A novel stochastic method for the solution of direct and inverse exterior elliptic problems.

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Abstract

A new method, in the interface of stochastic differential equations with boundary value problems, is developed in this work, aiming at representing solutions of exterior boundary value problems in terms of stochastic processes. The main effort concerns exterior harmonic problems but furthermore special attention has been paid on the investigation of time-reduced scattering processes (involving the Helmholtz operator) in the realm of low frequencies. The method, in principle, faces the construction of the solution of the direct versions of the aforementioned boundary value problems but the special features of the method assure definitely the usefulness of the approach to the solution of the corresponding inverse problems as clearly indicated herein.

1 Introduction

Solving boundary value problems in several scientific areas is traditionally connected with the implication of global methods that are based on analytical or numerical approaches involving simultaneously the determination of the sought field in a large number of observation points. As an example, the integral equation methods ([1], [2]) involve integral representations along a whole manifold - which classically has to be discretized totally in the framework of the numerical implementation via the boundary element methodology - while in the realm of the finite element methods [3], the features of the discretization of the continuum introduces a perplexity incorporating the structure of the whole region in which the problem is solved. It is well known however that in several applications we are mainly interested in determining the solution of the problem at a specific restricted region of the domain instead of having a global knowledge, which is usually an expensive goal. A representative example is the investigation of boundary value problems aiming at the determination of the elastic displacement field inside an elastic specimen enclosing crack defects. Then it is clear that only the field near the crack endpoints really matters and it is usually annoying that the traditional methods require by default the global determination of the whole elastic field.

In contrast to the traditional methods, the local methods give the solution of a partial differential equation at an arbitrary point in the domain directly, instead of extracting the response value at this point from the whole field solution. These methods are based on probabilistic interpretations of certain partial differential equations. The relationship between stochastic processes and parabolic and elliptic differential equations was demonstrated a long time ago by Lord Rayleigh [4] and Courant [5], respectively. The development of the probabilistic methods is based on the Itô calculus, properties of Itô diffusion processes, and Monte Carlo simulations. The theoretical considerations supporting the probabilistic methods involve random processes and stochastic integrals. An elaborate presentation of this framework can be encountered in ([6]- [12]) and the references cited therein. The main idea of these approaches concerns boundary value problems in bounded domains and is included here for reference reasons in Appendix A.2. The concept is the following: We are interested in the determination of the solution of a direct elliptic boundary value problem at a specific point x of a bounded domain D. The probabilistic manner is to create a large number of trajectories emanating from this particular point x and obeying evolutionary to a system of stochastic differential equations, which are driven by a drift and diffused by a Wiener process connected both directly with the coefficients of the differential operator under investigation[1]. These

[1]As an example, in case of the Laplace operator, this stochastic differential system disposes zero driving term while the randomness is purely Brownian.

trajectories hit for the first time the boundary ∂D of the bounded domain in finite time. So every trajectory has a finite lifetime during its journey inside the domain D. All these paths are gathered and exploited as follows: The points of the boundary on which the first exit occurs - the traces of the trajectories on the boundary - are selected and offer a set of points on which the average of the values of the boundary data of the boundary value problem is calculated formatting so a first accumulation term. In addition, on every trajectory a stochastic integral is calculated where the integrand is the inhomogeneous term of the underlying differential equation. The mean value of these integrals over the large number of trajectories forms a second accumulator which is superposed to the first one[2] leading to the construction of an extended mean value term. When the number of the trajectories increases, the aforementioned total mean value converges to the corresponding probabilistic expectation value of the underlying fields, which by its turn coincides with the sought, from the beginning, value of the solution of the B.V. problem at the starting point x. The description above refers rather to the Dirichlet problem, which is the main subject of investigation in the present work but similar arguments are encountered in the Neumann boundary value problem ([14]-[15]). One of the main advantages of the probabilistic approach is that it is based on very stable and accurate Monte Carlo simulations.

Two main remarks should be made here. First, the boundedness of the domain D is necessary for the first exit time to constitute a stochastic variable taking almost surely finite values. The finiteness of the duration of the trajectories traveling inside D is a prerequisite for the applicability of the probabilistic approach and this can not be guaranteed in unbounded domains. Secondly, the whole setting of the stochastic approach is appropriate only for the solution of the direct boundary value problem, in which the domain D hosting the paths and its boundary ∂D being hit by them are known.

The aim of the present work is to develop local stochastic methods to face a class of direct and inverse exterior boundary value problems involving the Laplace operator and taking place in unbounded domains. It is immediately apparent that the fundamental prerequisites for the applicability of the stochastic calculus, as presented above, are not valid any more. Indeed, it is well known that the Brownian motion is recurrent in R^2 but transient [7] in R^n, $n > 2$. Consequently, the first drawback of the classical stochastic approach is that the generated trajectories have a strong probability to travel to infinity without hitting the boundary of the domain. Actually, even if some paths cross the boundary, their travel time could be very large creating strong difficulties to the application of the Monte-Carlo simulation. Consequently, we need a reformulation of the probabilistic settlement in exterior domains assuring first of all the certainty for the trajectories to hit in finite time the boundary of the unbounded domain and establishing good adaptation to the specific characteristics of the boundary value problems under investigation.

In addition, our strong motif is to create a methodological framework handling not only the direct but also the corresponding inverse problem where measurements of the involved fields are given but the boundary ∂D, where the strikes of the trajectories occur is unknown and constitutes the target of our investigation! It is obvious that in the framework of the inverse problem a strong resettlement of the method is needed so that the expectation values of the involved fields over the unknown surface obtain a specific exploitable meaning.

The structure of the paper is the following: In Section 2 we gather all the exterior boundary value problems related with the Laplace operator that attract our interest and pertain to harmonic and low frequency scattering processes outside a bounded region. These problems are handled in the sequel via the novel stochastic approach. Before proceeding to the subsequent sections, the reader who is not familiar with the stochastic calculus described so far, is encouraged to follow all the necessary probabilistic background presented in Appendix A, where special focus is devoted to the stochastic processes and their connection with expectation values representing solutions of differential equations subject to suitable boundary conditions.

In Section 3, the stochastic representation for exterior boundary value problems is presented gradually. More precisely, in Section 3.1, we reformulate the stochastic process so that the first goal is assured: the trajectories emanating from the observation point x, are forced to hit the boundary ∂D in a finite period of time. This is accomplished via a suitable conditioning of the stochastic process - on the basis of evoking an auxiliary harmonic function from a mono-parametric suitably selected set of functions- having as support the intersection of the exterior region with a critical cone whose vertex ξ is a kind of attractor for the stochastic trajectories while the conical surface plays the role of a repulsing reflector. The fundamental concept presented in Section 3.1 relies on the selection of the attractor point inside the bounded component of the problem assuring the secure attraction of the stochastic process towards the surface of the boundary. In order to clarify the situation, we focus, for three-dimensional problems, on the selection of a driving term[3] of the form $\frac{\nabla h_m(X_t-\xi)}{h_m(X_t-\xi)}$, where[4] $h_m(y) = \frac{P_m(\hat{n}\cdot y/|y|)}{|y|^{m+1}}$ is a harmonic function and ξ plays the role of the attractor. So the trajectories stem from the point x belonging to the axis of the cone, are repulsed by the singular lateral surface and are attracted - with "magnitude" increasing with the order m- by the source ξ which could be located outside the exterior

[2]This second term is absent in case of a homogeneous differential equation.

[3]The driving term of the stochastic differential system governing the evolution of the trajectories X_t.

[4]P_m is the Legendre polynomial of order $m \in N$, while $\hat{n} = \frac{x-\xi}{|x-\xi|}$ defines the orientation of the aforementioned critical cone, the lateral surface of which is characterized by a zero of P_m. The driving term blows up on this lateral surface.

space to force, at last, hitting on the boundary. There exist several alternative settlements as will be apparent in Section 3. In addition, several issues emerge that must be appropriately faced. The most important is the necessity to avoid singular behavior of the driving term of the process on the lateral surface of the cone. This imposes the implication of an interior conical surface protecting the involved fields to obtain infinite values. This protecting cone absorbs however a critical amount of trajectories while still repulses the most part of them. This antagonism between the "successes" (hits on the boundary) and "fails" (strikes on the lateral conical surface) gives birth to a stochastic calculus, which on the basis of Dynkin's formula, offers the possibility to obtain stochastic representations for exterior fields of elliptic boundary value problems. It is worthwhile to notice that this specific conditional driving imposed by h_m leads to concrete expectation value and covariance for the travel time of the trajectories dependent on the distance between the observation point and the attractor as well as on the parameter m.

In Section 3.2 two different stochastic representations are constructed. The first one (the mildly conditioned stochastic method) allows the attractor point to exit in the exterior region and the cones to become thick in order to create a large space for the mobility of the stochastic process, serving at solving the direct boundary value problem since in this case all the surface points have the opportunity to serve as strike absorbers and equivalently participate to the accumulation buffers. Theorem 4 contains the principal results of this approach. Its usefulness is dependent on the specific partition of the strikes of the trajectories over the possible escaping boundaries confining the region of the evolution of the trajectories. If the strikes over the lateral surface of the cone and its remote cup are considerably less than the strikes over the surface ∂D then a useful stochastic representation for the solution of the direct problem arises disposing though a remainder dependent explicitly on the fraction of the "fails" over the "successes".

The second method (the strongly conditioned stochastic method) is closer to the initial concept working with attractor points inside the bounded component of the space and with thin cones detaching only a small portion of the boundary. The ultimate goal of this approach is the treatment of the inverse problem as presented later in Section 4.2. However the method has applicability also to the direct problem. The theorems 11 and 12 present the stochastic representations for the Dirichlet and Neumann boundary value problems. In the same paragraph, it is demonstrated how a local stochastic characterization of the Dirichlet to Neumann operator can be constructed. The advantage of the method is that the modified field which is subject to stochastic analysis vanishes over the protective interior cone and so the lost strikes do not hide information from the total accumulator. The disadvantage of the strongly conditioned stochastic method is that it offers representations for the far or the very near field, while the involved reminder is not controllable[5] for intermediate locations of the observation point x (always with respect to the location of the auxiliary attractor point ξ).

In the interface of the two methods, a striking result has been proved giving an exact stochastic representation to the simple spherical symmetric potential outside a sphere (see Proposition 9). Although disposing the character of a benchmark solution, this exact representation is very helpful in connecting, in stochastic terms, the local curvature of the boundary with the measured field at points inside narrow cones cropping small portions of the boundary, at least in the direct problem framework. In addition, it merits its own interest as an exact theoretical result establishing a probabilistic representation for the simplest exterior three dimensional harmonic function.

Section 3.3 is devoted to reveal the efficiency of the strongly conditioned stochastic method to represent stochastically the far field pattern of the acoustical scattering problem in low-frequencies. The representation offered by Corollary 17 can be considered the main outcome of this section giving birth to the development of the method solving the inverse low-frequency acoustical problem presented in Section 4.2.

The numerical implementation of the problem is presented in Section 4 where both direct and inverse exterior boundary value problems are encountered mainly in the realm of low-frequency acoustic scattering. Although the theoretical arsenal of Section 3.1 has been constructed for both 2-D and 3-D problems, the numerical investigation has been confined to 3-D dimensional problems, mainly due to the fact that they constitute the suitable cradle for the low-frequency processes. The numerical ingredients of the investigation of the set of the stochastic differential equations governing the trajectories formation are first discussed in order to reveal the complexity level of this effort. In Section 4.1 the mildly conditioned method is used to solve the multiple low frequency scattering problem referring to a system of two spheres. The results are in perfect agreement with the already existed analytical results met in [18]. In Section 4.2, the strongly conditioned method is tested and proved to comply with the rigorous aforementioned result of the probabilistic representation of the spherically symmetric potential. In addition, it is used to solve the inverse low-frequency scattering problem of plane waves by a specific ellipsoidal surface on the basis of data furnished by the analytical results found in [21]. The architecture of the inversion algorithm is based on the construction and implication of several auxiliary cones connecting a set of measurement points with a corresponding set of interior attractors, being assumed that we have knowledge about a core interior region of the scatterer where attractors can be safely located.

[5] The mildly conditioned stochastic method works for all possible distances $|x - \xi|$, fact which in combination with the freedom of trajectory moves supported by the thin cones render this method more appropriate for the solution of the direct problem not only in asymptotic regions, but in all the exterior domain.

We form then an appropriate functional forcing the stochastically determined - via the strongly conditioned method - values of the far field pattern (at the observation points) to comply with the data of the problem. This functional is implicitly dependent on the intersection points (rather small patches) of the thin cones with the unknown scatterer. The optimization of this functional with respect to the aforementioned intersection points for several states of plane incidence offers the reconstruction of the scatterer as the assembly of these minimizing intersection points.
The outcomes of the present work constitute a first attempt to develop stochastic methods for the solution of exterior direct and inverse boundary value problems pertaining to the Laplace operator. Our investigation extends to the study of Helmholtz operator with small wave number k, in the realm of low-frequency scattering but actually the stochastic trajectories are built via the methodology induced by the Laplace operator.
Our next goal is to work with the whole Helmholtz operator in order mainly to confront the scattering processes stemmed from acoustics or electromagnetism in the time harmonic regime with arbitrary frequency. We can see [16] that when trying to construct stochastic paths with drift and diffusion pertaining to the Helmholtz operator, then these paths must live in a complex $(2n)-$dimensional space in contrast to the Laplace equation where the trajectories belong to the real $n-$dimensional space. This increases the complexity of the problem, which is under current investigation by the authors [23]. An intermediate stage, which is also under current investigation, is the extension of the method to provide representations of the solution of the modified Helmholtz equation in exterior domains. Given that this operator is the negative of a strongly elliptic operator, the trajectory space remains embedded in the real $n-$dimensional space and the trajectories have more deterministic behavior due to the diffusive character of the Green's function - pertaining to the modified Helmholtz operator - involving in the driving force of the stochastic differential system.

2 Exterior boundary value problems

Let us consider an open bounded region D, confined by a smooth[6] surface ∂D, standing for a hosted inclusion inside the surrounding medium $D^e = R^n \setminus \bar{D}$. We state the problem generally in the $n-$dimensional space having in mind though that in applications the cases $n = 2, 3$ are of special interest. Let us consider the following exterior Dirichlet boundary value problem:

$$Lu(x) = 0, \quad x \in D^e \tag{2.1}$$
$$u(x) = f(x), \quad x \in \partial D \tag{2.2}$$
$$\mathcal{M}u(x) = 0, \quad x \in R^n \tag{2.3}$$

where L is an elliptic - or better the opposite of an elliptic - differential operator of second order, which in the realm of the present work is expressed generally by the formula[7] $L = \sum_i b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$, $f(x)$ represents the Dirichlet data on the surface ∂D, while the function $\mathcal{M}u(x)$ is the more intricate term of the scheme and is constructed via the application of the second Green's identity to the pair of functions consisted of the solution $u(x)$ and the fundamental free space solution $G(x, y)$ corresponding to the operator L. The definition formula of $\mathcal{M}u(x)$ is a global relation of integral type on a sphere of a sufficiently large radius but it is not necessary to be presented here and can be found for example in [1]. Its significance lies on the fact that its specification is equivalent to imposing constraints on the behavior of the solution $u(x)$ as the distance $r = |x|$ increases. So in case that $\mathcal{M}u(x) = 0$, it is proved [1] that any solution of the boundary value problem (2.1)-(2.3), being locally square integrable (along with its derivatives of first order), can be represented as a superposition of single and double layer potentials and so the solution has specific asymptotic decaying behavior at infinity with known convergence rate.
The situation is clarified in the case of the Laplace operator ($L = \Delta$), in which the condition $\mathcal{M}u(x) = 0,\ x \in R^n$ holds if and only if $u(x) = \mathcal{O}(|x|^{2-n})$ as $|x| \to \infty$ for $n \geq 3$ and $u(x) = b \log(|x|) + \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$ for $n = 2$. Moreover, in the case $n = 2$, demanding solutions remaining finite for large values of r, is equivalent with the existence of a constant b such that $\mathcal{M}u(x) = b$ and $u(x) = b + \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$. We are then in position to state the well known exterior Dirichlet boundary value problem for Laplace operator concerning bounded solutions for $n = 2, 3$:

$$\Delta u(x) = 0, \quad x \in D^e \tag{2.4}$$
$$u(x) = f(x), \quad x \in \partial D \tag{2.5}$$
$$u(x) = b + \mathcal{O}(|x|^{-1}), \quad r = |x| \to \infty \tag{2.6}$$

[6] with continuous curvature to support the classical version of the probabilistic calculus though there exist improvements allowing Lipschitz domains [13].
[7] We will allow that the operator L disposes also an additional multiplicative term of the form $c_0(x)$ but only the derivative terms will be connected, in this work, with the subsequent stochastic analysis.

with $b = 0$ necessarily in case $n = 3$ while in the 2-dimensional case the relation $\mathcal{M}u(x) = b$ connects the solution $u(x)$ with the remote dominant term b. However, by considering as unknown function the field $u(x) - b$ instead of $u(x)$ itself, it is equivalent to take $b = 0$ for both dimensional cases.
The boundary value problem above is possible to stem from different interesting physical processes. As an example, in three dimensions, the field $u(x)$ could be the potential part of the low-frequency expansion of the acoustic scattered field connected on the soft scatterer with the first component $(-f(x))$ of the corresponding frequency expansion of the incident field. Alternatively, $u(x)$ could be the electrostatic potential induced by the interference of the perfect conductor $\bar{D}$ with a specific exciting field (stemmed by the potential $\Phi^0(x) = -f(x)$). Equivalently, the boundary value problem above might represent the temperature distribution -in the steady state- outside a region $\bar{D}$, whose surface is held in specific temperature $f(x)$. For $n = 2$, the potential problem could lead to solutions expanding logarithmically far away the domain D. However, if we are interested in bounded solutions, the problem (2.4)-(2.6) adequately serves as the suitable model to describe static processes as mentioned above.

