An analytic algorithm for shape reconstruction from low frequency moments

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In the present work, a novel method, concerning the solution of the inverse scattering problem, is developed and implemented, in the realm of low-frequency acoustics. The method is based on the suitable exploitation of the low-frequency moments, which are the structural pieces of the far-field pattern. The stimulus for the present method has been offered by a recent accomplishment permitting the extraction of the moments from the far-field pattern via a systematic, direct and stable manner. The aim of the method is to reconstruct polynomial scatterers and to approximate general scatterers by polynomial surfaces. This is accomplished via the formulation of suitable objective functionals involving the unknown coefficients of the cartesian representation of the sought polynomial surface along with the low-frequency moments. These functionals are constructed by forcing the target polynomial surface to comply with the moments extracted from real data. The minimization of these functionals provides the optimized coefficients of the polynomial manifold, while stability is inherent in the nature of the minimization process. The method has been implemented to the reconstruction of second and fourth order polynomial scatterers as well as to fitting of general scatterers by polynomial surfaces.

I. INTRODUCTION

A linear inversion methodology was suggested in a sequence of publications[1–4] aiming at reconstructing the shape of the scatterer given the set of the low frequency moments. The method stated therein concerned the case of polynomial scatterers or smooth scatterers fitted appropriately by polynomial manifolds. The data of the inversion were the generalized moments, stemmed from the low frequency moments, which participate in the low frequency expansion of the far field pattern. Restricting our analysis to the case of soft acoustic scatterers[1], we mention that the low frequency moments under discussion are surface integrals - on scatterer's surface - involving linearly the normal derivatives of the low frequency components of the total exterior field, multiplied with powers of the inner product between the observation direction and the scatterer's surface position vector. So the unknown shape is involved in several implicit or explicit ways in the form of moments and the aim was to decode this implication. The really amazing outcome of this approach consisted of the fact that the exploitation of the Rayleigh approximation alone was sufficient for recovering the shape of the scatterer at least in theoretical terms. More precisely, it had been proved that taking advantage of all the zero-order low-frequency moments (up to a number firmly dependent on the surface degree) leads to the formulation of a linear algebraic system, whose solution is unique and provides the coefficients of the representation of the scattering surface in terms of spherical harmonics.

The suggested so far inversion methodology considered as data the low frequency moments and was restricted to give only a theoretical argumentation about the ability in principle to determine the moments starting from

being provided with the low-frequency expansion of the far-field pattern. This algorithm required data over all possible observation directions and undesirable differential operations over the acquired data. In a recent work[5], a strong application stimulus has been applied on the inversion method of moments. In that work, the starting point has been the far field itself, while the moments are considered as a subsequent product. So the first effort was to construct a stabilized technique to extract the low frequency components from the far field data, while the second step was the investigation of the ability to mine the moments that are hidden in these low frequency components. Especially the second attempt was really demanding with very interesting results. Referring to this second task[5], it has been proved that there exist two classes of moments. The first one consists of members that can be immediately obtained from measurements taken at specific observation directions, in the framework of scattering processes stimulated by specific plane wave excitations. This class contains all the moments pertaining to the Rayleigh approximation in conjunction with harmonic kernels but is not restricted to them. The second class contains moments that participate in measured structures, which can not be decomposed with measurement techniques. Instead, a new integral equation calculus incorporating the novel concept of the double moments has been constructed in order to offer estimations for the moments that are hidden in measurements and are not immediately accessible. This calculus is analytic but very technical and demanding for the reconstruction of the second class of moments. As a consequence of that work, two possibilities emerge. The first one is to insist on the previous algorithm[1] - and the consequent works - and construct the necessary data following the new results[5]. Three drawbacks characterize this approach: First we can not avoid the use of all the "tough" moments of the second kind. In addition in order to formulate linear systems with equal number of equa-

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tions and unknowns (the coefficients of the polar representation of the polynomial surface) we need much more low-frequency components (data) than the theoretically expected and this lacks optimality. Finally the stability is not inherent property but is imposed by a Tikhonov regularization process which must be supplemented to a suitable process[5] applied to stabilize the extraction of low-frequency moments from the far-field measurements. Alternatively, the second possibility is to formulate a new inversion algorithm exploiting the new status quo to the mining of moments and avoiding as much as possible the disadvantages stated above.

The present work aims at renovating the concept of inverse scattering via the low-frequency moments. In Section II, we formulate the acoustic scattering problem in connection with the regime of low-frequency moments. Furthermore, we create the necessary calculus of moments, suitably adapted to the application framework of the present work. More precisely, as far as the first set of accessible moments is concerned, we generalize our previous results[5], by giving a generic characterization for the structure of the specific subset of the first group of moments, which is useful in the inversion schemes that follow in the next part of the work. Moreover, referring to the second class of moments that behave as measurably unaccessible participants in measured quantities, we develop an alternative integral equation framework based mainly on the scientific area of the Null-Field Equations[6] in order to obtain estimations for these moments. We state also qualitative results establishing embeddings of second class moments between bounds that constitute known moments of first class. In Section III, we develop the new inversion algorithm. Two main cases are considered: The first one concerns polynomial scatterers and in the present work we investigate second and fourth degree surfaces. The second case, attracting mainly our interest, concerns general scatterers and the purpose is to approximate the scatterer's unknown surface by a polynomial one in a optimal manner. This optimality is accomplished by constructing minimization functionals, which force the approximating polynomial shapes to support the moments corresponding to the real scatterer. The polynomial closed surfaces are fully characterized by the coefficients of their cartesian representation, which constitute the unknowns to be estimated. Projecting functionally these cartesian representations on measures originated by the low-frequency components of the total exterior field leads to algebraic systems having as structural terms the moments. These systems are linear regarding the aforementioned coefficients of the polynomial cartesian expressions. Nevertheless, the new idea is to treat differently the two classes of moments. So the second class of moments provide with elements that may considered as additional unknowns of the problem. As discussed above these supplementary parameters are constrained due to the estimations offered by the N.F.Eq.-framework introduced in Section II. However there exist very interesting cases where inherent symmetries allow to express the moments of second class in terms of the first class moments and so we are in position to detour both the double moments method[5] and the current N.F.Eq.-method. The first indicative case is that of an ellipsoidal scatterer or the case of the optimal fitting ellipsoidal surface, where only accessible moments of the first kind are necessary and it is possible not to touch the arsenal with the "tough" integral calculus for determining moments of second class. In contrast, these moments can be interrelated with moments of first kind and these connections can be involved as additional constraints to the global minimization scheme. The same situation is met in several cases of fourth order scatterers (or fitting surfaces) as extensively presented in Section III. Nevertheless, there exist cases where no inner symmetry can be detected to allow the interrelation between the two classes of moments. In these cases the straight implication of the second class moments is obligatory. In the last part of Section III, a new approach is suggested even with these cases lacking inner symmetry, which avoids the aforementioned disadvantages of the old version of the inversion[1].

In Section IV, we present the numerical investigation interrelated with the reconstruction method presented herein. We expose the process and the results of the minimization of several objective functionals. More precisely we examine extensively the case of the reconstruction of the piloting ellipsoidal scatterer for several eccentricities and mainly we investigate the influence of several levels of measurement error to the stability of the reconstruction. In addition, we examine the inverse scattering problem for general scatterers paying attention to non smooth scatterers as the characteristic shape of the parallelepiped. We present the process of approximating such kinds of non smooth surfaces by polynomial surfaces and pay attention on the influence of the geometrical anisotropy or degeneracy to the robustness of the inversion.

This last example justifies the applicability of the method and reveals its main characteristics and functionality, considered in the context of alternative already well known robust inversion methodologies. More precisely we mention here the linear sampling method or the factorization method, which constitute very efficient inversion techniques[7–10]. These methods evidently are well established, generic and powerful working in a wide range of frequencies and for a variety of geometrical configurations. The present method relies on its simplicity and highlights some special features of scattering theory, which are valuable from the theoretical and application point of view. First, one of the most important difficulties of the aforementioned inversion methods is the necessity to stimulate the scatterer and subsequently take measurements - at least theoretically - in all possible orientations, independently of the scatterer's regularity. The case of restricted data has been considered of course in the context of the precedent methods with very successful results. However the present work is characterized by

defining - via an inherent constitutive process - the sufficient and necessary set of directions at which stimulation and measurement are required, in order to perform the inversion. The characteristics of this set - dimension and specific directions - depend explicitly on the degree and the orientation of the approximating polynomial surface. So the restriction of the data or the possible incidences is not a drawback any more but reveals - as accompanying byproduct - the solution of a very interesting physical problem stated as follows: Which are the necessary stimuli and measurements that are invoked by an underlying geometrical structure, in order for this structure to be estimated optimally by a smooth manifold of specific regularity?

On the other hand, the present method has restricted the ill-posed part of it to the mining of moments from measurements, matter that has been faced recently[5], where Tikhonov regularization techniques played the most important role. Apart from this stage, the method develops a stable optimization process by minimizing simple objective functionals. This simplicity reflects the gain of the effort to work with estimating polynomial surfaces instead of the general scatterer itself.

II. ACOUSTIC SCATTERING AND THE LOW-FREQUENCY MOMENTS

We consider the three dimensional exterior boundary value problem concerning the scattering of an acoustic time-harmonic plane wave from an impenetrable inhomogeneity situated inside an infinite environment hosting acoustic propagating waves. The obstacle is considered to occupy the bounded region D, whose soft interface is a star-shaped, smooth surface ∂D having at least continuous curvature. The incident plane wave - after being subject to time reduction - is expressed through the time-reduced potential field $u^{inc}(\mathbf{r}; \mathbf{k}) = \exp(i\mathbf{k} \cdot \mathbf{r})$. Notice that $\mathbf{k} = k\hat{\mathbf{k}}$ where k represents the wave number of the process and the unit vector $\hat{\mathbf{k}}$ indicates the direction of the incidence of the plane wave.

Scattering of the incident field from the obstacle gives birth to the secondary scattering time harmonic wave, with corresponding time-reduced potential field $u^{sc}(\mathbf{r}; \mathbf{k})$ satisfying the following exterior boundary value problem

$$(\Delta + k^2)u^{sc}(\mathbf{r}; \mathbf{k}) = 0, \quad \mathbf{r} \in R^3 \backslash \bar{D} \quad (1)$$

$$u^{sc}(\mathbf{r}; \mathbf{k}) + exp(i\mathbf{k} \cdot \mathbf{r}) = 0, \quad \mathbf{r} \in \partial D \quad (2)$$

$$\frac{\partial}{\partial r} u^{sc}(\mathbf{r}; \mathbf{k}) - iku^{sc}(\mathbf{r}; \mathbf{k}) = O(\frac{1}{r^2}), \quad r = |\mathbf{r}| \to \infty. \quad (3)$$

The scattered field as well as the incident one satisfy the well known Helmholtz equation (1) outside the scatterer. The boundary condition (2) reflects the free pressure behavior of the interface, while radiation Sommerfeld's condition (3) ensures the outgoing orientation of the scattered field and also determines its energy rate at large distances. In this asymptotic region $r \to \infty$, the

scattered field obtains the asymptotic expansion

$$u^{sc}(\mathbf{r}; \mathbf{k}) = \frac{\exp(ikr)}{ikr} f_\infty(\hat{\mathbf{r}}; \mathbf{k}) + O(\frac{1}{r^2}), \quad r = |\mathbf{r}| \to \infty, \quad (4)$$

where the normalized scattering amplitude $f_\infty(\hat{\mathbf{r}}; \mathbf{k})$ describes the response of the scatterer in the direction of observation $\hat{\mathbf{r}}$ when it is excited by a plane wave propagating in the direction $\hat{\mathbf{k}} = \mathbf{k}/k$. This function is usually the data of the classical inverse scattering problem and the aim is the exploitation of it to reconstruct the shape of the scatterer. The total acoustic field $u^{tot} = u^{inc} + u^{sc}$ is the superposition of the stimulating and the secondary field, satisfies also Helmholtz equation outside the scatterer, while vanishes over the surface ∂D. The low-frequency treatment[1,11–13] of the scattering problem under discussion reduces the above exterior boundary value problem to an infinite sequence of exterior boundary value problems for the Laplace's operator, which can be solved iteratively. Specifically, the total field assumes the expansion

$$u^{tot}(\mathbf{r}; \mathbf{k}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Phi_n(\mathbf{r}; \hat{\mathbf{k}}). \quad (5)$$

The n-th order low-frequency approximation $\Phi_n(\mathbf{r}; \hat{\mathbf{k}})$ vanishes on the scatterer's surface ∂D and satisfies the following integral representation, which expresses the interrelation of Φ_n with the lower order terms:

$$\Phi_n(\mathbf{r}; \hat{\mathbf{k}}) = (\hat{\mathbf{k}} \cdot \mathbf{r})^n - \frac{1}{4\pi} \sum_{\rho=0}^{n} \binom{n}{\rho} \int_{\partial D} |\mathbf{r} - \mathbf{r}'|^{\rho-1}$$

$$\times \frac{\partial \Phi_{n-\rho}}{\partial n'}(\mathbf{r}'; \hat{\mathbf{k}}) ds(\mathbf{r}'), \quad \mathbf{r} \in R^3 \backslash D. \quad (6)$$

As a consequence, the scattering amplitude obtains its own low-frequency approximation:

$$f_\infty(\hat{\mathbf{r}}; \mathbf{k}) = -\sum_{n=0}^{\infty} \frac{(ik)^{n+1}}{n!} \mathcal{H}_n(\hat{\mathbf{r}}; \hat{\mathbf{k}}) =$$

$$\sum_{n=0}^{\infty} \frac{(ik)^{n+1}}{n!} \sum_{\rho=0}^{n} \binom{n}{\rho}(-1)^{\rho+1} M_{n-\rho}^{\rho}(\hat{\mathbf{r}}; \hat{\mathbf{k}}). \quad (7)$$

In Eq.(7) we meet the low-frequency moments $M_l^m(\hat{\mathbf{r}}; \hat{\mathbf{k}})$, given by

$$M_l^m(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = \frac{1}{4\pi} \int_{\partial D} (\hat{\mathbf{r}} \cdot \mathbf{r}')^m \frac{\partial \Phi_l}{\partial n'}(\mathbf{r}'; \hat{\mathbf{k}}) ds(\mathbf{r}') \quad \hat{\mathbf{r}}, \hat{\mathbf{k}} \in S^2. \quad (8)$$

We notice here that the dependence of the moments on the arguments is simplified in the cases (i) $M_0^m(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = M_0^m(\hat{\mathbf{r}})$ (the zeroth-order approximation does not depend on the excitation direction) and (ii) $M_l^0(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = M_l^0(\hat{\mathbf{k}})$ (the observation point is irrelevant). In the inverse scattering realm, a set of measurements of the far-field pattern $f_\infty(\hat{\mathbf{r}}; \mathbf{k})$ for several possible observations (vectors $\hat{\mathbf{r}}$) and excitations (wave numbers k and propagation directions $\hat{\mathbf{k}}$) is given. We have already explained[5] how this

information is decoded in order to provide stable estimations for the far-field components $\mathcal{H}_n(\hat{\mathbf{r}};\hat{\mathbf{k}})$. In addition in the same work, we find the general methodology to estimate the moments that are hidden in the far-field components. As an example, the fundamental moments are very easily deducible from the far-filed components:

$$M_0^0 = \mathcal{H}_0, \quad M_0^1(\hat{\mathbf{r}}) = \frac{1}{2}[\mathcal{H}_1(-\hat{\mathbf{r}};\hat{\mathbf{k}}) - \mathcal{H}_1(\hat{\mathbf{r}};\hat{\mathbf{k}})],$$

$$M_1^0(\hat{\mathbf{k}}) = \frac{1}{2}[\mathcal{H}_1(\hat{\mathbf{r}};\hat{\mathbf{k}}) + \mathcal{H}_1(-\hat{\mathbf{r}};\hat{\mathbf{k}})]. \tag{9}$$

Moments of higher order need special treatment to be mined in measurements and relevant mentioning will be reported in the sequel. The special form of the inverse problem encountered in the present work is the determination of the surface ∂D, given the low-frequency moments. It is helpful to recapitulate some fundamental issues concerning the low-frequency expansion approximations of the total field together with the resultant low-frequency moments. First we notice that the low-frequency terms Φ_n are decomposed in terms of spherical harmonics with respect to the excitation direction $\hat{\mathbf{k}}$. This emerges if the non homogeneous term $(\hat{\mathbf{k}}\cdot\mathbf{r})^n$ of representation (6) is expanded in terms of these harmonics. We present the following decomposition outcomes:

Proposition 1 *The three expansion terms* Φ_n, $n = 1,2,3$ *of the total field are represented in terms of* $\hat{\boldsymbol{k}}$-*spherical harmonics as follows:*

(i) $\Phi_1(\boldsymbol{r};\hat{\boldsymbol{k}}) = \Phi_1^{(0)}(\boldsymbol{r}) + \Phi_1^{(1)}(\boldsymbol{r};\hat{\boldsymbol{k}}) = \Phi_1^{(0)}(\boldsymbol{r}) + \hat{\boldsymbol{k}}\cdot\boldsymbol{A}_1(\boldsymbol{r}) = -M_0^0\Phi_0(\boldsymbol{r}) + \hat{\boldsymbol{k}}\cdot\boldsymbol{A}(\boldsymbol{r})$ $\qquad(10)$

(ii) $\Phi_2(\boldsymbol{r};\hat{\boldsymbol{k}}) = \Phi_2^{(0)}(\boldsymbol{r}) + \Phi_2^{(1)}(\boldsymbol{r},\hat{\boldsymbol{k}}) + \Phi_2^{(2)}(\boldsymbol{r},\hat{\boldsymbol{k}}) = \Phi_2^{(0)}(\boldsymbol{r}) + \hat{\boldsymbol{k}}\cdot\boldsymbol{A}_2(\boldsymbol{r}) + \frac{2}{3}\sum_{m=-2}^{2}\frac{(2-|m|)!}{(2+|m|)!}\overline{Y_2^m(\hat{\boldsymbol{k}})}A_2^{(2,m)}(\boldsymbol{r})$ $\quad(11)$

(iii) $\Phi_3(\boldsymbol{r};\hat{\boldsymbol{k}}) = \Phi_3^{(0)}(\boldsymbol{r}) + \Phi_3^{(1)}(\boldsymbol{r},\hat{\boldsymbol{k}}) + \Phi_3^{(2)}(\boldsymbol{r},\hat{\boldsymbol{k}}) + \Phi_3^{(3)}(\boldsymbol{r},\hat{\boldsymbol{k}})$

$= \Phi_3^{(0)}(\boldsymbol{r}) + \frac{3}{5}\hat{\boldsymbol{k}}\cdot\boldsymbol{A}_3(\boldsymbol{r}) + \frac{2}{3}\sum_{m=-2}^{2}\frac{(2-|m|)!}{(2+|m|)!}\overline{Y_2^m(\hat{\boldsymbol{k}})}A_3^{(2,m)}(\boldsymbol{r}) + \frac{2}{5}\sum_{m=-3}^{3}\frac{(3-|m|)!}{(3+|m|)!}\overline{Y_3^m(\hat{\boldsymbol{k}})}A_3^{(3,m)}(\boldsymbol{r}),$ $\qquad(12)$

where the components $\Phi_n^{(0)}$, $\boldsymbol{A}_n$, $A_n^{(l,m)}$ vanish on ∂D and satisfy specific integral representations produced after substituting the decompositions (i)-(iii) in Eq.(6).

