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On the dyadic scattering problem in three-dimensional gradient elasticity: an analytic approach

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Abstract

The investigation of the direct scattering problem of an elastic dyadic incident field from a spherical inclusion, is the main outcome of this work, in the case where the scatterer and the host environment dispose microstructure. The framework of the method is based on the implication of Mindlin's gradient theory. The development of the method is fully analytic and gives successively several byproducts, which are indispensable for the solution of the scattering problem but constitute also independent results of their own theoretical and practical value. So the numerable set of Navier eigendyadics is constructed, which is proved to be a basis for every dyadic field obeying the dynamic gradient elasticity equation. This permits the construction of a useful spectral representation for every gradient elasticity field. Furthermore, the set of dyadic spherical harmonics is built, which stands for the extension of the well-known spherical vector harmonics to the dyadic realm. Every dyadic field restricted on the unit sphere can be expanded in terms of these spherical dyadic harmonics. The orthogonality relations of these functions are determined in close form and this is the prerequisite for the fully analytic treatment of the boundary conditions involving the scattering problem under consideration.

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1. Introduction

The classical theory is proved inadequate to predict the suitable mechanical behavior of linear elastic materials when they exhibit a specific type of microstructure. Granular materials, polymers, liquid crystals, porous media, solids with micro cracks, dislocations and disclinations, composites are characteristic cases in which micro continuum field theories are necessary to be implemented to give the appropriate investigation framework.

The concept of the microcontinuum naturally arises when wave phenomena are studied in materials with microstructure. This concept is interrelated with the notion of length and time scale in the involved wave fields. The response of an elastic body to the incidence of an elastic field is influenced heavily by the ratio of the characteristic wavelength λ, which is a special feature of the external wave stimulus, to the internal characteristic length l. In the case that $\frac{\lambda}{l} \gg 1$, the classical elasticity theory gives reliable predictions since a large number of particles act in collaboration. In contrast, when $\frac{\lambda}{l} \approx 1$, the response of the several subcontinua (particles) becomes important, so that the axiom of locality underlying classical theory fields fails. As an example, the classical theory of elasticity is well known to predict two non-dispersive waves (the longitudinal and the transverse one) whose short wavelength behavior departs essentially from experimental observations.

To remedy the non-locality of the involved elastic fields, several approaches have been followed modeling the microstructural effects in a macroscopic manner by introducing higher-order strain gradient, micropolar and couple-stress theories. We note here the principal contributions of Mindlin and co-workers [1–3], Aifantis and co-workers [4–7] and Vardoulakis and co-workers [8, 9] in connection with the higher-order strain theories. In addition one may pay attention to the contribution of Eringen and co-workers [10–12] pertaining to the micropolar theories, as well as [13–16] in connection with the couple-stress theories.

The implication of non-classical theories in dynamic problems handling wave propagation in beams and half-space has been proved very promising [17–19] by achieving the elimination of singularities or discontinuities of classical theory and capturing the expected size effects and wave dispersion in cases where, as mentioned before, this was not possible by classical elasticity methods means. A special study of elastic waves propagation incorporating enhanced theories is attributed to Georgiadis and his co-authors [20, 21].

In the present work, we examine the scattering problem arising when an arbitrary elastic field (plane wave in preference) is traveling in a medium with microstructure, which hosts a spherical elastic inclusion with different elastic macro and micro-parameters. The framework of the method is based on the implication of Mindlin's gradient theory [1]. We mention here a relevant study, instead concerning the micropolar materials case, exposed in [22] and the relevant references therein. Apart from the different field model, the framework followed in this work is totally diversified due to the fact that the methodology is developed in the dyadic framework which suits perfectly to the elasticity realm [23]. As a consequence, the stimulus of the scattering process as well as the scattered field are supposed to be dyadic elastic fields. As a first consequence, the dyadic incident field incorporates simultaneously all the possible excitation orientations and polarizations. As a second consequence, the setting of this work can be used unaltered to provide with the construction of the dyadic Green function of the system: host medium-spherical inclusion.

