the above weak formulation we denote by $v_+^n = \lim_{s\to 0^+} v(t^n + s)$, $n = 0,\dots,N-1$ the trace from above, by $v^n \equiv v_-^n = \lim_{s\to 0^+} v(t^n - s), n = 1,\dots,N$ the trace from below, and by $[v^n] = v_+^n - v^n, n = 0,\dots,N-1$ the corresponding jump term. Note that the embedding $L^2(0,T;V) \cap H^1(0,T;V^*) \subset C(0,T;L^2(\Omega))$ justifies the existence of point-wise values. Therefore, subtracting (4.2) from (4.1) we obtain the following "orthogonality condition", i.e., $e = u - \tilde{u}^h$ satisfies:

$$\begin{aligned}(e^n, v^n) &+ \int_{t^{n-1}}^{t^n} \big(- \langle e_h, v_t^h\rangle + a(u, v^h) - \tilde{a}(\tilde{u}^h, v^h)\big) \\ &= (e^{n-1}, v_+^{n-1}) \quad \forall v^h \in \mathcal{P}_k\big(t^{n-1}, t^n; U_h^n\big).\end{aligned} \tag{4.3}$$

Following the work of [12], we split the error into two terms, i.e., $e = u - \tilde{u}^h = (u - \bar{P}^h u) + (\bar{P}^h u - \tilde{u}^h) = e_p + e_h$, where $\bar{P}^h$ is defined as follows:

DEFINITION 4.1.

(1) *The projection* $\bar{P}^n_{loc} : C(t^{n-1}, t^n; L^2(\Omega)) \to \mathcal{P}_k(t^{n-1}, t^n; V^n_h)$ *satisfies* $(\bar{P}^n_{loc} u)^n = P^n u(t^n)$, *and*

$$\int_{t^{n-1}}^{t^n} \big(u - \bar{P}^n_{loc} u, v_h\big) = 0, \qquad \forall\, v_h \in \mathcal{P}_{k-1}\big(t^{n-1}, t^n; V^n_h\big).$$

Here we have used the convention $(\bar{P}^n_{loc} u)^n \equiv (\bar{P}^n_{loc} u)(t^n)$ *and* $P^n : L^2(\Omega) \to V^n_h$ *is the projection operator onto* $V^n_h \subset L^2(\Omega)$.

(2) *The projection* $\bar{P}^h : C(0, T; L^2(\Omega)) \to \mathcal{U}_h$ *satisfies*

$$\bar{P}^h u \in \mathcal{U}_h \text{ and } (\bar{P}^h u)|_{(t^{n-1}, t^n]} = \bar{P}^n_{loc}(u|_{(t^{n-1}, t^n]}).$$

We also define the local "consistency projection" into each V^n_h, i.e., $I^n_h : V \to V^n_h$ satisfying the consistency properties of Assumption (3.1). In addition we denote by $\bar{I}^h|_{(t^{n-1}, t^n]} \equiv I^n_h$.

The projection $\bar{P}^h$ satisfies standard approximation properties (see e.g. [12]). The projection $\bar{I}^h$ also satisfies the standard approximation properties (3.1). Now, we are ready to state and prove the main result. The proof uses techniques from [6, Section 2], appropriately modified to handle the perturbed bilinear form.

THEOREM 4.2. *Let* $u, \tilde{u}^h$ *be the solutions of* (4.1)–(4.2) *respectively. Suppose that the spaces* V^n_h *consist of piecewise polynomials of degree* $p \geq 0$, *satisfying approximation properties* (3.1) *and the perturbed bilinear form satisfy the consistency Assumption 3.1. Let* $e_h = \bar{P}^h u - \tilde{u}^h$, *where* $\bar{P}^h$ *is the projection of Definition 4.1. Then the following estimate holds:*

$$\begin{aligned} \|e^N_h\|^2_{L^2(\Omega)} &+ \sum_{n=0}^{N-1} \|e^n_h - e^n_{h+}\|^2_{L^2(\Omega)} + \int_0^T \|e_h\|^2_V \\ &\leq C\Big(\|e^0_h\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \|(I - P^{n-1}) u(t^{n-1})\|^2_{L^2(\Omega)} \\ &\qquad + \int_0^T \big(\|u - \bar{P}^h u\|^2_V + \|u - \bar{I}^h u\|^2_V + \big|\tilde{a}\big(\bar{I}^h u, e_h\big) - a\big(\bar{I}^h u, e_h\big)\big|\big)\Big). \end{aligned}$$