Another elliptic boundary value problem of great importance is the one involving Helmholtz equation, which is produced after imposing time harmonic dependence in scattering and vibrating processes in acoustics, electromagnetism and elasticity. As an example, the acoustic scattering field $u(x)\exp(-i\omega t)$ emanated from the interference of an incident time harmonic wave $u^{in}(x,t) = \exp(i(k\hat{k}\cdot x - \omega t))$ with the soft scatterer $\bar{D} \subset R^n$ satisfies the following boundary value problem

$$(\Delta + k^2)u(x) = 0, \quad x \in D^e \tag{2.7}$$

$$u(x) = -\exp(ik\hat{k}\cdot x), \quad x \in \partial D \tag{2.8}$$

$$\lim_{r\to\infty} r^{\frac{1-n}{2}} \left(\frac{\partial u(x)}{\partial r} - iku(x) \right) = 0, \tag{2.9}$$

where we recognize the wave number $k \neq 0$, the unit vector $\hat{k}$ indicating the direction of the incident wave and the angular frequency ω of the scattering process. Now the relation $\mathcal{M}u(x) = 0$ leads to the Sommerfeld radiation condition (2.9) which holds uniformly over all possible direction $\hat{x} = \frac{x}{r}$ and is valid for every dimension n. This condition not only gives information about the asymptotic behavior of the scattered wave but also incorporates the physical property according to which the whole energy of the scattered wave travels outwards leaving behind the scatterer from whom emanates. For example for $n = 3$ we obtain

$$u(x) = \frac{1}{|x|} u_\infty(\hat{x}; \hat{k}, k) + u_1(x), \quad |x| u_1(x) \to 0, \text{ as } |x| \to \infty \tag{2.10}$$

where we recognize the far-field pattern $u_\infty(\hat{x}; \hat{k}, k)$ totally characterizing the behavior of the wave field $u(x)$ a few wave-lengths away the scatterer D.
The Helmholtz operator $L = \Delta + k^2$ (with k real) can be considered as a perturbation of the general formula $L = \sum_i b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$, encountered above, due to the k^2-term. Although this term does not belong to the principal part of the elliptic operator, it introduces qualitative differences in handling the corresponding boundary value problem via stochastic calculus. We can see [16] that when trying to construct stochastic paths with drift and diffusion pertaining to the Helmholtz equation, then these paths must live in a complex $(2n)-$dimensional space in contrast to the Laplace equation where the trajectories belong to the real $n-$dimensional space.

This is not the case if we deal with the modified Helmholtz equation $L = \Delta - k^2$, which is the opposite of a bounded below elliptic operator [1], and the stochastic methodology applying for the Laplace operator can also be evoked in this case. This operator could naturally appear if we studied damped, in time, solutions of the wave equation in the exterior space of a bounded domain and the corresponding boundary value can be generated from problem (2.7)-(2.9) by the assumption $k \in C$, $\Im k = \frac{\pi}{2}$. We obtain then the model

$$(\Delta - k^2)u(x) = 0, \quad x \in D^e \quad (k > 0) \tag{2.11}$$

$$u(x) = f(x), \quad x \in \partial D \tag{2.12}$$

$$\lim_{r\to\infty} r^{\frac{1-n}{2}} \left(\frac{\partial u(x)}{\partial r} + ku(x) \right) = 0, \tag{2.13}$$

where the modified Sommerfeld condition excludes solutions of space exponential growth and after this is accomplished, the Green's operator of the problem is so regular that every term participating in the condition (2.13) decays exponentially in space as $r = |x|$ increases.
All the previous examples of boundary value problems involve boundary conditions of Dirichlet type but almost the same arguments could be developed in case that instead Neumann boundary conditions were present. In that case, we have knowledge about the normal derivative of the field on the surface ∂D.
In all cases, the direct exterior boundary value problem consists in the determination of the field $u(x)$ outside

D when boundary data (i.e the function f) and geometry (i.e the shape of ∂D) are given. In fact, in most applications, we are interested in determining the remote pattern of this field far away the bounded domain D. For example, in the case of the BVP (2.7)-(2.9) in three dimensions, it would be sufficient to determine the far field pattern $u_\infty(\hat{x};\hat{k})$ participating in the representation (2.10) if we deal with an application in which we do not have access near the domain D.
The inverse exterior boundary value problem aims at determining the shape of the surface ∂D when the boundary data is known and the remote pattern is measured. Equivalently, instead of considering as data the measured remote field, it is usual to have at hand the Dirichlet to Neumann (DtN) operator on a sphere - or part of it - surrounding[8] the domain D. In other words a large class of interesting inverse boundary value problem are based on data incorporating both the measured field along with its normal derivative on a given surface belonging to the near field region[9].
The investigation of the above mentioned direct and inverse boundary value problems will be accomplished in the forthcoming sections via stochastic calculus for two specific differential operators: a) the Laplace operator and b) the Helmholtz operator in the low frequency realm where the wave number k is small compared with the geometric characteristics of the problem. In the present work the differential operator L is taken to be at most a small perturbation of the Laplace operator and the stochastic analysis will be built on the structure of the Laplace operator itself. The forthcoming section offers the basic prerequisite material concerning the bridge between the stochastic calculus and the classical theory of boundary value problems mainly referred to interior domains.

3 The stochastic representation for exterior boundary value problems.

3.1 The construction of conditioned stochastic processes for Laplace operator.

In the stochastic framework under discussion, we encountered the first exit time τ_D from the open set D. At that time the stochastic process X_t, obeying to Eq.(A.2) with $X_0 = x$ and very large T, "hits" the boundary ∂D. This particular exit process brings into the light the boundary itself and a crucial connection is established between the solution of the differential equation and the points of the boundary on which data are given. It is questionable how this situation could be exploitable in the service of the direct and (or) inverse boundary value problem.

We start by studying the stochastic formulation of the direct static problem (2.4)-(2.6), which is defined on the unbounded open domain $D^e := R^n\backslash\bar{D}$. The first idea is to evoke the stochastic process, whose infinitesimal generator is exactly the Laplace operator Δ, participating in the boundary value problem (2.4)-(2.6). This is exactly the Brownian motion and the crucial point is that this random walk takes place in the unbounded domain D^e. So even if the starting point x of the process X_t is very close to the boundary $\partial D^e = \partial D$ and even if we work in R^2 it is not guaranteed that the process hits, in finite time, the boundary. A different idea has been implemented in the present work inspired by [10] and involving the concept of *conditioning* stochastic processes to have a specific directivity. More precisely, we select a point ξ inside the bounded component D. This point could be the coordinate origin O or could be selected according to the specific features of the problem. Let $x \in D^e$ be once again the initial point of the stochastic process under construction or equivalently the point at which the solution of the B.V.P (2.4)-(2.6) is sought. We consider the unit vector $\hat{n}_{x,\xi} := \frac{x-\xi}{|x-\xi|}$. For simplicity we denote $\hat{n}_{x,\xi}$ as $\hat{n}$ since the points x, ξ are assumed as fixed parameters, though the same procedure might be profitable to be applied for several pairs (x, ξ). We introduce now a family of harmonic functions that have singular behavior at $y = 0$: For the two dimensional case we select the harmonic functions $h_m(y) := \frac{1}{|y|^{m+1}} \cos\left((m+1)\cos^{-1}(\hat{n}\cdot y/|y|)\right)$, $m = 0, 1, 2, ...$, where the range of $\cos^{-1}$ is selected to be $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In R^3 we evoke the well known Legendre polynomial functions[10] $P_m(\cos\theta)$ and introduce the family of singular at origin harmonic functions $h_m(y) = \frac{P_m(\hat{n}\cdot y/|y|)}{|y|^{m+1}}$, $m = 0, 1, 2, ...$. We consider the stochastic processes X_t, Y_t where $X_t = Y_t + \xi$ and

$$dY_t = \frac{\nabla h_m(Y_t)}{h_m(Y_t)}dt + dB_t, \quad 0 \le t \le T, \quad Y_0 = x - \xi \tag{3.1}$$

or equivalently

$$dX_t(= dY_t) = \frac{\nabla h_m(X_t - \xi)}{h_m(X_t - \xi)}dt + dB_t, \quad 0 \le t \le T, \quad X_0 = x. \tag{3.2}$$

[8]This is rather connected with the form (2.1-2.3) of the boundary value problem and especially with the condition $\mathcal{M}u(x) = 0$.
[9]Pertaining to Helmholtz operator, we refer to [3] (section 3.2) as an excellent reference relevant to the construction of the DtN mapping.
[10]It is essential to select the normalization condition $P_m(1) = 1$.

Both processes X_t, Y_t depend on the adopted member h_m of the family $\{h_j,\ j=0,1,2,...\}$ but this dependence is ignored in the symbolism, for simplicity. The existence and uniqueness of the process X_t will be examined further later but these issues are expected to be easily established before the process enters the domain D since the point ξ is isolated from the trajectory of X_t and the usual Lipschitz requirements are satisfied, at least locally near the starting point of the process.
To face uniformly both dimensions $n=2,3$, we put

$$h_m(y) := \frac{1}{|y|^{m+1}} \mathcal{Q}_m(\hat{n}\cdot y/|y|), \tag{3.3}$$

where

$$\mathcal{Q}_m(\hat{n}\cdot y/|y|) = \left\{ \begin{array}{c} \cos\left((m+1)\cos^{-1}(\hat{n}\cdot y/|y|)\right), \quad n=2 \\ P_m(\hat{n}\cdot y/|y|), \quad n=3 \end{array} \right. , \quad m=0,1,2,.... \tag{3.4}$$

We also define $\Theta := \cos^{-1}(\hat{n}\cdot y/|y|)$, representing so the azimuthal coordinate φ in case of the two dimensional space (where the x positive semi-axis starts from ξ and points to x) and the polar coordinate θ in the three dimensional case (where the pair (ξ, x) defines now in the same manner the $z-$axis of the coordinate system). In y-terminology, the origin of the coordinates coincides with the point ξ.

We are likely to determine the behavior of the distance $|X_t-\xi| = |Y_t|$ as time passes or more generally the expected values of the power $|Y_t|^k$ for a general integer k.

Lemma 1 *For every integer $k\in N$, it holds that*

$$d|Y_t|^k = \frac{k}{2}(k+n-2m-4)|Y_t|^{k-2}dt + k|Y_t|^{k-2}Y_t\cdot dB_t \tag{3.5}$$

Proof. We apply the Itô formula (A.4) to the function $F(t,\omega) = f(Y_t(\omega)) = |Y_t(\omega)|^k$ and obtain in tensor form

$$d|Y_t|^k = \nabla|Y_t|^k\cdot dY_t + \frac{1}{2}\nabla\nabla|Y_t|^k : dY_t dY_t \tag{3.6}$$

We find that $\nabla|Y_t|^k = k|Y_t|^{k-2}Y_t$ and $\nabla\nabla|Y_t|^k = k(k-2)|Y_t|^{k-4}Y_tY_t + k|Y_t|^{k-2}I$, where I is the $n\times n$ identity tensor. Consequently, Eq.(3.6) becomes

$$d|Y_t|^k = k|Y_t|^{k-2}(Y_t\cdot dY_t) + \frac{1}{2}\left[k(k-2)|Y_t|^{k-4}(Y_t\cdot dY_t)^2 + k|Y_t|^{k-2}(dY_t\cdot dY_t)\right] \tag{3.7}$$

The products $(Y_t\cdot dY_t)$ and $(dY_t\cdot dY_t)$ must be determined via the stochastic differential equation (3.1). The drift term of the process Y_t can be written as

$$\begin{aligned} \frac{\nabla h_m(Y_t)}{h_m(Y_t)} &= \left(-(m+1)\frac{Y_t}{|Y_t|^{m+3}}\mathcal{Q}_m(\cos(\Theta_t)) + \nabla\mathcal{Q}_m(\cos(\Theta_t))\frac{1}{|Y_t|^{m+1}}\right) \\ &\quad \times\frac{|Y_t|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_t))} \end{aligned} \tag{3.8}$$

Clearly $\nabla\mathcal{Q}_m(\cos(\Theta_t)) = \frac{\hat{\Theta}_t}{|Y_t|}\frac{\partial}{\partial\Theta_t}\mathcal{Q}_m(\cos(\Theta_t))$, where $\hat{\Theta}_t$ is a unit vector always perpendicular to the polar unit vector $\frac{Y_t}{|Y_t|}$. Consequently

$$Y_t\cdot dY_t = Y_t\cdot\left(\frac{\nabla h_m(Y_t)}{h_m(Y_t)}dt + dB_t\right) = -(m+1)dt + Y_t\cdot dB_t, \tag{3.9}$$

$$dY_t\cdot dY_t = \left(\frac{\nabla h_m(Y_t)}{h_m(Y_t)}dt + dB_t\right)\cdot\left(\frac{\nabla h_m(Y_t)}{h_m(Y_t)}dt + dB_t\right) = ndt, \text{ and} \tag{3.10}$$

$$(Y_t\cdot dY_t)^2 = (-(m+1)dt + Y_t\cdot dB_t)(-(m+1)dt + Y_t\cdot dB_t) = |Y_t|^2dt, \tag{3.11}$$

after using the basic infinitesimal product relations of Itô calculus. As a result, Eq.(3.7) after some straightforward manipulations, becomes

$$d|Y_t|^k = \frac{k}{2}(k+n-2m-4)|Y_t|^{k-2}dt + k|Y_t|^{k-2}Y_t\cdot dB_t \tag{3.12}$$

■
The aforementioned regime of stochastic calculus will be accordingly exploited after returning to the postponed issue of investigating the existence and uniqueness of the process Y_t (or X_t) as well as to the determination of the active domain of the process. These subjects are drastically influenced by the specific singular form of the

driving force $\frac{\nabla h_m}{h_m}$. Two sources of singularity arise. The first one has been already mentioned and concerns the $(m+1)$-order pole of h_m at zero. However, this singularity is activated only after the process X_t leaves the unbounded domain D^e and so does not make harm to the regularity of the process before the - potential - entrance of X_t in D. The second source of singularity stems from the zeros of h_m, i.e. the zeros of the function $\mathcal{Q}_m(\cos\Theta)$. Denoting $\tilde{h}_m(|y|,\Theta) = h_m(y)$, we easily remark that $\tilde{h}_m(|y|,0) = \frac{1}{|y|^{m+1}} > 0$, for both cases $n = 2, 3$. In the 2-dimensional case the harmonic function h_m sustains its positiveness for $\varphi \in I_m = (-\frac{\pi}{2(m+1)}, \frac{\pi}{2(m+1)})$, while vanishes on the lines $\partial K_m \cap D^e$, where $K_m = \{y \in R^2 : \varphi \in I_m\}$. Similarly, in the 3-dimensional case, we see that $h_m > 0$ inside the cone $K_m = \{y \in R^3 : \theta \in [0, \theta_{m,1})\}$, where $\chi_{m,1} = \cos(\theta_{m,1})$ is the closest root of $P_m(\chi)$ to the right endpoint of its domain $[-1, 1]$. Again, touching the boundary ∂K_m leads to explosion of $\frac{1}{h_m}$. For both dimensions, we remark that the cone K_m becomes narrower as the parameter m increases. We consider the corresponding cones with vertex at ξ, i.e. the sets $\mathcal{K}_m = \xi + K_m$. So in physical terms the process X_t is generated in $\mathcal{K}_m \cap D^e$ - at the point x - and is attracted by the singularity ξ at the same time that it is repelled by the boundary $\partial\mathcal{K}_m \cap D^e$. This descriptive allegation must of course be justified rigorously.

Theorem 2 *There exists only one stochastic process X_t satisfying strongly the S.D.E (3.2). This process enters the domain D through the portion $\partial D \cap \mathcal{K}_m$ of the surface ∂D in finite time, almost surely. The expectation of the first exit time (from the exterior space D^e) is bounded above by $\frac{|x-\xi|^2}{(2m+2-n)}$.*

Proof. We select an increasing sequence $\{D^e_{m,l}\}$ of open bounded subsets of $\mathcal{K}_m \cap D^e$ such that $\overline{D^e_{m,l}} \subset \mathcal{K}_m \cap D^e$ and $\cup_{l=1}^{\infty} D^e_{m,l} = \mathcal{K}_m \cap D^e$. More precisely, we select a very small positive number η and an increasing unbounded sequence η_l - whose particular form will be selected later - so that a sequence of cones $\mathcal{K}_{m,l} = \xi + K_{m,l}$, where $K_{m,l} = \{y \in R^2 : \varphi \in (-\frac{\pi}{2(m+1)} + \frac{\eta}{\eta_l}, \frac{\pi}{2(m+1)} - \frac{\eta}{\eta_l})\}$ in the 2-dimensional case and $K_{m,l} = \{y \in R^3 : \theta \in [0, \theta_{m,1} - \frac{\eta}{\eta_l})\}$ in the 3-dimensional case is formatted. Then we construct the aforementioned nested sequence as $D^e_{m,l} = D^e \cap \mathcal{K}_{m,l} \cap \{z \in R^n : |z - \xi| \leq 2^l |x - \xi|\}$. For each l, the equation (3.2) can be solved (strongly) for $t < \tau_{D^e_{m,l}}$. This gives in a natural way a solution for $t < \tau := \lim_{l\to\infty} \tau_{D^e_{m,l}}$. Let us consider Eq.(3.12) with $k = 2$ and integrate over the time interval $(0, \sigma_r)$, where $\sigma_r = \min(r, \tau_{D^e_{m,l}})$. We obtain

$$|Y_{\sigma_r}|^2 = |x - \xi|^2 - (2m + 2 - n)\sigma_r + 2\int_0^{\sigma_r} Y_t \cdot dB_t \tag{3.13}$$

Taking expectation values to both sides, we obtain

$$E^x[|X_{\sigma_r} - \xi|^2] = |x - \xi|^2 - (2m + 2 - n)E^x[\sigma_r] + 2E^x[\int_0^{\sigma_r} Y_t \cdot dB_t] \tag{3.14}$$

The last Itô integral encountered in Eq.(3.13) can be written as $\int_0^r \mathcal{X}_{\{\tau_{D^e_{m,l}} \geq t\}} Y_t \cdot dB_t$. Given that $|Y_t|$ is bounded for $t \in (0, \sigma_r)$, one of the fundamental properties of the Itô integral is that the expectation of the last integral of the equation above attains zero value (see for example Theorem 3.2.1 of [7]). The physical explanation of this is based on the independence of $\mathcal{X}_{\{\tau_{D^e_{m,l}} \geq t\}} Y_t$ and dB_t and to the zero mean value of the increment dB_t. Consequently Eq.(3.14) gives easily that

$$E^x[\sigma_r] \leq \frac{|x - \xi|^2}{(2m + 2 - n)} \quad \text{for all} \quad r. \tag{3.15}$$

So letting $r \to \infty$ we conclude that $\tau_{D^e_{m,l}} = \lim \sigma_r < \infty$ a.s. and

$$E^x[\tau_{D^e_{m,l}}] \leq \frac{|x - \xi|^2}{(2m + 2 - n)}. \tag{3.16}$$

Consequently taking the limit $l \to \infty$, we obtain

$$E^x[\tau] \leq \frac{|x - \xi|^2}{(2m + 2 - n)}, \tag{3.17}$$

from where we infer that $\tau < \infty$ a.s.