Several byproducts arise along the present work, which constitute generalizations of well-known important features of classical elasticity. The paper is organized as follows. In section 2, the boundary value problem in dynamic gradient elasticity is studied in the dyadic formulation. The ultimate result is the examination of the aforementioned BVP in the time harmonic framework and under the introduction of spherical coordinates. The outcomes are several and we pay attention here to the construction of the set of dyadic Navier eigenfunctions which constitutes the generalization of the Navier eigenvectors. This set is important since every interior or exterior (radiating or attenuated) dyadic field of gradient elasticity can be expressed as a countable expansion in terms of the elements of this set. This fact is a fruitful spectral representation of solutions of the dynamic equation of gradient elasticity in spherical coordinates. Special issues of these eigensolutions concerning mainly their independence are presented in appendix B. It is important to note that the restriction of the Navier eigendyadics on spherical surfaces gives birth to the dyadic harmonics, which constitute the generalization of the well-known vector spherical harmonics. This set is also a basis in the sense that the trace of every dyadic field on a spherical surface can be represented as a countable superposition of dyadic spherical harmonics. In section 3, the scattering problem by a spherical penetrable elastic body with microstructure is studied. The incident, interior and scattered fields are expanded in terms of the Navier eigendyadics and are forced to satisfy the boundary conditions of gradient elasticity on the scatterer surface. Exploiting the orthogonality relations of the dyadic spherical harmonics, which are exposed in appendix A, we are in position to decouple the countable expansions of the involved fields and to acquire for every particular index of the spectral representation a well-formed linear system with coefficients presented in appendix C and unknowns the expansion coefficients of the relevant fields corresponding to the specific index. Potentially then we are in position to determine the expansions of the participating fields in the scattering process by calculating the coefficients of the spectral representations by solving finite linear algebraic systems whose production is fully analytical and exact and is not based on truncation processes, thanks to the exploitation of orthogonality results of the herein constructed basis of dyadic special functions.

2. Dyadic formulation and spectral analysis in dynamic gradient elasticity

Before introducing the dyadic formalism, let us recall, for the reader's convenience, some fundamental concepts related to kinematics of elastic bodies with microstructure, selecting as a guide reference the famous work of Mindlin [1]. Let us consider a three-dimensional linear, gradient elastic body, in which we pay attention on a material volume V confined by a surface S, which is geometrically characterized by its normal vector $\widehat{\mathbf{n}}$, which for simplicity is taken to be a continuous vector field, fact reflecting the smoothness of the boundary S. As in [1], we denote by X_i, $i = 1, 2, 3$, the rectangular components of the material position vector, measured from a fixed origin, and by x_i the components, in the same rectangular frame, of the spatial position vector. The components of displacement of a material particle are defined as

$$u_i \equiv x_i - X_i. \tag{1}$$

Embedded in each material particle there is assumed to be a micro-volume V' in which X_i' and x_i' are the components with respect to a Cartesian system, having parallel axis with the unprimed one and moving with the displacement. The micro-displacement is so introduced with components

$$u_i' \equiv x_i' - X_i'. \tag{2}$$

The absolute values of the displacement gradients are assumed to be small in comparison with unity. These gradients play an important role in the dynamic behavior of the elastic medium. We mention here the usual strain (now the macro-strain)

$$\epsilon_{ij} \equiv \tfrac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{3}$$

in accordance with the micro-strain

$$\psi_{(ij)} \equiv \tfrac{1}{2}(\psi_{ij} + \psi_{ji}), \tag{4}$$

where ψ_{ij} stands for the micro-deformation $\psi_{ij} \equiv \partial_i' u_j'$, which is assumed to be homogeneous in the micro-medium V' and non-homogeneous in the macro-medium V. The antisymmetric part of the micro-deformation is the micro-rotation $\psi_{[ij]} \equiv \frac{1}{2}(\psi_{ij} - \psi_{ji})$, reminiscent of the well-known macro-rotation $\omega_{ij} \equiv \frac{1}{2}(\partial_i u_j - \partial_j u_i)$. We finally introduce the relative deformation $\gamma_{ij} \equiv \partial_i u_j - \psi_{ij}$ and the micro-deformation gradient (the macro-gradient of the micro-deformation) as $\kappa_{ijk} = \partial_i \psi_{jk}$. All three of the tensors $\epsilon_{ij}, \gamma_{ij}$ and κ_{ijk} are proved to constitute the independent arguments of the potential energy function U (potential energy per unit-macrovolume) referring to the elastic medium under consideration, i.e.,