In addition, let $u \in L^2(0, T; \mathcal{H}^{p+1}(\Omega))$, $u_t \in L^2(0, T; \mathcal{H}^k(\Omega))$ *where* $k \geq 0$. *Then for a time step size* $\tau \approx h$ *the following estimate holds:*

$$\begin{aligned} \|e^N\|_{L^2(\Omega)} + \|e\|_{L^2(0,T;V)} \leq C(&h^{p+1-(\rho/2)} |\log h|^\nu + \max\{h^{p+(1/2)}, h^{p+1-(\rho/2)}\} \\ &+ \tau^{k+1-(\rho/2)}), \end{aligned}$$

where $\nu > 0$ *is a constant depending on the compressed bilinear form* $\tilde{a}(.,.)$ *and* C *is a constant depending on* $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, \delta$ *and on the domain.*

PROOF. Recall, that we split the error in two terms, i.e., $e = u - \tilde{u}^h = e_p + e_h$. Then adding and subtracting appropriate terms, (4.3) implies:

$$
\begin{aligned}
(4.4)\quad & \big(e_h^n, v^n\big) + \int_{t^{n-1}}^{t^n} \big(- \langle e_h, v_t^h \rangle + a\big(e_h, v^h\big)\big) - \big(e_h^{n-1}, v_+^{n-1}\big) \\
&= \big(e^n, v^n\big) + \int_{t^{n-1}}^{t^n} \big(- \langle e, v_t^h \rangle + a(e, v^h)\big) - \big(e^{n-1}, v_+^{n-1}\big) \\
&\quad - \Big(\big(e_p^n, v^n\big) + \int_{t^{n-1}}^{t^n} \big(- \langle e_p, v_t^h \rangle + a\big(e_p, v^h\big)\big) - \big(e_p^{n-1}, v_+^{n-1}\big)\Big) \\
&= \big(e^n, v^n\big) + \int_{t^{n-1}}^{t^n} \big(- \langle e, v_t^h \rangle + a(e, v^h)\big) - \big(e^{n-1}, v_+^{n-1}\big) \\
&\quad - \big(e_p^{n-1}, v_+^{n-1}\big) - \int_{t^{n-1}}^{t^n} a\big(e_p, v^h\big)
\end{aligned}
$$

where at the last equality we have used the definition of $\bar{P}^h$ which implies that the first two e_p terms of the right hand side vanish. Note that the orthogonality condition (4.3) together with the elementary observation,

$$
a(e, v^h) = a(u, v^h) - \tilde{a}(\tilde{u}^h, v^h) + \tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h)
$$

imply that (4.4) is equivalent to:

$$
\begin{aligned}
(4.5)\quad & \big(e_h^n, v^n\big) + \int_{t^{n-1}}^{t^n} \big(- \langle e_h, v_t^h \rangle + a\big(e_h, v^h\big)\big) - \big(e_h^{n-1}, v_+^{n-1}\big) \\
&= -\big(e_p^{n-1}, v_+^{n-1}\big) + \int_{t^{n-1}}^{t^n} (\tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h))dt - \int_{t^{n-1}}^{t^n} a\big(e_p, v^h\big).
\end{aligned}
$$

Similar to the proof of Theorem 3.3, adding and subtracting appropriate terms, we denote by $\mathrm{Con}_1 = \tilde{a}(\bar{I}^h u, v^h) - a(\bar{I}^h u, v^h)$ and rewrite

$$
\begin{aligned}
\tilde{a}(\tilde{u}^h, v^h) - a(\tilde{u}^h, v^h) &= \tilde{a}(\tilde{u}^h - \bar{I}^h u, v^h) - a(\tilde{u}^h - \bar{I}^h u, v^h) + \mathrm{Con}_1 \\
&= -\tilde{a}\big(e_h, v^h\big) + \tilde{a}(\bar{P}^h u - \bar{I}^h u, v^h) \\
&\quad + a\big(e_h, v^h\big) - a(\bar{P}^h u - \bar{I}^h u, v^h) + \mathrm{Con}_1.
\end{aligned}
$$