Proof. As mentioned above, the non homogeneous term $(\hat{\mathbf{k}}\cdot\mathbf{r})^n$, after being expanded in spherical harmonics of $\hat{\mathbf{k}}$ forces every component Φ_n to adopt a consequent expansion. Some of the induced integral representations are proved to be the following:

$$\Phi_0(\mathbf{r}) = 1 - \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Phi_0}{\partial n'}(\mathbf{r}')ds(\mathbf{r}'), \tag{13}$$

$$\Phi_1(\mathbf{r},\hat{\mathbf{k}}) = -M_0^0 + \hat{\mathbf{k}}\cdot\mathbf{r}$$
$$- \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Phi_1}{\partial n'}(\mathbf{r}',\hat{\mathbf{k}})ds(\mathbf{r}'), \tag{14}$$

while the subcomponents of the function Φ_2 satisfy

$$\Phi_2^{(0)}(\mathbf{r}) = \frac{r^2}{3} - \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Phi_2^{(0)}}{\partial n'}(\mathbf{r}')ds(\mathbf{r}')$$
$$- \frac{1}{2\pi}\int_{\partial D}\frac{\partial\Phi_1^{(0)}}{\partial n'}(\mathbf{r}')ds(\mathbf{r}') - \frac{1}{4\pi}\int_{\partial D}|\mathbf{r}-\mathbf{r}'|\frac{\partial\Phi_0}{\partial n'}(\mathbf{r}')ds(\mathbf{r}')$$
$$\tag{15}$$

$$\Phi_2^{(1)}(\mathbf{r},\hat{\mathbf{k}}) = -\frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Phi_2^{(1)}}{\partial n'}(\mathbf{r}',\hat{\mathbf{k}})ds(\mathbf{r}')$$
$$- \frac{1}{2\pi}\int_{\partial D}\frac{\partial\Phi_1^{(1)}}{\partial n'}(\mathbf{r}',\hat{\mathbf{k}})ds(\mathbf{r}') \tag{16}$$

$$A_2^{(2,m)}(\mathbf{r}) = r^2 Y_2^m(\hat{\mathbf{r}}) - \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial A_2^{(2,m)}}{\partial n'}(\mathbf{r}')ds(\mathbf{r}')$$
$$|m| \le 2. \tag{17}$$

(all for $\mathbf{r} \in R^3\backslash D$)

The role of the integral representations is crucial for establishing interrelations between the integral components and the relevant moments. As an example, Eq.(14), considered on ∂D, multiplied with the measure $\frac{\partial\Phi_0}{\partial n}$ and integrated over ∂D gives $\int_{\partial D}\frac{\partial\mathbf{A}}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \int_{\partial D}\mathbf{r}\frac{\partial\Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r})$. As a consequence, Eq.(16) establishes that $\mathbf{A}_2(\mathbf{r}) = -\left\{\frac{1}{2\pi}\int_{\partial D}\mathbf{r}'\frac{\partial\Phi_0}{\partial n'}(\mathbf{r}')ds(\mathbf{r}')\right\}\Phi_0(\mathbf{r}) = -\mathbf{M}_0^1\Phi_0(\mathbf{r})$, where $\mathbf{M}_0^1 = \int_{\partial D}\mathbf{r}\frac{\partial\Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r})$. Applying the same process to Eq.(17) we obtain that the moments $\int_{\partial D}\frac{\partial A_2^{(2,m)}}{\partial n}ds$ equal to the moments $\frac{1}{4\pi}\int_{\partial D}Y_2^m\,r^n\frac{\partial\Phi_0}{\partial n}ds$, referring to te Rayleigh component Φ_0. ∎

The results of Proposition 1 can be extended to terms Φ_n of order $n > 3$, but this will not be needed in the context of the present work. The essence of the result above is that every moment $M_l^m(\hat{\mathbf{r}};\hat{\mathbf{k}})$ can be analyzed further providing moments of a deeper level. As an example, the moment $M_2^1(\hat{\mathbf{r}};\hat{\mathbf{k}})$ gives birth to three separate

set of moments: the scalar function $\frac{1}{4\pi}\int_{\partial D}(\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial\Phi_2^{(0)}}{\partial n'}ds'$, the vector fuction $\frac{1}{4\pi}\int_{\partial D}(\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial\mathbf{A}}{\partial n'}ds'$ and the five scalar functions $\frac{1}{4\pi}\int_{\partial D}(\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial A_2^{(2,m)}}{\partial n'}ds'$, $m=-2,-1,...,2$. We present the following theorem as a first attempt to characterize a large subclass of the first set of moments:

Theorem 2 *For every* $n = 0,1,2,...$ *and every given harmonic function* h_n *of order* n, *the corresponding generalized zeroth-order moment*

$$\mathcal{M}_0(h_n) = \frac{1}{4\pi}\int_{\partial D} h_n(\mathbf{r})\frac{\partial\Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}), \qquad (18)$$

can be computed from the knowledge of $\mathcal{H}_n(\hat{\mathbf{r}}\,;\hat{\mathbf{k}})$ *for all unit vectors* $\hat{\mathbf{r}}$ *and one value of* $\hat{\mathbf{k}}$.

Proof. The primitive result, from which stems the statement of the present theorem, is that the moments

$$\mathcal{M}_0^{(n,m)} = \frac{1}{4\pi}\int_{\partial D} Y_n^m(\hat{\mathbf{r}})r^n\frac{\partial\Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}), \; |m|\le n, \; (19)$$

where $Y_n^m(\hat{\mathbf{r}})$ stand for the well known spherical harmonics, are reconstructible once the far-field low-frequency component $\mathcal{H}_n(\hat{\mathbf{r}}\,;\hat{\mathbf{k}})$ is given, since all the other terms participating in the form of $\mathcal{H}_n$ contain harmonics of lower degree. ■

Remark 3 *The symbolism adopted in the previous theorem is compact and concerns, as stated, every harmonic polynomial of order* n. *However in many places of the present work, we invoke simultaneously several specific polynomials of the same degree* n *since we need the exploitation of several moments at the same time. This necessitates the introduction of a new parameter* j *running over* $j = 1,2,...(2n+1)$, *specifying which member of all the possible independent harmonic functions of order* n *we refer to. In addition the most profitable form of the harmonic functions* h_n *is proved to be the cartesian representation instead of the spherical one. So the generic form* $\mathcal{M}_0(h_n)$ *is frequently replaced by the more intricate but flexible form* $\widetilde{\mathcal{M}}_0^{(n,j)}$ *introduced in the primitive work[5]. The "tilda" symbol indicates that the harmonic kernel is expressed in cartesian coordinates.*

As examples, we remark that the generalized moments $\frac{1}{4\pi}\int_{\partial D}(x^2-y^2)\frac{\partial\Phi_0}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}xyz\frac{\partial\Phi_0}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}(x^4+y^4-6x^2y^2)\frac{\partial\Phi_0}{\partial n}ds$, e.t.c., belong to the first class of moments whose extraction from measurements is amenable. The crucial distinction between the two classes of moments is clarified through a simple example. Although the moments $\frac{1}{4\pi}\int_{\partial D}(x^2-y^2)\frac{\partial\Phi_0}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}(x^2-z^2)\frac{\partial\Phi_0}{\partial n}ds$ participate in the measurably reconstructible set of moments, the building elements $\frac{1}{4\pi}\int_{\partial D}x^2\frac{\partial\Phi_0}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}y^2\frac{\partial\Phi_0}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}z^2\frac{\partial\Phi_0}{\partial n}ds$ (along with their sum $\frac{1}{4\pi}\int_{\partial D}r^2\frac{\partial\Phi_0}{\partial n}ds$) fail separately to belong to the same privileged class. There is a specific characterization of

the moments $\mathcal{M}_0(h_n)$ for $n=1,2,3$ in terms of the measured data in specific observation directions[5]. Remark that the excitation direction is irrelevant and this reflects the independence of Φ_0 on $\hat{\mathbf{k}}$.

The members $\mathcal{M}_0(h_n)$ do not exhaust the privileged class of measurable moments.

Proposition 4 *(i) The vector moments* $\mathbf{M}_1(h_{n-1})=\frac{1}{4\pi}\int_{\partial D}h_{n-1}(\mathbf{r})\frac{\partial\mathbf{A}}{\partial n}ds$, $n\ge 1$ *are deducible from the set of data* $\mathcal{H}_n(\hat{\mathbf{r}}\,;\hat{\mathbf{k}})$.
(ii) The scalar moments $\mathcal{M}_2^m(h_{n-2}) = \frac{1}{4\pi}\int_{\partial D}h_{n-2}(\mathbf{r})\frac{\partial A_2^{(2,m)}}{\partial n}ds$, $n\ge 2$; $|m|\le 2$ *are deducible from the set of data* $\mathcal{H}_n(\hat{\mathbf{r}}\,;\hat{\mathbf{k}})$.

Proof. We expand the definition relation of the far-field components as follows

$$\mathcal{H}_n(\hat{\mathbf{r}};\hat{\mathbf{k}}) = \sum_{\rho=0}^{n}\binom{n}{\rho}(-1)^\rho M_{n-\rho}^\rho(\hat{\mathbf{r}};\hat{\mathbf{k}}) = (-1)^n\left[M_0^n(\hat{\mathbf{r}})\right.$$

$$\left.-nM_1^{n-1}(\hat{\mathbf{r}};\hat{\mathbf{k}})+...+(-1)^nM_n^0(\hat{\mathbf{k}})\right]. \qquad (20)$$

In addition, using arguments in the spirit of Proposition 1 above, we easily verify that $M_{l'}^l$, expanded as a function of $\hat{\mathbf{k}}$, contains spherical harmonics of maximal order l'. The only member of the expansion (20) containing simultaneously a $(n-1)$-order $\hat{\mathbf{r}}$-spherical harmonic and first order $\hat{\mathbf{k}}$-spherical harmonic term is the second one, while this specific term provides the vector moments $\mathbf{M}_1(h_{n-1})$. So at least theoretically, these moments can be extracted from measurements. Part (ii) is proved if we focus on the only member containing a (n-2,2)-order spherical harmonic term w.r.t the pair $(\hat{\mathbf{r}}, \hat{\mathbf{k}})$. It is clear that we may find several pairs (l,l') of $(\hat{\mathbf{r}}, \hat{\mathbf{k}})$-spherical harmonics involved in only one member of the expansion (20) and so able to be extracted from $\mathcal{H}_n(\hat{\mathbf{r}};\hat{\mathbf{k}})$ by double "projection" on $Y_l^m(\hat{\mathbf{r}})$ $Y_{l'}^{m'}(\hat{\mathbf{k}})$ and consequent change in cartesian coordinates. So we are in position to define a large family of moments composing the first class of moments, exceeding a lot the class of Rayleigh moments provided by Theorem 2, but we give herein only the necessary members of this set, in terms of measurements in specific directions. ■

Remark 5 *Similarly as before, the generic set of moments introduced in Proposition 4 contain exactly the members* $\widetilde{\mathbf{M}}_1^{(n-1,j)}$, $j = 1,2,...,2n-1$; $n\ge 1$ *and* $\widetilde{\mathcal{M}}_2^{(m,n-2,j)}$, $j = 1,2,...,2n-3$; $n\ge 2$; $|m|\le 2$ *when we introduce all the possible independent harmonic cartesian polynomials.*

In the introductory work[5], we did not confined ourselves to the theoretical possibility to extract moments from measurements by projecting on spherical harmonics since this requires integration over $S^2\times S^2$, which necessitates the knowledge of $\mathcal{H}_n(\hat{\mathbf{r}};\hat{\mathbf{k}})$ for every observation and excitation orientation. This rich information is scarcely at

hand. Mostly we have measurements only in a few observation directions in scattering processes corresponding to a few plane wave excitations. An extended constructive analysis was necessary to establish an optimal characterization of these moments in terms of the smallest number of the necessary directions of observation and stimulation. This optimal analysis lead to the determination of the scalar moments $\mathcal{M}_0(h_n)$, $n = 0, 1, 2, 3$, of the vector moments $\mathbf{M}_1(h_{n-1})$, $n = 1, 2, 3, 4$ as well as of the moments $\mathcal{M}_2^m(h_{n-2})$, $|m| \leq 2$, $n = 2, 3, 4$. The last two sets need several but specific plane wave excitations. In Appendix, we present these results to reveal the interrelation between moments and the relevant set of measurements at specific observations and stimulations. In contrast to the first class of moments, the second one contains all the remaining moments participating in the far-field low-frequency components $\mathcal{H}_n(\hat{\mathbf{r}}; \hat{\mathbf{k}})$, which can not be determined immediately from measurements. To give some light into the distinction between the first and second class of moments and to the kind

of the induced perplexity, we introduce symmetric and antisymmetric parts of measurements: $\mathcal{H}_n^{sym(ant)}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = \frac{1}{2}[\mathcal{H}_n(\hat{\mathbf{r}}; \hat{\mathbf{k}}) + (-)\mathcal{H}_n(-\hat{\mathbf{r}}; \hat{\mathbf{k}})]$ and we evoke[5] the representation of the symmetric part of the component $\mathcal{H}_2$

$$\mathcal{H}_2^{sym}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = M_0^2(\hat{\mathbf{r}}) + M_2^0(\hat{\mathbf{k}}) = \mathcal{B} + 2\mathcal{H}_0\mathcal{H}_1^{ant}(\hat{\mathbf{k}}; \hat{\mathbf{k}}) +$$
$$\frac{2}{3}\sum_{m=-2}^{2}\frac{(2-|m|)!}{(2+|m|)!}(\overline{Y_2^m(\hat{\mathbf{r}})} + \overline{Y_2^m(\hat{\mathbf{k}})})\mathcal{M}_0^{(2,m)} \tag{21}$$

where $4\pi\mathcal{B}$ is equal to $\int_{\partial D}\frac{\partial \Phi_2^{(0)}}{\partial n'}(\mathbf{r}')ds(\mathbf{r}') + \frac{1}{3}\int_{\partial D}r'^2\frac{\partial \Phi_0}{\partial n'}(\mathbf{r}')ds(\mathbf{r}')$.

For the first class of moments, we remark indicatively that the representation (21) is easily exploited to provide $\mathcal{M}_0^{(2,m)}$, $|m| \leq 2$ or equivalently the moments $\mathcal{M}_0(h_2)$, in terms of measurements. Selecting orientations $\hat{\mathbf{r}}$ parallel to the unit cartesian vectors and combining suitably the produced equations, we find

$$\frac{1}{4\pi}\int_{\partial D}(x^2-y^2)\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \mathcal{H}_2^{sym}(\hat{\mathbf{x}}; \hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{y}}; \hat{\mathbf{k}})$$

$$\frac{1}{4\pi}\int_{\partial D}(x^2-z^2)\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \mathcal{H}_2^{sym}(\hat{\mathbf{x}}; \hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{z}}; \hat{\mathbf{k}})$$

$$\frac{1}{4\pi}\int_{\partial D}xy\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \mathcal{H}_2^{sym}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}); \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{x}}; \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{y}}; \hat{\mathbf{k}})$$

$$\frac{1}{4\pi}\int_{\partial D}xz\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \mathcal{H}_2^{sym}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{z}}); \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{x}}; \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{z}}; \hat{\mathbf{k}})$$

$$\frac{1}{4\pi}\int_{\partial D}yz\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \mathcal{H}_2^{sym}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}}); \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{y}}; \hat{\mathbf{k}}) - \frac{1}{2}\mathcal{H}_2^{sym}(\hat{\mathbf{z}}; \hat{\mathbf{k}}) \tag{22}$$

Staying a little more with the second component of the far-field pattern we deduce clearly from the definition of $\mathcal{H}_2(\hat{\mathbf{r}}; \hat{\mathbf{k}})$ that $M_1^1(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = -\frac{1}{2}\mathcal{H}_2^{ant}(\hat{\mathbf{r}}; \hat{\mathbf{k}})$. Denoting generally $\hat{\mathcal{H}}_n^{ant}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = \frac{1}{2}\left(\mathcal{H}_n^{ant}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) - \mathcal{H}_n^{ant}(\hat{\mathbf{r}}; -\hat{\mathbf{k}})\right)$ and using that $M_1^1(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = -M_0^0 M_0^1(\hat{\mathbf{r}}) + \hat{\mathbf{r}} \cdot \widetilde{\mathbf{C}}_1 \cdot \hat{\mathbf{k}}$ (where $\widetilde{\mathbf{C}}_1 = \frac{1}{4\pi}\int_{\partial D}\mathbf{r}\frac{\partial \mathbf{A}}{\partial n}(\mathbf{r})ds(\mathbf{r})$), we find that

$$\hat{\mathbf{r}} \cdot \widetilde{\mathbf{C}}_1 \cdot \hat{\mathbf{k}} = -\frac{1}{2}\hat{\mathcal{H}}_n^{ant}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) \tag{23}$$

It can be shown that the matrix $\widetilde{\mathbf{C}}_1$ is symmetric due to reciprocity. Eq.(23) just states the ability to express all the elements of the tensor $\widetilde{\mathbf{C}}_1$ in terms of measurements. So to obtain for example the moment $\frac{1}{4\pi}\int_{\partial D}x\frac{\partial A_2}{\partial n}ds$, we need measurements in the observation directions $\hat{\mathbf{x}}$, $-\hat{\mathbf{x}}$ for two scattering processes stimulated by plane wave excitations in the directions $\hat{\mathbf{y}}$, $-\hat{\mathbf{y}}$.

We pass now to the moments of second kind offered by the representation (21). The quantity $\mathcal{B}$ can not be separated further. In other words there is not a purely measurement process decomposing $\mathcal{B}$, which appears as

a measured entity involving two terms referring to two low-frequency approximations of different order. The moments $\int_{\partial D}\frac{\partial \Phi_2^{(0)}}{\partial n}ds$ and $\frac{1}{3}\int_{\partial D}r^2\frac{\partial \Phi_0}{\partial n}ds$ participate in a strong interrelation, which can not be broken with assistance from the measurement arsenal. On the other hand, the decomposition is feasible alternatively via the activation of the analytic calculus based on the novel concept of the double moments[5]. Then we may estimate the inaccessible moment $\frac{1}{4\pi}\int_{\partial D}\frac{\partial \Phi_2^{(0)}}{\partial n}(\mathbf{r})ds(\mathbf{r})$ by the number $[\frac{3}{2}\mathcal{B} - (M_0^0)^3]$. We would like here to give some light to this tough analysis by an alternative independent simpler methodology, which is inspired by the fundamental concept of Null-Field equations[6].