$$U = U(\epsilon_{ij}, \gamma_{ij}, \kappa_{ijk}). \tag{5}$$

The variation of the potential with respect to the aforementioned arguments gives birth to the corresponding stresses. In fact, we obtain the 'macroscopic' symmetric Cauchy stress τ_{ij} by differentiating the energy with respect to the macro-strain ϵ_{ij} (i.e., $\tau_{ij} \equiv \frac{\partial U}{\partial \epsilon_{ij}}$) and similarly we define the relative stress as $\sigma_{ij} \equiv \frac{\partial U}{\partial \gamma_{ij}}$ and the double sress as $\mu_{ijk} \equiv \frac{\partial U}{\partial \kappa_{ijk}}$. Then the variation of the potential energy obtains the form

$$\delta U = \tau_{ij}\delta\epsilon_{ij} + \sigma_{ij}\delta\gamma_{ij} + \mu_{ijk}\delta\kappa_{ijk}. \tag{6}$$

This is the most general form for the variation of potential energy. However, in the framework of the long-wavelength approximation, the simplifying form II of Mindlin [1] emerges. In this approach, and using arguments pertaining to similarities with the wave propagation in homogeneous plates, Mindlin considers vanishing of the antisymmetric relative deformation $\gamma_{[ij]}$ and of the symmetric relative stress $\sigma_{(ij)}$. In addition, using the underlying constitutive relations, the variable κ_{ijk} can be expressed (and so replaced) by the new tensor $\hat{\kappa}_{ijk} \equiv \partial_i \epsilon_{jk}$, whose 'dual' double stress tensor is now $\hat{\mu}_{ijk} \equiv \frac{\partial U}{\partial \hat{\kappa}_{ijk}}$. So the variation of the energy becomes

$$\delta U = \tau_{ij}\delta\epsilon_{ij} + \hat{\mu}_{ijk}\delta\hat{\kappa}_{ijk}. \tag{7}$$

We adopt now the dyadic elastic formulation dealing with dyadic displacements instead of vector ones. We introduce the dyadic displacement field $\widetilde{u} = \mathbf{u}_l \otimes \widehat{\mathbf{x}}_l$, where $\mathbf{u}_l = u_{jl}\widehat{\mathbf{x}}_j$ are the Cartesian vector components. The macro-strain tensor $\widetilde{e} = \frac{1}{2}(\nabla\widetilde{u} + (\nabla\widetilde{u})^{213})$ is now a triadic and can be expanded in the form $e_{ijl} = \frac{1}{2}(\partial_i u_{jl} + \partial_j u_{il})$. The symbol $(\circ)^{213}$ denotes a specific permutation of vector arguments, i.e. $(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})^{213} = \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c}$ and similar notation holds for tensors of higher order. The Cauchy stress tensor $\widetilde{\boldsymbol{\tau}}$, which is the dual tensor to a strain field in the energy functional is also a triadic. The gradient of the strain $(\nabla\widetilde{e})$ becomes now a dyadic of fourth order and this property is inherited by its dual dyad in the energy dual pairing, i.e. the double stress tetradic: $\widetilde{\boldsymbol{\mu}} = \mu_{ijkl}\widehat{\mathbf{x}}_i \otimes \widehat{\mathbf{x}}_j \otimes \widehat{\mathbf{x}}_k \otimes \widehat{\mathbf{x}}_l$. (We omit the hat symbol over the double stresses for simplicity). The constitutive relations connecting strains and stresses are given in [1] and reflect the simplest possible assumption of a homogeneous, isotropic, quadratic (in terms of its arguments) energy density. More precisely, omitting in the sequel the juxtaposition symbol $\otimes$ in tensor notation, we express the Cauchy and double stresses as follows:

$$\widetilde{\boldsymbol{\tau}} = 2\mu\widetilde{e} + \lambda\widetilde{\mathbf{I}}(\nabla \cdot \widetilde{u}) = \mu(\nabla\widetilde{u} + (\nabla\widetilde{u})^{213}) + \lambda\widetilde{I}(\nabla \cdot \widetilde{u}), \tag{8}$$