Combining the last inequality and (4.5) and setting $v^h = e_h$, we obtain

$$
\begin{aligned}
& \big\|e_h^n\big\|_{L^2(\Omega)}^2 - \big\|e_h^{n-1}\big\|_{L^2(\Omega)}^2 + \big\|e_h^{n-1} - e_{h+}^{n-1}\big\|_{L^2(\Omega)}^2 + \tilde{\beta} \int_{t^{n-1}}^{t^n} \|e_h\|_a^2 \\
&\le C(\alpha, \tilde{\alpha}, \tilde{\beta}) \Big(\big| \big((I - P^{n-1}) u(t^{n-1}), e_{h+}^{n-1}\big) \big| + \int_{t^{n-1}}^{t^n} \|\bar{P}^h u - \bar{I}^h u\|_a \|e_h\|_a \\
&\qquad + \int_{t^{n-1}}^{t^n} (\|e_p\|_a \|e_h\|_a + \mathrm{Con}_1) \Big).
\end{aligned}
$$

It remains to bound the first term of the right hand side. For that purpose note that using the definition of projection P^{n-1},

$$\begin{aligned}\big|\big((I-P^{n-1})u(t^{n-1}),e_{h+}^{n-1}\big)\big| &= \big|\big((I-P^{n-1})u(t^{n-1}),e_{h+}^{n-1}-e_{h}^{n-1}\big)\big| \\ &\leq \|(I-P^{n-1})u(t^{n-1})\|^2_{L^2(\Omega)}+\frac{1}{4}\big\|e_{h+}^{n-1}-e_{+}^{n-1}\big\|_{L^2(\Omega)}\end{aligned}$$

since $e_h^{n-1}\in V_h^{n-1}$. The first estimate now follows by standard techniques after noting the equivalence of $\|.\|_V$, $\|.\|_a$ norms. For the second estimate note that the regularity of solution $u\in L^2(0,T;\mathcal{H}^{p+1}(\Omega))$ and the consistency assumption (3.6) implies that $|\mathrm{Con}_1|\leq h^{p+1-(\rho/2)}|\log h|^{\nu}\|u\|_{\mathcal{H}^{p+1}(\Omega)}\|e_h\|_V$, so we may hide all $\|e_h\|_V$ to the right. Note that using the stability of the $L^2(\Omega)$ projection, and the approximation property (3.1) for $s=0$, $t=p+1$, we may bound the error due to the projection at the jumps as:

$$\begin{aligned}\sum_{n=0}^{N-1}\big\|(I-P_{loc}^{n-1})u(t^{n-1})\big\|^2_{L^2(\Omega)} &\leq CNh^{2p+2}\big\|D_x^{p+1}u\big\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C(T/\tau)h^{2p+2}\big\|D_x^{p+1}u\big\|^2_{L^\infty(0,T;L^2(\Omega))}\end{aligned}$$

which clearly implies that for $\tau\approx h$ the above term is of order $\mathcal{O}(h^{2p+1})$. Standard approximation properties imply that

$$\|e_p\|_{L^2(0,T;L^2(\Omega))}\leq C\big(h^{p+1}\big\|D_x^{p+1}u\big\|_{L^2(0,T;L^2(\Omega))}+\tau^{k+1}\big\|D_t^{k+1}u\big\|_{L^2(0,T;L^2(\Omega))}\big)$$

and using an inverse estimate (3.2) for $s=\rho/2$,

$$\begin{aligned}&\|e_p\|_{L^2(0,T;V)} \\ &\quad\leq C\left(h^{p+1-(\rho/2)}\big\|D_x^{p+1}u\big\|_{L^2(0,T;L^2(\Omega))}+\frac{\tau^{k+1}}{h^{(\rho/2)}}\big\|D_t^{k+1}u\big\|_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}$$

Using triangle inequality, we finally arrive at

$$\begin{aligned}\|e^N\|^2_{L^2(\Omega)}+\|e\|^2_{L^2(0,T;V)} &\leq C\big(\|e^0\|^2_{L^2(\Omega)}+\big\|e_p^N\big\|^2_{L^2(\Omega)}+\|e_p\|^2_{L^2(0,T;H^1(\Omega))} \\ &\qquad+\big\|e_h^N\big\|^2_{L^2(\Omega)}+\|e_h\|^2_{L^2(0,T;H^1(\Omega))}\big) \\ &\leq C\big(h^{2(p+1)-\rho}(|\log h|^{2\nu})+\max\{h^{2p+1},h^{2(p+1)-\rho}\} \\ &\qquad+\tau^{2k+2-\rho}\big).\end{aligned}$$