The crucial question is what happens to the process X_t as $t \to \tau$. To answer this question, we determine the infinitesimal generator of the process X_t. We use the harmonicity of h_m and we find that

$$Af = \frac{\nabla h_m}{h_m} \cdot \nabla f + \frac{1}{2}\Delta f = \frac{h_m \Delta f + 2\nabla f \cdot \nabla h_m + f \Delta h_m}{2h_m} = \frac{\Delta(f h_m)}{2h_m} \tag{3.18}$$

Working in the bounded set $D^e_{m,l}$, we adopt as f the C^2-function $\frac{1}{h_m}$ and obtain that $A(\frac{1}{h_m}) = 0$. We apply the Dynkin's formula (A.5) to obtain

$$E^x\left[\frac{1}{h_m(X_{\tau_{D^e_{m,l}}})}\right] = \frac{1}{h_m(x)} \tag{3.19}$$

The process X_t is a continuous function of t and we focus now on the limit $\lim_{t\to\tau} X_t$. Eq.(3.19) implies that $P[\lim_{t\to\tau} X_t \in \partial\mathcal{K}_m \cap D^e] = 0$, given that the function h_m vanishes on the lateral surface of the cone $\mathcal{K}_m$. In addition $\lim_{t\to\tau} X_t$ can not escape to infinity. Indeed, let us consider the auxiliary relation (3.12) with $k = 2m + 4 - n$. We integrate in the time interval $(0, \tau_{D^e_{m,l}})$ and take the expectation value. As before, the Itô integral gives zero contribution to the final result and we finally obtain

$$E^x\left[|X_{\tau_{D^e_{m,l}}} - \xi|^{2m+4-n}\right] = |x-\xi|^{2m+4-n} \tag{3.20}$$

We deduce easily that

$$\begin{aligned} R^{2m+4-n}\, P\left[|X_{\tau_{D^e_{m,l}}} - \xi| > R\right] &\le E^x\left[|X_{\tau_{D^e_{m,l}}} - \xi|^{2m+4-n}\right] \\ \Rightarrow P\left[|X_{\tau_{D^e_{m,l}}} - \xi| > R\right] &\le \left(\frac{|x-\xi|}{R}\right)^{2m+4-n}, \end{aligned} \tag{3.21}$$

for every large radius R. Consequently

$$\begin{aligned} P\left[|\lim_{l\to\infty} X_{\tau_{D^e_{m,l}}} - \xi| \ge R\right] &\le \left(\frac{|x-\xi|}{R}\right)^{2m+4-n} \\ \Rightarrow P\left[|\lim_{t\to\tau} X_t - \xi| \ge R\right] &\le \left(\frac{|x-\xi|}{R}\right)^{2m+4-n} \end{aligned} \tag{3.22}$$

We infer that

$$P\left[|\lim_{t\to\tau} X_t| < \infty\right] = 1 \tag{3.23}$$

So the almost sure escape of X_t from the unbounded set $D^e \cap \mathcal{K}_m$ takes place exclusively from the surface portion $\partial D \cap \mathcal{K}_m$. ■

We determined an upper bound for the mean value of the exit time τ via relation (3.17). We are in position to estimate the covariance in the estimation of the exit time.

Proposition 3 *It holds that*

$$E^x\left[(\tau - E^x[\tau])^2\right] \le \frac{|x-\xi|^4}{(2m+2-n)^2}. \tag{3.24}$$

Proof. We begin by applying the Schwartz inequality and using Eq.(3.20) to obtain

$$E^x\left[|X_{\tau_{D^e_{m,l}}} - \xi|^q\right] \le \left(E^x\left[|X_{\tau_{D^e_{m,l}}} - \xi|^{2m+4-n}\right]\right)^{\frac{q}{2m+4-n}} = |x-\xi|^q, \quad q \in \mathbb{N}. \tag{3.25}$$

We consider Eq.(3.12) for $k = 4$ and integrate over the time interval $(0, \tau_{D^e_{m,l}})$. We obtain

$$|Y_{\tau_{D^e_{m,l}}}|^4 = |x-\xi|^4 - 2(2m-n)\int_0^{\tau_{D^e_{m,l}}} |Y_s|^2 ds + 4\int_0^{\tau_{D^e_{m,l}}} |Y_s|^2 Y_s \cdot dB_s \tag{3.26}$$

We have

$$|Y_s|^2 = |x-\xi|^2 - (2m+2-n)s + 2\int_0^s Y_{s'} \cdot dB_{s'} \tag{3.27}$$

with $s \le \tau_{D^e_{m,l}}$. Inserting this expression into Eq.(3.26), we find

$$\begin{aligned} |Y_{\tau_{D^e_{m,l}}}|^4 &= |x-\xi|^4 - 2(2m-n)|x-\xi|^2\tau_{D^e_{m,l}} + (2m-n)(2m+2-n)(\tau_{D^e_{m,l}})^2 \\ &\quad -4(2m-n)\int_0^{\tau_{D^e_{m,l}}} \left(\int_0^s Y_{s'} \cdot dB_{s'}\right) ds + 4\int_0^{\tau_{D^e_{m,l}}} |Y_s|^2 Y_s \cdot dB_s. \end{aligned} \tag{3.28}$$

We take expectation values on both sides of the equation above. Following the argument stated after Eq. (3.14) we prove that $E^x\left[\int_0^{\tau_{D^e_{m,l}}} |Y_s|^2 Y_s \cdot dB_s\right] = 0$. The same is valid for the double time integral since

$$\int_0^{\tau_{D^e_{m,l}}} \left(\int_0^s Y_{s'} \cdot dB_{s'}\right) ds = \int_0^{\tau_{D^e_{m,l}}} (\tau_{D^e_{m,l}} - s') Y_{s'} \cdot dB_{s'}. \tag{3.29}$$

We conclude to

$$E^x[|Y_{\tau_{D^e_{m,l}}}|^4] = |x-\xi|^4 - 2(2m-n)|x-\xi|^2 E^x[\tau_{D^e_{m,l}}] + (2m-n)(2m+2-n)E^x[(\tau_{D^e_{m,l}})^2]. \quad (3.30)$$

Combining the last result with the bound (3.25) (for $q=4$), we obtain

$$E^x[(\tau_{D^e_{m,l}})^2] \leq \frac{2}{(2m+2-n)}|x-\xi|^2 E^x[\tau_{D^e_{m,l}}]. \quad (3.31)$$

The covariance of the exit time is

$$E^x\left[(\tau_{D^e_{m,l}} - E^x[\tau_{D^e_{m,l}}])^2\right] = E^x[(\tau_{D^e_{m,l}})^2] - (E^x[\tau_{D^e_{m,l}}])^2, \quad (3.32)$$

which due to the inequality (3.31) provides that

$$E^x\left[(\tau_{D^e_{m,l}} - E^x[\tau_{D^e_{m,l}}])^2\right] \leq \frac{2}{(2m+2-n)}|x-\xi|^2 E^x[\tau_{D^e_{m,l}}] - (E^x[\tau_{D^e_{m,l}}])^2. \quad (3.33)$$

Using Eq.(3.16), we are in position to determine an upper bound for the covariance:

$$E^x\left[(\tau_{D^e_{m,l}} - E^x[\tau_{D^e_{m,l}}])^2\right] \leq \frac{|x-\xi|^4}{(2m+2-n)^2}$$

and taking the limit $l \to \infty$ we acquire the estimation (3.24). ■

We have presented some crucial characteristics of the exit time τ. Its expectation value is proportional, as is typical in diffusion processes, to the square of a characteristic distance of the problem. This is the distance of the observation point x and the artificial point ξ, which lies in the bounded component D and forces, as an attractor, the process X_t to escape the exterior domain through a specific small portion of the boundary ∂D. The larger the parameter m (and the oscillating behavior of h_m) is, the smaller is the expected time of boundary hitting of the stochastic process and the narrow is the surface portion of escaping. It becomes clear that when the time-parameter $t_0 := \frac{|x-\xi|^2}{(2m+2-n)}$ is small enough the stochastic process hits the boundary rapidly at the expected time t_0 with covariance equal to t_0^2. The process X_t constructed above is very helpful for the stochastic representation of the solution of boundary value problems of the type (2.4)-(2.6).

3.2 On the stochastic representation of the solution of the exterior Dirichlet problem for Laplace operator.

After having constructed the appropriate stochastic process in the previous section, we shall proceed to obtain the suitable stochastic representation for the solution of the exterior boundary value problem pertaining to Laplace operator. A primitive approach is to apply one more time the Dynkin's formula in the domain $D^e_{m,l}$ - where everything is regular - to the function $f(x) = \frac{u(x)}{h_m(x-\xi)}$. Following the same arguments to those leading to Eq. (3.19) and just exploiting that the function $u(x)$ satisfies Laplace equation in D^e, we obtain that

$$\begin{aligned} Af = \frac{\nabla h_m}{h_m}\cdot\nabla f + \frac{1}{2}\Delta f &= \frac{h_m\Delta f + 2\nabla f\cdot\nabla h_m + f\Delta h_m}{2h_m} \\ &= \frac{\Delta(fh_m)}{2h_m} = \frac{\Delta u}{2h_m} = 0 \end{aligned}$$

and so

$$\frac{u(x)}{h_m(x-\xi)} = E^x\left[\frac{u(X_{\tau_{D^e_{m,l}}})}{h_m(X_{\tau_{D^e_{m,l}}}-\xi)}\right], \quad x \in D^e_{m,l}. \quad (3.34)$$

The representation of $u(x)$ given by Eq.(3.34) seems a priori promising since it is based on the expectation of $f(.) = \frac{u(.)}{h_m(.-\xi)}$ restricted on exit points from $D^e_{m,l}$.

Two separate methodologies having different strategy in orientating the trajectories in the exterior domain will be exposed in the sequel. In both approaches, the interior cone $\mathcal{K}_{m,int}$, which constitutes a protective cloak excluding the proximity with the exterior main cone, is selected to be a fixed cone. In cases that the parameter m is large enough - and this will be the case mainly in the second approach - this interior cone will be selected as the cone $\mathcal{K}_{m+1}$ that keeps a considerable closeness with the lateral surface of $D^e \cap \mathcal{K}_m$ and offers some very useful advantages to the underlying analysis[11]. In case that m is selected to be a small integer - situation adapting mainly to the first approach - then this interior cone $\mathcal{K}_{m,int}$ will be chosen as a fixed member of the family $\mathcal{K}_{m,l}$. In any case, the exterior space in which the trajectories are confined will be denoted as the region $D^e_{m,int} := D^e \cap \mathcal{K}_{m,int} \cap \{z \in R^n : |z-\xi| \leq T|x-\xi|\}$, where $T(>1)$ is a fixed parameter[12] serving at defining the spherical cup of the domain of the paths[13].

3.2.1 The mildly conditioned stochastic method for the solution of harmonic exterior boundary value problems

The first suggested method is based on the representation (3.34) adapted to the fixed domain $D^e_{m,int}$ instead of the sequence of domains $D^e_{m,l}$, $l = 1, 2, 3, ...$, which is written again in the form

$$u(x) = \frac{1}{|x-\xi|^{m+1}} E^x \left[\frac{u(X_{\tau_{D^e_{m,int}}})}{h_m(X_{\tau_{D^e_{m,int}}} - \xi)} \right], \quad x \in D^e_{m,int}, \tag{3.35}$$

given that $h_m(x-\xi) = \frac{1}{|x-\xi|^{m+1}}$.
The name of the method will be completely justified later but the nomination is ought to the fact that the crucial parameter m, which expresses the grade of conditioning of the underlying stochastic processes, is kept small in this approach and so the trajectories obey to a moderate driving status. The method under presentation is based on the fundamental concept of the Monte Carlo [17] methodology according to which the expectation $E[g(Z)]$ of a known function g of a random variable Z can be approximated by the average $\frac{1}{N}\sum_{i=1}^N g(z_i)$, where z_i, $i = 1, 2, ..., N$ represent N independent outcomes concerning the random variable Z. Taking larger number of experiments N leads to better accuracy of the approximation $E[g(Z)] \approx \frac{1}{N}\sum_{i=1}^N g(z_i)$.
Adapting this cornerstone concept to our framework, the main idea is to perform a large number N of experiments, each one of which (with label i, $i = 1, 2, ..., N$) consists of the construction of the path obeying to the non linear system of stochastic differential equations (3.1) (or 3.2) and offers as outcome the first exit time τ_i and mainly the position $X_i = Y_i + \xi$ of escaping the region $D^e_{m,int}$. Based on all these exit positions, we form the average $\frac{1}{N}\sum_{i=1}^N \frac{u(X_i)}{h_m(Y_i)}$, which according to formula (3.35) must be a very good approximation of $u(x)|x-\xi|^{m+1}$ in case that N is large and the experiments are independent. The number of experiments N can be decomposed as $N_{\partial D} + N_{\partial \mathcal{K}_{m,int}} + N_{cup}$ with evident terminology. Only the $N_{\partial D}$ incidences offer terms in the Monte Carlo summation involving known values of $u(X_i)$ (points of the surface where data are given). The crucial idea is that if the numbers $N_{\partial \mathcal{K}_{m,int}}, N_{cup}$ are significantly smaller than $N_{\partial D}$ then a promising perspective emerges permitting the formulation of the following theorem.

Theorem 4 *We consider the boundary value problem*

$$\begin{aligned} &\Delta u(x) = 0, \qquad x \in D^e \subset R^n, \ n = 2, 3 \\ &\lim_{\substack{x \to y \\ x \in R^n \setminus \bar{D}}} u(x) = \phi(y), \quad y \in \partial D, \\ &u(x) = \mathcal{O}(|x|^{-1}), \qquad r = |x| \to \infty. \end{aligned}$$

We perform N independent experiments consisted of generations of trajectories satisfying the system of stochastic differential equations (3.1) and terminated on the boundary of the region $D^e_{m,int}$ defined above on the basis of the surface ∂D, the interior cone $\mathcal{K}_{m,int}$ (with polar angle $\theta_{m,int}$) and the parameter $T > 1$. Depending on the region of first evasion, the number of the experiments is divided as follows: $N = N_{\partial D} + N_{\partial \mathcal{K}_{m,int}} + N_{cup}$. The parameter $m \in N$ is selected to be greater than $\frac{3n-5}{2}$. Let us assume there exists a real $\gamma \in (\frac{\frac{n}{2}-1}{2m+4-n}, \frac{1}{2})$ such that the fraction $\frac{N_{\partial \mathcal{K}_{m,int}}}{N}$ is less or equal to the bound $\frac{1}{T^{(\frac{m+3-n}{2}+\gamma(3m+5-n))}}$. Then the unique solution $u(x)$ of the stated above problem is given by the following representation

$$u(x) = \frac{1}{|x-\xi|^{m+1}} E^x_{\partial D \cap \mathcal{K}_{m,int}} \left[\frac{\phi(X_{\tau_{\partial D}})|Y_{\tau_{\partial D}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial D}}))} \right] + \mathcal{B}(x) \tag{3.36}$$

where the residual $B(x)$ is bounded above as follows:

$$|\mathcal{B}(x)| \leq 2\frac{L}{\mathcal{Q}_m(\cos(\theta_{m,int}))} \frac{1}{T^{1-\frac{n}{2}+\gamma(2m+4-n)}}, \tag{3.37}$$

where L is a constant dependent on the data of the boundary value problem.

[11] The new cone is constructed on the basis of the function $\mathcal{Q}_{m+1}$.
[12] We stabilize the position of the cup in accordance with the specification of the protective interior cone.
[13] We simply denote $S_{cup} = D^e_{m,int} \cap \{z \in R^n : |z-\xi| = T|x-\xi|\}$.

Proof. Due to its technical nature, the proof is given in Appendix B. ■

Remark 5 *A rephrase of the previous result, adapted to the Monte Carlo simulation terminology, is that the solution $u(x)$ can be calculated as follows:*

$$u(x) \approx \frac{1}{|x-\xi|^{m+1}} \frac{1}{N} \sum_{i=1}^{N_{\partial D}} \frac{\phi(X_i)|Y_i|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_i))}. \tag{3.38}$$

The points X_i are the hitting points of the trajectories on the portion of the surface ∂D, which is captured inside $N_{\partial \mathcal{K}_{m,int}}$. In addition $Y_i = X_i - \xi$ has the polar angle Θ_i.

Remark 6 *The previous theorem offers the possibility to determine the solution of the boundary value problem via a representation with a residual term that accumulates all the "fails" of the experiment, i.e. all the trajectories that escape from the domain wherever except the surface of the domain D. The method is useful if the term $\mathcal{B}$ can become small enough. It is logical to assume that this residual term can be suppressed as much as possible if the outcomes of Theorem 2 are exploited. More precisely, selecting the interior cone $\mathcal{K}_{m,int}$ to be very close to the repulsive original cone $\mathcal{K}_m$, the crossings of the paths with this auxiliary lateral surface drastically diminish. Furthermore, choosing a very large T makes escaping from the cup S_{cup} almost impossible. Then everything seems at a first sight amenable and feasible at handling the behavior of the residual term $\mathcal{B}$. However, the choice of an interior cone just being a slight perturbation of the original cone has some other disadvantages. Indeed the function of the polar coordinate $\mathcal{Q}_m$ appearing in the denominator of Eq.(3.37) is defined on the specific argument $\cos(\theta_{m,int})$. The closer is the interior cone selected to the original cone, the closer is the argument $\cos(\theta_{m,int})$ located to the root $\chi_{m,1} = \cos(\theta_{m,1})$ of the function $\mathcal{Q}_m$. So there is a tug of war between the terms appearing in the residual term $\mathcal{B}$ as the interior cone or the cup change position. The forthcoming numerical results designate all these features. What is necessary to state is that this first method can be characterized as a method of a posteriori type. This means that the method can be employed after making a selection of the appropriate geometry on which the Monte Carlo experiments take place with main concern to ensure the validity of the criterion $\frac{N_{\partial \mathcal{K}_{m,int}}}{N} \leq \frac{1}{T^{(\frac{m+3-n}{2}+\gamma(3m+5-n))}}$ as stated in the previous theorem.*

Remark 7 *The efficiency of the method is expected a priori in a specific asymptotic range of physical and geometrical parameters. More precisely, when m is small, the two coaxial cones of the structure are thick and then the most part of the trajectories, which begin their journey from points on the axis of the cones far apart the lateral walls, do not reach the surface of the interior cone before meeting ∂D since there exists enough space for the draw from the attractor ξ to overwhelm the randomness of the Brownian motion. This is obviously comprehensive in case that we are looking for determining the near field at points x close to the surface ∂D: The trajectories are expected to leave the exterior space through ∂D in short times and the crossings on the wall of the interior cone are going to be rare.*

Remark 8 *At first sight, the representation (3.38) mystifies some readers, who might expect a result of the form[14] $u(x) \approx \frac{\mathcal{A}}{|x-\xi|} + \mathcal{O}(\frac{1}{|x-\xi|^2})$, which constitutes a paraphrase of harmonic expansions in exterior regions for $n = 3$. First, this consensus is not violated in the near field region where the ratio $\frac{|Y_i|}{|x-\xi|}$ is slightly less than 1 and its $(m+1)$-power provides no negligible contribution. Even when x comes away from D, the auxiliary attractor ξ, which also has the freedom to pull away[15], is responsible to keep the ratio $\frac{|Y_i|}{|x-\xi|}$ in the vicinity of unity. As far as the last assertion according to the remoteness of ξ is concerned, it is worthwhile to mention that in the regime of the mildly conditioned stochastic method, the pair of coaxial cones must be thick[16] and centered at a distant point ξ so that the whole conical structure embodies globally the region D. Then all the points of the surface ∂D have the opportunity to contribute to the mean value (3.38) as the well posedness of the direct b.v.p dictates. Finally the role of $\mathcal{Q}_m(\cos(\Theta_i))$ in the denominator of (3.38) must not be underestimated since, for crossings on ∂D close to the cone walls, the function $\frac{1}{\mathcal{Q}_m}$ might take large values balancing the smallness of $(\frac{|Y_i|}{|x-\xi|})^{m+1}$.*

3.2.2 The strongly conditioned stochastic method for the solution of harmonic exterior boundary value problems

The cornerstone consideration of the mildly conditioned stochastic method was the avoidance of crossings of the stochastic process along the lateral surface of the protective cone. The starting point of the second method

[14] $\mathcal{A}$ absorbs dependence of the remaining coordinates except the radial distance in a coordinate system centered at ξ.