and

$$\begin{aligned}\widetilde{\boldsymbol{\mu}} = \tfrac{1}{2}a_1[(\widetilde{\mathbf{I}}\Delta\widetilde{u})^{3124} + \widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u} + (\widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u})^{3124} + (\widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u})^{1324}]\\ + 2a_2(\widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u})^{3124} + \tfrac{1}{2}a_3[\widetilde{\mathbf{I}}\Delta\widetilde{u} + \widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u} + (\widetilde{\mathbf{I}}\Delta\widetilde{u})^{1324} + (\widetilde{\mathbf{I}}\nabla\nabla\cdot\widetilde{u})^{1324}]\\ + a_4[\nabla\nabla\widetilde{u} + (\nabla\nabla\widetilde{u})^{2314}] + \tfrac{1}{2}a_5[2(\nabla\nabla\widetilde{u})^{3124} + \nabla\nabla\widetilde{u} + (\nabla\nabla\widetilde{u})^{2314}],\end{aligned} \tag{9}$$

where $\mu = \mu_{\text{mac}} - \frac{2g_2^2}{b_2+b_3}$, $\lambda + 2\mu = \lambda_{\text{mac}} + 2\mu_{\text{mac}} - \frac{8g_2^2}{3(b_2+b_3)} - \frac{(3g_1+2g_2)^2}{3(3b_1+b_2+b_3)}$. The parameters $\lambda_{\text{mac}}, \mu_{\text{mac}}$ are the Lame constants of the macroscopic elastic material, while $a_i, i = 1, \ldots, 5, g_i, i = 1, 2$ and $b_i, i = 1, 2, 3$ are constitutive parameters due exclusively to the presence of the microstructure. The operators $\widetilde{\boldsymbol{\tau}}$ and $\widetilde{\boldsymbol{\mu}}$ merit symmetry properties. More precisely, we have that $\tau_{ijk} = \tau_{jik}$ and $\mu_{ijkl} = \mu_{ikjl}$. Apart from the artificial last index introduced via the dyadic formulation, the specific symmetry of the other indices is typical in the framework of Mindlin theory referring to form II.

The variation of the potential energy U_V stored in region V is given by the formula

$$\delta U_V = \int_V \delta U \, \mathrm{d}\mathbf{r} = \int_V [\widetilde{\boldsymbol{\tau}} \,\dot{:}\, \delta \widetilde{\mathbf{e}} + (\widetilde{\boldsymbol{\mu}})^{4321} :: \delta \nabla \widetilde{\mathbf{e}}] \, \mathrm{d}\mathbf{r}, \tag{10}$$

where the dots represent tensor contractions.

Adapting suitably the already established vectorial analysis in [24] to the dyadic displacement case, it is proved that the variation of the energy functional is

$$\begin{aligned}\delta U_V = &- \int_V [\nabla \cdot (\widetilde{\boldsymbol{\tau}} - \nabla \cdot \widetilde{\boldsymbol{\mu}})] : \delta \widetilde{u}^T \mathrm{d}\mathbf{r} + \int_S [\widehat{\mathbf{n}} \cdot \widetilde{\boldsymbol{\mu}}^{1243} \cdot \widehat{\mathbf{n}}] : [\widehat{\mathbf{n}} \cdot \nabla(\delta \widetilde{u}^T)] \, \mathrm{d}S \\ &+ \int_S \left[\widehat{\mathbf{n}} \cdot \widetilde{\boldsymbol{\tau}} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} : \frac{\partial \widetilde{\boldsymbol{\mu}}}{\partial n} - \widehat{\mathbf{n}} \cdot (\nabla_S \cdot \widetilde{\boldsymbol{\mu}}) - \widehat{\mathbf{n}} \cdot (\nabla_S \cdot \widetilde{\boldsymbol{\mu}}^{2134}) \right] : \delta \widetilde{u}^T \, \mathrm{d}S \\ &+ \int_S [(\nabla_S \cdot \widehat{\mathbf{n}}) \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - (\nabla_S \widehat{\mathbf{n}})] : \widetilde{\boldsymbol{\mu}} : \delta \widetilde{u}^T \, \mathrm{d}S, \end{aligned} \tag{11}$$

where $\nabla_S = (\widetilde{\mathbf{I}} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}}) \cdot \nabla = \nabla - \widehat{\mathbf{n}} \frac{\partial}{\partial n}$ is the surface gradient operator.