□

REMARK 4.3. Note that for $1\leq\rho\leq 2$ the estimate of Theorem 4.2 leads to an error of order $\mathcal{O}(h^{p+1-(\rho/2)}|\log h|^{\nu}+h^{p+(1/2)}+\tau^{k+1-(\rho/2)})$ for the natural energy norm, while for $0\leq\rho<1$ an estimate of order $\mathcal{O}(h^{p+1-(\rho/2)}(|\log h|^{\nu}+1)+\tau^{k+1-(\rho/2)})$ is valid.

Remark 4.4. For the diffusion operator A_D, $\rho = 2$, using the projection properties of P^n, P^{n-1} and using an inverse estimate, we obtain

$$\begin{aligned}
&\big|\big((I - P^{n-1})u(t^{n-1}), e_{h+}^{n-1}\big)\big| \\
&\qquad = \big|\big(P^n(I - P^{n-1})u(t^{n-1}), e_{h+}^{n-1}\big)\big| \\
&\qquad \leq \|P^n(I - P^{n-1})u(t^{n-1})\|_{H^{-1}(\Omega)} \big\|e_{h+}^{n-1}\big\|_{H^1(\Omega)} \\
&\qquad \leq (C_k/\tau)\|P^n(I - P^{n-1})u(t^{n-1})\|_{H^{-1}(\Omega)} \int_{t^{n-1}}^{t^n} \|e_h\|^2_{H_0^1(\Omega)} \\
&\qquad \leq (C_k/\tau)\|P^n(I - P^{n-1})u(t^{n-1})\|^2_{H^{-1}(\Omega)} + \frac{\tilde{\beta}}{4}\int_{t^{n-1}}^{t^n} \|e_h\|^2_{H^1(\Omega)}.
\end{aligned}$$

Note that last term can moved to right, while the first bound leads to an improved estimate for $\tau \approx h$ of the form,

$$\begin{aligned}
\sum_{n=0}^{N-1}(C_k/\tau)\|P^n(I - P^{n-1})u(t^{n-1})\|^2_{H^{-1}(\Omega)} &\leq \big(C_k\tau^2\big)h^{2p+4}\big\|D_x^{p+1}u\big\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leq C(C_k, u)h^{2p+2}.
\end{aligned}$$

Similar estimates are also valid for the integro-differential equations, provided that the enhanced approximation properties for the dual norms $\|.\|_{V^*}$ are valid. For finite element discretizations such estimates on negative norms are indeed true for sufficiently regular triangulations.

Remark 4.5. Note that using the triangle inequality the first estimate reveals that the error estimate consists of three distinct components:

$$\|e^N\|^2_{L^2(\Omega)} + \|e\|_{L^2(0,T;V)} \leq C(\|\text{approx. error}\| + \|\text{cons. error}\| + \|\text{subsp. error}\|).$$

In particular, we emphasize that the term $\sum_{n=0}^{N} |(I - P^{n-1})u(t^{n-1})|^2_{L^2(\Omega)}$ is due to the use of different subspaces in each time step. In a computational scheme, measures whether or not a new subspace, and hence remeshing, is needed.

Remark 4.6. As stated in [12], [20, Theorem 12.3] (see also references within), a super-convergent estimate for $k \geq 2$, at nodal points can be derived for the semi-discrete in time approximations, using duality techniques. However, severe regularity assumptions have to be imposed. For a posteriori analysis, and its relation to adaptive schemes for parabolic PDE's one may consult [11].

Remark 4.7. In [18] the hp-discontinuous Galerkin scheme based on wavelet basis is analyzed for parabolic equations in higher dimensions. A key ingredient of the main proof is the use of sparse tensor product spaces to reduce the number of degrees of freedom. The main result requires higher regularity on the initial data, or a solution operator that admits the parabolic smoothing effect.

4.2 Error estimates at arbitrary times.

In addition to estimates on the natural energy norm, we derive error estimates at arbitrary times. Our technique is based on techniques of [6, Section 2.3, 2.4] appropriately modified in order to handle the auxiliary perturbed bilinear form. An important element of the technique of [6] is the construction of approximations of discrete characteristic functions. Below, we quote the main definitions and results.