Proposition 6 *The unmeasurable moment $\frac{1}{4\pi}\int_{\partial D}\frac{\partial \Phi_2^{(0)}}{\partial n}ds$ can be analytically estimated by the measurable quantity $[\frac{3}{2}\mathcal{B} - (M_0^0)^3]$.*

Proof. We pay attention on the integral representations (6) interconnecting the low-frequency components. It is easily proved that these relations can be extended to be

valid unaltered to the whole space R^3, if the components Φ_n are extended continuously to be zero in D and this is accomplished due to vanishing of Φ_n on ∂D. So we consider relation (6), for $n = 0, 2$, evaluated exactly at the coordinate origin. For the case $n = 2$, we restrict our attention at the zeroth order term with respect to the excitation direction $\hat{\mathbf{k}}$. We obtain the null-field equations

$$\frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_0}{\partial n} ds = 1 \qquad (24)$$

$$\frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_2^{(0)}}{\partial n} ds - 2(M_0^0)^2 + \frac{1}{4\pi} \int_{\partial D} r \frac{\partial \Phi_0}{\partial n} ds = 0 \qquad (25)$$

We mention that Eq.(24) is recognizable since it can be immediately deduced after applying the Green theorem for the pair of harmonic functions $\Phi_0, \frac{1}{r}$ in the exterior region $R^3 \backslash \bar{D}$. We define the auxiliary function $h(\mathbf{r}) = \frac{\frac{\partial \Phi_2^{(0)}}{\partial n}(\mathbf{r})}{\frac{\partial \Phi_0}{\partial n}(\mathbf{r})}$ and its mean value $\bar{h}$ on ∂D. In addition we consider the mean value $\overline{r^2}$ of r^2 over the scatterer's surface. We obtain easily that $\frac{1}{4\pi} \int_{\partial D} \frac{\partial \Phi_2^{(0)}}{\partial n} ds = \bar{h} M_0^0$, $\frac{1}{4\pi} \int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds = \overline{r^2} M_0^0$. Consequently

$$\bar{h} + \frac{1}{3} \overline{r^2} = \frac{\mathcal{B}}{M_0^0} \qquad (26)$$

In addition, using Eq.(24), we find that $\frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_2^{(0)}}{\partial n} ds = \bar{h} \frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_0}{\partial n} ds = \bar{h}$ and $\frac{1}{4\pi} \int_{\partial D} r \frac{\partial \Phi_0}{\partial n} ds = \overline{r^2} \frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_0}{\partial n} ds = \overline{r^2}$. Using Eq.(25) we obtain

$$\bar{h} - 2(M_0^0)^2 + \overline{r^2} = 0 \qquad (27)$$

Combining Eqs.(26,27) we find that $\bar{h} = \frac{3}{2} \frac{\mathcal{B}}{M_0^0} - (M_0^0)^2$ and so $\frac{1}{4\pi} \int_{\partial D} \frac{\partial \Phi_2^{(0)}}{\partial n} ds = \bar{h} M_0^0 = [\frac{3}{2} \mathcal{B} - (M_0^0)^3]$. As an immediate consequence, $\frac{1}{4\pi} \int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds = 3[(M_0^0)^3 - \frac{1}{2} \mathcal{B}]$. ∎

Nevertheless, the determination of specific bounds for the moments under consideration is feasible. More precisely, we present a first generic result and secondly an interesting theorem, valid at least for convex scatterers sharing the property of inversion symmetry.

Proposition 7 *For every $n = 0, 1, 2, 3, ...$, we have*

$$(M_0^0)^{n+1} \leq \frac{1}{4\pi} \int_{\partial D} r^n \frac{\partial \Phi_0}{\partial n} ds \qquad (28)$$

Proof. For $n = 0$ we just have a trivial equality. We apply the Cauchy-Schwartz inequality to obtain

$$(M_0^0)^2 = \left(\frac{1}{4\pi} \int_{\partial D} \frac{\partial \Phi_0}{\partial n} ds \right)^2 \leq \left(\frac{1}{4\pi} \int_{\partial D} r \frac{\partial \Phi_0}{\partial n} ds \right)$$
$$\times \left(\frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_0}{\partial n} ds \right) \qquad (29)$$

and using the null field equation (24) we obtain easily Eq. (28) for $n = 1$. We treat similarly the case $n = 2$:

$$(M_0^0)^2 \leq \frac{1}{4\pi} \int_{\partial D} r \frac{\partial \Phi_0}{\partial n} ds \leq \left(\frac{1}{4\pi} \int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds \right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{4\pi} \int_{\partial D} \frac{\partial \Phi_0}{\partial n} ds \right)^{\frac{1}{2}} \Rightarrow (M_0^0)^3 \leq \left(\frac{1}{4\pi} \int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds \right)$$

In the realm of mathematical induction, for an odd power $n = m + 1$, we have

$$(M_0^0)^{\frac{m+2}{2}} \leq \left(\frac{1}{4\pi} \int_{\partial D} r^{\frac{m}{2}} \frac{\partial \Phi_0}{\partial n} ds \right)$$
$$\leq \left(\frac{1}{4\pi} \int_{\partial D} r^{m+1} \frac{\partial \Phi_0}{\partial n} ds \right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{\partial \Phi_0}{\partial n} ds \right)^{\frac{1}{2}}$$
$$\Rightarrow (M_0^0)^{n+1} \leq \left(\frac{1}{4\pi} \int_{\partial D} r^n \frac{\partial \Phi_0}{\partial n} ds \right),$$

while for every even power n

$$(M_0^0)^{\frac{n+2}{2}} \leq \left(\frac{1}{4\pi} \int_{\partial D} r^{\frac{n}{2}} \frac{\partial \Phi_0}{\partial n} ds \right)$$
$$\Rightarrow (M_0^0)^{\frac{n+2}{2}} \leq \left(\frac{1}{4\pi} \int_{\partial D} r^n \frac{\partial \Phi_0}{\partial n} ds \right)^{\frac{1}{2}} (M_0^0)^{\frac{1}{2}}$$
$$\Rightarrow (M_0^0)^{n+1} \leq \left(\frac{1}{4\pi} \int_{\partial D} r^n \frac{\partial \Phi_0}{\partial n} ds \right).$$

∎

We would like to insist a while on investigating the matrix $\widetilde{\mathbf{C}}_1$ and more precisely to reveal a very interesting geometrical property of its trace. This property concerns the case of convex scatterers with inversion symmetry, but this restriction does not spoil the importance of the characterization under discussion:

Theorem 8 *For every convex scatterer sharing the property of inversion symmetry, it holds that*

$$(i) \quad r^2 \frac{\partial \Phi_0}{\partial n} \leq \left(\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial n} \right), \quad \mathbf{r} \in \partial D \quad and \quad (30)$$
$$(ii) \quad \mathbf{r} \cdot \nabla \Phi_0 \leq \nabla \cdot \mathbf{A}, \quad \mathbf{r} \in \partial D \quad (31)$$

Proof. We report first the well known property $\Phi_0 \geq 0$ outside D as well as the positiveness of the measure $\frac{\partial \Phi_0}{\partial n}$. Let us introduce the auxiliary function $w(\mathbf{r}) = z \Phi_0(\mathbf{r}) - A_3(\mathbf{r})$, defined on $R^3 \backslash D$. It is clearly deduced that

$$\Delta w(\mathbf{r}) = 2 \frac{\partial \Phi_0}{\partial z}, \quad \mathbf{r} \in R^3 \backslash \bar{D}. \qquad (32)$$

In addition the r.h.s $w_1(\mathbf{r}) = 2 \frac{\partial \Phi_0}{\partial z}$ is harmonic in $R^3 \backslash \bar{D}$. Let us consider the open region $B_{R,D}^+$ confined by the scatterer ∂D, the surface ∂B_R^+ of a large sphere B_R of radius R - centered at the coordinate origin O - for non negative z and the plane $z = 0$. Introducing the coordinate θ of the spherical system we remark that $w_1|_{\partial B_R^+} = 2\cos\theta M_0^0 \frac{1}{R^2} + o(\frac{1}{R^2})$, while $w_1|_{\partial D} = 2(\hat{\mathbf{z}} \cdot \hat{\mathbf{n}}) \frac{\partial \Phi_0}{\partial n}$.

In the upper half-plane $\cos\theta \geq 0$ and so $w_1|_{\partial B_R^+} \geq 0$ (for every R sufficiently large). In addition, due to convexity, $(\hat{\mathbf{z}} \cdot \hat{\mathbf{n}}) \geq 0$ on $\partial D^+ = \partial D \cap \{z \geq 0\}$, which in combination with the positiveness of $\frac{\partial \Phi_0}{\partial n}$ provides that $w_1|_{\partial D^+} \geq 0$. Due to the inversion symmetry, the function Φ_0 is an even function and so $w_1|_{\{z=0\}} = 0$. We apply then the maximum principle for the harmonic function $w_1(\mathbf{r})$ in $B_{R,D}^+$ and obtain that $w_1(\mathbf{r}) \geq 0$ in $B_{R,D}^+$ for every large R and so in the whole region $\{R^3 \backslash \bar{D}\} \cap \{z \geq 0\}$. Consequently the r.h.s of Eq. (32) is not negative in $R^3 \backslash \bar{D}$ and so by definition[14], the function $w(\mathbf{r})$ is a C^2-subsolution relative to the Laplacian operator Δ and to the bounded domain $B_{R,D}^+$. According to the weak maximum principle for subsolutions (Theorem 2.5 of the introductory work[14]),

$$max \; \overline{B_{R,D}^+} \; w = max_{\partial D^+ \cup \partial B_R^+ \cup \{z=0\}} w \qquad (33)$$

It holds that $w(\mathbf{r}) = 0$ on ∂D as well as on $\{z = 0\}$ (due to the inversion symmetry $A_3 = 0$ for $z = 0$), while $w(\mathbf{r})|_{\partial B_R^+} = -\cos\theta M_0^0 + O(\frac{1}{R}) \leq 0$. We infer from Eq. (33) that $\frac{\partial w}{\partial n} \leq 0$ on ∂D^+ and consequently due to the inherent parity that $z\frac{\partial w}{\partial n} \leq 0$ on ∂D, relation coinciding with $z^2\frac{\partial \Phi_0}{\partial n} \leq z\frac{\partial A_3}{\partial n}$ on ∂D. Repeating the same arguments with the directions $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ we find that $x^2\frac{\partial \Phi_0}{\partial n} \leq x\frac{\partial A_1}{\partial n}$ and $y^2\frac{\partial \Phi_0}{\partial n} \leq y\frac{\partial A_2}{\partial n}$ on ∂D. Adding over the components of $\mathbf{A}$, we prove the first part of the theorem. In addition, starting once again with $\frac{\partial w}{\partial n} \leq 0$ on ∂D^+, we find that $n_3 z\frac{\partial \Phi_0}{\partial n} \leq n_3\frac{\partial A_3}{\partial n}$ on ∂D. Working similarly with the components A_2, A_3 and adding the resulting equations we find that $\mathbf{r} \cdot \hat{\mathbf{n}}\frac{\partial \Phi_0}{\partial n} \leq \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{A}}{\partial n}$, relation which coincides with Eq.(31), due to vanishing of Φ_0, $\mathbf{A}$ on ∂D. ∎

Remark 9 *Last Theorem implies that* $\int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds \leq \int_{\partial D} \left(\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial n}\right) ds$. *It can be shown that the l.h.s of this inequality is representative of the volume of the scatterer and so the trace of the tensor* $4\pi\widetilde{\mathbf{C}}_1$ *is an upper volumetric bound for the scatterer. In addition* $trace(\widetilde{\mathbf{C}}_1)$ *is measurable quantity, which constitutes an upper bound for the member* $\frac{1}{4\pi}\int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds$ *of the measurable entity* $\mathcal{B}$. *A lower bound for the same moment is offered by Proposition 7 and coincides with* $(M_0^0)^3$. *Consequently, based on integral mean value theorem, we acquire the relation* $\frac{1}{4\pi}\int_{\partial D} r^2 \frac{\partial \Phi_0}{\partial n} ds = \xi^2 M_0^0$, *where* $(M_0^0)^2 \leq \xi^2 \leq \frac{1}{M_0^0} trace(\widetilde{\mathbf{C}}_1)$. *So for convex scatterers with inversion symmetry, we could roughly trust the outcome of Theorem 8, instead of using the double moments calculus to decompose second class moments.*

III. THE INVESTIGATION OF THE INVERSE SCATTERING PROBLEM

We have already presented an inversion algorithm[1] for the reconstruction of polynomial surfaces of generic even degree p, based on the exploitation of all the possible moments pertaining exclusively to the Rayleigh approximation Φ_0 up to order $2p$. More precisely, the surface was represented via a polar representation in terms of spherical harmonics and the coefficients of this representation were the unknowns of the problem, formating the unknown vector $\boldsymbol{\chi}(p)$ of dimension $N(p) = \frac{1}{6}p(p^2+6p+11)$. A linear system of $N(p)$ equations with $N(p)$ unknowns of the type $\mathbf{A}(p)\boldsymbol{\chi}(p) = \mathbf{b}(p)$ was constructed, where the matrices $\mathbf{A}(p), \mathbf{b}(p)$ were composed from generalized Rayleigh moments. The positiveness of the measure $\frac{\partial \Phi_0}{\partial n}$ was proved to be the crucial argument establishing the invertibility of the matrix $\mathbf{A}(p)$. On the other hand a Tikhonov regularization technique was suggested to guarantee the stability of the inversion. The generalized moments composing the matrices $\mathbf{A}(p), \mathbf{b}(p)$ were proved to be expressed via the moments M_0^l, $l = 1, 2, ..., 2p$. Nevertheless, the algorithm extracting the elements of these matrices from the moments was an extended analytical process with several intermediate steps. This algorithm involved repeated applications of Beltrami type differential operators over the moments as well as use of data over all possible observation directions. This was of course acceptable from the theoretical point of view but undesirable as far as the realistic implementation is concerned since it is strongly unstable to differentiate data in order to build the necessary inversion elements. The method was extended[4] to face surfaces with continuous curvature, suggesting a systematic way to approximate, to the desired accuracy, the scatterer's surface by suitable approximating polynomial surfaces. However, the necessity to differentiate data in order to construct the inversion algorithm remained present and was the main drawback for the limited exploitation of the method.

The motif of the present work is to revise the inversion algorithm by adapting the new approach[5], developed also herein, concerning the efficient and optimal exploitation of the data. We examine two different inverse scattering problems: The first case concerns simple polynomial scatterers and the second one concerns approximation of more general shape scatterers by polynomial surfaces.

A. The solution of the inverse scattering problem for polynomial scatterers

The basic assumption of this section is that the scatterer is fitted perfectly (actually can be represented appropriately) by a polynomial closed surface. The next section faces the problem of not polynomial scatterers. In both cases we try to develop algorithms that do not require intermediate solutions of the direct problem. First we give the main ideas of our approach in the simple case of the ellipsoid, although several concepts encountered in the forthcoming subsection are generic.

1. The ellipsoidal case

There exist a lot of works facing the direct and inverse scattering problem corresponding to ellipsoidal and spheroidal surfaces. In a series of innovative works[15–19] for the establishment of low frequency techniques in scattering by spheroids. In addition we pay attention on the contribution of G. Dassios - some primitive and fundamental results are presented in[20,21] - to the study of scattering problems involving ellipsoidal scatterers. His methodology[22] is very important for the geometrical characterization of the moments and in this paper has been stated for the first time the conjecture that the low-frequency moments $M_0^m(\hat{\mathbf{r}})$ can give all the necessary information to recover the surface of the scatterer. This claim has been approved in[1] for the case of polynomial scatterers in the theoretical basis, as already commented extensively in the current work. In this work, we borrow some intuitive ideas[22] but give a new prospect to the solution of the inverse scattering problem. More precisely we find in that paper comments on the basic role of the "capacity" $M_0^0(\hat{\mathbf{r}})$, of the moment $M_0^1(\hat{\mathbf{r}})$ - which determines the "center" of the scatterer - and the moment of second order $M_0^2(\hat{\mathbf{r}})$, whose knowledge defines the orientation of the scatterer through the principal axes. All these concepts are reformulated in this section for the case of the reconstruction of an ellipsoid.

First, we implement the well known result that a shift in the phase of the incident field can be "assigned" to shifting of the "center" of the scatterer More precisely, let us stimulate the scatterer by the incident field $\exp(i\mathbf{k} \cdot \mathbf{r} - i\mathbf{k} \cdot \mathbf{d})$ instead of $\exp(i\mathbf{k} \cdot \mathbf{r})$, where $\mathbf{d}$ is some specific displacement. The low-frequency series of the new total field obtains the form

$$u_{new}^{tot}(\mathbf{r}; \mathbf{k}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \widetilde{\Phi}_n(\mathbf{r}; \hat{\mathbf{k}}). \qquad (34)$$

where the new components are expressed via the old ones $(\Phi_n(\mathbf{r}; \hat{\mathbf{k}}))$ as follows

$$\widetilde{\Phi}_n(\mathbf{r}; \hat{\mathbf{k}}) = \sum_{\rho=0}^{n} (-1)^\rho \binom{n}{\rho} (\hat{\mathbf{k}} \cdot \mathbf{d})^\rho \Phi_{n-\rho}(\mathbf{r}; \hat{\mathbf{k}}) \qquad (35)$$

The important remark is that if we change (translate) variables and consider the new coordinate origin O'' as the trace of the position vector $\mathbf{d}$, then the new position variable is $\mathbf{r}'' = \mathbf{r} - \mathbf{d}$ and clearly $u_{new}^{tot}(\mathbf{r}''; \mathbf{k}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Phi_n(\mathbf{r}''; \hat{\mathbf{k}}, O'')$. Then Eq.(35) can be used to give the relation

$$\Phi_n(\mathbf{r}''; \hat{\mathbf{k}}, O'') = \sum_{\rho=0}^{n} (-1)^\rho \binom{n}{\rho} (\hat{\mathbf{k}} \cdot \mathbf{d})^\rho \Phi_{n-\rho}(\mathbf{r}; \hat{\mathbf{k}}) \quad (36)$$

We remark that Φ_0 is not affected by the phase shift (or the equivalent coordinate translation) but all the other components are affected. So the initial moment $M_0^1(\hat{\mathbf{r}})$ is

transformed to $M_{0,\text{new}}^1(\hat{\mathbf{r}}) = M_0^1(\hat{\mathbf{r}}) - (\hat{\mathbf{r}} \cdot \mathbf{d}) M_0^0$. Selecting

$$\mathbf{d} = \frac{1}{M_0^0} \frac{1}{4\pi} \int_{\partial D} \mathbf{r}' \frac{\partial \Phi_0}{\partial n'}(\mathbf{r}') ds(\mathbf{r}') = \frac{1}{M_0^0} \mathbf{M}_0^1 \quad (37)$$

we find $M_{0,\text{new}}^1(\hat{\mathbf{r}}) = 0$. So, this particular $\mathbf{d}$ stands for the "physical" center of the scatterer with respect to which the moment M_0^1 vanishes, fact reflecting the annihilation of the weighted mean value of the locations of the scatterer's points $(\frac{1}{4\pi} \int_{\partial D} \mathbf{r}'' \frac{\partial \Phi_0}{\partial n''}(\mathbf{r}'') ds(\mathbf{r}'') = 0)$. We mention that in case that the scatterer has inversion symmetry then the "physical" center $\mathbf{d}$ coincides with the geometrical center of it. We reveal so a method to locate the scatterer's center by selecting the phase of the incident field leading to vanishing of the moment M_0^1, which is immediately deducible from measurements. According to these comments we suppose in the sequel that $M_0^1(\hat{\mathbf{r}}) = 0$, implying that we have already defined the center of the scatterer coinciding with the origin O of the coordinate system.