The differential equations along with the accompanying boundary conditions for the dynamic gradient elasticity problem are produced via the application of Hamilton's principle

$$\delta \int_{t_0}^{t_1} (K_V - U_V) \, \mathrm{d}t + \int_{t_0}^{t_1} \delta W_V \, \mathrm{d}t = 0. \tag{12}$$

In the previous relation, K_V is the total kinetic energy of the medium occupying the volume V, while δW_V is the variation of the work done by external forces expressed as [1, 3]

$$\delta W_V = \int_V \widetilde{f} : \delta \widetilde{u}^T \, \mathrm{d}\mathbf{r} + \int_S \widetilde{R} : \widehat{\mathbf{n}} \cdot \nabla(\delta \widetilde{u}^T) \, \mathrm{d}S + \int_S \widetilde{P} : \delta \widetilde{u}^T \, \mathrm{d}S, \tag{13}$$

where $\widetilde{f}$ denotes body forces, $\widetilde{P}$ surface tractions and $\widetilde{R}$ stands for surface double stresses.

Combining equations (11)–(13) and adopting a suitable form for the kinetic energy, we can construct, from the variational formulation, the governing differential equations along with the accompanying boundary conditions. There are several models for the kinetic energy depending on the nature of the microstructure. Following here the approach of Mindlin, where there exists a non-negligible contribution to the macro-velocity tensor from the micro-velocity field, whose contribution is actually proportional to the macro-velocity gradients, we infer that the differential equation of the gradient elasticity problem is

$$\nabla \cdot (\widetilde{\boldsymbol{\tau}} - \boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{\mu}}) + \widetilde{f} = \rho \frac{\partial^2 \widetilde{u}}{\partial t^2} - \nabla \cdot \left(\rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \nabla \widetilde{u} \right), \qquad \mathbf{r} \in V, \tag{14}$$

where ρ' stands for the mass of micro-material per unit macro-volume and $\widetilde{D}$ is a specific tensor of fourth order depending on the physical and geometrical parameters of the microstructure.

The corresponding boundary conditions are the following:

(i) classical BCs

$$\begin{aligned}\widetilde{P}(\mathbf{r}) &= \widehat{\mathbf{n}} \cdot \widetilde{\tau}(\mathbf{r}) - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} : \frac{\partial \widetilde{\boldsymbol{\mu}}}{\partial n}(\mathbf{r}) - \widehat{\mathbf{n}} \cdot (\nabla_S \cdot \widetilde{\boldsymbol{\mu}}(\mathbf{r})) - \widehat{\mathbf{n}} \cdot (\nabla_S \cdot \widetilde{\boldsymbol{\mu}}^{2134}(\mathbf{r})) \\ &\quad + [(\nabla_S \cdot \widehat{\mathbf{n}}) \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - (\nabla_S \widehat{\mathbf{n}})] : \widetilde{\boldsymbol{\mu}}(\mathbf{r}) + \widehat{\mathbf{n}} \cdot \rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \left(\widehat{\mathbf{n}} \frac{\partial}{\partial n} \widetilde{u}(\mathbf{r}) + \nabla_S \widetilde{u}(\mathbf{r}) \right) \\ &= \widetilde{P}_0, \qquad \mathbf{r} \in S \end{aligned} \tag{15}$$

and/or

$$\widetilde{u}(\mathbf{r}) = \widetilde{u}_0, \qquad \mathbf{r} \in S. \tag{16}$$

(ii) non-classical BCs

$$\widetilde{R}(\mathbf{r}) = \widehat{\mathbf{n}} \cdot \widetilde{\boldsymbol{\mu}}^{1243}(\mathbf{r}) \cdot \widehat{\mathbf{n}} = \widetilde{R}_0, \qquad \mathbf{r} \in S \tag{17}$$

and/or

$$\frac{\partial \widetilde{u}}{\partial n}(\mathbf{r}) = \widetilde{q}_0, \qquad \mathbf{r} \in S. \tag{18}$$

The fields $\widetilde{P}_0, \widetilde{u}_0, \widetilde{R}_0$ and $\widetilde{q}_0$ denote prescribed values. Furthermore, assigning to equation (14) suitable initial conditions, we have the well-posed initial boundary value problem of dynamic gradient elasticity.