Note that to compute the error at arbitrary times we would have liked to set $v_h = \chi_{[t^{n-1},t)} u_h$ into the orthogonality condition. However, this is not a function in $\mathcal{U}_h$ therefore we need to construct approximations of such functions. The approximations are constructed on the interval $(0,\tau)$, and it is invariant under translations.

We consider polynomials $s \in \mathcal{P}_k(0,\tau)$, and we denote the discrete approximation of $\chi_{[0,t)} s$ of s the polynomial $\hat{s} \in \{\hat{s} \in \mathcal{P}_k(0,\tau), \hat{s}(0) = s(0)\}$ which satisfies

$$\int_0^\tau \hat{s} q = \int_0^t s q \quad \forall\, q \in \mathcal{P}_{k-1}(0,\tau).$$

The above construction is motivated by the elementary observation that for $q = s'$ we obtain $\int_0^\tau s' \hat{s} = \int_0^t s s' = \frac{1}{2}(s^2(t) - s^2(0))$.

The construction can be extended to approximations of $\chi_{[0,t)} v$ for $v \in \mathcal{P}_k(0,\tau;V)$ where V is a linear space. The discrete approximation of $\chi_{[0,t)} v$ in $\mathcal{P}_k(0,\tau;V)$ is defined by $\hat{v} = \sum_{i=0}^k \hat{s}_i(t) v_i$ and if V is a semi-inner product space then,

$$\hat{v}(0) = v(0), \quad \text{and} \quad \int_0^\tau (\hat{v}, w)_V = \int_0^t (v,w)_V \quad \forall w \in \mathcal{P}_{k-1}(0,\tau;V).$$

Finally, we quote the main result from [6].

PROPOSITION 4.8. *Let V be a semi-inner product space, then the mapping $\sum_{i=0}^k s_i(t) v_i \to \sum_{i=0}^k \hat{s}_i(t) v_i$ on $\mathcal{P}_k(0,\tau;V)$ is continuous in $\|.\|_{L^2(0,\tau;V)}$. In particular,*

$$\|\hat{v}\|_{L^2(0,\tau;V)} \leq C_k \|v\|_{L^2(0,\tau;V)}, \qquad \|\hat{v} - \chi_{[0,t)} v\|_{L^2(0,\tau;V)} \leq C_k \|v\|_{L^2(0,\tau;V)}$$

where C_k is a constant depending on k.

PROOF. See [6, Lemma 2.4]. ☐

THEOREM 4.9. *Let $u, \tilde{u}^h$ be the solutions of (4.1)–(4.2) respectively. Suppose that the projection $\bar{P}^h$ is defined as in Definition 4.1 and let $e = e_p + e_h$ where $e_h = \bar{P}^h u - \tilde{u}_h$, $e_p = u - \bar{P}^h u$ respectively. Then the following estimate holds:*

$$\begin{aligned}
&\sup_{t^{n-1} \leq t \leq t^n} \|e_h\|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} \left\|e_h^n - e_{h+}^n\right\|^2_{L^2(\Omega)} + \int_0^T \|e_h\|^2_V \\
&\quad \leq C\Big(\left\|e_h^0\right\|^2_{L^2(\Omega)} + \sum_{n=1}^N \|(I - P^{n-1}) u(t^{n-1})\|^2_{L^2(\Omega)} \\
&\qquad + \int_0^T \left(\|u - \bar{P}^h u\|^2_V + \|u - \bar{I}^h u\|^2_V + |\tilde{a}(\bar{I}^h u, \hat{e}) - a(\bar{I}^h u, \hat{e})| \right) \Big)
\end{aligned}$$

where $\hat{e}$ is the discrete approximation of $\chi_{[t^{n-1},t^n)}e_h$. In addition, let $u \in L^2(0,T;\mathcal{H}^{p+1}(\Omega))$ $u_t \in L^2(0,T;\mathcal{H}^k(\Omega))$ with $k \geq 0$, then for a time step size $\tau \approx h$ the following estimate holds:

$$\begin{aligned}&\|e\|_{L^\infty(0,T;L^2(\Omega))}\\ &\qquad \leq C(h^{p+1-(\rho/2)}|\log h|^\nu + \max\{h^{p+(1/2)}, h^{p+1-(\rho/2)}\} + \tau^{k+1-(\rho/2)}),\end{aligned}$$

where $\nu > 0$ is a constant depending on the compression operator and C is a constant depending on $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, \delta, C_k$ and on the domain.