[15] In case that the attractor point is taken in the exterior domain of the problem, the cones structure must be supplied with a small spherical cup encountering and excluding the vertex ξ to belong to the excursion field of the trajectories in order for all the good properties guaranteeing the unique solution of the stochastic differential system to hold.

[16] not only to avoid lateral crossings as mentioned before.

is the formulation of a model not sensitive to the incidences of the trajectories on the lateral surface. We will describe the methodology for three-dimensional problems ($n = 3$), although identical results hold in two dimensions. The first step of the method is to select as interior cone the conical surface corresponding to the parameter $m + 1$, i.e the cone $\mathcal{K}_{m+1}$. We have already stated that the strongly conditioned stochastic method works efficiency for large values of m and so the system of cones involves closely adjacent surfaces. The second selection - in accordance with the first one - concerns the function which is going to be subject to the application of Dynkin's formula. We chose $f(x) := \frac{|x-\xi|u(x)h_{m+1}(x-\xi)}{h_m(x-\xi)}$ as the candidate function with the crucial property of vanishing on the wall of $\mathcal{K}_{m+1}$. Thus, the conical surface offers zero contribution to the stochastic expectation under construction. There are more intrinsic advantageous properties of this particular f that will be revealed later on. The parameter T defining the cup will be selected to be a considerably large constant, just to assure that the domain $D^e_{m,int}$ of our experiments is not infinite. To apply Dynkin's formula (A.5) to f in the domain $D^e_{m,int}$ we first calculate Af using the harmonicity of h_m:

$$\begin{aligned} Af(x) &= \frac{\nabla h_m(x) \cdot \nabla f(x)}{h_m(x)} + \frac{1}{2}\Delta f(x) = \frac{\Delta(f(x)h_m(x))}{2h_m(x)} \\ &= \frac{\Delta(|x-\xi|u(x)h_{m+1}(x-\xi))}{2h_m(x)}. \end{aligned} \tag{3.39}$$

Then we find easily that

$$\begin{aligned} u(x) = {} & E^x\left[u(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right] \\ & -E^x\left[\int_0^{\tau_{D^e_{m,int}}}\left(\frac{\Delta\left(u(X_s)|X_s-\xi|h_{m+1}(X_s-\xi)\right)}{2h_m(X_s-\xi)}\right)ds\right] \end{aligned} \tag{3.40}$$

Thus, an immediate consequence of our choice for f is the appearance of an expectation term involving time integrals along the trajectories of the stochastic process. Our main effort is to investigate this term. Before doing so let us notice that the expectation value $E^x\left[u(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right]$ is practically equal to $E^x_{\partial D\cap\mathcal{K}_{m+1}}\left[\phi(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right]$ given that the conical surface does not provide contribution as mentioned before, while the participation of the cup is negligible for large T thanks to relation (3.21) and the asymptotic condition at infinity:

$$\left|E^x_{S_{cup}}\left[u(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right]\right| \leq \frac{C}{P_m(\chi_{m+1,1})}\frac{1}{|x-\xi|T^{2m+2}}, \tag{3.41}$$

where C is proportional to the capacity of the field.
The first result is really striking although refers to the simplest possible geometry and stimulation. Actually this pilot problem has a well known solution for which we do not need the stochastic calculus but can be used as a benchmark example for our purposes.

Proposition 9 *Let the region D be a sphere of radius a and center the coordinate origin O. Let us suppose that the attractor point ξ is situated at the center of the sphere. Let $D^e_{m+1} = D^e \cap \mathcal{K}_{m+1}$ be the portion of the exterior space confined by the cone[17] $\mathcal{K}_{m+1}$. Then the well known solution of the exterior boundary value problem*

$$\Delta u(x) = 0, \qquad x \in D^e \subset R^3 \tag{3.42}$$

$$\lim_{\substack{x\to y \\ x\in R^3\setminus\bar{D}}} u(x) = 1, \quad y \in \partial D, \tag{3.43}$$

$$u(x) = \mathcal{O}(|x|^{-1}), \quad r = |x| \to \infty, \tag{3.44}$$

which is given by the spherical symmetric function $u(x) = \frac{a}{|x|}$ has the following exact stochastic representation

$$\frac{a}{|x|} = E^x_{\partial D}\left[\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right], \quad x \in D^e. \tag{3.45}$$

This result is valid for every possible parameter $m \in \mathbb{N}$.

[17] adapted as usually to the axis formatted by the attractor $\xi \equiv O$ and the point of observation x.

Proof. The first exit time for every bounded set $D^e_{m,int} = D^e_{m+1} \cap G_T$ (with $G_T = \{z \in R^3 : |z| \le T|x-\xi|\}$) is finite and less than the first exit time $\tau_{D^e_m}$, which has been investigated in Theorem 2 and been proved to have finite expectation value. We apply the Dynkin's formula (3.40) in $D^e_{m,int}$ and make the crucial remark that for this simple harmonic field $u(x) = \frac{a}{|x|}$, we have

$$\Delta\left(u(X_s)|X_s - \xi| h_{m+1}(X_s - \xi)\right) = \Delta\left(a h_{m+1}(X_s)\right) = 0, \tag{3.46}$$

due to the harmonicity of h_{m+1}. The formula (3.40) becomes

$$\begin{aligned} u(x) &= E^x\left[u(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right] \\ &= E^x_{\partial D}\left[\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right] + E^x_{S_{cup}}\left[u(X_{\tau_{D^e_{m,int}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m,int}}})}{P_m(\Theta_{\tau_{D^e_{m,int}}})}\right]. \end{aligned}$$

Using Eq.(3.41) and taking the limit as $T \to \infty$ we obtain the stated result. ■

Remark 10 *This result reveals an interesting harmonic field that can be generated by the conditioned expectation value of the ratio of two Legendre polynomials of consecutive order with driving term generated by the denominator and conical surface pertaining to the nominator. This fraction is positive and always less than unity since the argument of the Legendre polynomials lies in the interval* $(\chi_{m+1,1}, 1]$. *Only when the starting point x falls*[18] *on the surface ∂D, the expectation value is trivially equal to one, in accordance with the boundary condition. For a starting point $x \in D^e$, several trajectories will cross the conical surface but this is no longer a problem. Only the successful escapes via the surface ∂D offer quantitative contributions to (3.45) and the fails are just responsible for the reduction of the field as x moves away the surface ∂D, given that the non contributing fails increase. This can be expressed in the Monte Carlo terminology via the expression* $\frac{a}{|x|} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N_{\partial D}} \frac{P_{m+1}(\cos(\Theta_i))}{P_m(\cos(\Theta_i))}$.

Let us return to the case of a general body D. We state the next main outcome of this paragraph along with the next theorems and propositions without presenting their profs since they are very technical, involved and rather elongated. So in order not to disorientate the reader, we have placed the proofs of all the subsequent statements of this paragraph in Appendix B.

Theorem 11 *We consider the C^2-solution $u(x)$ of the Dirichlet problem*

$$\begin{aligned} &\Delta u(x) = 0, \qquad x \in D^e = R^3 \backslash \overline{D}, \\ &\lim_{\substack{x\to y \\ x\in R^3\backslash \bar{D}}} u(x) = \phi(y) \quad y \in \partial D, \\ &u(x) = \mathcal{O}(|x|^{-1}), \qquad r = |x| \to \infty \end{aligned}$$

Let $b_\xi = dist\{\xi, \partial D \cap \mathcal{K}_{m+1}\} \ge 1$. Then the field $u(x)$ obtains the representation

$$u(x) = E^x_{\partial D}\left[\phi(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x), \tag{3.47}$$

with

$$|\mathcal{E}(x)| \le C\frac{1}{|x-\xi|^2} \quad (\textit{in large distances } |x-\xi|) \quad \textit{and} \quad |\mathcal{E}(x)| \le \left(\frac{1}{b_\xi|x-\xi|} - \frac{1}{|x-\xi|^2}\right), \quad \textit{in the near field.}$$

The constant C dependents only on the domain D and the data ϕ. In both cases $|x| \to \infty$ and $x \to \partial D$, the residual satisfies the asymptotic relation $|x-\xi||\mathcal{E}(x)| \to 0$.

So far, we faced exclusively the Dirichlet problem. However the Neumann exterior boundary value problem can be handled in the same manner. We state the following theorem.

Theorem 12 *We consider the C^2-solution $u(x)$ of the Neumann problem*

$$\begin{aligned} &\Delta u(x) = 0, \qquad x \in D^e = R^3 \backslash \overline{D}, \\ &\frac{\partial u}{\partial n}(y) = g(y) \quad y \in \partial D, \\ &u(x) = \mathcal{O}(|x|^{-1}), \qquad r = |x| \to \infty. \end{aligned}$$

[18]remaining on the axis connecting ξ, x.

The axis $\hat{n}_{x,\xi}$ of the cone $\mathcal{K}_{m+1}$ is now selected to be parallel to the exterior unit normal vector $\hat{n}_{y_\xi}$ on ∂D at the point y_ξ, which is the intersection of the conical axis with the surface ∂D. Then the following representation holds

$$u(x) = -E^x_{\partial D}\left[|Y_{\tau_{D^e_{m+1}}}|g(X_{\tau_{D^e_{m+1}}})\cos(\Theta_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] + \mathcal{O}\left(\frac{1}{|x-\xi|^2}+\frac{\sin(\theta_{m+1,1})}{|x-\xi|}\right). \quad (3.48)$$

The obtained so far results obey to the asymptotic rate $\mathcal{O}(\frac{1}{|x-\xi|^2})$ and the arising question is whether is possible to improve the rate of convergence. The following Proposition ameliorates in some aspect the rigorousness of the previous theorem.

Proposition 13 *Adopting the same assumptions made in Theorem 11 about the exterior boundary value problem under consideration, we state that*

$$\begin{aligned} 2u(x) + (x-\xi)\cdot\nabla u(x) &= E^x_{\partial D}\left[2u(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] \\ &+ E^x_{\partial D}\left[Y_{\tau_{D^e_{m+1}}}\cdot\nabla u(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \tilde{\mathcal{E}}(x), \end{aligned} \quad (3.49)$$

with $|\tilde{\mathcal{E}}(x)| \le \frac{1}{|x-\xi|^3}$ for $|x|\to\infty$.

The result presented in Proposition 13 is not exploitable at first sight since it consists of a representation involving simultaneously boundary values of the field $u|_{\partial D}$ and its gradient $\nabla u|_{\partial D}$. However, this is exactly the property that discloses the advantages of the formula by offering the possibility to produce a stochastic characterization of the Dirichlet to Neumann mapping.

Theorem 14 *Let u be the exterior solution of the Laplace operator, obeying to the asymptotic relation $u(x) = \mathcal{O}(\frac{1}{|x-\xi|})$. The parameter m is selected suitably large so that $\sin(\theta_{m+1,1}) < \frac{1}{|x-\xi|}$. The cone axis $\hat{n}_{x,\xi}$ is chosen parallel to the normal vector $\hat{n}_{y_\xi}$ at the intersection point y_ξ of the surface ∂D and the cone. Then the Dirichlet to Neumann map $\Lambda : u|_{\partial D} \to \frac{\partial u}{\partial n}\big|_{\partial D}$ obtains the stochastic representation*

$$\begin{aligned} A_0 &= |x-\xi|E^x_{\partial D}\left[2u(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] \\ &+ |x-\xi|E^x_{\partial D}\left[|Y_{\tau_{D^e_{m+1}}}|(\Lambda u)(X_{\tau_{D^e_{m+1}}})\cos(\Theta_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] + \mathcal{O}(\frac{1}{|x-\xi|^2}), \quad |x-\xi|\to\infty \end{aligned} \quad (3.50)$$

Remark 15 *The Theorem above gives merely a local character to the DtN mapping given that only a small portion $\partial D \cap \mathcal{K}_{m+1}$ of the surface ∂D is participating every time in the stochastic calculus. This is a quantitative advantage of the current approach given that the non locality of the operator Λ is one of the cornerstone difficulties in the framework of integral equation methodology.*

3.3 Stochastic calculus and low-frequency scattering processes

The low-frequency scattering problem merits special interest in case that the geometric characteristics of the problem are significantly smaller than the wavelength of the acoustic stimulation. The three dimensional case is handled here again, given that this is the suitable cradle for the low-frequency theory. We examine the plane wave excitation problem presented via Eqs.(2.7-2.9). The stochastic technique in the regime of the strongly conditioned stochastic process applies again leading to the following theorem:

Theorem 16 *Let $b_\xi = dist\{\xi, \partial D \cap \mathcal{K}_{m+1}\} \ge 1$ and $u(x)$ the classical solution of the boundary value problem*

$$(\Delta + k^2)u(x) = 0, \quad x \in D^e \subset R^3 \quad (3.51)$$

$$u(x)(= g(x, k\hat{k})) = -\exp(ik\hat{k}\cdot x), \quad x \in \partial D \quad (3.52)$$

$$\lim_{r\to\infty}\frac{1}{r}\left(\frac{\partial u(x)}{\partial r} - iku(x)\right) = 0. \quad (3.53)$$

Then

$$e^{-ik|x-\xi|}u(x) = -E^x_{\partial D}\left[e^{-ik\hat{k}\cdot X_{\tau_{D^e_{m+1}}}}e^{-ik|Y_{\tau_{D^e_{m+1}}}|}\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x), \quad (3.54)$$

with

$$|\mathcal{E}(x)| \quad \leq \quad C\frac{1}{|x-\xi|}\min\left(\frac{1}{|x-\xi|},\left(\frac{1}{b_\xi}-\frac{1}{|x-\xi|}\right)\right)+\mathcal{O}(kb_\xi) \tag{3.55}$$

Proof. We evoke the Atkinson-Wilcox expansion for the scattering field with respect to the suitable spherical coordinate system (r_ξ,θ,ϕ) centered at ξ, which is valid outside a sphere circumscribing the region D.

$$\tilde{u}=\frac{e^{ikr_\xi}}{r_\xi}\sum_{n=0}^{\infty}\frac{F_n(\theta,\phi,k,\hat{k})}{r_\xi^n} \tag{3.56}$$

We apply Dynkin's formula to the function $f(x):=\frac{|x-\xi|e^{-ik|x-\xi|}u(x)h_{m+1}(x-\xi)}{h_m(x-\xi)}$. After extended manipulations we find that

$$\begin{aligned}\frac{1}{2h_m}\Delta\left(\tilde{u}e^{-ikr_\xi}r_\xi h_{m+1}\right) \quad &= \quad -(m+1)\left[\frac{\tilde{u}}{r_\xi^2}+\frac{1}{r_\xi}\frac{\partial\tilde{u}}{\partial r_\xi}-ik\frac{\tilde{u}}{r_\xi}\right]e^{-ikr_\xi}\frac{P_{m+1}}{P_m}-ik\left[\frac{\partial\tilde{u}}{\partial r_\xi}-ik\tilde{u}\right]e^{-ikr_\xi}\frac{P_{m+1}}{P_m}\\ &+ \quad \left[\frac{1}{{r_\xi}^2}\frac{\mathbb{D}\tilde{u}\cdot\mathbb{D}P_{m+1}}{P_m}\right]e^{-ikr_\xi}.\end{aligned} \tag{3.57}$$

We easily obtain the estimate

$$\left|\left[\frac{\tilde{u}}{r_\xi^2}+\frac{1}{r_\xi}\frac{\partial\tilde{u}}{\partial r_\xi}-ik\frac{\tilde{u}}{r_\xi}\right]e^{-ikr_\xi}\right|\leq\frac{C}{{r_\xi}^4}. \tag{3.58}$$

Exploiting the integral representation of the far field pattern F_0, we find that[19]

$$\begin{aligned}\hat{\theta}\cdot\mathbb{D}F_0(\theta,\phi,k,\hat{k}) \quad &= \quad -\frac{ik}{4\pi}\hat{\theta}\cdot\int_{\partial D}\tilde{u}(y)\hat{n}_y e^{-ik\hat{x}_\xi\cdot y}ds_y+\frac{ik}{4\pi}\hat{\theta}\cdot\int_{\partial D}\frac{\partial\tilde{u}}{\partial n_y}(y)\; y e^{-ik\hat{x}_\xi\cdot y}ds_y\\ &= \quad ik[(\mathcal{A}\cos(\phi)+\mathcal{B}\sin(\phi))\cos(\theta)+\mathcal{C}\sin(\theta)]+\mathcal{O}(k^2).\end{aligned} \tag{3.59}$$

So the interesting term of $\left[\frac{1}{{r_\xi}^2}\frac{\mathbb{D}\tilde{u}\cdot\mathbb{D}P_{m+1}}{P_m}\right]e^{-ikr_\xi}$ as far as the formation of expectation values is concerned obtains an upper bound:

$$\left|\left[\frac{1}{{r_\xi}^2}\frac{\mathbb{D}\tilde{u}\cdot\mathbb{D}P_{m+1}}{P_m}\right]e^{-ikr_\xi}\right|\leq C\frac{k}{{r_\xi}^3}\frac{\sin^2(\theta)P'_{m+1}(\cos(\theta))}{P_m(\cos(\theta))}+\mathcal{O}(k^2). \tag{3.60}$$