The next effort is to find the principal directions of the scatterer. The methodology is based again on previous ideas[22], with a slight but essential difference. Let us gather the information offered by the low-frequency component $\mathcal{H}_2(\hat{\mathbf{r}}, \hat{\mathbf{k}})$ and more precisely from the symmetric part $\mathcal{H}_2^{sym}(\hat{\mathbf{r}}; \hat{\mathbf{k}}) = M_0^2(\hat{\mathbf{r}}) + M_2^0(\hat{\mathbf{k}})$ written in dyadic form as $\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} : \widetilde{\mathbf{C}}$, where

$$\widetilde{\mathbf{C}} = \frac{1}{4\pi} \int_{\partial D} \mathbf{r}' \otimes \mathbf{r}' \frac{\partial \Phi_0}{\partial n'}(\mathbf{r}') ds(\mathbf{r}') + \frac{\widetilde{\mathbf{I}}}{4\pi} \int_{\partial D} \frac{\partial \Phi_2}{\partial n'}(\mathbf{r}') ds(\mathbf{r}').$$
$$(38)$$

The diagonal elements of $\widetilde{\mathbf{C}}$ are equal to $\mathcal{H}_2^{sym}(\hat{\mathbf{x}}_i; \hat{\mathbf{k}})$, $i = 1, 2, 3$, while the non diagonal ones are equal to $\frac{1}{4\pi} \int_{\partial D} x_i x_j \frac{\partial \Phi_0}{\partial n} ds$ and belong to the subclass of moments $\widetilde{\mathcal{M}}_0^{(2,j)}$, which stem easily from measurements. What remains is just the diagonalization of tensor $\widetilde{\mathbf{C}}$, in order to obtain the principal directions of the scatterer. It is interesting that in the old approach[22], this diagonalization is performed on the - hidden in measurements - matrix $\frac{1}{4\pi} \int_{\partial D} \mathbf{r}' \otimes \mathbf{r}' \frac{\partial \Phi_0}{\partial n'}(\mathbf{r}') ds(\mathbf{r}')$ alone, but the difference of this matrix with $\widetilde{\mathbf{C}}$ is a diagonal one, which does not alter the direction of the principal axes.

All the above arguments are valid for arbitrary scatterers. Let us suppose now that we have an ellipsoidal surface. First we note that due to inversion symmetry, $M_0^{(2n+1)}(\hat{\mathbf{r}}) = 0$ for $n = 0, 1, 2, \dots$. In addition, we are in position, as explained, to determine the principal axes and to rotate the coordinate system in order to have the following simple form for the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (39)$$

where a, b, c are the unknowns of the problem. Due to

the symmetry of Eq.(39), we have

$$\frac{1}{a^2}\int_{\partial D} x^2 \frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \frac{1}{b^2}\int_{\partial D} y^2 \frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r})$$
$$= \frac{1}{c^2}\int_{\partial D} z^2 \frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \frac{1}{3}4\pi M_0^0 \qquad (40)$$

This relation can be verified through the solution of

$$(\frac{1}{a^2}-\frac{1}{b^2})\int_{\partial D} x^2 \frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \frac{1}{b^2}\int_{\partial D}(y^2-x^2)\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) \Rightarrow (b^2-a^2)\frac{1}{a^2}\int_{\partial D} x^2\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r})$$
$$= \int_{\partial D}(y^2-x^2)\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}) \Rightarrow b^2-a^2 = \frac{3}{4\pi M_0^0}\int_{\partial D}(y^2-x^2)\frac{\partial \Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r}),$$

and in combination with the set of relations (22), we find

$$b^2-a^2 = \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{y}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right] \qquad (41)$$

Similarly

$$c^2-a^2 = \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{z}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]. \qquad (42)$$

The relations above are valid independently of the excitation polarization. Having the relative differences $h_2^2 = a^2 - c^2$, $h_3^2 = a^2 - b^2$ at hand, what remains is a third condition connecting these coefficients. But this is provided exactly by the moment M_0^0, which, in ellipsoidal coordinates[20], is expressed as

$$M_0^0 \ (= \mathcal{H}_0) = \frac{1}{I^0(a)} \qquad (43)$$

where we meet the elliptic integral $I^0(\rho) = \int_\rho^\infty \frac{du}{[(u^2-h_2^2)(u^2-h_3^2)]^{\frac{1}{2}}}$. Consequently Eq.(43) provides the value of the semi-axis a, by inverting the function of the elliptic integral and constitutes the suitable complementary relation to Eqs.(41,42). The last three equations define the ellipsoid and can be used in the construction of the following functional

$$f(a,b,c) = \left(I^0(a) - \frac{1}{M_0^0}\right)^2$$
$$+ \left(b^2-a^2 - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{y}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2$$
$$+ \left(c^2-a^2 - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{z}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2 \ (44)$$

The minimization of this functional leads of course to the determination of the ellipsoid via the relations (41,42,43). Moreover, the applicability of the objective function (44) is broader since - as it will be clarified in subsequent sections - it is offered as the starting point for the approximation of more general scatterers by ellipsoidal surfaces.

the direct problem in ellipsoidal geometry but, as mentioned before, we avoid evoking this approach, to keep the method as much as generic, free of the technicalities of special coordinate systems. Based on Eq.(40), we easily find that

Remark 10 *Comparing the revised version of the method presented herein with the old form of the method, we verify easily the simplification introduced by the new approach. In addition there is a qualitative difference. The old approach[1] drives the reconstruction attempt to the formulation of a linear 9×9-system ($N(2) = 9$), which contains "easy" and "tough" moments of orders 0,2 and 4. So that approach needed even the component $\mathcal{H}_4$ of the far-filed pattern at the same time that the present work makes use only of $\mathcal{H}_2$, which is expected to be an optimal and reasonable situation for the case of reconstructing just an ellipsoid. Generally the old methodology[1] needed more data than the expected - roughly speaking moments of "double" order than the degree of the surface - in order to establish by any means the desired linearity.*

We proceed now to polynomial surfaces of higher degree.

2. The case of a scatterer of fourth degree

Let us consider a polynomial scatterer with the representation

$$\frac{x^4}{a^4} + \frac{y^4}{b^4} + \frac{z^4}{c^4} = 1. \qquad (45)$$

It is necessary to evoke now the moments $\widetilde{\mathcal{M}}_0^{(4,j)}$, $j = 1,...,9$, which are recoverable from measurements (particularly stem from $\mathcal{H}_4$) as stated in Theorem 2. These last quantities can be assigned to special combinations of measurements $\mathcal{H}_4(\hat{\mathbf{r}};\hat{\mathbf{k}})$ for particular observation and excitation directions. We focus on the moments $\widetilde{\mathcal{M}}_0^{(4,1)} = \frac{1}{4\pi}\int_{\partial D}(x^4+y^4-6x^2y^2)\frac{\partial \Phi_0}{\partial n}ds$, $\widetilde{\mathcal{M}}_0^{(4,2)} = \frac{1}{4\pi}\int_{\partial D}(y^4+z^4-6y^2z^2)\frac{\partial \Phi_0}{\partial n}ds$ and $\widetilde{\mathcal{M}}_0^{(4,3)} = \frac{1}{4\pi}\int_{\partial D}(x^4+z^4-6x^2z^2)\frac{\partial \Phi_0}{\partial n}ds$, where a clear simplification in the symbolism has been adopted by omitting the dependence on the argument $\mathbf{r}$. Taking advantage of the symmetric form (45) of the

surface, we state the obvious relations

$$\frac{1}{a^4}\int_{\partial D}x^4\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{b^4}\int_{\partial D}y^4\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{c^4}\int_{\partial D}z^4\frac{\partial\Phi_0}{\partial n}ds$$
$$(46)$$

together with

$$\frac{1}{a^2}\int_{\partial D}x^2y^2\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{c^2}\int_{\partial D}y^2z^2\frac{\partial\Phi_0}{\partial n}ds, \quad (47)$$

$$\frac{1}{a^2}\int_{\partial D}x^2z^2\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{b^2}\int_{\partial D}y^2z^2\frac{\partial\Phi_0}{\partial n}ds. \quad (48)$$

We multiply Eq.(45) with $\frac{1}{4\pi}\frac{\partial\Phi_0}{\partial n}$ and integrate over the scatterer's surface. We find easily that every one of the common terms in Eq.(46) is expressed as

$$\frac{1}{a^4}\int_{\partial D}x^4\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{3}4\pi M_0^0. \quad (49)$$

Expanding suitably the moments $\widetilde{\mathcal{M}}_0^{(4,j)}$, $j=1,2,3$ and using Eqs.(46,49), we obtain

$$\frac{1}{3}(a^4+b^4)M_0^0 - \widetilde{\mathcal{M}}_0^{(4,1)} = 6\frac{1}{4\pi}\int_{\partial D}x^2y^2\frac{\partial\Phi_0}{\partial n}ds \quad (50)$$

$$\frac{1}{3}(b^4+c^4)M_0^0 - \widetilde{\mathcal{M}}_0^{(4,2)} = 6\frac{1}{4\pi}\int_{\partial D}y^2z^2\frac{\partial\Phi_0}{\partial n}ds \quad (51)$$

$$\frac{1}{3}(a^4+c^4)M_0^0 - \widetilde{\mathcal{M}}_0^{(4,3)} = 6\frac{1}{4\pi}\int_{\partial D}x^2z^2\frac{\partial\Phi_0}{\partial n}ds \quad (52)$$

Exploiting the symmetry equations (47,48) we transform the equations above in two new relations connecting the three unknowns a^2,b^2,c^2 exclusively with the moments $\widetilde{\mathcal{M}}_0^{(4,j)}$, $j=1,2,3$. Indeed

$$c^2\frac{1}{3}(a^4+b^4)M_0^0 - c^2\widetilde{\mathcal{M}}_0^{(4,1)} = a^2\frac{1}{3}(b^4+c^4)M_0^0$$

$$-a^2\widetilde{\mathcal{M}}_0^{(4,2)} = b^2\frac{1}{3}(a^4+c^4)M_0^0 - b^2\widetilde{\mathcal{M}}_0^{(4,3)}. \quad (53)$$

These two equalities give easily

$$(a^2-c^2)\frac{1}{3}(c^2a^2-b^4)M_0^0 = c^2\widetilde{\mathcal{M}}_0^{(4,1)} - a^2\widetilde{\mathcal{M}}_0^{(4,2)}(54)$$

$$(b^2-a^2)\frac{1}{3}(a^2b^2-c^4)M_0^0 = a^2\widetilde{\mathcal{M}}_0^{(4,2)} - b^2\widetilde{\mathcal{M}}_0^{(4,3)}(55)$$

The third necessary relation completing the non linear system comes from the moments of second order. More precisely we evoke again the coherent symmetry to obtain

$$\frac{1}{a^2}\int_{\partial D}x^2\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{b^2}\int_{\partial D}y^2\frac{\partial\Phi_0}{\partial n}ds = \frac{1}{c^2}\int_{\partial D}z^2\frac{\partial\Phi_0}{\partial n}ds$$
$$(56)$$

where in contrast to Eq.(40), these common terms are not longer known. However, we easily deduce that

$$(b^2-a^2)\left(\mathcal{H}_2^{sym}(\hat{\mathbf{z}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right) =$$

$$(c^2-a^2)\left(\mathcal{H}_2^{sym}(\hat{\mathbf{y}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right). \quad (57)$$

It is not the goal of our approach to investigate completely the induced non linearity and the consequent solvability of the equations (54,55,57) since the final aim is to suggest - in the next session - a more generic approach to the inverse problem solution. In the general case these relations are expected to provide with the solution (a,b,c) but this is not always the case as can be proved in some degenerate cases. In fact in the non degenerate case $a\neq b, b\neq c, a\neq c$ we can exploit relation (57) to express b^2 in terms of the remaining coefficients a^2 and c^2 and then replace in the relations (54,55) to obtain an algebraic non linear system of polynomial non linearity. However the situation changes when, for example, we recognize in the data the coincidence $\widetilde{\mathcal{M}}_0^{(4,2)} = \widetilde{\mathcal{M}}_0^{(4,3)}$. Then we infer from the definition of the moments that $a=b$. This actually can be supported also by Eq.(55). The point is that Eq.(57) becomes useless, while Eq.(54) obtains the form

$$\frac{1}{3}a^2(a^2-c^2)^2 M_0^0 + c^2\widetilde{\mathcal{M}}_0^{(4,1)} - a^2\widetilde{\mathcal{M}}_0^{(4,2)} = 0, \quad (58)$$

which reveals a polynomial non linearity of third order w.r.t the involved (square) coefficients. A further possible symmetry $\widetilde{\mathcal{M}}_0^{(4,1)} = \widetilde{\mathcal{M}}_0^{(4,2)}$ would impoverish the equation above, replacing it by the additional ascertainment $a=c$. In this case, the complete determination of all the coefficients demands not only the implicit but also the explicit use of the basic moment M_0^0 as the measure of the radius of the equivalent sphere. The volume of the scatterer is proved to be $Aabc$, where A is a constant incorporating several Gamma functions of specific arguments, independent of the coefficients ($A = -\frac{6\sqrt{2}\Gamma(\frac{5}{4})^2}{\Gamma(\frac{-1}{4})\Gamma(\frac{7}{4})}$). Equating this expression with $\frac{4\pi}{3}(M_0^0)^3$, we impose volume equivalence of the scatterer with the equivalent sphere and provide with the relation establishing the size determination of the scatterer. In Section III B 2, we will develop an alternative minimization process assuring mainly the stability of the problem and avoiding the solution of non linear systems.

B. Approximation of the scatterer's surface by polynomial manifolds

The more interesting aspect of the inverse problem in the low-frequency regime is to approximate the boundary of the scatterer by a polynomial surface fitting suitably with the original one. The methodology must be stable, optimal and exploit as much as necessary information provided by the low-frequency approximation of the far-field pattern.

1. The determination of the best fitting ellipsoid

The problem under consideration has attracted the scientific interest for several years. We mention here an interesting approach[23], where for a hard acoustic scatterer,

a suitable functional is formulated, whose minimization leads to the construction of the fitting ellipsoid. Here we reformulate again the problem according to the developed herein theory and suggest a different optimization scheme. More precisely, we begin again following the same arguments met in the previous subsection in order to define the center and the principle directions of the scatterer. However since in this case the scatterer is not a real ellipsoid, we try to make more stable the process of the determination of the principal directions. This task[23] is fulfilled globally in the unified functional. Here we proceed differently under the following concept. In case of a real ellipsoid the principal directions diagonalize simultaneously the tensor $\widetilde{\mathbf{C}}$ defined by (38) and the tensor $\widetilde{\mathbf{C}}_1 = \frac{1}{4\pi} \int_{\partial D} \mathbf{r} \frac{\partial \mathbf{A}}{\partial n}(\mathbf{r}) ds(\mathbf{r})$. Indeed, this last tensor is symmetric as it is proved via reciprocity arguments and so is diagonalizable. In the ellipsoidal case $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{C}}_1$ become diagonal only through the principal axes. As an example, the physical meaning of $\hat{\mathbf{x}} \cdot \widetilde{\mathbf{C}}_1 \cdot \hat{\mathbf{z}} = 0$ is that the observation direction $\hat{\mathbf{x}}$ is "blind" when we stimulate in $\hat{\mathbf{z}}$-direction, as far as the contribution of $\Phi_1^{(1)}$ is concerned. In algebraic terms this common diagonalization is inscribed in the tensor relation $\widetilde{\mathbf{C}} \cdot \widetilde{\mathbf{C}}_1 = \widetilde{\mathbf{C}}_1 \cdot \widetilde{\mathbf{C}}$. In the case of an arbitrary scatterer this coincidence of the eigenvector systems is probably distorted together with the satisfaction of the matrix equation above. This divergence of eigenvector systems becomes a criterion measuring the compatibility of the ellipsoidal geometry with all the data acquired exclusively from the low-frequency component $\mathcal{H}_2$. In addition this divergence could be assigned to poor accuracy of data (unsatisfactory level of noise δ). Especially in the last case, lack of symmetrization of $\widetilde{\mathbf{C}}_1$ is equivalent to violation of the reciprocity theorem and then special care must be assigned to the measurement process. The method develops as follows: When even approximately the relation $\widetilde{\mathbf{C}} \cdot \widetilde{\mathbf{C}}_1 = \widetilde{\mathbf{C}}_1 \cdot \widetilde{\mathbf{C}}$ is satisfied, we calculate the principal directions as in case (i) of the previous subsection. In the opposite case, we minimize the following quadratic functional G over all possible unitary (res. diagonal) matrices $\widetilde{\mathbf{P}}$ (res. $\mathbf{\Lambda}, \mathbf{\Lambda}_1$):

$$G(\widetilde{\mathbf{P}}, \mathbf{\Lambda}, \mathbf{\Lambda}_1) = (\widetilde{\mathbf{C}} - \widetilde{\mathbf{P}} \cdot \mathbf{\Lambda} \cdot \widetilde{\mathbf{P}}^T) : (\widetilde{\mathbf{C}} - \widetilde{\mathbf{P}} \cdot \mathbf{\Lambda} \cdot \widetilde{\mathbf{P}}^T)$$
$$+ (\widetilde{\mathbf{C}}_1 - \widetilde{\mathbf{P}} \cdot \mathbf{\Lambda}_1 \cdot \widetilde{\mathbf{P}}^T) : (\widetilde{\mathbf{C}}_1 - \widetilde{\mathbf{P}} \cdot \mathbf{\Lambda}_1 \cdot \widetilde{\mathbf{P}}^T), \quad (59)$$

where tensor instead of matrix symbolism has been adopted. The minimizing matrix $\widetilde{\mathbf{P}}_{\mathrm{m}}$ offers, as columns, the optimum estimation of the directions of the principal axes.