PROOF. *(Sketch)* Note that V defines a semi-inner product space with norm $\|.\|_a = a(.,.)$. Fix $t \in [t^{n-1}, t^n)$ and substitute $v_h = \hat{e}$ into Equation (4.5), where $\hat{e}$ is the discrete approximation of $\chi_{[t^{n-1},t)}e_h$, with $e_h = \bar{P}^h u - \tilde{u}_h$. Therefore, using similar considerations as in Theorem (4.2),

$$\begin{aligned}&\frac{1}{2}\|e(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|e_h^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|e_h^{n-1} - e_{h+}^{n-1}\|^2_{L^2(\Omega)}\\
&\quad \leq C\Big(\big|((I-P^{n-1})u(t^{n-1}), e_{h+}^{n-1})\big| + \int_{t^{n-1}}^{t^n} \tilde{a}(e_h,\hat{e}) + \int_{t^{n-1}}^{t^n} \|\bar{P}^h u - \bar{I}^h u\|_a \|\hat{e}\|_a\\
&\qquad\quad + \int_{t^{n-1}}^{t^n} \big(\|e_p\|_a\|\hat{e}\|_a + \mathrm{Con}_2\big)\Big)\\
&\quad \leq C(C_k)\Big(\big|((I-P^{n-1})u(t^{n-1}), e_{h+}^{n-1})\big| + \int_{t^{n-1}}^{t^n} \|e_h\|_a^2\\
&\qquad\quad + \int_{t^{n-1}}^{t^n} \|\bar{P}^h u - \bar{I}^h u\|_a^2 + \|e_h\|_a^2 + \int_{t^{n-1}}^{t^n} \big(\|e_p\|_a^2 + \mathrm{Con}_2\big)\Big)\end{aligned}$$

where we have used Cauchy–Schwarz inequality, Proposition (4.8). Here, $\mathrm{Con}_2 = \tilde{a}(\bar{I}^h u, \hat{e}) - a(\bar{I}^h u, \hat{e})$ and $C(c_k)$ depends on $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, C_k, \Omega$. The rest of proof easily follows from Theorem (4.2). ◻

5 Compression via wavelet basis functions.

The applicability of the main results of Theorems 3.3–4.2 are examined. In particular, we are interested in applying the results of Sections 3 and 4 for perturbed bilinear forms $\tilde{a}(.,.)$ obtained by wavelet compression techniques. First, we describe the basic definitions regarding basis constructed by wavelets and quote the approximation properties of the corresponding projections. Then, we quote results from [16, 17], which demonstrate that the compressed matrix $\tilde{\mathbf{A}}$, resulting from the perturbed bilinear forms $\tilde{a}(.,.)$ has the desired number of $\mathcal{O}(N \log N)$ nonzero elements.

5.1 Triangulations and subspaces.

Throughout this section we assume that Ω is a polygonal domain. Let $\mathcal{T}_0$ be the initial coarse triangulation of the domain and we define the triangulation $\mathcal{T}_j$ as a suitable subdivision of $\mathcal{T}_{j-1}$ (for example, subdivide the triangles of $\mathcal{T}_{j-1}$

in four congruent subtriangles). The triangulation denoted by $\mathcal{T}_h$, is obtained by this subdivision procedure and it is assumed that corresponds to $\mathcal{T}_J$ with $h = C2^{-J}$ for some $C > 0$. If $1 \leq \rho \leq 2$ we denote by V^h the space of continuous piecewise polynomials of degree $p \geq 1$ on the triangulation with zero boundary values, while for $0 < \rho < 1$, we denote by V^h the spaces of piecewise polynomials of degree $p \geq 0$.

To summarize, in a similar way we may define spaces V^j corresponding to the triangulation $\mathcal{T}_j$ so that

$$V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^J = V^h.$$

Here, $N^j \equiv dimV^j$ and $M^j := N^j - N^{j-1}$, and hence $N := dimV^h = N^J = C2^J$.