So the residual term $\mathcal{E}(x)$ in the current setting[20] obtains the estimate

$$\begin{aligned}|\mathcal{E}(x)| \quad &\leq \quad C(m+1)E^x\left[\int_0^{\tau_{D^e_{m,int}}}\frac{1}{|Y_s|^4}ds\right]+C(m+1)kE^x\left[\int_0^{\tau_{D^e_{m,int}}}\frac{1}{|Y_s|^3}ds\right]\\ &+ \quad kE^x\left[\int_0^{\tau_{D^e_{m,int}}}\left|\frac{\partial\tilde{u}(Y_s)}{\partial|Y_s|}-ik\tilde{u}(Y_s)\right|ds\right].\end{aligned} \tag{3.61}$$

The first term of the right hand side of Eq.(3.61) has been already estimated. Furthermore, it is proved that

$$\begin{aligned}&(m+1)kE^x\left[\int_0^{\tau_{D^e_{m,int}}}\frac{1}{|Y_s|^3}ds\right]=k\left(E^x\left[\frac{1}{|Y_{D^e_{m,int}}|}\right]-\frac{1}{|x-\xi|}\right)\\ &\leq k\left(\left(E^x\left[\frac{1}{|Y_{D^e_{m,int}}|^2}\right]\right)^{\frac{1}{2}}-\frac{1}{|x-\xi|}\right)\leq kB_m\min\left(\frac{1}{|x-\xi|},(\frac{1}{b_\xi}-\frac{1}{|x-\xi|})\right),\end{aligned} \tag{3.62}$$

where B_m is a bounded constant. In addition it holds that

$$\begin{aligned}&E^x\left[\int_0^{\tau_{D^e_{m+1}}}\left|\frac{\partial\tilde{u}(Y_s)}{\partial|Y_s|}-ik\tilde{u}(Y_s)\right|ds\right]\leq E^x\left[\left(\int_0^{\tau_{D^e_{m+1}}}\left|\frac{\partial\tilde{u}(Y_s)}{\partial|Y_s|}-ik\tilde{u}(Y_s)\right|^2ds\right)^{\frac{1}{2}}\left(\int_0^{\tau_{D^e_{m+1}}}1ds\right)^{\frac{1}{2}}\right]\\ &\leq CE^x\left[\left(\int_0^{\tau_{D^e_{m+1}}}\frac{1}{|Y_s|^4}ds\right)^{\frac{1}{2}}\left(\tau_{D^e_{m+1}}\right)^{\frac{1}{2}}\right]\leq C\left(E^x\left[\left(\int_0^{\tau_{D^e_{m+1}}}\frac{1}{|Y_s|^4}ds\right)\right]\right)^{\frac{1}{2}}\left(E^x\left[\tau_{D^e_{m+1}}\right]\right)^{\frac{1}{2}}\\ &\leq C\frac{\sqrt{|x-\xi|}}{\sqrt{(m+1)(2m-1)}}\min\left(\frac{1}{\sqrt{|x-\xi|}},\left(\frac{1}{b_\xi}-\frac{1}{|x-\xi|}\right)^{\frac{1}{2}}\right)\end{aligned} \tag{3.63}$$

[19] We set $\hat{x}_\xi=(\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta))$.
[20] after recalling the Y_s-terminology.

Consequently the Dynkin's formula applied to f leads to

$$e^{-ik|x-\xi|}u(x) = -E^x_{\partial D}\left[e^{-ik\hat{k}\cdot X_{\tau_{D^e_{m+1}}}} e^{-ik|Y_{\tau_{D^e_{m+1}}}|} \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x), \tag{3.64}$$

with

$$\begin{aligned} |\mathcal{E}(x)| &\leq C\frac{1}{|x-\xi|}\min\left(\frac{1}{|x-\xi|}, \left(\frac{1}{b_\xi} - \frac{1}{|x-\xi|}\right)\right) + C(kb_\xi)\frac{1}{b_\xi}\min\left(\frac{1}{|x-\xi|}, (\frac{1}{b_\xi} - \frac{1}{|x-\xi|})\right) \\ &+ C\frac{(kb_\xi)}{\sqrt{(m+1)(2m-1)}}\min\left(\frac{1}{b_\xi}, \frac{1}{b_\xi}\left(\frac{|x-\xi|}{b_\xi} - 1\right)^{\frac{1}{2}}\right) \end{aligned} \tag{3.65}$$

Taking a sufficiently large $m \approx |x-\xi|^2$ we obtain the stated result. ■
The following Corollary gives the adequate stochastic characterization of the far field pattern.

Corollary 17 *The far field pattern $F_0(\theta, \phi, k, \hat{k})$ of the exterior scattering problem obtains the low-frequency stochastic representation*

$$F_0(\theta, \phi, k, \hat{k}) = -|x-\xi| E^x_{\partial D}\left[e^{-ik\hat{k}\cdot X_{\tau_{D^e_{m+1}}}} e^{-ik|Y_{\tau_{D^e_{m+1}}}|} \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] + \mathcal{O}\left((kb_\xi) + \frac{1}{|x-\xi|}\right). \tag{3.66}$$

4 Numerical Implementation of the two conditioned stochastic methods

The methods presented so far constitute two different approaches revealing their efficiency in complementary areas of interest. The mildly conditioned stochastic method is an exact representation, as far as the Dynkin's formula is concerned, but is based on throwing away the contributions of the escaping points belonging on the lateral conical surface. So the validity of the method relies on the assumption that the most part of the trajectories cross the boundary instead of the lateral surface of the cone and this demands an attentive setting of the geometric features of the trajectory domain. This prerequisite has as immediate consequence the selection of spacious cones averting the frequency of lateral escapes. As a result this method aims better at the solution of the direct problem given that the cone encloses the domain D.
The strongly conditioned stochastic method is by nature an approximated Dynkin's formula, which is though exact in omitting the indifferent lateral escapes. But in order to control the error of the approximation, it is necessary to work with narrow cones corresponding to large values of the parameter m. These narrow cones excise a small part of the domain D or its boundary ∂D and are more appropriate to give results in the service of the inverse problem.
In both approaches, the first step is the numerical solution of the stochastic ordinary differential equations (3.1). These equations lead - after some simple manipulations - to the discretized Euler scheme

$$\begin{aligned} &Y_0 = (Y_0^{(1)}, Y_0^{(2)}, Y_0^{(3)}) = x \\ &Y_{n+1}^{(i)} = Y_n^{(i)} - \frac{(2m+1)}{|Y_n|^2}Y_n^{(i)}\Delta t_n - \frac{P'_{m-1}(\cos\theta_n)}{P_m(\cos\theta_n)}\frac{1}{|Y_n|^2}Y_n^{(i)}\Delta t_n + \Delta B_n^{(i)}, \quad i = 1, 2 \\ &Y_{n+1}^{(3)} = Y_n^{(3)} - \frac{(2m+1)}{|Y_n|^2}Y_n^{(3)}\Delta t_n + \frac{mP_{m-1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)}\frac{1}{|Y_n|^2}Y_n^{(3)}\Delta t_n + \Delta B_n^{(3)}. \end{aligned} \tag{4.1}$$

This scheme is relatively sufficient in the case of spacious cones. But when the parameter m increases, the space of evolution of the trajectories becomes restricted and approaching the lateral walls brings into the light significant terms of higher order that have been ignored on the basis of selecting the Euler approximation. The more adequate approach is to apply the so called [20] *order 1.5 strong Taylor scheme*, which is proved to be very efficient since it includes terms involving interference between the time increment Δt_n and the vector Wiener increments $W_n = (\Delta B_n^{(1)}, \Delta B_n^{(2)}, \Delta B_n^{(3)})$. Extended manipulations lead to the following corrections of the discretized scheme :

$$
\begin{aligned}
Y_{n+1}^{(i)} &= Y_{n+1}^{(i),\mathrm{Euler}} + 2\frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)}\frac{1}{|Y_n|^4}(Y_n\cdot W_n)Y_n^{(i)}\Delta t_n - \frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)}\frac{1}{|Y_n|^2}W_n^{(i)}\Delta t_n \\
&+\cos\theta_n\left[2\frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)} - \left((m+1)\frac{P_{m+1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)} + (m+1)^2\right) - \sin^2\theta_n\left(\frac{P'_m(\cos\theta_n)}{P_m(\cos\theta_n)}\right)^2\right] \\
&\frac{1}{|Y_n|^3}\frac{Y_n^{(i)}}{\sin\theta_n}(\hat{\theta}_n\cdot W_n)\Delta t_n, \quad i=1,2
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
Y_{n+1}^{(3)} &= Y_{n+1}^{(3),\mathrm{Euler}} + \left[\frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)} + (m+1)\frac{P_{m+1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)}\right]\frac{1}{|Y_n|^4}(Y_n\cdot W_n)Y_n^{(3)}\Delta t_n \\
&-\frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)}\frac{1}{|Y_n|^2}W_n^{(3)}\Delta t_n - \left[\frac{P'_{m+1}(\cos\theta_n)}{P_m(\cos\theta_n)} - (m+1)^2 - \sin^2\theta_n\left(\frac{P'_m(\cos\theta_n)}{P_m(\cos\theta_n)}\right)^2\right] \\
&\frac{1}{|Y_n|^3}\frac{\sin\theta_n Y_n^{(3)}}{\cos\theta_n}(\hat{\theta}_n\cdot W_n)\Delta t_n
\end{aligned}
\tag{4.3}
$$

The total time of flight of the trajectories is estimated by the value $\frac{|x-\xi|^2}{2m-1}$ as the theoretical analysis brought out and the time step $h = \Delta t_n$ is an important parameter of the scheme. The adopted scheme provides an accuracy of order $\mathcal{O}(h^{1.5})$, which means that the expectation value of the distance between the real trajectory paths and the numerically constructed ones is controlled by the estimate $Ch^{1.5}$. In principle, this time step is diminished when we approach the lateral surface of the cone, where the driving term becomes large enough but alternatively it is possible to design uniform time steps influencing though the rapidity of the numerical experiments. In any case the stability of the method must be investigated by sensitivity analysis with respect to the time step size.

4.1 On solving direct exterior boundary value problems via the mildly conditioned stochastic method

The pilot exterior problem is the potential or low-frequency exterior problem referring to the corresponding processes outside two spheres of unequal radii R_1 and R_2. The analytic solution of this problem can be found

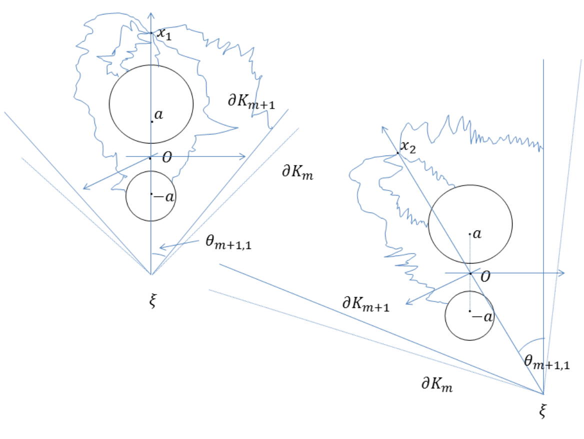

Figure 1: Solving the direct exterior boundary value problem outside two spheres of unequal radii.

in [18] and is based on the exploitation of the bispherical system of coordinates. We select here a pair of spheres with radii $R_1 = 1$ and $R_2 = 2$ while the distance of their centers, located on the axis at the points $z_1 = -1.625$ and $z_2 = 2.375$, is chosen to be equal to $d = 4$ (length units). We introduce the bispherical system [18] of coordinates (ζ, θ, ϕ), which on the basis of our assumptions disposes as focii distance the length $2a = 2 \times 1.28 = 2.56$. The small sphere is assigned to the coordinate value $\zeta_1 = -1.067$, while the large sphere defines the surface $\zeta_2 = 0.603$. We use the result presented in [18] for the Rayleigh approximation of the exterior field satisfying the problem (3.51)-(3.53).

The first component of the low-frequency expansion u_0 has the exact representation [18]

$$u_0(\zeta,\theta) = -1 + \sqrt{\cosh\zeta - \cos\theta}\sum_{n=0}^{\infty} I_n(\zeta)P_n(\cos\theta), \tag{4.4}$$

with

$$\begin{aligned} I_n(\zeta) \;=\; \sqrt{2}\Bigg[& e^{-(n+\frac{1}{2})|\zeta|} - \frac{e^{(2n+1)\zeta_1}-1}{e^{(2n+1)(\zeta_1+\zeta_2)}-1}e^{(n+\frac{1}{2})\zeta} \\ & -\frac{e^{(2n+1)\zeta_2}-1}{e^{(2n+1)(\zeta_1+\zeta_2)}-1}e^{-(n+\frac{1}{2})\zeta}\Bigg]. \end{aligned} \tag{4.5}$$

We define a set of observation points of interest on which we would like to test the mildly conditioned stochastic method. In Figure 1 we depict two points of observation: The first one x_1 is situated on the axis connecting the centers of the two spheres (with coordinates[21] $\zeta(x_1) = 0.55, \theta(x_1) = 0$). The associated attractor point $\xi = \xi_1$ for this experiment is located at the position $(0, 0, -3.5)$. The second observation point is located at the eccentric position[22] $\zeta(x_2) = 0.33, \theta(x_2) = \frac{\pi}{3}$ and $\phi(x_2) = \frac{7\pi}{6}$ and its dual attractor point is placed at the opposite direction position $\xi_2 = -4x_2$. Additional observation points have been considered, revealing the special behavior of this specific geometric structure. We focus on the point x_3 situated almost[23] at the coordinate origin $O = (0, 0, 0)$ with corresponding attractor point $\xi_3 = (0, 0, -7)$. The representation (3.38) is used to exploit the stochastic experiments and all the calculations have been performed on the basis of the selection $m = 2$ or $m = 3$. The cone $\mathcal{K}_{m+1}$ presented in Figure 1 is selected to be alternatively a slight perturbation of $\mathcal{K}_m$ in the context of the approach followed in the mildly conditioned stochastic method or just the cone corresponding to the next Legendre parameter level, as occurs definitely in the strongly conditioned stochastic case. The deviation of the results due to the cone selection has been proved uniformly negligible in all cases that the measurement point is located at the axis of the cones. The results have also only slight differences between the cases $m = 2$ and $m = 3$ and are presented in Table 1 in comparison with the exact theoretical outcomes. In all cases the percentage of the the trajectories crossing the conical surface is very small and confined to represent just 0.5% of the total number of trajectories.

Observation point	**The attractor point**	**The exact analytic field**	**The mild stochastic field**
$(0, 0, 4.77)$	$(0, 0, -3.5)$	-0.839	-0.85
$(-1.73, -1, 0.78)$	$(6.92, 4, -3.12)$	-0.828	-0.81
$(0, 0, 0)$	$(0, 0, -7)$	-0.966	-0.95
$(10.3, 0, 20.4)$	$(-2.06, 0, -4.08)$	-0.107	-0.095
$(0, 0, 8.6)$	$(0, 0, -8)$	-0.338	-0.312
$(0, 0, 6.49)$	$(0, 0, -8)$	-0.502	-0.54
$(2.175, 0, 2.14)$	$(0, 0, -8)$	-0.922	-0.94

Table 1: Testing the mildly conditioned stochastic method via the Rayleigh approximation of the low-frequency scattering problem by two spheres.

4.2 On solving direct and inverse exterior boundary value problems via the strongly conditioned stochastic method

4.2.1 The exact spherical case

The first task of this section is of course the numerical verification of the exact representation (3.45), solving the specific exterior harmonic problem (3.42)-(3.44) outside a spherical body of radius a. We perform experiments with cones of different thickness and observation points in the near and far field[24]. The main results are tabulated in Table 2 and reveal the exactness of the stochastic model in assuring the validity of the relation (3.45). Actually only cases corresponding to large values of m are displayed since these results are orientated to be exploited in the inverse problem realm where the localization of the points of ∂D requires narrow cones. The number of experiments N is confined to a thousand of paths and stability of the results is fulfilled uniformly for the most part of the cases. It is worthwhile to notice that when the parameter m increases so that only a small portion of the body ∂D participates in the stochastic calculus, then the Euler scheme becomes poor while

[21] In cartesian terms $x_1^{(1)} = x_1^{(2)} = 0, x_1^{(3)} = 4.77$.
[22] corresponding to the cartesian representation $(-1.73, -1, 0.78)$.
[23] We select $\theta = \pi$ but $\zeta = 0.001$ and not zero to avoid convergence instability of formula (4.4).
[24] The attractor is stabilized in the center of the sphere in this case, while the cone $\mathcal{K}_{m+1}$ corresponds definitely to the first root of the Legendre polynomial P_{m+1}.

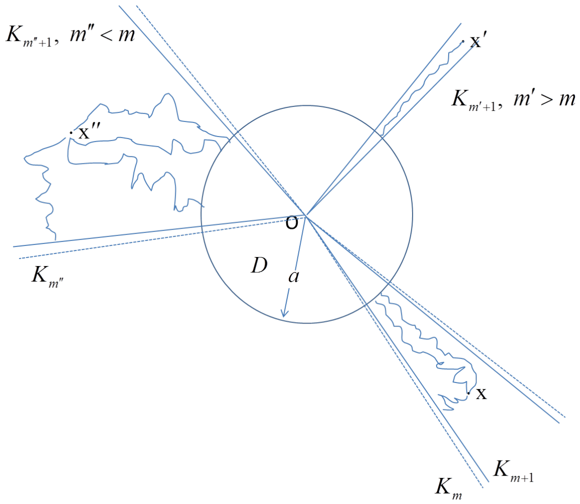

Figure 2: The strong stochastic solution for the spherical case.

the *order 1.5 strong Taylor scheme* (4.3) displays robustness in ensuring stable results. This happens since the majority of the trajectories move closely to the lateral conical surface and in this region the additional terms of the scheme (4.3) obtain comparable size to the terms of the Euler scheme. It is worthwhile to mention that the first and the last cases of Table 2 present some faintly deviation between theoretical and stochastic results. In all the other cases the starting point x has been slightly moved from the conical axis and then the exact result has been acquired. This behavior is not isolated and leads to the conclusion that when the trajectories stem from points of the conical axis, then the driving term of the stochastic process has a prominent role in comparison with the Brownian evolution and the trajectories reveal a more deterministic behavior by collapsing abruptly on surface ∂D. In addition the Brownian dynamics has not the space and the suitable time to be

Case	radius a	observation point	parameter m	exact field	$\frac{1}{N}\sum_{i=1}^{N_{\partial D}} \frac{P_{m+1}(\cos(\Theta_i))}{P_m(\cos(\Theta_i))}$
1	1	$(0, 0, 2)$	80	0.5	0.55
2	1	$(0.059, 0, 1.999)$	80	0.5	0.51
3	2	$(0.118, 0, 3.998)$	80	0.5	0,51
4	2	$(0, 0.118, 3.998)$	90	0.5	0,49
5	1	$(0.089, 0, 2.998)$	80	0.33	0.34
6	1	$(0, 0.089, 2.998)$	80	0.33	0.34
7	1	$(0, 0, 3)$	80	0.33	0.39

Table 2: The strong stochastic solution for the spherical case.

regularly unfolded. This is justified if the formulae (4.1) are taken into consideration and in particular the fact that the perpendicular to the axis movements of the trajectories are strongly suppressed for large m, in case that the starting point x lies on the axis. Transferring slightly[25] the position of the generation point x offers to the Brownian dynamics the possibility to play its role in the short time of the movement of every trajectory.