After the determination of the orientation of the fitting ellipsoid, we proceed to the optimal determination of its semi-axes. If the scatterer was known to be just a perturbation of an ellipsoid then the minimization of the objective function (44) would be the optimal process to estimate the semi-axes of the ellipsoid. However we have to confess that the functional (44) is privileged by the fact that we already know the solution of the direct scattering problem for the ellipsoidal geometry and so we are in position to build its first stabilizing term, guiding safely the estimation of the semi-axis a. It is well known that alternatively we could detour the implication of the direct problem solution by considering the interpretation of M_0^0 as the radius of the equivalent sphere of the scatterer. This introduces a volumetric equivalence and the formulation of the simpler objective function

$$h(a,b,c) = \left(abc - (M_0^0)^3\right)^2$$
$$+ \left(b^2 - a^2 - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{y}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2$$
$$+ \left(c^2 - a^2 - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{z}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2 \quad (60)$$

However this functional fails to provide with a reasonable reconstruction in case of an elongated ellipsoid where the aspect ratios a/b and (or) a/c differ significantly from unity.

Our purpose is to establish a systematic method to approximate scatterers by polynomial surfaces, able to be generalized in a methodological manner from the ellipsoidal to higher degree approximating surfaces and independent of the - usually unavailable - solution of the corresponding direct scattering problem. So we keep from the functionals (44,60) the two common terms expressing the relative correlation of the unknown semi-axes and pay attention to construct the first term, which is responsible to establish the size of the ellipsoid. This will be accomplished in the general framework that it is not appropriate to treat anymore Eqs.(41,42,43) as equalities, since these relations are not expected to be exact but only approximate. To obtain accurate results relying on a broader class of measurements, we could construct and minimize a functional incorporating terms that force the ellipsoid to obey optimally to a set of data involving as much as possible moments participating in the fundamental and higher order low-frequency components $\mathcal{H}_n$. As an example $\mathcal{H}_4$ brings very rich information involving fourth degree moments pertaining to Φ_0 , third degree moments refered to Φ_1 e.t.c. When the scatterer is an ellipsoid, this extra information is just compatible with the data offered by $\mathcal{H}_2$ as far as the determination of the semi-axes is concerned. But in the general case, we must pay attention on the deviation between the data hidden in higher order terms of the far field and the corresponding fields estimated via the fitting ellipsoid. So it could be profitable to take into account the first order approximation Φ_1 of the field or even higher order terms.

The methodology is initialized by considering the ellipsoidal representation (39), "pretending" that it represents the scatterer surface, multiplying it with the basic surface measure $\frac{\partial \Phi_0}{\partial n}$ and integrating over the scatterer. We obtain - after symbolizing $\alpha = \frac{1}{a^2}$, $\beta = \frac{1}{b^2}$, $\gamma = \frac{1}{c^2}$ - the following relation

$$\mathcal{N}_0^{(2,1)}\alpha + \mathcal{N}_0^{(2,2)}\beta + \mathcal{N}_0^{(2,3)}\gamma + \frac{1}{3}(\alpha+\beta+\gamma)N_{0,2} \rightleftharpoons M_0^0$$
$$\Rightarrow$$
$$\mathcal{N}_0^{(2,1)}(\alpha-\gamma) + \mathcal{N}_0^{(2,2)}(\beta-\gamma) + \frac{1}{3}(\alpha+\beta+\gamma)N_{0,2} \rightleftharpoons M_0^0,$$

where the symbol "$\rightleftharpoons$" replaces the "virtual" equality to indicate that this relation is candidate to participate in a minimization scheme and can not be considered as an equality and that the two terms of the relation are expected to be close to each other but are not identical. We meet in relation (61) the moments $\mathcal{N}_0^{(2,i)} = \frac{1}{4\pi}\int_{\partial D}(x_i^2 - \frac{1}{3}r^2)\frac{\partial\Phi_0}{\partial n}(\mathbf{r})ds(\mathbf{r})$, $x_1 = x, x_2 = y, x_3 = z$, which are expressed directly in terms of the deducible from measurements moments $\mathcal{M}_0(h_2)$, met in Theorem 2. We encounter also in (61) a special case of the quantities $N_{n,l} = \frac{1}{4\pi}\int_{\partial D}r^l\frac{\partial\Phi_n}{\partial n}(\mathbf{r})ds(\mathbf{r})$, which are not measurable moments. Consequently, in the relation above there are just four unknowns: the coefficients of the fitting scatterer and the moment $N_{0,2}$. As mentioned several times before, the moment $N_{0,2}$ is firmly melted with

$\int_{\partial D}\frac{\partial\Phi_2^{(0)}}{\partial n}(\mathbf{r})ds(\mathbf{r})$ in the formulation of the measured moment $\mathcal{B}$ and technically inseparable from this structure. Nevertheless, every moment that can not be determined from measurements, must be determined or estimated a priori via the geometrical representation of the fitting polynomial surface. For the ellipsoidal case under consideration, we exploit Eqs. (40) - rather in equivalence and not equality form - to obtain

$$N_{0,2} \rightleftharpoons \frac{1}{3}(a^2 + b^2 + c^2)M_0^0 = \frac{1}{3}(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})M_0^0 \quad (61)$$

The first attempt to construct the optimization functional is based on approximations (61, 61) and on the estimations (41,42), concerning the interrelations of the unknowns, leading to

$$g_0(\alpha, \beta, \gamma, N_{0,2}) = \left(\mathcal{N}_0^{(2,1)}(\alpha - \gamma) + \mathcal{N}_0^{(2,2)}(\beta - \gamma) + \frac{1}{3}(\alpha + \beta + \gamma)N_{0,2} - M_0^0\right)^2 + \left(\frac{1}{3}(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})M_0^0 - N_{0,2}\right)^2$$

$$+ \left(\frac{1}{\beta} - \frac{1}{\alpha} - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{y}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2 + \left(\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{3}{M_0^0}\left[\mathcal{H}_2^{sym}(\hat{\mathbf{z}};\hat{\mathbf{k}}) - \mathcal{H}_2^{sym}(\hat{\mathbf{x}};\hat{\mathbf{k}})\right]\right)^2. \quad (62)$$

We are immediately tempted to eliminate $N_{0,2}$ in the functional above but this is avoided and we insist on the form (62) so that the moment $N_{0,2}$ remains active to be inerrelated - as a component of the known moment $\mathcal{B}$ - with optimization terms involving the component $\Phi_2^{(0)}$. In addition, this elimination would be equivalent to considering Eq.(61) as a strict equality, which is not generally accurate. In the spirit of the documentation introducing this section, we are in position to supplement to g_0 additional terms, forcing the ellipsoid to obey to rules imposed by other moments at hand. As an example, we apply the same methodology using the surface integrals $\frac{\partial A_2^{(2,m)}}{\partial n}$ instead of $\frac{\partial\Phi_0}{\partial n}$. However we have first to construct a rotation invariant measure and this may be realized easily using the following members of the family generated easily by the functions $A_2^{(2,m)}$:

$$A_{2,i} = x_i^2 - \frac{1}{3}r^2 - \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial A_{2,i}}{\partial n'}ds', \ i = 1,2,3. \quad (63)$$

So we obtain

$$\psi(\mathbf{r}) = 1 - \frac{1}{3}(\alpha + \beta + \gamma)r^2 - \frac{1}{4\pi}\int_{\partial D}\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial\psi}{\partial n'}ds' \quad (64)$$

where $\psi = \alpha A_{2,1} + \beta A_{2,2} + \gamma A_{2,3}$. We find, based on the last remark of Proposition 1, that

$$\frac{1}{4\pi}\int_{\partial D}\frac{\partial\psi}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \alpha\mathcal{N}_0^{(2,1)} + \beta\mathcal{N}_0^{(2,2)} + \gamma\mathcal{N}_0^{(2,3)} \quad (65)$$

Then we are in position to exploit symmetry in order to obtain

$$\frac{1}{4\pi}\int_{\partial D}r^2\frac{\partial\psi}{\partial n}ds \rightleftharpoons \frac{1}{3}(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})\left[\alpha\mathcal{N}_0^{(2,1)} + \beta\mathcal{N}_0^{(2,2)} + \gamma\mathcal{N}_0^{(2,3)}\right]. \quad (66)$$

We multiply the ellipsoidal representation (39) with $\frac{\partial\psi}{\partial n}$, integrate over the surface and use (66) to give birth to the following additional minimization term

$$g_2^{(2)}(\alpha, \beta, \gamma) = \left((\alpha - \gamma)[\alpha\mathcal{N}_2^{(2,1,1)} + \beta\mathcal{N}_2^{(2,2,1)} + \gamma\mathcal{N}_2^{(2,3,1)}] + (\beta - \gamma)[\alpha\mathcal{N}_2^{(2,1,2)} + \beta\mathcal{N}_2^{(2,2,2)}\right.$$

$$\left. + \gamma\mathcal{N}_2^{(2,3,2)}] + \frac{1}{9}(\alpha + \beta + \gamma)(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})[\alpha\mathcal{N}_0^{(2,1)} + \beta\mathcal{N}_0^{(2,2)} + \gamma\mathcal{N}_0^{(2,3)}] - [\alpha\mathcal{N}_0^{(2,1)} + \beta\mathcal{N}_0^{(2,2)} + \gamma\mathcal{N}_0^{(2,3)}]\right)^2, \quad (67)$$

where the moments $\mathcal{N}_2^{(2,j,i)} = \frac{1}{4\pi}\int_{\partial D}(x_i^2 - \frac{1}{3}r^2)\frac{\partial A_{2,j}}{\partial n}ds$ are directly deduced from the known moments $\mathcal{M}_2^m(h_2)$

determined in Proposition 4. All these moments stem of course from the far-field component $\mathcal{H}_4$. We would also

exploit information hidden in $\mathcal{H}_3$ as well. For example, proceeding as above at the basis of the surface densities $x\frac{\partial A_1}{\partial n}, y\frac{\partial A_2}{\partial n}, z\frac{\partial A_3}{\partial n}$, we obtain after extended manipulations that

$$\left[\frac{C_1}{3\alpha+\beta+\gamma}+\frac{C_2}{\alpha+3\beta+\gamma}+\frac{C_3}{\alpha+\beta+3\gamma}\right]$$
$$-\frac{1}{15}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})\int_{\partial D}\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial n}ds \rightleftharpoons 0 \qquad (68)$$

where

$$C_i = \int_{\partial D} x_i \frac{\partial A_i}{\partial n}ds + \alpha\mathcal{D}_{i1}+\beta\mathcal{D}_{i2}+\gamma\mathcal{D}_{i3},\ i=1,2,3 \qquad (69)$$

and

$$\mathcal{D}_{ij} = \int_{\partial D} x_i(\frac{1+2\delta_{ij}}{5}r^2 - x_j^2)\frac{\partial A_i}{\partial n}ds,\ i,j=1,2,3 \qquad (70)$$

where we recognize the Kronecker's symbol δ_{ij}. The integrals $\mathcal{D}_{ij}$ have harmonic kernel and are trivially determined from the moments $\widetilde{\mathbf{M}}_1^{(3,j)}$ constructed in Proposition 4. In addition we recognize in Eqs.(68,69) the diagonal terms of the measurable tensor $\widetilde{\mathbf{C}}_1$. Eq.(68) gives birth to the additional minimization term

$$g_1(\alpha,\beta,\gamma) = \left(\left[\frac{C_1}{3\alpha+\beta+\gamma}+\frac{C_2}{\alpha+3\beta+\gamma}\right.\right.$$
$$\left.\left.+\frac{C_3}{\alpha+\beta+3\gamma}\right] - \frac{1}{15}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})\int_{\partial D}\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial n}ds\right)^2 (71)$$

Similar techniques could be applied to terms pertaining to surface measures $r^2\frac{\partial\Phi_0}{\partial n}, \frac{\partial\Phi_2^{(0)}}{\partial n}, x_i\frac{\partial A_{3,i}}{\partial n}, \frac{\partial\Phi_2^{(2)}}{\partial n}(\mathbf{r},\hat{\mathbf{r}})$ e.t.c, to produce more - rotationally invariant - functionals in the realm of the measurement components $\mathcal{H}_i$, $1 = 1,2,3,4$. It is interesting to present how the component $\Phi_2^{(0)}$ influences the structure of the scheduled algorithm. More precisely, it is possible to establish a "chain" of interrelated equations involving the sequence

of the "even" moments, which play an important role, since they are built upon the fundamental component Φ_0. When we multiply Eq.(39) with $\frac{1}{4\pi}\frac{\partial\Phi_2^{(0)}}{\partial n}$ and integrate over the surface, we obtain

$$\mathcal{N}_2^{(0,2,1)}\alpha + \mathcal{N}_2^{(0,2,2)}\beta + \mathcal{N}_2^{(0,2,3)}\gamma + \frac{1}{3}(\alpha+\beta+\gamma)N_{2,2}^{(0)}$$
$$\rightleftharpoons \mathcal{B} - \frac{1}{3}N_{0,2},\ \text{ or}$$

$$\mathcal{N}_2^{(0,2,1)}(\alpha-\gamma) + \mathcal{N}_2^{(0,2,2)}(\beta-\gamma) + \frac{1}{3}(\alpha+\beta+\gamma)N_{2,2}^{(0)}$$
$$\rightleftharpoons \mathcal{B} - \frac{1}{3}N_{0,2}. \qquad (72)$$

In addition $\mathcal{N}_2^{(0,2,i)} = \frac{1}{4\pi}\int_{\partial D}(x_i^2 - \frac{1}{3}r^2)\frac{\partial\Phi_2^{(0)}}{\partial n}ds$. In contrast to $\mathcal{N}_0^{(2,j)}$, the quantities $\mathcal{N}_2^{(0,2,j)}$ are not measurable but participate in measured quantities in the same manner that $N_{0,2}$ or $\frac{1}{4\pi}\int_{\partial D}\frac{\partial\Phi_2^{(0)}}{\partial n}(\mathbf{r})ds(\mathbf{r})$ participate in $\mathcal{B}$. More precisely, projecting on spherical harmonics of order two, we detect in $\mathcal{H}_4$ the measurable quantities $\mathcal{B}_{4,2,j} = \mathcal{N}_2^{(0,2,j)} + \frac{1}{7}\frac{1}{4\pi}\int_{\partial D}(x_j^2 - \frac{1}{3}r^2)r^2\frac{\partial\Phi_0}{\partial n}ds$, $j=1,2,3$. Clearly we are in position, as often in this work, to use Eq. (39) in combination with the surface measure $\frac{1}{7}r^2\frac{\partial\Phi_0}{\partial n}$ to produce a moment equation, which added to relation (72) gives

$$\mathcal{B}_{4,2,1}(\alpha-\gamma) + \mathcal{B}_{4,2,2}(\beta-\gamma) + \frac{1}{3}(\alpha+\beta+\gamma)$$
$$\times[N_{2,2}^{(0)} + \frac{1}{7}N_{0,4}] \rightleftharpoons \mathcal{B} - \frac{4}{21}N_{0,2}. \qquad (73)$$

The ellipsoidal symmetry aids at obtaining

$$N_{0,4} \rightleftharpoons \frac{1}{3}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})N_{0,2},$$
$$N_{2,2}^{(0)} \rightleftharpoons \frac{1}{3}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})\frac{1}{4\pi}\int_{\partial D}\frac{\partial\Phi_2^{(0)}}{\partial n}$$
$$= \frac{1}{3}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})[\mathcal{B} - \frac{1}{3}N_{0,2}]$$

So we construct the functional

$$g_2^{(0)}(\alpha,\beta,\gamma,N_{0,2},N_{0,4},N_{2,2}^{(0)}) = \left(\mathcal{B}_{4,2,1}(\alpha-\gamma) + \mathcal{B}_{4,2,2}(\beta-\gamma) + \frac{1}{3}(\alpha+\beta+\gamma)[N_{2,2}^{(0)} + \frac{1}{7}N_{0,4}] - \mathcal{B} + \frac{4}{21}N_{0,2}\right)^2$$
$$+ \left(\frac{1}{3}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})N_{0,2} - N_{0,4}\right)^2 + \left(\frac{1}{3}(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})[\mathcal{B} - \frac{1}{3}N_{0,2}] - N_{2,2}^{(0)}\right)^2 \qquad (74)$$

The enriched functional obtains the form

$$g = g_0 + \epsilon_1 g_1 + \epsilon_2^{(0)}g_2^{(0)} + \epsilon_2^{(2)}g_2^{(2)} \qquad (75)$$

where $\epsilon_1, \epsilon_2^{(0)}, \epsilon_2^{(2)}$ are non negative weight coefficients. Working with just g_0 means simply that the scatterer is expected to be a perturbation of an ellipsoid and the

component $\mathcal{H}_2$ of the far-field is considered adequate to provide a good estimation of the shape of the scatterer.