Equivalently, if we repeat the process in a two-phase domain V, confined by its boundary S and surrounding a matrix relation hosting an inclusion D with boundary ∂D and different physical parameters, then we obtain two differential equations of type (14) valid in the interior domain D and its exterior complement differing only in the involved physical parameters. Moreover, the boundary condition on the interface ∂D impose now continuity of the fields $\widetilde{u}$, $\frac{\partial \widetilde{u}}{\partial n}$, $\widetilde{p}$ and $\widetilde{R}$ across ∂D, while on exterior surface S, the BCs remain unaltered. If we are interested in transmission scattering processes, the exterior surface S is removed to infinity and the adoption of the outgoing radiation character of the elastic waves is compatible with vanishing of the contribution of all relevant fields to the 'infinite' surface S integrals.

The contribution $\widetilde{s} = -\nabla \cdot \widetilde{\boldsymbol{\mu}}$ to the classical stress tensor in equation (14) is given by

$$\widetilde{s} = -\nabla \cdot \widetilde{\mu} = -[2\mu c_3 \Delta \widetilde{e} + \lambda c_1 \widetilde{\mathbf{I}} \Delta(\nabla \cdot \widetilde{u}) + \lambda c_2 \nabla\nabla(\nabla \cdot \widetilde{u})], \tag{19}$$

where $2\mu c_3 = a_3 + 2a_4 + a_5$, $\lambda c_1 = a_1 + 2a_2$, $\lambda c_2 = a_1 + a_3 + a_5$.

The fourth-order operator $\widetilde{D}$ entering equation (14) is written as [1]

$$\widetilde{D} = \tfrac{1}{2} d_{pkmn} \widehat{\mathbf{x}}_p \widehat{\mathbf{x}}_k \widehat{\mathbf{x}}_m \widehat{\mathbf{x}}_n, \tag{20}$$

where

$$d_{pkmn} = \tfrac{1}{2} d^2 [\delta_{pn}\delta_{km} - \delta_{pm}\delta_{kn} + 2\alpha(3\alpha + 2\beta)\delta_{pk}\delta_{mn} + \beta^2(\delta_{pm}\delta_{kn} + \delta_{pn}\delta_{km})]. \tag{21}$$

In this relation, the constants α and β are constitutive parameters referring to the micro-deformation field while d is the characteristic dimension of the representative cell of the microstructure.

Introducing the parameters

$$h_1^2 = \frac{\rho' d^2 [2\alpha^2 + (\alpha + \beta)^2]}{3\rho}, \qquad h_2^2 = \frac{\rho' d^2 [1 + \beta^2]}{6\rho}, \tag{22}$$

we express the inertia term $\nabla \cdot \left(\rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \nabla \widetilde{u}\right)$ as follows:

$$\nabla \cdot \left(\rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \nabla \widetilde{u}\right) = \rho \frac{\partial^2}{\partial t^2} \left(h_1^2 \nabla\nabla \cdot \widetilde{u} - h_2^2 \nabla \times \nabla \times \widetilde{u}\right), \tag{23}$$

while the surface contribution $\widehat{\mathbf{n}} \cdot \rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \left(\widehat{\mathbf{n}} \frac{\partial}{\partial n} \widetilde{u} + \nabla_S \widetilde{u}\right)$ in surface traction $\widetilde{P}$ becomes

$$\widehat{\mathbf{n}} \cdot \rho' \widetilde{D} : \frac{\partial^2}{\partial t^2} \left(\widehat{\mathbf{n}} \frac{\partial}{\partial n} \widetilde{u} + \nabla_S \widetilde{u}\right) = \rho \frac{\partial^2}{\partial t^2} \left[\left(h_2^2 - \frac{\rho' d^2}{3\rho}\right) \widehat{\mathbf{n}} \times (\nabla \times \widetilde{u}) + \left(2h_2^2 - \frac{\rho' d^2}{3\rho}\right) \frac{\partial \widetilde{u}}{\partial n} + \left(h_1^2 - 2h_2^2 + \frac{\rho' d^2}{3\rho}\right) \widehat{\mathbf{n}} (\nabla \cdot \widetilde{u}) \right]. \tag{24}$$