Furthermore, the comparison of the cases 2 and 3 verify the expected principle of the corresponding states, since a simple rescaling of all the characteristic lengths of the problem does not alter of course the result.

4.2.2 The inverse problem in the framework of the strongly conditioned stochastic methodology.

Let us consider the low-frequency problem (3.51)-(3.53). Assume that the scatterer is a star shaped body with respect to the interior coordinate origin O and that the far field is measured at several points of a surface which encircles the region Ω in which the scatterer D is searched. Every observation point x_n is connected with the origin or an interior point of the scatterer in order to form an axis for one of the cones serving at the solution of the inverse problem. This axis is terminated at one of the attractor points of the problem ξ_n, which is supposed to have a distance b_{ξ_n} from the unknown scatterer, greater than one. The cones are considered to be narrow enough so that their intersection with the surface ∂D specifies a very definite small region of the scatterer's surface. The value of the involved fields at the central point z of this small piece of surface - the intersection of the cone axis with the scatterer - is representative for the mean value of the contribution of all the points on it. For every triple (x, z, ξ) we calculate the expectation value of the right hand side of Eq.(3.66) via Monte Carlo

[25]Slightly in comparison to the geometric dimensions of the conical structure but considerably with respect to the Brownian step $\sqrt{\Delta t_n}$.

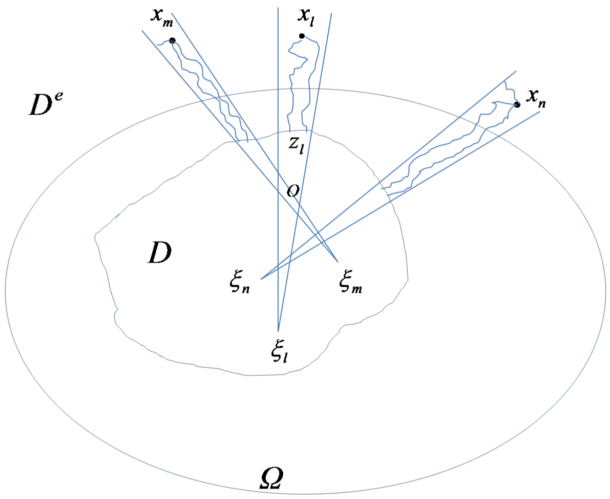

Figure 3: The narrow cones in the service of the inverse low-frequency scattering problem.

mean value as follows:

$$
|x-\xi| E^x_{\partial D}\left[e^{-ik\hat{k}\cdot X_{\tau_{D^e_{m+1}}}} e^{-ik|Y_{\tau_{D^e_{m+1}}}|} \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})} \right] \approx |x-\xi| \frac{1}{N} \sum_{j=1}^{N_{\partial D}} \frac{e^{-ik\hat{k}\cdot X_j} e^{-ik|Y_j|} P_{m+1}(\theta_j)}{P_m(\theta_i)}
$$

$$
\approx |x-\xi| e^{-ik\hat{k}\cdot z} e^{-ik|z-\xi|} \frac{1}{N} \sum_{j=1}^{N_{\partial D}} \frac{P_{m+1}(\theta_j)}{P_m(\theta_j)}. \tag{4.6}
$$

We define so the functional

$$
J(\tilde{z}; k, \hat{k}, \tilde{\xi}) = \sum_{n=1}^{\mathcal{M}_p} \left| F_0(\hat{x}_n, k, \hat{k}) + |x_n - \xi_n| \mathcal{G}_n(z_n, k, \hat{k}, \xi_n) \right|^2 \tag{4.7}
$$

where $\mathcal{M}_p$ is the number of the observation points, $\tilde{\xi}$ gathers all the attractors ξ_n while $\tilde{z}$ assembles all the intersection points of the axis of the cones with the scatterer. In addition

$$
\mathcal{G}_n(z_n, k, \hat{k}, \xi_n) := e^{-ik\hat{k}\cdot z_n} e^{-ik|z_n-\xi_n|} \frac{1}{N^{(n)}} \sum_{j=1}^{N^{(n)}_{\partial D}} \frac{P_{m+1}(\theta_{j,n})}{P_m(\theta_{j,n})}. \tag{4.8}
$$

The minimization of $J(\tilde{\xi}; k, \hat{k}, \tilde{\xi})$ over all the selections $\tilde{z}$ with $|z_n - \xi_n| \geq b_\xi \geq 1$, in case that the residual term of the Corollary 17 is small, leads to the determination of all the structural points z_n of the surface of the scatterer and offers so a reconstruction of the body D. Actually even in the case of restricted data, a portion of the surface can be built on the basis of exploitation of a part of the measurements $F_0(\hat{x}_n, k, \hat{k})$. The situation becomes more enriched in information in case that we consider several incidences $\hat{k}_j$. The form of the function G_n and the previous analysis about the exact spherical case are very tempting to the adoption of the reasonable formula $\frac{1}{N^{(n)}} \sum_{j=1}^{N^{(n)}_{\partial D}} \frac{P_{m+1}(\theta_{j,n})}{P_m(\theta_{j,n})} \approx \frac{|z_n-\xi_n|}{|x_n-\xi_n|}$ leading to the result $|x_n - \xi_n| \mathcal{G}_n(z_n, k, \hat{k}, \xi_n) \approx e^{-ik\hat{k}\cdot z_n} e^{-ik|z_n-\xi_n|}$, detouring the necessity to calculate Monte Carlo estimations in order to construct the functional under consideration. However, this is not the case as Figure 4 clarifies: The intersection of the measurement cone with the surface ∂D generates a central point z, around which the surface portion $\mathcal{E}_m := \partial D \cap \mathcal{K}_{m+1}$ has a local mean curvature $\kappa_z = \frac{1}{R_z}$, by means of which a new cone can be constructed with vertex at the center of the spherical surface fitting locally to the small surface element $\mathcal{E}_m$. Only if the measurement cone coincided with the curvature cone - and the attractor point ξ with the center ζ_z of the locally perfectly fitting sphere -, we could evoke the formula $\frac{1}{N} \sum_{j=1}^{N} \frac{P_{m+1}(\theta_j)}{P_m(\theta_j)} \approx \frac{|z-\zeta_z|}{|x_z-\zeta_z|}$. But this is only accidental and can not be designed in the framework of the inverse boundary value problem, where the curvature cones are unknown[26]. So the Monte Carlo simulations are indispensable for the formation of the particular terms of the functional $J(\tilde{z}; k, \hat{k}, \tilde{\xi})$.

These concepts have been applied to the calculation of the low frequency Rayleigh component of the scattered field outside an ellipsoidal scatterer ([21], [22]) with semiaxes $a_1 = 3, a_2 = 2, a_3 = 1$. The Rayleigh field obtains

[26] In the realm of the direct problem, the curvature cones are known and can be used to give the solutions at the points x_z but in this work we insist on the efficiency of the mildly conditioned stochastic approach to the solution of the direct problem, due to its global character.

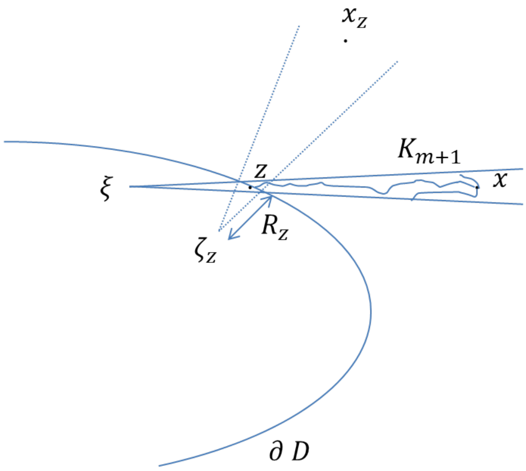

Figure 4: Every surface point induces a specific cone adapted to the local curvature, which differs from the actual cone of the measurement process

the closed form $u_0(\rho) = -\frac{I_0^1(\rho)}{I_0^1(a_1)}$, where $I_0^1(\rho)$ stands for a well known elliptic integral [22] depending on ρ, which is the first of the three ellipsoidal coordinates (ρ, μ, ν). The surface of the scatterer is given by the equation $\rho = a_1$. Using this formula for the synthetic data of the inverse problem and being restricted[27] at the meridian level $x^{(1)} = 0$, we apply the strongly conditioned stochastic method by minimizing the functional $J(\tilde{z}; k, \hat{k}, \tilde{\xi})$. We work with $\mathcal{M}_p = 180$ observation points $x_n, n = 1, 2, ..., \mathcal{M}_p$, distributed uniformly on the portion of the circle $r = \sqrt{\left(x^{(2)}\right)^2 + \left(x^{(3)}\right)^2} = 5$ belonging to the quadrant $x^{(2)} \geq 0, x^{(3)} \geq 0$. All these observation points are connected with the same attractor located at the coordinate origin to create a number of $\mathcal{M}_p = 180$ narrow cones $\mathcal{K}_{m+1,n}, n = 1, 2, ..., \mathcal{M}_p$ corresponding all to the parameter $m = 80$. The minimization of $J(\tilde{z}; k, \hat{k}, \tilde{\xi})$ consists in assembling the values of this functional for arguments $z_n, n = 1, 2, ..., \mathcal{M}_p$ belonging (each one of z_n) correspondingly to the cone $\mathcal{K}_{m+1,n}$ and to the circular annulus $|z_n| \in [0.4, 3]$, which is considered as an a priori information about the hosting region Ω. For every cone we apply a different Monte Carlo simulation but all these experiments are executed in parallel and are of course independent. Interpolating and coloring the level sets of this assembling process, we acquire Figure 5, where the small values of the functional define the intersection of the scatterer with the plane $x^{(1)} = 0$. So we obtain some kind of a plane tomography of the scatterer, which predicts the anticipated curve accurately. The lack of sharpness in defining the curve $\frac{\left(x^{(2)}\right)^2}{4} + \left(x^{(3)}\right)^2 = 1$ reflects the well known fact that the potential inversion (or the low-frequency inversion) disposes an intrinsic vagueness in the determination of the exact shape of the interface. Furthermore, we remark a slight underestimation of the axis $a_2 = 2$, but this is mainly due to the fact that the measurement surface radius $r = 5$ is not so remote to the particular portion of the surface near the $x^{(2)}$-axis, as should be in order for the asymptotic terms of the stochastic analysis to offer small contributions.

The inverse problem presented herein is of course a very primitive application of the general concept supported by the implemented stochastic analysis. There are several degrees of freedom that could be exploited in order to ameliorate the numerical treatment. As an example the assumption to take a common attractor for all the cones should be reconsidered on the basis of a multi-parametric analysis. Actually, the idea of a common attractor is expected to give reasonable results in case of star shaped connected scatterers, while in different case the flexibility of choosing several suitably distributed attractors seems to be necessary.

Furthermore, the concept to confine the minimization process in 2-D planes intersecting the scatterer is justified by the purpose to restrict the numerical burden of the underlying Monte Carlo processes as far as possible but leads to partial results of reconstruction although this can not be considered as a drawback but as a structural characteristic in the framework of the local methodologies under discussion. In the present work the aim is to give the primitive approach to the topic. It is expected that working with the Helmholtz operator in the resonance region - with wave numbers k of significant value - will reveal the special features of the direct and inverse boundary value problem [23], which are anticipated to be strongly connected with the wave number of the process.

[27] We denote here the cartesian coordinates as $x^{(i)}, i = 1, 2, 3$.

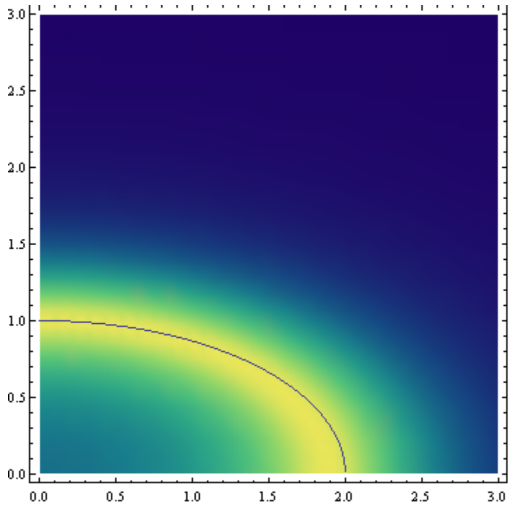

Figure 5: The reconstruction of the intersection of the ellipsoidal scatterer with the plane $x^{(1)} = 0$.

A The stochastic calculus in the service of differential equations

A.1 Basic Preliminaries of stochastic processes

The aim of this section is not to cover of course the very extended area of stochastic differential equations but to give the necessary notions that are indispensable in the stochastic analysis. The introductory features of this section are mainly inspired by a very instructive book [7] and some references cited therein.

Given a set Ω and a σ-algebra $\mathcal{F}$ constructed by subsets of Ω, we consider the probability space $(\Omega, \mathcal{F}, P)$ consisted of the measurable space $(\Omega, \mathcal{F})$ and a probability measure P defined on it. A random variable X is a $\mathcal{F}$-measurable function $X : \Omega \to R^n$. Every random variable induces a probability measure μ_X on R^n, called the distribution of X and defined by

$$\mu_X(B) = P(X^{-1}(B)),$$

for every Borel set B, member of the Borel σ-algebra $\mathcal{B}$ on R^n.
If $\int_\Omega |X(\omega)|dP(\omega) < \infty$, then the number

$$E[X] := \int_\Omega X(\omega)dP(\omega) = \int_{R^n} x d\mu_X(x)$$

is called the expectation of X (w.r.t P). This relation is generalized for every Borel measurable function $f : R^n \to R$ as follows:

$$E[f(X)] := \int_\Omega f(X(\omega))dP(\omega) = \int_{R^n} f(x)d\mu_X(x)$$

provided that $\int_\Omega |f(X(\omega))|dP(\omega)$ is bounded.
The σ-algebra $\mathcal{H}_X$ *generated* by X is defined to be the smallest σ-algebra on Ω containing all the sets $X^{-1}(U)$, with $U \in R^n$ open. (Actually, restriction of U's on the Borel sets $B \in \mathcal{B}$ is sufficient). The random variable X is clearly $\mathcal{H}_X$-measurable and no smaller σ-algebra exists with this property. A collection of random variables $X_i;\ i = 1, 2, ..., n$ is characterized as *independent* if the collection of the corresponding generated σ-algebras $\mathcal{H}_{X_i}$ is independent. If two random variables X, Y are independent then $E[XY] = E[X]E[Y]$, provided that $E[|X|]$ and $E[|Y|]$ are bounded.

A *stochastic process* is a parametrized collection of random variables $\{X_t\}_{t\in T}$, defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in R^n. T is usually the halfline $[0, \infty)$ or an interval in this halfline, or the non negative integers etc. For every specific time t, the family $\{X_t\}_{t\in T}$ just offers the concrete random variable $\omega \to X_t(\omega);\ \ \omega \in \Omega$. For every specific sapling element $\omega \in \Omega$, the family $\{X_t\}_{t\in T}$ provides the function $t \to X_t(\omega);\ t \in T$, which is called a *path* of X_t. Intuitively, t is conceived as "time" and each ω as an individual "experiment". Sometimes we write $X(t, \omega)$ instead of $X_t(\omega)$ considering the process as a function from $T \times \Omega$ into R^n. The process $\{X_t\}_{t\in T}$ gives birth to the so-called *finite dimensional distributions* of the process, which coincide with the measures $\mu_{t_1,...t_k}$ defined on R^{nk}, $k = 1, 2, ...$, by

$$\mu_{t_1,...t_k}(F_1 \times F_2 \times ... \times F_k) = P[X_{t_1} \in F_1, ..., X_{t_k} \in F_k];\quad t_i \in T \tag{A.1}$$

The family of all finite-dimensional distributions generally gives rich information about the process although not exhausting. The converse problem is very important: Given the probability measures $\nu_{t_1,...t_k}$ find a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}$ such that Eq.(A.1) is satisfied (where μ is replaced by ν). The Kolmogorov's extension theorem asserts the existence of the probability space and the corresponding process under some natural consistency conditions for $\nu_{t_1,\ldots t_k}$ [8]. It is the application of last theorem to a specific family of probability measures $\nu_{t_1,\ldots t_k}$, consistent with consecutive gaussian transitions starting from an initial point x, that establishes the existence of a probability space $(\Omega, \mathcal{F}, P^x)$ and the well known accompanying Brownian motion B_t (starting at x) disposing the same finite-dimensional distributions with the given measures (mention here that $P^x(B_0 = x) = 1$). The Brownian motion is not unique in the sense that several quadruples $(B_t, \Omega, \mathcal{F}, P^x)$ exist referring to the same probability measures ν. However we are always able to adopt the canonical Brownian motion, which is just the space $C([0,\infty), R^n)$ equipped with certain probability measures P^x as described above. Under the law P^x, the Brownian motion merits interesting qualitative and quantitative properties. We mention here that $E^x[B_t] = x$, for all $t \geq 0$, $E^x[(B_t - B_s)^2] = n(t-s)$, for $t \geq s$ and that B_t has *independent increments* B_{t_1},$B_{t_2} - B_{t_1}$,...,$B_{t_k} - B_{t_{k-1}}$ for all $0 \leq t_1 < t_2 < \ldots < t_k$.