5.2 Wavelet basis functions.

We will use a wavelet basis that allows to represent the bilinear form $a(.,.)$ as a matrix with "many negligable" elements. The application of this basis yields the "compressed" (approximate) bilinear form $\tilde{a}(.,.)$. The construction of this basis is based on biorthogonal wavelets. Here, we describe the main principles. We refer the reader to [7, 9] for an excellent exposition. Let $\psi_l^j, l = 1, \ldots, M^j$ and $j = 0, \ldots$ denote a hierchical basis functions satisfying the following three properties:

1. **Locality (L)** The functions ψ_l^j are local, with a decreasing width of support (denoted by supp here) for growing discretization level j i.e.,
$$\text{diam}\left(\text{supp}\,\psi_l^j\right) \approx 2^{-j}$$
2. **Cancellation property (CP)** The wavelets ψ_l^j with $(\text{supp}\,\psi_l^j) \cap \partial\Omega = \mathbf{O}$ have vanishing moments up to order p, i.e., $(\psi_l^j, q) = 0\ \forall q \in \mathcal{P}_p$.
3. **Riesz basis property (R)** Every function $v \in V^h$ can be represented as
$$v = \sum_{j=0}^{J} \sum_{l=1}^{M^j} v_l^j \psi_l^j$$
with $v_l^j = (v, \tilde{\psi}_l^j)$ where $\tilde{\psi}_l^j$ denotes the "dual wavelets" (see e.g. [7, 9]).

Note also that for $v \in V$ the infinite series, $v = \sum_{j=0}^{\infty} \sum_{l=1}^{M^j} v_l^j \psi_l^j$ with $v_l^j \equiv (v, \tilde{\psi}_l^j)$ converges in $\bar{H}^s$, for all $0 \leq s \leq (\rho/2)$ and the following norm equivalence is valid

$$c_e \|v\|^2_{H^s(\Omega)} \leq \sum_{j=0}^{\infty} \sum_{l=1}^{M^j} |v_l^j|^2 2^{2js} \leq C_e \|v\|^2_{H^s(\Omega)}. \tag{5.1}$$

For $(\rho/2) < s \leq p+1$ the following one-sided bound holds for the truncated series:

$$\sum_{j=0}^{J} \sum_{l=1}^{M^j} \left|v_j^l\right|^2 2^{2js} \leq C_E J^\nu \|v\|^2_{H^s(\Omega)}, \quad \text{with } \nu = 0 \text{ if } s < p+1 \text{ and } \nu = 1 \text{ if } s = p+1, \tag{5.2}$$

where C_E is independent of J.

For examples of wavelet basis as well as the “wavelet analogue” of the standard approximation properties (3.1), we refer the reader to [7, 9]. Here, we state the main property: for all $v \in \mathcal{H}^t(\Omega)$, $t \geq (\rho/2)$,

(5.3)
$$\inf_{v^h \in V^h} \|v - v^h\|_{\mathcal{H}^s(\Omega)} \leq C 2^{-J(t-s)} \|v\|_{\mathcal{H}^t(\Omega)}, \quad 0 \leq s \leq (\rho/2), \ (\rho/2) \leq t \leq p+1.$$

5.3 Verification of assumptions.

In order to utilize the estimates of the previous section, one need to satisfy the assumption 3.1. First note that the truncated wavelet expansion plays the role of the basic “interpolation projector”. Therefore, we may define, $I^h := V \to V^h$ by

$$I^h v := \sum_{j=0}^{J} \sum_{l=1}^{M^j} v_l^j \psi_l^j.$$

Note that $I^h v$ satisfies approximation properties (5.3) (see e.g. [7, 9]). In addition to the truncated projector I^h we define the standard $L^2(\Omega)$ projection, by requiring $(P^h v - v, v^h) = 0 \quad \forall v^h \in V^h$. Note that the approximation properties of P^h can be easily deduced from I^h. One may use standard techniques to derive similar approximation properties in the time-space spaces (see e.g. [5]). The projections of Definition 4.1 can be constructed similar to [12, 20]. It remains to show the “consistency” property, of the compressed bilinear form $\tilde{a}(.,.)$. This is proven in [17]. For completeness, we quote the main results. First, note that the bilinear form $a(.,.) : V^h \times V^h$ corresponds to a matrix $\mathbf{A}$ with entries $A_{(j,l),(j',l')} = a(\psi_l^j, \psi_{l'}^{j'})$. The natural way to define the compressed matrix $\tilde{\mathbf{A}}$, corresponding to $\tilde{a}(.,.)$ is to neglect “small” elements of $\mathbf{A}$. In particular, following the work [16, 17], specialized to our notation, we define by