2. The reconstruction of polynomial fitting surfaces of fourth degree

We consider first the simple case of a surface expressed via the representation (45). First we evoke the symmetry relations, met already in the case of the exact fourth degree scatterer, referring to Rayleigh moments located in $\mathcal{H}_2$. More precisely we have

$$h_1(a,b,c) := (b^2 - a^2)\widetilde{\mathcal{M}}_0^{(2,2)} - (c^2 - a^2)\widetilde{\mathcal{M}}_0^{(2,1)} \rightleftharpoons 0 \quad (76)$$

where $\widetilde{\mathcal{M}}_0^{(2,1)} = \frac{1}{4\pi}\int_{\partial D}(y^2 - x^2)\frac{\partial \Phi_0}{\partial n}ds$ and $\widetilde{\mathcal{M}}_0^{(2,2)} = \frac{1}{4\pi}\int_{\partial D}(z^2 - x^2)\frac{\partial \Phi_0}{\partial n}ds$. The functional h_1 will participate in the formulation of the minimization objective function. We proceed exploiting the information offered by $\mathcal{H}_4$. We evoke here the harmonicity of the functions $x^4 + y^4 - 6x^2y^2$, $y^4 + z^4 - 6y^2z^2$, $x^4 + z^4 - 6x^2z^2$ and introduce the harmonic also functions $35x^4 - 30x^2r^2 + 3r^4$, $35y^4 - 30y^2r^2 + 3r^4$ and $35z^4 - 30z^2r^2 + 3r^4$ of fourth degree. The three first harmonics give rise to the moments $\widetilde{\mathcal{M}}_0^{(4,j)}$, $j = 1,2,3$ met already in this work, while the new comers after multiplied with $\frac{1}{4\pi}\frac{\partial \Phi_0}{\partial n}$ and integrated

over ∂D give birth to the moments $\mathcal{N}_0^{(4,j)}$, $j = 1,2,3$ which are also deducible from $\mathcal{H}_4$. Exploiting as usually the symmetry of the fitting surface, we easily find after "projecting" x_j^4 onto the measure $\frac{\partial \Phi_0}{\partial n}$, that

$$35\int_{\partial D} x_j^4 \frac{\partial \Phi_0}{\partial n}ds = 4\pi\mathcal{N}_0^{(4,j)} + 30\int_{\partial D} x_j^2 r^2 \frac{\partial \Phi_0}{\partial n}ds$$
$$-3\int_{\partial D} r^4 \frac{\partial \Phi_0}{\partial n}ds, \quad j = 1,2,3, \quad (77)$$

Due to the inherent symmetry

$$\frac{1}{a_i^4}\int_{\partial D} x_i^4 \frac{\partial \Phi_0}{\partial n}ds \rightleftharpoons \frac{4\pi}{3}M_0^0, \quad i = 1,2,3 \quad (78)$$

$$\frac{1}{a_1^2}\int_{\partial D} x_1^2 r^2 \frac{\partial \Phi_0}{\partial n}ds \rightleftharpoons \frac{1}{a_2^2}\int_{\partial D} x_2^2 r^2 \frac{\partial \Phi_0}{\partial n}ds \rightleftharpoons$$
$$\frac{1}{a_3^2}\int_{\partial D} x_3^2 r^2 \frac{\partial \Phi_0}{\partial n}ds \quad (79)$$

where $a_1 = a$, $a_2 = b$ and $a_3 = c$. Clearly Eq.(79) gives

$$\int_{\partial D} r^4 \frac{\partial \Phi_0}{\partial n}ds \rightleftharpoons (1 + \frac{b^2}{a^2} + \frac{c^2}{a^2})\int_{\partial D} x^2 r^2 \frac{\partial \Phi_0}{\partial n}ds. \quad (80)$$

Alternatively, we find

$$\frac{1}{4\pi}\int_{\partial D} r^4 \frac{\partial \Phi_0}{\partial n}ds = \frac{1}{4\pi}\int_{\partial D}(x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2x^2z^2)\frac{\partial \Phi_0}{\partial n}ds = \frac{1}{4\pi}\int_{\partial D}(x^4 + y^4 + z^4$$

$$+\frac{2}{3}x^4 + \frac{2}{3}y^4 + \frac{2}{3}z^4)\frac{\partial \Phi_0}{\partial n}ds - \frac{1}{3}\sum_{n=1}^{3}\widetilde{\mathcal{M}}_0^{(4,n)} \rightleftharpoons \frac{5}{3}(a^4 + b^4 + c^4)\frac{1}{3}M_0^0 - \frac{1}{3}\sum_{n=1}^{3}\widetilde{\mathcal{M}}_0^{(4,n)} \quad (81)$$

Combining Eqs.(77,79,80,81) we find

$$h_{2,j}(a,b,c) := 35a_j^4 M_0^0 - 3\mathcal{N}_0^{(4,j)}$$
$$-\left(10\frac{a_j^2}{(a^2 + b^2 + c^2)} - 1\right)\left[5(a^4 + b^4 + c^4)M_0^0\right.$$
$$\left.-3\sum_{n=1}^{3}\widetilde{\mathcal{M}}_0^{(4,n)}\right] \rightleftharpoons 0, \quad j = 1,2,3 \quad (82)$$

It is mentioned that $\sum_{n=1}^{3}\widetilde{\mathcal{M}}_0^{(4,n)} = \frac{1}{7}\sum_{n=1}^{3}\widetilde{\mathcal{N}}_0^{(4,n)}$. We are now in position to construct the simpler functional corresponding to our fourth degree approximating surface, which incorporates information taken from $\mathcal{H}_n$, $n = 0,2,4$ and involving only the Rayleigh approximation:

$$h(a,b,c) = \epsilon_0 h_0^2(a,b,c) + \epsilon_1 h_1^2(a,b,c)$$
$$+\epsilon_2 \sum_{j=1}^{3} h_{2,j}^2(a,b,c). \quad (83)$$

The term $h_0(a,b,c) = Aabc - \frac{4\pi}{3}(M_0^0)^3$, with $A = -\frac{6\sqrt{2}\Gamma(\frac{5}{4})^2}{\Gamma(\frac{-1}{4})\Gamma(\frac{7}{4})}$, imposes the volumetric equivalence and its

stabilizing role is important in degenerate cases as will be clarified in Section IV. In the majority of cases taking $\epsilon_0 = 0$ does not alter the minimization outcome.

Additional terms would also incorporated to the functional above. These terms are provided by moments hidden in components $\mathcal{H}_n$, $n > 4$. We avoid, as in the case of the ellipsoid, "odd" moments since for scatterers fitted well by manifolds with inversion symmetry, these moments are small enough -due to their almost odd integrands- and are not expected to offer strong constraints to the total functional. Then the component $\mathcal{H}_6$ is our next target for mining useful moments. The participant term M_2^4 expanded in spherical harmonics, with respect to both arguments $\hat{\mathbf{r}}, \hat{\mathbf{k}}$, offers the -recoverable from the far-field pattern- set of moments $\mathcal{N}_2^{(2,j,i)} = \frac{1}{4\pi}\int_{\partial D} h_{4,i}(\mathbf{r})\frac{\partial A_{2,j}}{\partial n}(\mathbf{r})ds(\mathbf{r})$, $i,j = 1,2,3$ where $h_{4,i}(\mathbf{r})$ are the harmonic kernels $35x_i^4 - 30x_i^2 r^2 + 3r^4$ along with the moments $\widetilde{\mathcal{M}}_2^{(2,j,n)} = \frac{1}{4\pi}\int_{\partial D}(x_n^4 + x_{n+1}^4 - 6x_n^2 x_{n+1}^2)\frac{\partial A_{2,j}}{\partial n}(\mathbf{r})ds(\mathbf{r})$, $j,n = 1,2,3$ (adopting modulo 3 index symbolism). We construct the auxiliary field $\psi = \sum_j \frac{A_{2,j}}{a_j^2}$, which involves in symmetric equiv-

alence the terms $\frac{x_i^2}{a_i^2}$, as can be easily verified by evoking the integral representation (63). We easily find that $\frac{1}{4\pi}\int_{\partial D}\frac{\partial\psi}{\partial n}(\mathbf{r})ds(\mathbf{r}) = \sum_{j=1}^{3}\frac{\mathcal{N}_0^{(2,j)}}{a_j^2}$. The integration process based on (45) and the measure $\frac{\partial\psi}{\partial n}$, combined with

the implication of the usual symmetry arguments for handling the a priori involved non measurable moments, leads to the additional functional

$$h_{2,i}^{(2)}(a,b,c) := 35a_i^4\sum_{j=1}^{3}\frac{\mathcal{N}_0^{(2,j)}}{a_j^2} - 3\sum_{j=1}^{3}\frac{\mathcal{N}_2^{(2,j,i)}}{a_j^2}$$

$$-\left(10\frac{a_i^2}{(a^2+b^2+c^2)}-1\right)\left[5(a^4+b^4+c^4)\sum_{j=1}^{3}\frac{\mathcal{N}_0^{(2,j)}}{a_j^2} - 3\sum_{j=1}^{3}\sum_{n=1}^{3}\frac{\widetilde{\mathcal{M}}_2^{(2,j,n)}}{a_j^2}\right] \rightleftharpoons 0,\ j=1,2,3 \qquad (84)$$

The enhanced functional, subject to minimization, becomes

$$\tilde{h}(a,b,c) = \epsilon_0 h_0^2(a,b,c) + \epsilon_1 h_1^2(a,b,c)$$
$$+\epsilon_2\sum_{j=1}^{3}h_{2,j}^2(a,b,c) + \epsilon_3\sum_{i=1}^{3}(h_{2,i}^{(2)}(a,b,c))^2. \qquad (85)$$

Several additional minimizing terms would be implemented, extracted all from $\mathcal{H}_6$, but it is not the aim of this work to present all this stuff here. As a matter of fact, only quantitative burden emerges while no significant qualitative difference arises.

Special attention must be paid in case that the fitting surface does not have the simple form (45) but is selected to be a general close surface of fourth degree. Several possible representation arise, among whom some are handled via only the first set of measurable moments while the rest demand the implication of the second set of moments. We start with the indicative case of a closed surface, whose cartesian representation is based on a homogeneous polynomial of fourth degree, as follows:

$$\left(\frac{x^2}{a_{11}^2}+\frac{y^2}{a_{12}^2}\right)^2 + \left(\frac{y^2}{a_{22}^2}+\frac{z^2}{a_{23}^2}\right)^2 + \left(\frac{z^2}{a_{33}^2}+\frac{x^2}{a_{31}^2}\right)^2 = 1 \qquad (86)$$

Clearly the representation (86) includes the studied above form (45) as a special case and can be considered as a generalization of it. However in this more general case, the interrelation between coefficients and moments becomes more complicated since the exploitation of the inherent symmetries is much more demanding. We are in position again to avoid the evocation of the second set of moment. We present here the induced implication and the process of the corresponding functional construction. We introduce index symbolism modulo 3 and work first with Rayleigh moments of order two. Exploiting the equivalence of the terms $\frac{x_i^2}{a_{ii}^2} + \frac{x_{i+1}^2}{a_{i,i+1}^2}$ for $i=1,2,3$ as participants in the constructed moments, we find that the moments $\frac{1}{4\pi}\int_{\partial D}\left(\frac{x_i^2}{a_{ii}^2}+\frac{x_{i+1}^2}{a_{i,i+1}^2}\right)\frac{\partial\Phi_0}{\partial n}ds$ are considered

to be equal for $i=1,2,3$. This equality leads to the following building term of the functional to be minimized:

$$h_1(a_{ij}) = \gamma_1\left[(\frac{1}{a_{22}^2}-\frac{1}{a_{12}^2})\widetilde{\mathcal{M}}_0^{(2,1)} + \frac{1}{a_{23}^2}\widetilde{\mathcal{M}}_0^{(2,2)}\right]$$
$$-\gamma_2\left[\frac{1}{a_{33}^2}\widetilde{\mathcal{M}}_0^{(2,2)} - \frac{1}{a_{12}^2}\widetilde{\mathcal{M}}_0^{(2,1)}\right] \rightleftharpoons 0, \qquad (87)$$

where $\gamma_1 = [\frac{1}{a_{11}^2}+\frac{1}{a_{12}^2}-\frac{1}{a_{31}^2}-\frac{1}{a_{33}^2}]$ and $\gamma_2 = [\frac{1}{a_{11}^2}+\frac{1}{a_{12}^2}-\frac{1}{a_{22}^2}-\frac{1}{a_{23}^2}]$.

Working similarly with the terms $r^2(\frac{x_i^2}{a_{ii}^2}+\frac{x_{i+1}^2}{a_{i,i+1}^2})$, we find that

$$\frac{1}{4\pi}\int_{\partial D}r^2\left(\frac{x_i^2}{a_{ii}^2}+\frac{x_{i+1}^2}{a_{i,i+1}^2}\right)\frac{\partial\Phi_0}{\partial n}ds \rightleftharpoons$$
$$\frac{1}{4\pi}\int_{\partial D}r^2\left(\frac{x_{i+1}^2}{a_{i+1,i+1}^2}+\frac{x_{i+2}^2}{a_{i+1,i+2}^2}\right)\frac{\partial\Phi_0}{\partial n}ds,$$
$$i=1,2 \qquad (88)$$

Beginning with the equation above, we can construct obviously two simple rational parametric functions $\mathcal{L}_n(a_{ij})$, $n=1,2$ such that

$$\int_{\partial D}r^2x_n^2\frac{\partial\Phi_0}{\partial n}ds \rightleftharpoons \mathcal{L}_n(a_{ij})\int_{\partial D}r^2x_1^2\frac{\partial\Phi_0}{\partial n}ds =$$
$$\mathcal{L}_n(a_{ij})\int_{\partial D}r^2x^2\frac{\partial\Phi_0}{\partial n}ds,\ n=2,3. \qquad (89)$$

Setting $\mathcal{L}_1(a_{ij})=1$, we remark that

$$\int_{\partial D}r^4\frac{\partial\Phi_0}{\partial n}ds \rightleftharpoons \left(\sum_{n=1}^{3}\mathcal{L}_n(a_{ij})\right)\int_{\partial D}r^2x^2\frac{\partial\Phi_0}{\partial n}ds \qquad (90)$$

Keeping on treating moments of fourth order, we remark that the integrals $\frac{1}{4\pi}\int_{\partial D}\left(\frac{x_i^4}{a_{ii}^4}+\frac{x_{i+1}^2}{a_{i,i+1}^4}\right.$ $\left.+2\frac{x_i^2}{a_{ii}^2}\frac{x_{i+1}^2}{a_{i,i+1}^2}\right)\frac{\partial\Phi_0}{\partial n}ds$ behave equivalently for $i=1,2,3$

and so referring to Eq.(86), we find that

$$\frac{1}{4\pi}\int_{\partial D}\left(\frac{x_i^4}{a_{ii}^4}+\frac{x_{i+1}^4}{a_{i,i+1}^4}+2\frac{x_i^2}{a_{ii}^2}\frac{x_{i+1}^2}{a_{i,i+1}^2}\right)\frac{\partial\Phi_0}{\partial n}ds\rightleftharpoons\frac{1}{3}M_0^0,$$
$$i=1,2,3 \qquad (91)$$

Consequently

$$(3\frac{1}{a_{ii}^4}+\frac{1}{a_{ii}^2a_{i,i+1}^2})\frac{1}{4\pi}\int_{\partial D}x_i^4\frac{\partial\Phi_0}{\partial n}ds+(3\frac{1}{a_{i,i+1}^4}$$
$$+\frac{1}{a_{ii}^2a_{i,i+1}^2})\frac{1}{4\pi}\int_{\partial D}x_{i+1}^4\frac{\partial\Phi_0}{\partial n}ds-\frac{1}{a_{ii}^2a_{i,i+1}^2}\widetilde{\mathcal{M}}_0^{(4,i)}\rightleftharpoons M_0^0,$$
$$i=1,2,3. \qquad (92)$$

Due to the positivity of the coefficients $(3\frac{1}{a_{ii}^4}+\frac{1}{a_{ii}^2a_{i,i+1}^2})$, $(3\frac{1}{a_{i,i+1}^4}+\frac{1}{a_{ii}^2a_{i,i+1}^2})$ and of the structure of the system (92), it is easily shown that the determinant of this system can not be zero. So there exist again simple rational parametric functions $\mathcal{R}_n(a_{ij})$ and $\mathcal{Q}_{nl}(a_{ij})$ such that

$$\int_{\partial D}x_n^4\frac{\partial\Phi_0}{\partial n}ds\rightleftharpoons\mathcal{R}_n(a_{ij})4\pi M_0^0$$
$$+\sum_{l=1}^3\mathcal{Q}_{nl}(a_{ij})\frac{1}{a_{ll}^2a_{l,l+1}^2}4\pi\widetilde{\mathcal{M}}_0^{(4,l)},\quad n=1,2,3. \quad (93)$$

Adding Eqs.(93) over n and doing simple analysis provides with

$$4\pi\sum_{n=1}^3\left[5\mathcal{R}_n(a_{ij})M_0^0+5\sum_{l=1}^3\mathcal{Q}_{nl}(a_{ij})\frac{1}{a_{ll}^2a_{l,l+1}^2}\widetilde{\mathcal{M}}_0^{(4,l)}\right.$$
$$\left.-\widetilde{\mathcal{M}}_0^{(4,n)}\right]-3\int_{\partial D}r^4\frac{\partial\Phi_0}{\partial n}ds\rightleftharpoons0 \qquad (94)$$

Alternatively, it holds that

$$\int_{\partial D}x_n^4\frac{\partial\Phi_0}{\partial n}ds\rightleftharpoons\frac{6}{7}\int_{\partial D}r^2x_n^2\frac{\partial\Phi_0}{\partial n}ds-\frac{3}{35}\int_{\partial D}r^4\frac{\partial\Phi_0}{\partial n}ds$$
$$+4\pi\frac{1}{35}\mathcal{N}_0^{(4,n)},\ n=1,2,3. \qquad (95)$$

Eq.(94) provides with the integral $\int_{\partial D}r^4\frac{\partial\Phi_0}{\partial n}ds$ in terms of measured moments and so the moments $\int_{\partial D}r^2x_n^2\frac{\partial\Phi_0}{\partial n}ds$ are reconstructible via Eqs.(89,90) as

$$\int_{\partial D}r^2x_n^2\frac{\partial\Phi_0}{\partial n}ds\rightleftharpoons\frac{\mathcal{L}_n(a_{ij})}{\sum_l\mathcal{L}_l(a_{ij})}\int_{\partial D}r^4\frac{\partial\Phi_0}{\partial n}ds \qquad (96)$$

Combining Eqs.(93,94,95,96), we determine the second class of three building terms of the minimization functional:

$$h_{2,n}(a_{ij})=\left(40-50\frac{\mathcal{L}_n(a_{ij})}{\sum_l\mathcal{L}_l(a_{ij})}\right)\left[\ \mathcal{R}_n(a_{ij})M_0^0\right.$$
$$\left.+\sum_{l=1}^3\mathcal{Q}_{nl}(a_{ij})\frac{1}{a_{ll}^2a_{l,l+1}^2}\widetilde{\mathcal{M}}_0^{(4,l)}\right]+\left(10\frac{\mathcal{L}_n(a_{ij})}{\sum_l\mathcal{L}_l(a_{ij})}\right.$$
$$\left.-1\ \right)\widetilde{\mathcal{M}}_0^{(4,n)}-\mathcal{N}_0^{(4,n)},\quad n=1,2,3 \qquad (97)$$

The suggested functional has the form

$$h(a_{ij})=\epsilon_0h_0^2(a_{ij})+\epsilon_1h_1^2(a_{ij})+\epsilon_2\sum_{n=1}^3h_{2,n}^2(a_{ij}).\ (98)$$

The situation becomes even harder in the most general case of a fourth degree closed fitting surface. We present here the most general form of a fourth degree surface disposing inversion symmetry. This manifold shares the following hybrid form

$$\alpha_1h_{4,1}(\mathbf{r})+\alpha_2h_{4,2}(\mathbf{r})+\alpha_3h_{4,3}(\mathbf{r})+\delta_1r^2(x^2-\frac{1}{3}r^2)+\delta_2r^2(y^2-\frac{1}{3}r^2)+\zeta r^4+\eta_1(x^2-\frac{1}{3}r^2)+\eta_2(y^2-\frac{1}{3}r^2)+\theta r^2=1$$
$$(99)$$

very reminiscent of the so called polar representation form[1]. Every term in this expression contains harmonic terms or products of harmonic terms with powers of the distance r. It is apparent that no inner symmetry can be detected any more in order to estimate suitably the non measurable moments. So we are obliged to follow the alternative but tough methodology based on the double moments calculus[5] in combination with the methodology described herein and connected with the N.F.Eq. regime.