In addition we define as $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated by the random variables $\{B_i(s)\}_{1\leq i\leq n; 0\leq s\leq t}$. We can think $\mathcal{F}_t$ as "the history of B_s up to time t". It is clear that $\{\mathcal{F}_t\}$ is increasing and that $\mathcal{F}_t \subset \mathcal{F}$. In addition, mainly due to independence of increments, the Brownian motion B_t is proved to be a martingale w.r.t to the (increasing) σ-algebras $\mathcal{F}_t$ [7]. This property reflects the reasonable characteristic of the Brownian motion that the conditional expectation of the Brownian motion at a future time $s(> t)$ given the past information till the present time t is just the random variable B_t. For every increasing family of σ-algebras $\{\mathcal{N}_t\}$ of subsets of Ω, a process $g(t,\omega) : [0,\infty) \times \Omega \to R^n$ is called $\{\mathcal{N}_t\}$-adapted if for each $t \geq 0$, the function $\omega \to g(t,\omega)$ is $\{\mathcal{N}_t\}$-measurable. By construction the Brownian motion is of course $\{\mathcal{F}_t\}$-adapted. Furthermore, a function $\tau : \Omega \to [0,\infty)$ is called a stopping time w.r.t. $\{\mathcal{N}_t\}$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{N}_t$, for all $t \geq 0$. As an example the *first exit time* $\tau_U := inf\{t > 0; B_t \notin U\}$ of a Brownian motion from an open set $U \subset R^n$ is a stopping time w.r.t $\{\mathcal{F}_t\}$.

In the core of the present work lie the stochastic differential equations of the type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, \quad X_0 = Z. \tag{A.2}$$

In the equation above, $T > 0$ while $b(.,.) : [0,\infty) \times R^n \to R^n$ and $\sigma(.,.) : [0,\infty) \times R^n \to R^{n\times m}$ are measurable functions. The Brownian motion is m-dimensional while the initial state random variable Z is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by the Brownian motion at all times. It is proved in [7] that under certain strict conditions on b and σ, the stochastic differential equation (A.2) has a unique t-continuous solution $X_t(\omega)$, which is adapted to the filtration (increasing family) $\mathcal{F}_t^Z$ generated by Z and $B_s; s \leq t$. In addition $E[\int_0^T |X_t|^2 dt < \infty]$. We may integrate obtaining

$$X_t = X_0 + \int_0^T b(t, X_t)dt + \int_0^T \sigma(t, X_t)dB_t \tag{A.3}$$

where we recognize [7] the Itô integral $\int_0^T \sigma(t, X_t)dB_t$, which is well defined given that the solution X_t involving in the integrand is $\mathcal{F}_t^Z$-adapted. The strict conditions mentioned above impose at most linear growth and Lipschitz behavior of the coefficients both in terms of the second spatial argument, uniformly over time.
The unique solution X_t, generated by the arguments above, is called *strong* solution, because the version B_t of the Brownian motion is given in advance and the solution constructed from it is $\mathcal{F}_t^Z$-adapted. The price we pay to obtain such a good and unique solution is the restriction on the coefficients b and σ. In general terms, the linear growth excludes the appearance of explosive solutions while the Lipschitz condition establishes uniqueness. However it is fruitful to consider a more general class of solutions of Eq.(A.2), which allow more general forms of the coefficients b and σ. More precisely, suppose that we are only given the functions $b(t,x)$ and $\sigma(t,x)$. We ask for a pair of processes $((\tilde{X}_t, \tilde{B}_t), \mathcal{H}_t)$ on a probability space $(\Omega, \mathcal{F}, P)$ with the following properties: i) $\mathcal{H}_t$ is an increasing family of σ-algebras such that $\tilde{X}_t$ is $\mathcal{H}_t$-adapted, ii)$\tilde{B}_t$ is a Brownian motion and simultaneously a martingale w.r.t $\mathcal{H}_t$ and iii) Eq.(A.2) is satisfied with $(\tilde{X}_t, \tilde{B}_t)$ in place of (X_t, B_t). Then the pair $(\tilde{X}_t, \tilde{B}_t)$ is called a *weak* solution of the stochastic differential equation (A.2). We mention that the martingale property permits to define the Itô integral on the right hand side of Eq.(A.3) exactly as before, even though $\tilde{X}_t$ need not be $\mathcal{F}_t^Z$-adapted.
The uniqueness mentioned in the framework of strong solutions is called *strong* or *path-wise* uniqueness and has a very clear interpretation. There is also the notion of *weak* uniqueness according to which any two solutions (weak or strong) are identical in law.
The time-homogeneous Itô diffusions X_t are of special importance and obey to the rule (A.2) with time-homogeneous coefficients $b(X_t)$ and $\sigma(X_t)$. The coefficient $b(X_t)$ is known as the drift of the process. In the absence of the random term, the drift is exclusively responsible for the evolution of the dynamical system X_t and so "drives" the vector X_t. It clearly retains this basic property in case of small randomness, induced by small $\sigma(X_t)$, and the trajectory of the process keeps its orientation, while obtaining a fluctuating morphology due of course to the randomness. It is an issue of great importance to investigate the behavior of composite functions of the form $F(t,\omega) = f(t, X_t) = f(t, X(t))$, where $f(t,x) = (f_1(t,x), f_2(t,x), \ldots, f_p(t,x))$ is a C^2 map

from $[0,\infty) \times R^n$ into R^p. The method for this effort is provided by the well known multi-dimensional Itô formula, according to which $F(t,\omega)$ is again an Itô process with components F_k, $k = 1,2,...,p$, satisfying

$$dF_k = \frac{\partial F_k}{\partial t}(t,X)dt + \sum_i \frac{\partial F_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j \tag{A.4}$$

where the relations $dB_i dB_j = \delta_{ij}dt$, $dB_i dt = dt dB_i = 0$ span the calculus of products between infinitesimals.

For every Itô diffusion X_t in R^n, the infinitesimal *generator* A is defined by $Af(x) = lim_{t\downarrow 0}\frac{E^x[f(X_t)]-f(x)}{t}$, $x \in R^n$ and has a domain D_A including $C_0^2(R^n)$. More precisely, every $f \in C_0^2(R^n)$ belongs to D_A and satisfies $Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$. The infinitesimal generator offers the link between the stochastic processes and the partial differential equations.

A.2 Stochastic processes and interior boundary value problems.

The well known Dynkin's formula [7] connects the infinitesimal operator A with expectation values of suitable stochastic processes. Indeed, let $f \in C_0^2(R^n)$ and suppose that τ is a stopping time with $E^x[\tau] < \infty$. Then

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right] \tag{A.5}$$

The existence of a compact support for the functions f is not necessary if τ is the first exit time of a bounded set. The Dynkin's formula is very helpful in obtaining stochastic representations of boundary value problem solutions. Indeed, let D be a bounded domain in R^n and ϕ a bounded function on ∂D, while $u \in C^2(D)$ is supposed to be a solution of the boundary value problem

$$Lu(x) = 0 \quad x \in D \tag{A.6}$$

$$\lim_{\substack{x\to y \\ x\in D}} u(x) = \phi(y) \quad y \in \partial D \tag{A.7}$$

where L is the second order elliptic partial differential operator given by $L = \sum_i b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$. The concept of interrelating a stochastic process with the boundary value problem above begins with the selection of a matrix $\sigma(x) \in R^{n\times n}$ such that $\frac{1}{2}\sigma(x)\sigma^T(x) = [a_{ij}(x)]$ and goes on by considering the Itô diffusion X_t, satisfying the stochastic differential equation (A.2) with $X_0 = x$ (and $m = n$). The infinitesimal generator A of this process has the same expression with the differential operator L under discussion. To simplify the current presentation, suppose that $\tau_D < \infty$ a.s. P^x for all x. Then the Dynkin's formula imposes necessarily the stochastic representation

$$u(x) = E^x[\phi(X_{\tau_D})] \tag{A.8}$$

for the smooth solution of the boundary value problem, implying alternatively the well known uniqueness result for the Dirichlet problem. The last outcome (A.8) opens up the possibility to exploit the stochastic calculus in the service of the solution of boundary value problems. Several special issues, concerning the characteristics of the constructed Itô diffusion, arise that can not of course be examined here extensively. As a matter of fact, the aforementioned Lipschitz conditions for the coefficients of the operator A are sufficient in order to ensure the strong uniqueness of the process X_t and then to give clear sense to Eq.(A.8). However, as discussed in the previous section, a weak uniqueness result would be enough since two equivalent processes, in probabilistic law, lead to equal expectation values. So there are interesting cases of general (even singular) coefficients b_i, a_{ij} with very interesting applications. A second issue is the crucial condition $\tau_D < \infty$. As an example the Brownian motion itself is well known to be recurrent in R^2 (i.e. $P^x(\tau_D < \infty) = 1$) but transient in R^3 (i.e. $P^x(\tau_D < \infty) < 1$). So in R^3, a pure Brownian motion could ramble endlessly without hitting the boundary. However, the drift term is often responsible to guide the process toward the exit from the set D. Another crucial issue is that especially in applications involving processes in exterior domains, the requirement of the boundedness of the domain is not respected any more. It is expected that the drift term must dispose a very "attractive" character in order to force the stochastic process reaching the boundary of an unbounded domain. This is realized if the drift has a suitable singularity outside the domain. In the present work, special attention has been devoted to the investigation of specific type singular drift coefficients and to their influence in the interrelation between stochastic processes and boundary value problems.

B The technical proofs of the most part of the results concerning exterior stochastic representations

Proof of Theorem 4

By partitioning the expectation values, we transform Eq.(3.35) into the form

$$u(x) = \frac{1}{|x-\xi|^{m+1}} E^x_{\partial D\cap\mathcal{K}_{m,int}}\left[\frac{\phi(X_{\tau_{\partial D}})|Y_{\tau_{\partial D}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial D}}))}\right] +$$
$$\frac{1}{|x-\xi|^{m+1}} E^x_{\partial\mathcal{K}_{m,int}\cap\{|Y_\tau|<s|x-\xi|\}}\left[\frac{u(X_{\tau_{\partial\mathcal{K}_{m,int}}})|Y_{\tau_{\partial\mathcal{K}_{m,int}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial\mathcal{K}_{m,int}}}))}\right] +$$
$$\frac{1}{|x-\xi|^{m+1}} E^x_{\partial\mathcal{K}_{m,int}\cap\{|Y_\tau|\geq s|x-\xi|\}}\left[\frac{u(X_{\tau_{\partial\mathcal{K}_{m,int}}})|Y_{\tau_{\partial\mathcal{K}_{m,int}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial\mathcal{K}_{m,int}}}))}\right] +$$
$$\frac{1}{|x-\xi|^{m+1}} E^x_{S_{cup}}\left[\frac{u(X_{\tau_{S_{cup}}})|Y_{\tau_{S_{cup}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{S_{cup}}}))}\right], \quad \text{(B.1)}$$

where $s = T^{\frac{1}{2}+\epsilon}$ and $\epsilon \in (0, \frac{1}{2})$.
The last two terms of representation (B.1) can be compressed as follows

$$w(x) := \frac{1}{|x-\xi|^{m+1}} E^x_{\partial\mathcal{K}_{m,int}\cap\{|Y_\tau|\geq s|x-\xi|\}}\left[\frac{u(X_{\tau_{\partial\mathcal{K}_{m,int}}})|Y_{\tau_{\partial\mathcal{K}_{m,int}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial\mathcal{K}_{m,int}}}))}\right] +$$
$$\frac{1}{|x-\xi|^{m+1}} E^x_{S_{cup}}\left[\frac{u(X_{\tau_{S_{cup}}})|Y_{\tau_{S_{cup}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{S_{cup}}}))}\right] =$$
$$\frac{1}{|x-\xi|^{m+1}} E^x_{\partial D^e_{m,int}\cap\{T|x-\xi|\geq|Y_\tau|\geq s|x-\xi|\}}\left[\frac{u(X_{\tau_{\partial D^e_{m,int}}})|Y_{\tau_{\partial D^e_{m,int}}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial D^e_{m,int}}}))}\right].$$

Thanks to maximum principle for boundary value problems involving the Laplace operator, we have that $\sup_{z\in D^e}|u(z)| \leq L := \max_{\zeta\in\partial D}|\phi(\zeta)|$. Taking also into account that the function $\mathcal{Q}$ attains its positive minimum at $\theta = \theta_{m,int}$, we easily find that

$$|w(x)| \leq \frac{LT^{m+1}}{\mathcal{Q}_m(\cos(\theta_{m,int}))} E^x_{\partial D^e_{m,int}\cap\{T|x-\xi|\geq|Y_\tau|\geq s|x-\xi|\}}[1]. \quad \text{(B.2)}$$

The last expectation value of unity is less than the probability measure of the set $\{|Y_\tau| \geq s|x-\xi|\}$, which according[28] to Eq.(3.21) is further majorized suitably, implying that

$$|w(x)| \leq \frac{LT^{m+1}}{\mathcal{Q}_m(\cos(\theta_{m,int}))} P(|Y_\tau| \geq s|x-\xi|) \leq \frac{LT^{m+1}}{\mathcal{Q}_m(\cos(\theta_{m,int}))}\frac{1}{s^{2m+4-n}}$$
$$= \frac{L}{\mathcal{Q}_m(\cos(\theta_{m,int}))}\frac{1}{T^{1-\frac{n}{2}+\epsilon(2m+4-n)}} \quad \text{(B.3)}$$

It is necessary that $\epsilon > \frac{\frac{n}{2}-1}{2m+4-n}$ in order to guarantee lessening of $|w(x)|$ as the parameter T increases[29]. What remains is to handle the second term $v(x)$ of the sum of the r.h.s of the representation (B.1). It is immediate to obtain that

$$|v(x)| \leq \frac{Ls^{m+1}}{\mathcal{Q}_m(\cos(\theta_{m,int}))} E^x_{\partial\mathcal{K}_{m,int}\cap\{|Y_\tau|<s|x-\xi|\}}[1].$$

The expectation value $E^x_{\partial\mathcal{K}_{m,int}\cap\{|Y_\tau|<s|x-\xi|\}}[1]$, appearing in last inequality can be estimated via the aforementioned Monte Carlo approach and obviously represents - for a large number of independent experiments - the portion of the lateral surface[30] crossings among the total number of exits from the domain under investigation ${D^e}_{m,int}$. This percentage is then maximized by the ratio $\frac{N_{\partial\mathcal{K}_{m,int}}}{N}$ and consequently we obtain

$$|v(x)| \leq \frac{LT^{(\frac{1}{2}+\epsilon)(m+1)}}{\mathcal{Q}_m(\cos(\theta_{m,int}))}\frac{N_{\partial\mathcal{K}_{m,int}}}{N}. \quad \text{(B.4)}$$

[28]after just replacing $D^e_{m,l}$ with $D^e_{m,int}$.
[29]This condition does not violate the relation $\epsilon < \frac{1}{2}$ since $m > \frac{3n-5}{2}$.
[30]The additional restriction that $|Y_\tau| < s|x-\xi|$ is of course valid.

Thus, the difference $\mathcal{B}(x) := u(x) - u_{\partial D}(x)$ between the solution of the boundary value problem $u(x)$ and the contribution

$$u_{\partial D}(x) := \frac{1}{|x-\xi|^{m+1}} E^x_{\partial D \cap \mathcal{K}_{m,int}} \left[\frac{\phi(X_{\tau_{\partial D}})|Y_{\tau_{\partial D}}|^{m+1}}{\mathcal{Q}_m(\cos(\Theta_{\tau_{\partial D}}))} \right]$$

from the trajectories escaping region $D^e_{m,int}$ exclusively from the surface ∂D, is bounded above by the bounds appearing in the r.h.s of Equations (B.3) and (B.4). We summarize that

$$|\mathcal{B}(x)| \le \frac{L}{\mathcal{Q}_m(\cos(\theta_{m,int}))} \left(\frac{1}{T^{1-\frac{n}{2}+\epsilon(2m+4-n)}} + T^{(\frac{1}{2}+\epsilon)(m+1)} \frac{N_{\partial\mathcal{K}_{m,int}}}{N} \right) \tag{B.5}$$

Let know ϵ be selected to be the particular parameter γ for which the following balancing

$$\frac{N_{\partial\mathcal{K}_{m,int}}}{N} \le \frac{1}{T^{(\frac{m+3-n}{2}+\gamma(3m+5-n))}}, \tag{B.6}$$

of the trajectories is assured[31]. Eq.(B.5) based on Eq.(B.6) becomes

$$|\mathcal{B}(x)| \le 2 \frac{L}{\mathcal{Q}_m(\cos(\theta_{m,int}))} \frac{1}{T^{1-\frac{n}{2}+\gamma(2m+4-n)}}. \tag{B.7}$$

Proof of Theorem 11

We consider the representation (3.40) and handle the time integral term

$$\mathcal{E}(x) := E^x \left[\int_0^{\tau_{D^e_{m,int}}} \left(\frac{\Delta\left(u(X_s)|X_s-\xi|h_{m+1}(X_s-\xi)\right)}{2h_m(X_s-\xi)} \right) ds \right]. \tag{B.8}$$

The treatment of the term $\Delta\left(u(X_s)|X_s-\xi|h_{m+1}(X_s-\xi)\right)$ can be simplified after expressing everything in term of the variable Y_s, leading to the form $\Delta\left(\tilde{u}(Y_s)|Y_s|h_{m+1}(Y_s)\right)$, where $\tilde{u}(Y_s) := u(X_s)$. Setting $r_\xi = |Y_s|$ and $\nabla = \hat{r}_\xi \frac{\partial}{\partial r_\xi} + \frac{1}{r_\xi}\mathbb{D}$, we easily obtain

$$\begin{aligned} \frac{1}{2}\Delta\left(\tilde{u}|Y_s|h_{m+1}\right) &= (\frac{1}{r_\xi}\tilde{u} + \frac{\partial \tilde{u}}{\partial r_\xi})h_{m+1} + r_\xi \nabla\tilde{u}\cdot\nabla h_{m+1} + \tilde{u}\frac{\partial h_{m+1}}{\partial r_\xi} \\ &= -(m+1)(\frac{1}{r_\xi}\tilde{u} + \frac{\partial \tilde{u}}{\partial r_\xi})h_{m+1} + \frac{1}{r_\xi}\mathbb{D}\tilde{u}\cdot\mathbb{D}h_{m+1}. \end{aligned} \tag{B.9}$$

Consequently,

$$\begin{aligned} \mathcal{E}(x) = E^x \Bigg[\int_0^{\tau_{D^e_{m,int}}} \Bigg\{ -\frac{(m+1)}{|Y_s|} \left(\frac{1}{|Y_s|}\tilde{u}(Y_s) + \frac{\partial\tilde{u}(Y_s)}{\partial|Y_s|} \right) \\ \frac{P_{m+1}(\cos(\Theta_s))}{P_m(\cos(\Theta_s))} \Bigg\} ds \Bigg] + E^x \left[\int_0^{\tau_{D^e_{m,int}}} \frac{1}{|Y_s|^2} \frac{\mathbb{D}\tilde{u}\cdot\mathbb{D}P_{m+1}(\cos(\Theta_s))}{P_m(\cos(\Theta_s))} ds \right] \end{aligned} \tag{B.10}$$

The harmonic field $\tilde{u}$ can be expanded, in the exterior of a sphere, superscribing region D and centered at ξ, in a uniformly convergent series of inverse powers of the distance $|Y_s|$, as follows[32]:

$$\tilde{u}(Y_s) = \frac{A_0}{|Y_s|} + \frac{f_1(\Theta_s,\Phi_s)}{|Y_s|^2} + \sum_{j=3}^{\infty} \frac{f_j(\Theta_s,\Phi_s)}{|Y_s|^{j+1}}, \tag{B.11}$$

where all the involved functions f_n, $n \in N$ are regular and infinitely differentiable functions. It is straightforward to conclude that there is a constant C, independent of the parameter m of the process and depending only on the region D and the data ϕ such that[33]

$$\left| \frac{1}{|Y_s|}\tilde{u}(Y_s) + \frac{\partial\tilde{u}(Y_s)}{\partial|Y_s|} \right| \le \frac{C}{|Y_s|^3}, \quad |\mathbb{D}\tilde{u}| \le \frac{C}{|Y_s|^2}. \tag{B.12}$$

[31] This condition provides a necessary measure on the smallness of the fraction $\frac{N_{\partial\mathcal{K}_{m,int}}}{N}$ in conjunction with T.