$$(\tilde{A})_{(j,l),(j',l')} = \begin{cases} A_{(j,l),(j',l')} & \text{if } \operatorname{dist}(\operatorname{supp}\psi_l^j, \operatorname{supp}\psi_{l'}^{j'}) \leq \delta_{j,j'} \\ & \text{or } \operatorname{supp}(\psi_l^j) \cap \partial\Omega \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{5.4}$$

where the truncation parameter is defined, for some $c, \hat{a} > 0$, as

$$\delta_{j,j'} = c \max\{2^{J + \hat{a}(2J - j - j')}, 2^{-j}, 2^{-j'}\}.$$

The truncation parameter is related to ν (which appears in Theorems 3.3–4.2. Below, we state the main result which relates $\hat{a}$ and ν, specialized to our needs.

PROPOSITION 5.1. *If c is chosen sufficiently large then for all $J > 0$ Condition* (3.5) *holds. In addition, if $\hat{a} \geq \frac{2p+2}{2p+2+\rho}$ then Condition* (3.6) *holds with $\nu = 3/2$ if $\hat{a} = \frac{p+1}{p+2}$ and with $\nu = 1/2$ otherwise. Furthermore, the compressed matrix $\tilde{\mathbf{A}}$ has $\mathcal{O}(N \log N)$ nonzero elements if $\hat{a} < 1$ and $\mathcal{O}(N(\log N)^2)$ if $\hat{a} = 1$.*

PROOF. See [16, 17]. ◻

REMARK 5.2. In case of $0 < \rho \leq 2$ we may choose $\hat{a}$ such that $\nu = 1/2$ and consequently $\tilde{\mathbf{A}}$ has $\mathcal{O}(N \log N)$ nonzero elements. If $\rho = 0$ a similar considerations lead to $\mathcal{O}(N(\log N)^2)$ nonzero elements for the compressed stiffness matrix. See relevant discussion in [17].

6 Conclusions.

Combining the estimate of Theorem (3.3) with (5.1), and approximation properties (5.3), we obtain the following estimate: Suppose that $\rho > 0$ and that p, ρ are such that $\hat{a} \geq \frac{2p+2}{2p+2+\rho}$. If, $u \in L^2(0, T.\mathcal{H}^{p+1}(\Omega)) \cap H^1(0, T; H^{p-1}(\Omega))$ then,

$$\|u - \tilde{u}^J\|_{L^\infty(0,T;L^2(\Omega))} + \|u - \tilde{u}^J\|_{L^2(0,T;V)} \leq C 2^{-J(p+1-\rho/2)}(|\log 2^{-J}|^{1/2} + 1),$$

where $\tilde{u}$ denotes the discrete (in space) solution of (3.4). The constant C depends on $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, \delta, c_e, C_e, C_E$ and on the domain. In addition the compressed stiffness matrix contains $\mathcal{O}(N \log N)$ nonzero elements. A similar result also holds in the fully-discrete case. In particular under similar assumptions, and for $0 \leq \rho < 1$, $\tau \approx h \approx 2^{-J}$ using the discontinuous (in time) wavelet Galerkin scheme, we obtain an estimate of the form:

$$\|\tilde{u}^{J,N} - u(t^N)\|_{L^2(\Omega)} + \|u - \tilde{u}^J\|_{L^2(0,T;V)} \leq C(2^{-J(p+1-\rho/2)}(|\log 2^{-J}|^{1/2} + 1))$$

with a compressed matrix $\tilde{\mathbf{A}}$ that contains $\mathcal{O}(N \log N)$ nonzero elements. Here $\tilde{u}$ denotes the solution of (4.2). In addition, for $1 \leq \rho < 2$ and $\tau \approx h$ we obtain:

$$\begin{aligned} &\|\tilde{u}^{J,N} - u(t^N)\|_{L^2(\Omega))} + \|u - \tilde{u}^J\|_{L^2(0,T;V)} \\ &\qquad \leq C(2^{-J(p+1-\rho/2)}|\log 2^{-J}|^{1/2} + 2^{-J(p+(1/2))} + 2^{-J(p+1-\rho/2)}). \end{aligned}$$

Similar estimates also hold at arbitrary times for both cases.

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