We are not going to give all the details for the study of this general case since it is not possible to present all this stuff accompanied with the subsequent numerical implementation and since from now on the implication of the outcomes of the introductory work[5] becomes inevitable. However we would like to give the essence of the diversification of the method in the general case of the fitting surface (99) and explain how moments of second kind are incorporated in the algorithm. Our aim is not to stray to

moments of high degree (degree eight was the necessary upper bound in the previous approach[1]) but to design the algorithm with the minimum set of the required data. We begin, as usually, by "projecting" Eq.(99) on the surface measure $\frac{\partial \Phi_0}{\partial n}$ to obtain

$$g_1(\alpha_i, \delta_i, \zeta, \eta_i, \theta) := \alpha_1 \mathcal{N}_0^{(4,1)} + \alpha_2 \mathcal{N}_0^{(4,2)} + \alpha_3 \mathcal{N}_0^{(4,3)}$$
$$+\delta_1 \mathcal{N}_{0,2}^{(2,1)} + \delta_2 \mathcal{N}_{0,2}^{(2,2)} + \zeta N_{0,4}$$
$$+\eta_1 \widetilde{\mathcal{M}}_0^{(2,1)} + \eta_2 \widetilde{\mathcal{M}}_0^{(2,2)} + \theta N_{0,2} - M_0^0 \rightleftharpoons 0, \qquad (100)$$

where $\mathcal{N}_{0,2}^{(2,i)} = \frac{1}{4\pi} \int_{\partial D} r^2 (x_i^2 - \frac{1}{3} r^2) \frac{\partial \Phi_0}{\partial n} ds$, $N_{0,2}$ and $N_{0,4}$ are the undefined from data moments but estimated in[5].

As a result, in the minimization scheme (100) all the moments are measured or estimated and the unknown quantities are just the coefficients of the polynomial representation of the fitting surface. Usually, Eq.(100) has to be supplemented with similar relations giving light to the interrelation of these coefficients. The construction of these accompanying minimization terms is accomplished in this work in a totally different manner compared to the mechanism of formulation of the linear system[1]. The purpose is to restrict ourselves to information provided by the far-field coefficients $\mathcal{H}_n$, with order n not exceeding the degree 4 of the fitting surface. So instead of multiplying Eq.(99) with signed measures of the form $r^n Y_n^m \frac{\partial \Phi_0}{\partial n}$ ($n = 0, ...4 : |m| \le n$) and integrate over the scatterer to produce the remaining moment equations - as performed in the old approach[1] and activating even $\mathcal{H}_8$ - we apply exactly the "inverse" process: We divide the initial representation (99) by powers of distance r and perform then adequate functional projections over the surface. To this effort the arsenal of the null-field equations has to be exploited. As an example, dividing Eq.(99) with r^2, multiplying with $\frac{\partial \Phi_0}{\partial n}$ and integrating over ∂D, we obtain

$$g_2(\alpha_i, \delta_i, \zeta, \eta_i, \theta) := \sum_{i=1}^{3} \alpha_i \frac{1}{4\pi} \int_{\partial D} \frac{h_{4,i}(\mathbf{r})}{r^2} \frac{\partial \Phi_0}{\partial n} ds$$
$$+\sum_{i=1}^{2} \delta_i \widetilde{\mathcal{M}}_0^{(2,i)} + \zeta N_{0,2} + \sum_{i=1}^{2} \eta_i \frac{1}{4\pi} \int_{\partial D} \frac{x_i^2 - \frac{1}{3} r^2}{r^2} \frac{\partial \Phi_0}{\partial n} ds$$
$$+\theta M_0^0 - \frac{1}{4\pi} \int_{\partial D} \frac{1}{r^2} \frac{\partial \Phi_0}{\partial n} ds \rightleftharpoons 0.$$

It is apparent that $\frac{1}{4\pi} \int_{\partial D} \frac{x_i^2 - \frac{1}{3} r^2}{r^2} \frac{\partial \Phi_0}{\partial n} ds = (\chi_i - \frac{1}{3}) M_0^0$, where the dimensionless constants χ_i represent the mean values of $\frac{x_i^2}{r^2}$ over the surface and clearly $\sum_{i=1}^{3} \chi_i = 1$. These constants are estimated via the relations $\widetilde{\mathcal{M}}_0^{(2,i)} = (\chi_i - \frac{1}{3}) N_{0,2}$, i.e. $\chi_i = \frac{1}{3} + \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}}$. The integral $\mathcal{I}_1 = \frac{1}{4\pi} \int_{\partial D} \frac{1}{r^2} \frac{\partial \Phi_0}{\partial n} ds$ can be estimated as $\mathcal{I}_1 = \frac{1}{r}$ via the null-field equation (24). Furthermore the moments $\frac{1}{4\pi} \int_{\partial D} \frac{h_{4,i}(\mathbf{r})}{r^2} \frac{\partial \Phi_0}{\partial n} ds$ can be expressed, via the integral mean value theorem, as $\Gamma_i := 35 \chi_i^2 N_{0,4} - 30 \widetilde{\mathcal{M}}_0^{(2,i)} -$

$7 N_{0,2}$. Consequently g_2 becomes

$$g_2(\alpha_i, \delta_i, \zeta, \eta_i, \theta) := \sum_{i=1}^{3} \alpha_i \Gamma_i + \sum_{i=1}^{2} \delta_i \widetilde{\mathcal{M}}_0^{(2,i)} + \zeta N_{0,2}$$
$$+\sum_{i=1}^{2} \eta_i \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}} M_0^0 + \theta M_0^0 - (\frac{M_0^0}{N_{0,2}})^{1/2} \rightleftharpoons 0, \qquad (101)$$

where all the capital letter quantities are measured or constructed known terms. Further minimization components could be constructed. Indeed, dividing the representation (99) with r^4, projecting on the measure $\frac{\partial \Phi_0}{\partial n}$ and applying as usually the mean value integral calculus, we find that

$$g_3(\alpha_i, \delta_i, \zeta, \eta_i, \theta) := \sum_{i=1}^{3} \alpha_i E_i + \sum_{i=1}^{2} \delta_i \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}} M_0^0 + \zeta M_0^0$$
$$+\sum_{i=1}^{2} \eta_i \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}} (\frac{M_0^0}{N_{0,2}})^{1/2} + \theta (\frac{M_0^0}{N_{0,2}})^{1/2} - (\frac{M_0^0}{N_{0,2}})^{3/2} \rightleftharpoons 0,$$
$$(102)$$

where $E_i = [35(\frac{1}{3} + \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}})^2 - 30(\frac{1}{3} + \frac{\widetilde{\mathcal{M}}_0^{(2,i)}}{N_{0,2}}) + 3] M_0^0$. The global minimization functional till now obtains the form

$$g(\alpha_i, \delta_i, \zeta, \eta_i, \theta) := \sum_{j=1}^{3} \epsilon_j g_j^2(\alpha_i, \delta_i, \zeta, \eta_i, \theta). \quad (103)$$

Since we have nine unknowns, we expect six additional terms to participate in the form of the final objective function. There are several ways to fulfill this goal. These terms could, as example, emerge after "projecting" Eq.(99) on the signed measures $\frac{1}{r^{2k}}(x_j^2 - \frac{r^2}{3}) \frac{\partial \Phi_0}{\partial n}$ with $j = 1, 2$, $k = 1, 2, 3$ and treating similarly as above the produced surface integrals.

IV. NUMERICAL INVESTIGATION

In this section, we perform a sequence of shape reconstructions in the low-frequency regime, implementing the theoretical background that has been created herein. The investigation of the ellipsoidal case has an easy pillar in the existence of exact data concerning the far field components $\mathcal{H}_n$ themselves. In all the other cases we need to extract these data from the far-field pattern and then specific physical and geometrical parameters play an important role. We work in the range $k\alpha_c < 3.5$, where k is the acoustic wave number and α_c is a characteristic dimension of the scatterer. In physical terms, we work with frequencies between 8 and 80 kHz - a region involving audio and ultrasonic frequencies as well - and scatterers with characteristic dimensions of typical length not greater than 10 mm. We consider that the hosting environment is water, where the sound speed has the typical value of 1460 ms^{-1}. All the reconstructed sizes of crucial geometrical features of the problem, are

presented, for simplicity, as simple numbers but represent lengths measured in mm.

The first task of the numerical investigation is to testify the behavior of the suggested method in the case of exact data. This can be accomplished efficiently in the case of the reconstruction of the ellipsoidal scatterer. We formulate the objective functions f, h, g_0 and g given by Eqs.(44,60,62,75) respectively. We implement the minimization of these functionals in several indicative ellipsoidal cases using the Numerical Optimization process of Mathematica (NMinimize routine) [Wolfram Research (2004)]. More precisely, we have examined several cases among which we mention five characteristic examples: (i),(ii) the cases of the elongated ellipsoids with semi-axes $(a, b, c) = (6, 5, 2)$ and $(a, b, c) = (4, 3, 1)$ respectively, in order to test the efficiency of the method under the burden of the large aspect ratios, (iii) the moderate case $(a, b, c) = (4, 3, 2)$, (iv) the slightly perturbed spheroidal case $(a, b, c) = (2, 1 + 10^{-3}, 1)$ and (v) the "sphere" $(a, b, c) = (1 + 10^{-3}, 1, 1 - 10^{-3})$, the last two cases in order to examine geometrical stability. We begin by considering exact data that can be provided explicitly[20,24]. The optimization of the objective function $f(a, b, c)$ has been proved a very robust process, which is independent of the values of the semi-axes (extreme case or not). The minimization scheme gives always one and only minimizing solution, to any desired accuracy, independently of the declared range of the starting values of the variables or the extension of the region where the minimization searching takes place. Actually only the positiveness of the sought coefficients is necessary in the constraints of the NMinimize routine to guarantee obtaining one and only solution: the exact one. The same efficiency applies unaltered to the degenerate geometrical configurations (iv) and (v).

The investigation of the objective function $h(a, b, c)$ is sensitive to the magnitude of the aspect ratios, although it provides reliable results, without the need to predefine the minimization region. To validate the efficiency of this functional, we have examined the case of the ellipsoid $(a, b, c) = (4, 4 - 1/10, 4 - 2/10)$. The solution provided by the minimization is $(a_{min}, b_{min}, c_{min}) = (4.0006661, 3.9006832, 3.8007012)$, assigning zero value to the function h. We measure a satisfactory error of l^2-norm equal to 0.00118412 in the estimation of the semi-axes of the ellipsoid. However for the extreme case $(a, b, c) = (4, 3, 1)$, we obtain $(a_{min}, b_{min}, c_{min}) = (4.10995186, 3.14510799, 1.37539241)$ as the global minimum of the functional h with a considerable deviation in the estimation of the semi-axis c.

Special effort has been devoted to the numerical investigation of the functional $g_0(\alpha, \beta, \gamma, N_{0,2})$ introduced by Eq.(62) and constructed in principle for approximating general scatterers by ellipsoidal surfaces. All the numerical experiments - still working with exact data - have demonstrated the following interesting result. If we have at hand a good estimation for the radii of the inscribed and superscribed spheres of the ellipsoidal surface and impose this constraint to the variables α, β, γ of the NMinimize routine - the variable $N_{0,2}$ is simply assigned the lower bound $(M_0^0)^3$ by Proposition 7 - then the behavior of the objective function g_0 and the subsequent minimization results are exactly the same with that concerning the robust minimization process based on the functional f. To explain this special feature we consider the extreme case (i) with $(\alpha, \beta, \gamma) = (1/36, 1/25, 1/4)$. Demanding that $\alpha, \beta, \gamma \in [1/36, 1/4]$ leads to a minimization process with the characteristics encountered in the investigation of f. So imposing a 16-digit working precision, we obtain a minimum value 2.56261×10^{-24} of g_0 evaluated at the point $(\alpha_{min}, \beta_{min}, \gamma_{min}, (N_{0,2})_{min})$ with $\alpha_{min} = 0.027778$, $\beta_{min} = 0.039999$, $\gamma_{min} = 0.249999$ and $(N_{0,2})_{min} = 21.666667$, i.e. the expected exact solution. What really matters in the constraints is the radius of the inscribed sphere. Indeed, searching for example the coefficients in the interval $[1/3600, 1/4]$ instead of $[1/36, 1/4]$ and working with the same precision of 16 significant digits, we find just one minimization vector with elements differing from the previous ones only after the 13-digit position.

Till now all the information hidden in $\mathcal{H}_2$ has been used except the moment $\mathcal{B}$. This is interrelated with the implication of $\Phi_2^{(0)}$, which, in minimization terminology, activates the functional g (Eq.(75)) (with $\epsilon_1 = \epsilon_2^{(2)} = 0$) and introduces the additional variables $N_{0,4}, N_{2,2}^{(0)}$. Moreover, data offered by $\mathcal{H}_4$ are activated. It is interesting that the numerical investigation of g reveals the importance of the term $\mathcal{B}$. This moment imposes the inherent volume equivalence of the scatterer and does not share the weakness encountered in the treatment of g_0 alone, where the "size" of the inscribed sphere was a crucial parameter of the minimization process. So working with exact data for the case (iv) of the perturbed spheroid offers exactly one minimizing solution from which we pay attention on the three coefficients of the ellipsoidal surface that turn out to be $(\alpha_{min}, \beta_{min}, \gamma_{min}) = (0.25000243, 0.99800194, 1.00000134)$. This solution differs from the expected exact one only after the sixth important decimal point. No additional constraint is necessary except the positiveness of the coefficients α, β, γ and the usual restrictions imposed on $N_{0,2}, N_{0,4}$ by Proposition 7. Similarly, we treat the case (iii) and find again exactly one minimization solution offering the coefficients $\alpha_{min} = 0.062523$, $\beta_{min} = 0.11117$, $\gamma_{min} = 0.25001$, which apparently are slightly different from the original ones. Finally the treatment of the almost spherical case (v) has approved the geometrical stability of the method, since the solution obeys to any desired accuracy.

The same steps have been followed working with the complete functional g where the components $g_2^{(2)}$ and g_1 are included. Then further data from $\mathcal{H}_4$ are evoked, while the arsenal of $\mathcal{H}_3$-measurements is activated for the first time. The results are very similar in any particular case and only one interesting result should be mentioned here: The functional component g_1 is very

effective and could replace the term $g_2^{(0)}$ in the minimization functional. This is based on the special form of g_1, which incorporates the elements of the tensor $\widetilde{\mathbf{C}}_1 = \frac{1}{4\pi}\int_{\partial D}\mathbf{r}\frac{\partial \mathbf{A}}{\partial n}(\mathbf{r})ds(\mathbf{r})$. Especially the trace of this tensor substitutes the functionality of $\mathcal{B}$, since it also controls the volume of the scatterer via Theorem 8(i).

The next step is to testify the applicability of the method in the case of inexact data. In the introductory investigation[5], we developed a regularization technique extracting the far-field components $\mathcal{H}_n$ from the far-field measurements f_∞^δ in a stable manner. Here $\delta(>0)$ is the error level indicating the deviation of the measurements from the exact far-field pattern f_∞. As an example, in the simplified but instructive case of just one excitation $(\hat{\mathbf{r}}_0)$ and one observation direction $(\hat{\mathbf{k}}_0)$, the regularization technique leads to the construction of the elements $f_{n,\alpha}^\delta = \sum_{l=0}^{n}c_{n,l}^\alpha k^l$ - here we meet the stabilization parameter $\alpha = \alpha(n(\delta),\delta)$ entering the Tikhonov functional - such that $f_{n,\alpha}^\delta \rightarrow f_\infty$ and $(c_{n,0}^\alpha,...,c_{n,n}^\alpha,0,0,...) \rightarrow (\mathcal{H}_0,...,\mathcal{H}_i,\mathcal{H}_{i+1},...)$ as $\delta \rightarrow 0$. In this simple case, $\mathcal{H}_i$ stands for the single value $\mathcal{H}_i(\hat{\mathbf{r}}_0,\hat{\mathbf{k}}_0)$ for every $i = 1,2,3,....$ The relations in Appendix make clear that an error level δ is shifted from measurements to the first class moments, without significant change of order. The spherical case is the first test of stability. We pretend not recognizing the surface of the scatterer, not paying attention to the global symmetry of the data, and try to determine the best fitting ellipsoidal surface via minimizing $g = g_0 + g_2^{(0)}$ as well as the robust functional f. All the moments of first class can be analytically determined, a lot of them are zero (for example $\widetilde{\mathcal{M}}_0^{(2,i)} = 0,\ i = 1,2,3$) due to spherical symmetry, but we assign to all these - potentially accessible in measurements - moments an error of level δ as explained above. The performance of the minimization is illustrated for several noise levels. The minimization scheme gives always one and only vector minimizing solution independently of the declared range of the starting values of the variables or the extension of the region where the minimization searching takes place. The logarithm of the obtained accuracy is in linear dependence with the selected a priori working precision. In addition, focusing on the minimizing coefficients α,β,γ, we see that the relative error in their determination is proportional to the error level δ. What merits to be mentioned is that working generally with an error level $\delta = 10^{-n}$, it really matters, for example, if $\widetilde{\mathcal{M}}_0^{(2,1)}$ takes the value 10^{-n} or (-10^{-n}) but the difference of the minimizing solution is inscribed after the (n-1)-th significant digit of the coefficients α,β,γ. The same situation is met in the reconstruction of an ellipsoidal surface. For example, working with the functional f and referring to the ellipsoidal case (iii) we add a noise of level $\delta = 10^{-4}$ to the exact data (i.e. to the theoretically determined moments M_0^0 and $\widetilde{\mathcal{M}}_0^{(2,i)},\ i = 1,2,3$). Then running such a minimization routine with working precision 16, leads to the solution $(\alpha_{min},\beta_{min},\gamma_{min})= (0.06252, 0.11117, 0.25031)$ as-

signing a minimum value of 3.23055×10^{-20} to the objective function f. We see that the fourth decimal digit has been altered in γ, while α,β have been changed slightly after the fourth significant digit. Working with a noise of level $\delta = 10^{-3}$ we find the solution $(\alpha_{min},\beta_{min},\gamma_{min})= (0.06269, 0.11167, 0.25307)$ obeying to the same rule. The case $\delta = 10^{-2}$ corresponds to a significant level of noise, which ordinary should first be testified to satisfy the basic assumption that the signal to noise ratio is strictly bounded above one[5,25]. However, the minimization remains robust since it provides with the satisfactory solution $(\alpha,\beta,\gamma) = (0.064373, 0.11676, 0.28292)$ still complying with the general rule of accuracy. Till now, the exact data we have used are fully analytic. Moreover we repeated the same minimization methodology for scattering problems with synthetic data. We considered the inverse problem concerning the reconstruction of a spheroidal surface, whose data are outcome of numerical processes[26,27]. A multi-parametric analysis of the direct acoustic scattering problem has been presented therein, for a long range of wave numbers and a variety of eccentricities for the soft spheroidal surface. The advantage of the method developed in that works is the implication of arbitrary precision arithmetics, fact permitting to select the desired accuracy of the synthetic data in terms of the geometrical and physical characteristics of the problem. More precisely for the case of a spheroid with semi-axes $a = 4/3, b = c = 1$, we investigated the behavior of the objective function $g = g_0 + g_2^{(0)}$ for noise free synthetic data as well as for noisy measurements with several levels of noise: $\delta = 10^{-4}, 10^{-3}, 10^{-2}$. The same characteristics of the minimization process emerged, and the worst estimation for the semi-axes appeared for $\delta = 10^{-2}$, where we found $(a_{min}, b_{min}, c_{min}) = (1.3812, 1.0544, 1.023)$.