[32] Here we encounter for the first time the azimouthal coordinate Φ, which is absent in the harmonic functions h_m, h_{m+1} but present of course as argument of the field $\tilde{u}$ itself.

[33] The forthcoming relations are quite reasonable for every $|Y_s|$ grater than the radius R_1 of the sphere circumscribing the - possibly disconnected - region D. Moreover, the regularity of the involved fields allows to find a constant C assuring the validity of the equations even in the bounded intermediate region between D and the ball $B(\xi, R_1)$. Furthermore the condition $b_\xi = dist\{\xi, \partial D \cap \mathcal{K}_{m+1}\} \ge 1$ prevents the distance $|Y_s|$ to become smaller than unity, assuring a uniform validity of Eqs. (B.11, B.12) independent of the choice of ξ. The same assumption guarantees secure behavior of all the involved stochastic integrals including powers of the ratio $\frac{1}{|Y_s|}$.

As a matter of fact, it can be assumed for our purposes that the active contribution of the field $\tilde{u}$ in the current stochastic analysis obeys to the boundedness $\left|\frac{1}{\sin(\Theta_s)}\mathbb{D}\tilde{u}\right| \leq \frac{C}{|Y_s|^2}$. This is justified on the basis of the fact that $f_1(\Theta_s, \Phi_s)$ is necessarily a function of the form $(\mathcal{A}\cos(\Phi_s) + \mathcal{B}\sin(\Phi_s))\sin(\Theta_s) + \mathcal{C}\cos(\Theta_s)$ as easily comes out after making asymptotics to the Green's integral representation of the exterior harmonic function $\tilde{u}$. The term $(\mathcal{A}\cos(\Phi_s) + \mathcal{B}\sin(\Phi_s))\sin(\Theta_s)$ does not preoccupy our analysis given that every expectation value involving the azimuthal trigonometric linear combination $(\mathcal{A}\cos(\Phi_s) + \mathcal{B}\sin(\Phi_s))$ offers zero contribution since the driving term of the stochastic process is independent of Φ_s, while the domain of the evolution of the stochastic trajectories is invariant to azimuthal rotations. Based on this fact, exploiting the recurrence formula $(1-z^2)P'_{m+1}(z) = (m+1)P_m(z) - (m+1)zP_{m+1}(z)$ and using that $\frac{P_{m+1}}{P_m}$ is less that unity in the specific range of their argument, we find that

$$\begin{aligned}
\left|\frac{\mathbb{D}\tilde{u}\cdot\mathbb{D}P_{m+1}(\cos(\Theta_s))}{P_m(\cos(\Theta_s))}\right| &= \left|\frac{1}{\sin(\Theta_s)}\frac{\partial\tilde{u}}{\partial\Theta_s}\frac{\sin^2(\Theta_s)P'_{m+1}(\cos(\Theta_s))}{P_m(\cos(\Theta_s))}\right| \\
&\leq (m+1)\left|\frac{1}{\sin(\Theta_s)}\frac{\partial\tilde{u}}{\partial\Theta_s}\right| + (m+1)\left|\frac{1}{\sin(\Theta_s)}\frac{\partial\tilde{u}}{\partial\Theta_s}\frac{\cos(\Theta_s)P_{m+1}(\cos(\Theta_s))}{P_m(\cos(\Theta_s))}\right| \\
&\leq (m+1)\frac{C}{|Y_s|^2}. \qquad \text{(B.13)}
\end{aligned}$$

As a result, Eq.(B.10) gives

$$|\mathcal{E}(x)| \leq C(m+1)E^x\left[\int_0^{\tau_{D^e_{m,int}}}\frac{1}{|Y_s|^4}ds\right]. \qquad \text{(B.14)}$$

We evoke the fundamental relation (3.12) with $k=-2$ (and $n=3$), integrate over time in the interval $(0, \tau_{D^e_{m,int}})$ and take expectation values, as before in this work, to find that

$$E^x\left[\int_0^{\tau_{D^e_{m,int}}}\frac{1}{|Y_s|^4}ds\right] = \frac{1}{(2m+3)}\left(E^x\left[\frac{1}{|Y_{\tau_{D^e_{m,int}}}|^2}\right] - \frac{1}{|x-\xi|^2}\right). \qquad \text{(B.15)}$$

Based on Eqs.(3.40),(B.8),(B.14),(B.15) and on the process $T\to\infty$, described in Proposition 9, we find that

$$u(x) = E^x_{\partial D}\left[\phi(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x), \qquad \text{(B.16)}$$

with

$$|\mathcal{E}(x)| \leq C\frac{(m+1)}{(2m+3)}\left(E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right] - \frac{1}{|x-\xi|^2}\right). \qquad \text{(B.17)}$$

To handle the term $E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right]$, we apply Girsanov Theorem again to find that

$$E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right] = |x-\xi|^{m+1}E^x\left[\frac{1}{|B_{\tilde{\tau}_{D^e_{m+1}}}-\xi|^2}h_m(B_{\tilde{\tau}_{D^e_{m+1}}}-\xi)\right]. \qquad \text{(B.18)}$$

We apply then the recurrent relation $P_m(z) = \frac{2m+3}{m+1}zP_{m+1}(z) - \frac{m+2}{m+1}P_{m+2}(z)$ to last formula and obtain

$$\begin{aligned}
E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right] &= |x-\xi|^{m+1}\frac{2m+3}{m+1}E^x\left[\frac{1}{|B_{\tilde{\tau}_{D^e_{m+1}}}-\xi|^{m+3}}\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}})P_{m+1}(\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}}))\right] \\
&- |x-\xi|^{m+1}\frac{m+2}{m+1}E^x\left[\frac{1}{|B_{\tilde{\tau}_{D^e_{m+1}}}-\xi|^{m+3}}P_{m+2}(\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}}))\right]. \qquad \text{(B.19)}
\end{aligned}$$

Exploiting the vanishing of P_{m+1} on the lateral surface and Dynkin's formula for harmonic fields we find that

$$E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right] = |x-\xi|^{m+1}\frac{2m+3}{m+1}E^x_{\partial D}\left[\frac{1}{|B_{\tilde{\tau}_{D^e_{m+1}}}-\xi|^{m+3}}\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}})P_{m+1}(\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}}))\right]$$

$$-\quad\frac{m+2}{m+1}\frac{1}{|x-\xi|^2} \leq |x-\xi|^{m+1}\frac{1}{b_\xi}\frac{2m+3}{m+1}E^x_{\partial D}\left[\frac{1}{|B_{\tilde{\tau}_{D^e_{m+1}}}-\xi|^{m+2}}P_{m+1}(\cos(\Theta_{\tilde{\tau}_{D^e_{m+1}}}))\right] - \frac{m+2}{m+1}\frac{1}{|x-\xi|^2}$$

$$\leq \frac{2m+3}{m+1}\frac{1}{b_\xi}\frac{1}{|x-\xi|} - \frac{m+2}{m+1}\frac{1}{|x-\xi|^2} = \frac{2m+3}{m+1}\frac{1}{|x-\xi|}\left(\frac{1}{b_\xi}-\frac{1}{|x-\xi|}\right) + \frac{1}{|x-\xi|^2}. \tag{B.20}$$

Then the inequality (B.17) becomes

$$|\mathcal{E}(x)| \leq C\frac{1}{|x-\xi|}\left(\frac{1}{b_\xi}-\frac{1}{|x-\xi|}\right). \tag{B.21}$$

This relation assures that the residual term becomes small in the near field region, given that b_ξ and $|x-\xi|$ become comparable, especially when the point ξ is pulled back. In the far field region, we need a different treatment of the function $\mathcal{E}(x)$. We define $B_\xi = \max\{|\zeta-\xi|, \zeta \in \partial D \cap \mathcal{K}_{m+1}\}$. We handle the first term $|x-\xi|^{m+1}\frac{2m+3}{m+1}E^x_{\partial D}\left[\frac{1}{|\cdot|^{m+3}}zP_{m+1}(z)\right]$ of the right hand side of Eq.(B.20), applying once more the same recurrent formula for Legendre functions and the Dynkin's formula for harmonic functions to obtain that

$$|x-\xi|^{m+1}\frac{2m+3}{m+1}E^x_{\partial D\cap\mathcal{K}_{m+1}}\left[\frac{1}{|\cdot|^{m+3}}zP_{m+1}(z)\right] \leq B_\xi|x-\xi|^{m+1}\frac{2m+3}{m+1}\frac{2m+5}{m+2}$$

$$E^x_{\partial D\cap\mathcal{K}_{m+2}}\left[\frac{1}{|\cdot|^{m+4}}zP_{m+2}(z)\right] - B_\xi|x-\xi|^{m+1}\frac{2m+3}{m+1}\frac{m+3}{m+2}E^x_{\partial D\cap\mathcal{K}_{m+1}}\left[\frac{1}{|\cdot|^{m+4}}P_{m+3}(z)\right]$$

$$\leq \frac{B_\xi}{b_\xi}|x-\xi|^{m+1}\frac{2m+3}{m+1}\frac{2m+5}{m+2}E^x_{\partial D\cap\mathcal{K}_{m+2}}\left[\frac{1}{|\cdot|^{m+3}}P_{m+2}(z)\right] - B_\xi\frac{2m+3}{m+1}\frac{m+3}{m+2}\frac{1}{|x-\xi|^3} =$$

$$\frac{B_\xi}{b_\xi}\frac{2m+3}{m+1}\frac{2m+5}{m+2}\frac{1}{|x-\xi|^2} - B_\xi\frac{2m+3}{m+1}\frac{m+3}{m+2}\frac{1}{|x-\xi|^3}. \tag{B.22}$$

So

$$E^x\left[\frac{1}{|Y_{\tau_{D^e_{m+1}}}|^2}\right] \leq \frac{B_\xi}{b_\xi}\frac{2m+3}{m+1}\frac{2m+5}{m+2}\frac{1}{|x-\xi|^2} - B_\xi\frac{2m+3}{m+1}\frac{m+3}{m+2}\frac{1}{|x-\xi|^3} - \frac{2m+3}{m+1}\frac{1}{|x-\xi|^2}$$

$$+\frac{1}{|x-\xi|^2} \tag{B.23}$$

Combining Eqs.(B.17-B.20) and (B.23) we find that

$$|\mathcal{E}(x)| \leq C\left(\frac{B_\xi}{b_\xi}\left[\frac{2m+5}{m+2} - \frac{m+3}{m+2}\frac{b_\xi}{|x-\xi|}\right] - 1\right)\frac{1}{|x-\xi|^2}. \tag{B.24}$$

For large values of m, $\frac{B_\xi}{b_\xi} \cong 1$ and if we approach the boundary then $\frac{B_\xi}{b_\xi}\left[\frac{2m+5}{m+2} - \frac{m+3}{m+2}\frac{b_\xi}{|x-\xi|}\right] - 1 \to 0$. In both cases $|x| \to \infty$ and $x \to \partial D$, we remark that $|x-\xi||\mathcal{E}(x)| \to 0$.

Proof of Theorem 12

We apply theorem 11 to the function[34] $w(x) = (x-\xi)\cdot\nabla u(x)$ (instead of u itself), which satisfies the Laplace equation and behaves like $\frac{1}{|x-\xi|}$ in the far field region. In addition, using the expansion (B.11), we can show that the estimates (B.12) are valid for the function $\tilde{w}(Y_s)$ as well. Then, according to the analysis following before, we obtain

$$w(x) = E^x_{\partial D}\left[w(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x) \Rightarrow$$

$$(x-\xi)\cdot\nabla u(x) = E^x_{\partial D}\left[Y_{\tau_{D^e_{m+1}}}\cdot\nabla u(X_{\tau_{D^e_{m+1}}})\frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] - \mathcal{E}(x), \tag{B.25}$$

[34] giving birth to the function $\tilde{w}(Y_s) = Y_s\cdot\nabla_y\tilde{u}(Y_s)$.

where $\mathcal{E}(x)$ has been characterized asymptotically in Theorem 11. Evoking the expansion (B.11), we find that

$$w(x) = -u(x) + \mathcal{O}(\frac{1}{|x-\xi|^2}). \tag{B.26}$$

In addition we have

$$Y_{\tau_{D^e_{m+1}}} \cdot \nabla u(X_{\tau_{D^e_{m+1}}}) = Y_{\tau_{D^e_{m+1}}} \cdot \hat{n}_y \frac{\partial u}{\partial n}(X_{\tau_{D^e_{m+1}}}) + Y_{\tau_{D^e_{m+1}}} \cdot (\mathbb{I} - \hat{n}_y \otimes \hat{n}_y) \cdot \nabla u(X_{\tau_{D^e_{m+1}}}). \tag{B.27}$$

The second term of the right hand side offers an expectation value that can be estimated on the basis of the assumed regularity of the small region $\mathcal{K}_{m+1} \cap \partial D$ and the essence of Proposition 9:

$$E^x_{\partial D}\left[Y_{\tau_{D^e_{m+1}}} \cdot (\mathbb{I} - \hat{n}_y \otimes \hat{n}_y) \cdot \nabla u(X_{\tau_{D^e_{m+1}}}) \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] \leq$$

$$\sin(\theta_{m+1,1}) E^x_{\partial D}\left[|Y_{\tau_{D^e_{m+1}}}||\nabla u(X_{\tau_{D^e_{m+1}}})| \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] \leq \sin(\theta_{m+1,1}) \frac{C}{|x-\xi|}. \tag{B.28}$$

Combining Eqs. (B.25-B.28) the stated result comes out. Selecting m large enough[35], we may control the residual to be of rate $\mathcal{O}(\frac{1}{|x-\xi|^2})$.

Proof of Theorem 13

The field $v(x) = 2u(x) + (x-\xi) \cdot \nabla u(x)$ satisfies again Laplace equation and the asymptotic condition $v(x) = \mathcal{O}(\frac{1}{|x-\xi|})$. Moreover, on the basis of the expansion (B.11), it is easily deduced that the field $\tilde{v}(Y_s) = v(X_s)$ obeys to sharper boundedness relations:

$$\left|\frac{1}{|Y_s|}\tilde{v}(Y_s) + \frac{\partial \tilde{v}(Y_s)}{\partial |Y_s|}\right| \leq \frac{C}{|Y_s|^3}, \quad |\mathbb{D}\tilde{v}| \leq \frac{C}{|Y_s|^3}. \tag{B.29}$$

Then we apply Eq.(3.40) to v and notice that the new residual term (estimated before by Eq.(B.14)) is controlled now as follows

$$\left|\tilde{\mathcal{E}}(x)\right| \leq C(m+1) E^x\left[\int_0^{\tau_{D^e_{m,int}}} \frac{1}{|Y_s|^5} ds\right]. \tag{B.30}$$

We evoke the fundamental relation (3.12) with $k = -3$ (and $n = 3$), integrate over time in the interval $(0, \tau_{D^e_{m,int}})$ and take expectation values to find that

$$E^x\left[\int_0^{\tau_{D^e_{m,int}}} \frac{1}{|Y_s|^5} ds\right] = \frac{1}{3(m+2)}\left(E^x\left[\frac{1}{|Y_{\tau_{D^e_{m,int}}}|^3}\right] - \frac{1}{|x-\xi|^3}\right). \tag{B.31}$$

Working[36] as in Theorem 11 we show that

$$\left|\tilde{\mathcal{E}}(x)\right| \leq \frac{C(m+1)}{(m+2)}\left(E^x\left[\frac{1}{|Y_{\tau_{D^e_{m,int}}}|^3}\right] - \frac{1}{|x-\xi|^3}\right) = \mathcal{O}(\frac{1}{|x-\xi|^3}), \quad |x| \to \infty. \tag{B.32}$$

Proof of Theorem 14

The surface data $f := u|_{\partial D \cap \mathcal{K}_{m+1}}$ and $g := \nabla u|_{\partial D \cap \mathcal{K}_{m+1}}$ are connected via the stochastic rule

$$A_0 = |x-\xi| E^x_{\partial D}\left[2u(X_{\tau_{D^e_{m+1}}}) \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right]$$

$$+|x-\xi| E^x_{\partial D}\left[Y_{\tau_{D^e_{m+1}}} \cdot \nabla u(X_{\tau_{D^e_{m+1}}}) \frac{P_{m+1}(\Theta_{\tau_{D^e_{m+1}}})}{P_m(\Theta_{\tau_{D^e_{m+1}}})}\right] + \mathcal{O}(\frac{1}{|x-\xi|^2}), \quad |x-\xi| \to \infty \tag{B.33}$$

as easily deduced from Proposition 13.

On the basis of decomposition $\nabla u(y) = (\mathbb{I} - \hat{n}_y \otimes \hat{n}_y) \cdot \nabla u(y) + \hat{n}_y \hat{n}_y \cdot \nabla u(y)$, and via the arguments encountered in the construction of formula (B.28) we obtain the stated result.

[35] We recall that $\sin(\theta_{m+1,1}) \to 0$ as m incerases.

[36] The working steps are quite similar, although more extensive, and are not going to be presented here again.

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