The next and last part of this Section concerns the approximation of non smooth specific scatterers by polynomial manifolds. We focus on the reconstruction of scatterers having the shape of rectangular parallelepipeds. Two alternative ways have been followed to produce synthetic data. The first one is based on the boundary element method[28] and on using the code interrelated with the work[29]. We adopt the simplest version of the method with constant elements and exploit the coincidence of the elements with the real boundary. All this methodology refers of course to the solution of the second kind integral equation produced after representing the scattered wave in terms of the double layer potential. The nodes of the discretization coincide with the centers of the elements and so all the nodes belong to the smooth part of the scatterer. The second method is found in[30] in conjunction with the background encountered in the works[31] and[32]. More precisely, in[30], the field scattered by a soft impenetrable isotropic scatterer illuminated by a low-frequency acoustic plane wave is expressed in terms of a single polarizability tensor which is a function of only the geometry of the particle. The mathematical formulation is specialized to the case of a rectangular parallelepiped and the numerical technique suggested in[30] has been modified

for acoustics and implemented for computing the tensor elements. Both methodologies have provided with very compatible synthetic data. More precisely we present here the case (i) of the parallelepiped with dimensions $(a, b, c) = (1, 2, 3)$ in order to reveal the influence of the anisotropic geometry, the case (ii) of the parallelepiped with axes $(a, b, c) = (1, 1, 4)$ in order to examine the action of the inherent geometrical symmetry and finally the case (iii) of the cube $(a, b, c) = (1, 1, 1)$ to face stability in geometrical degeneracy. We restrict our attention to determine the best fitting ellipsoidal as well as the best fourth-order surface given by Eq.(45) that approximates the unknown scatterer.

In the case (i) we gathered the following necessary synthetic data: $M_0^0 = 1.31641$, $\widetilde{\mathcal{M}}_0^{(2,1)} = 1.36678$, $\widetilde{\mathcal{M}}_0^{(2,2)} = 3.52876$, $\widetilde{\mathcal{N}}_0^{(4,1)} = 79.13539$, $\widetilde{\mathcal{N}}_0^{(4,2)} = -163.15908$ and $\widetilde{\mathcal{N}}_0^{(4,3)} = 62.09735$ (ignoring units). Trying to find the best fitting ellipsoid and working with the functional f given by Eq.(44), we find the unsatisfactory minimum value 0.00293 evaluated for ellipsoidal semi-axes $a_{min} = 0.05422$, $b_{min} = 1.78782$ and $c_{min} = 2.89729$. It is observed that we fail to find a reasonable ellipsoidal approximation of the scatterer and this is more clarified if we try to minimize the functional g given by Eq.(75) for any selection of weight coefficients. The minimum value becomes larger than 10 implying the incapability of any ellipsoid to comply with the given data. In contrast to that, we find very important results in the case of searching the fourth-order approximating surface of the form (45). We use the objective function h given by (83), with weight coefficients equal to one. We provide this functional with the aforementioned data and apply as usually the NMinimize routine. We obtain the minimizing solution $(a_{min}, b_{min}, c_{min}) = (1.0999, 2.0099, 3.0001)$, assigning the minimum value 8.3039×10^{-12} to the functional h. We verify easily that this solution corresponds to a "smoothed" parallelepiped-like surface fitting perfectly with the original one everywhere except of course at edges and corners where an interesting mollification is encountered.

We proceed to the second case (ii) where the necessary data are gathered again and given by : $M_0^0 = 1.139$, $\widetilde{\mathcal{M}}_0^{(2,1)} = 10^{-10}$, $\widetilde{\mathcal{M}}_0^{(2,2)} = 5.6951$, $\widetilde{\mathcal{N}}_0^{(4,1)} = 203.6501$, $\widetilde{\mathcal{N}}_0^{(4,2)} = 203.6502$ and $\widetilde{\mathcal{N}}_0^{(4,3)} = 22.9465$, where we immediately recognize the hidden symmetry. Trying

to find the best fourth-order approximating surface via the minimization of the functional h leads to the solution $(a_{min}, b_{min}, c_{min}) = (0.994, 0.994, 3.9493)$ with corresponding error equal to 6.232×10^{-12}. We have again reconstructed a perfectly fitting parallelepiped mollified at edges and corners. If in addition we try to find the best ellipsoidal surface fitting the scatterer (ii), we reconstruct the spheroid with semi-axes $(a_{min}, b_{min}, c_{min}) = (0.2587, 0.2587, 3.8816)$, making a maximal error of level 3×10^{-1}. We remark that the inherent symmetry of the data is inscribed in the equality $a_{min} = b_{min}$ and that the third semi-axis c_{min} approximates reasonably the $z-$dimension of the scatterer. However the semi-axes a_{min}, b_{min} underestimate significantly the size of the original scatterer and this reflects the unsatisfactory error of the minimization, which by its turn expresses the unfitness of the given data with the ellipsoidal assumption.

Finally we examine the third case (iii) of the cube where we examine the influence of the geometric degeneracy. An error δ of level 10^{-2} has been added to the solution of the direct scattering problem to destroy the total symmetry of the problem. The necessary polluted data are $M_0^0 = 0.72469$, $\widetilde{\mathcal{M}}_0^{(2,1)} = 0.0049$, $\widetilde{\mathcal{M}}_0^{(2,2)} = 0.0098$, $\widetilde{\mathcal{N}}_0^{(4,1)} = -0.1$, $\widetilde{\mathcal{N}}_0^{(4,2)} = -0.0045$ and $\widetilde{\mathcal{N}}_0^{(4,3)} = 0.1$. Minimizing the functional h given by Eq.(83), we reconstruct the mollified cubic surface $\frac{x^4}{(1.0034)^4} + \frac{y^4}{(1.0133)^4} + \frac{z^4}{(1.0133)^4} = 1$, while h attains the minimal value 9.438×10^{-9}.

Appendix A: Expression of moments in terms of measurements

We supplement here Eqs.(9, 22, 23) with the relations expressing first class moments, that are needed in this work, in terms of measurements. The derivation of these equations is of course constructive and can be found in the introductory stuff[5]. However this constructive analysis is not needed, after the fulfillment of its aim, since the validity of the following relations is established by simple verification starting with the r.h.s, using the definition of $\mathcal{H}_n$ and ending to the l.h.s of each equation.

1. The moments $\widetilde{\mathcal{M}}_0^{(3,j)}$, $j = 1, 2, ..., 7$ (stemmed from $\mathcal{H}_3$)

$$\frac{1}{4\pi}\int_{\partial D}(x^3-3xy^2)\frac{\partial\Phi_0}{\partial n}ds = \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\hat{\mathbf{k}}) + \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\hat{\mathbf{k}})$$

$$-2\mathcal{H}_3^{ant}(\hat{\mathbf{x}};\hat{\mathbf{k}}),\quad \frac{1}{4\pi}\int_{\partial D}(y^3-3yx^2)\frac{\partial\Phi_0}{\partial n}ds = \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\hat{\mathbf{k}})$$

$$-\sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\hat{\mathbf{k}}) - 2\mathcal{H}_3^{ant}(\hat{\mathbf{y}};\hat{\mathbf{k}}),\quad \frac{1}{4\pi}\int_{\partial D}(y^3-3yz^2)\frac{\partial\Phi_0}{\partial n}ds = -2\mathcal{H}_3^{ant}(\hat{\mathbf{y}};\hat{\mathbf{k}})$$

$$\sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}});\hat{\mathbf{k}}) + \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}-\hat{\mathbf{z}});\hat{\mathbf{k}}),\quad \frac{1}{4\pi}\int_{\partial D}(z^3-3zy^2)\frac{\partial\Phi_0}{\partial n}ds =$$

$$\sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}});\hat{\mathbf{k}}) - \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}-\hat{\mathbf{z}});\hat{\mathbf{k}}) - 2\mathcal{H}_3^{ant}(\hat{\mathbf{z}};\hat{\mathbf{k}}),$$

$$\frac{1}{4\pi}\int_{\partial D}(z^3-3zx^2)\frac{\partial\Phi_0}{\partial n}ds = \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}});\hat{\mathbf{k}}) + \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}-\hat{\mathbf{x}});\hat{\mathbf{k}})$$

$$-2\mathcal{H}_3^{ant}(\hat{\mathbf{z}};\hat{\mathbf{k}}),\quad \frac{1}{4\pi}\int_{\partial D}(x^3-3xz^2)\frac{\partial\Phi_0}{\partial n}ds = \sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}});\hat{\mathbf{k}})$$

$$-\sqrt{2}\mathcal{H}_3^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}-\hat{\mathbf{x}});\hat{\mathbf{k}}) - 2\mathcal{H}_3^{ant}(\hat{\mathbf{x}};\hat{\mathbf{k}}),\quad \frac{1}{4\pi}\int_{\partial D}xyz\frac{\partial\Phi_0}{\partial n}ds =$$

$$\frac{\sqrt{3}}{4}\left\{\mathcal{H}_3^{ant}(\frac{1}{\sqrt{3}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}-\hat{\mathbf{z}});\hat{\mathbf{k}}) - \mathcal{H}_3^{ant}(\frac{1}{\sqrt{3}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}});\hat{\mathbf{k}}) + \mathcal{H}_3^{ant}(\frac{1}{\sqrt{3}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}});\hat{\mathbf{k}})\right.$$

$$\left. -\mathcal{H}_3^{ant}(\frac{1}{\sqrt{3}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}-\hat{\mathbf{z}});\hat{\mathbf{k}})\right\},$$

(for arbitrary excitation direction $\hat{\mathbf{k}}$).

from $\mathcal{H}_4$)

2. The vector moments $\widetilde{\mathbf{M}}_1^{(3,j)}$, $j = 1, 2$ (stemmed

$$\frac{1}{4\pi}\int_{\partial D}(x^3-3xy^2)\frac{\partial\mathbf{A}}{\partial n}ds = \frac{1}{8}\left\{\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}))(\hat{\mathbf{x}}+\hat{\mathbf{y}})\right.$$

$$+\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))(\hat{\mathbf{x}}-\hat{\mathbf{y}}) + \sqrt{2}\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\hat{\mathbf{z}})\hat{\mathbf{z}}$$

$$+\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}))(\hat{\mathbf{x}}+\hat{\mathbf{y}}) + \sqrt{2}\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\hat{\mathbf{z}})\hat{\mathbf{z}}$$

$$+\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))(\hat{\mathbf{x}}-\hat{\mathbf{y}}) - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{x}};\hat{\mathbf{x}})\hat{\mathbf{x}} - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{x}};\hat{\mathbf{y}})\hat{\mathbf{y}} - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{x}};\hat{\mathbf{z}})\hat{\mathbf{z}}\right\}$$

$$\frac{1}{4\pi}\int_{\partial D}(y^3-3yx^2)\frac{\partial\mathbf{A}}{\partial n}ds = \frac{1}{8}\left\{\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}))(\hat{\mathbf{x}}+\hat{\mathbf{y}})\right.$$

$$+\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))(\hat{\mathbf{x}}-\hat{\mathbf{y}}) + \sqrt{2}\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\hat{\mathbf{z}})\hat{\mathbf{z}}$$

$$-\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}))(\hat{\mathbf{x}}+\hat{\mathbf{y}}) - \sqrt{2}\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\hat{\mathbf{z}})\hat{\mathbf{z}}$$

$$-\hat{\mathcal{H}}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))(\hat{\mathbf{x}}-\hat{\mathbf{y}}) - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{y}};\hat{\mathbf{x}})\hat{\mathbf{x}} - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{y}};\hat{\mathbf{y}})\hat{\mathbf{y}} - 2\hat{\mathcal{H}}_4^{ant}(\hat{\mathbf{y}};\hat{\mathbf{z}})\hat{\mathbf{z}}\right\},$$

where $\hat{\mathcal{H}}_n^{ant}(\hat{\mathbf{r}};\hat{\mathbf{k}}) = \frac{1}{2}\left(\mathcal{H}_n^{ant}(\hat{\mathbf{r}};\hat{\mathbf{k}}) - \mathcal{H}_n^{ant}(\hat{\mathbf{r}};-\hat{\mathbf{k}})\right)$. Reciprocity relations for the low-frequency components $\hat{\mathcal{H}}_4^{ant}$ can be used to prove the mirror ($\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{y}}$) relationship between the representations above. Following cyclic permutations of $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, we can define also the moments ($j = 3, 4, 5, 6$) $\frac{1}{4\pi}\int_{\partial D}(y^3 - 3yz^2)\frac{\partial\mathbf{A}}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}(z^3 - 3zy^2)\frac{\partial\mathbf{A}}{\partial n}ds$, $\frac{1}{4\pi}\int_{\partial D}(z^3 - 3zx^2)\frac{\partial\mathbf{A}}{\partial n}ds$ and $\frac{1}{4\pi}\int_{\partial D}(x^3 - 3xz^2)\frac{\partial\mathbf{A}}{\partial n}ds$. The last moment ($j = 7$)

$\frac{1}{4\pi}\int_{\partial D}xyz\frac{\partial\mathbf{A}}{\partial n}(\mathbf{r})ds(\mathbf{r})$ is not of use in this work.

3. The moments $\widetilde{\mathcal{M}}_2^{(m,2,j)}$, $|m|\leq 2$, $j=1$ (stemmed from $\mathcal{H}_4$)

$$2\int_{\partial D}(x^2-y^2)\Re(\frac{\partial A_2^{(2,2)}}{\partial n})ds = (\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\hat{\mathbf{x}})-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\hat{\mathbf{x}}))-(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\hat{\mathbf{y}})$$

$$-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\hat{\mathbf{y}})),\quad 2\int_{\partial D}(x^2-y^2)\Im(\frac{\partial A_2^{(2,2)}}{\partial n})ds = (\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}))$$

$$-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})))-(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}))),$$

$$4\int_{\partial D}(x^2-y^2)\frac{\partial A_2^{(2,0)}}{\partial n}ds = \int_{\partial D}(x^2-y^2)\Re(\frac{\partial A_2^{(2,2)}}{\partial n})ds$$

$$+(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\hat{\mathbf{z}})-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\hat{\mathbf{z}}))-(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\hat{\mathbf{x}})-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\hat{\mathbf{x}})),$$

$$\int_{\partial D}(x^2-y^2)\Re(\frac{\partial A_2^{(2,1)}}{\partial n})ds = \left(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{z}}))\right.$$

$$\left.-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{z}}))\right)-\left(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{z}}))-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{z}}))\right),$$

$$\int_{\partial D}(x^2-y^2)\Im(\frac{\partial A_2^{(2,1)}}{\partial n'})ds = \left(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}}))\right.$$

$$\left.-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}}))\right)-\left(\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{x}};\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}-\hat{\mathbf{z}}))-\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{y}};\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}-\hat{\mathbf{z}}))\right),$$

where $\tilde{\mathcal{H}}_4^{sym}(\hat{\mathbf{r}};\hat{\mathbf{k}}) = \frac{1}{2}\left(\mathcal{H}_4^{sym}(\hat{\mathbf{r}};\hat{\mathbf{k}})+\mathcal{H}_4^{sym}(\hat{\mathbf{r}};-\hat{\mathbf{k}})\right)$. Cyclic permutation of the observation directions lead to the determination of moments representing "pro-jections" on the harmonic y^2-z^2 (the case $j=2$). The remaining cases ($j=3,4,5$) referring to the kernel functions xy, yz, zx are not used in the present work.

4. The moments $\widetilde{\mathcal{M}}_0^{(4,j)}$, $j=1,2,3$ (stemmed from $\mathcal{H}_4$)

$$\frac{1}{2}\frac{1}{4\pi}\int_{\partial D}(x^4+y^4-6x^2y^2)\frac{\partial\Phi_0}{\partial n}ds = \mathcal{H}_4^{ant}(\hat{\mathbf{x}};\hat{\mathbf{k}})+\mathcal{H}_4^{ant}(\hat{\mathbf{y}};\hat{\mathbf{k}})-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}});\hat{\mathbf{k}})$$

$$-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}});\hat{\mathbf{k}}),\quad \frac{1}{2}\frac{1}{4\pi}\int_{\partial D}(y^4+z^4-6y^2z^2)\frac{\partial\Phi_0}{\partial n}ds = \mathcal{H}_4^{ant}(\hat{\mathbf{y}};\hat{\mathbf{k}})+\mathcal{H}_4^{ant}(\hat{\mathbf{z}};\hat{\mathbf{k}})$$

$$-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}});\hat{\mathbf{k}})-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{y}}-\hat{\mathbf{z}});\hat{\mathbf{k}}),\quad \frac{1}{2}\frac{1}{4\pi}\int_{\partial D}(z^4+x^4-6z^2x^2)\frac{\partial\Phi_0}{\partial n}ds$$

$$= \mathcal{H}_4^{ant}(\hat{\mathbf{z}};\hat{\mathbf{k}})+\mathcal{H}_4^{ant}(\hat{\mathbf{x}};\hat{\mathbf{k}})-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}});\hat{\mathbf{k}})-\mathcal{H}_4^{ant}(\frac{1}{\sqrt{2}}(\hat{\mathbf{z}}-\hat{\mathbf{x}});\hat{\mathbf{k}}),$$

(for arbitrary excitation direction $\hat{\mathbf{k}}$ as all the Rayleigh moments).

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