On the basis of equations (19) and (23), equation (14) becomes

$$(\lambda+2\mu)\big(1-\xi_1^2\Delta\big)\nabla\nabla\cdot\widetilde{u}-\mu\big(1-\xi_2^2\Delta\big)\nabla\times\nabla\times\widetilde{u}$$
$$=\rho\frac{\partial^2}{\partial t^2}\big[\widetilde{u}-h_1^2\nabla\nabla\cdot\widetilde{u}+h_2^2\nabla\times\nabla\times\widetilde{u}\big], \tag{25}$$

where the involved parameters ξ_1 and ξ_2 are determined uniquely from the knowledge of $c_i, i=1,2,3$ appearing in equation (19) and of parameters λ,μ. Indeed, a simple calculation shows that

$$\xi_1^2=\frac{\lambda(c_1+c_2)+2\mu c_3}{\lambda+2\mu}, \qquad \xi_2^2=c_3. \tag{26}$$

In the degenerate case that $c_1=c_3=g^2$ and $c_2=0$, then $\xi_1^2=\xi_2^2=g^2$ and the differential equation (25) becomes

$$(1-g^2\Delta)[(\lambda+2\mu)\nabla\nabla\cdot\widetilde{u}-\mu\nabla\times\nabla\times\widetilde{u}]=\rho\frac{\partial^2}{\partial t^2}\big[u-h_1^2\nabla\nabla\cdot\widetilde{u}+h_2^2\nabla\times\nabla\times\widetilde{u}\big]. \tag{27}$$

This is the case when

$$\widetilde{\boldsymbol{\mu}}=g^2[2\mu\nabla\widetilde{e}+\lambda\widetilde{\mathbf{I}}\nabla(\nabla\cdot\widetilde{u})]^{3124}=g^2[2\mu\nabla\widetilde{e}+\lambda\nabla\widetilde{\mathbf{I}}(\nabla\cdot\widetilde{u})]=g^2\nabla\widetilde{\boldsymbol{\tau}}, \tag{28}$$

which is the dyadic version of the model proposed by Aifantis and co-workers [5, 6]. We return now to the investigation of the dynamic equation (25). The general dyadic solution $\widetilde{u}$ will be of the form [25]

$$\widetilde{u}=\nabla\nabla\phi+\nabla\nabla\times\mathbf{A}+\nabla\times\nabla\times\widetilde{G}, \tag{29}$$

with a general scalar field ϕ, a free divergence vector field $\mathbf{A}$ and a dyadic field $\widetilde{G}$. This is a general result for every dyadic field. Inserting equation (29) into equation (25) we find that

$$\big(1-\xi_1^2\Delta\big)(\lambda+2\mu)[\nabla\nabla\Delta\phi+\nabla\nabla\times\Delta\mathbf{A}]+\big(1-\xi_2^2\Delta\big)\mu\nabla\times\nabla\times\Delta\widetilde{\mathbf{G}}$$
$$=\rho[\nabla\nabla\ddot{\phi}+\nabla\nabla\times\ddot{\mathbf{A}}+\nabla\times\nabla\times\ddot{\widetilde{G}}]-\rho h_1^2[\nabla\nabla\Delta\ddot{\phi}+\nabla\nabla\times\Delta\ddot{\mathbf{A}}]$$
$$-\rho h_2^2\nabla\times\nabla\times\Delta\ddot{\widetilde{G}}, \tag{30}$$

where the dot above the functions denotes time derivative.

Doting two times from the left the equation above with ∇, which means applying the operator $\nabla\nabla$: on it, we find that

$$\Delta\big[\big(1-\xi_1^2\Delta\big)(\lambda+2\mu)\Delta^2\phi-\rho\big(\Delta\ddot{\phi}-h_1^2\Delta^2\ddot{\phi}\big)\big]=0. \tag{31}$$

We are interesting in time-harmonic dependence of the involved fields of the form $\exp(-\mathrm{i}\omega t)$. This reflects the physical assumption of time-harmonic waves with the specific frequency ω. Then the scalar field ϕ is assumed to be expressed as $\phi(\mathbf{r},t)=\phi^0(\mathbf{r})\exp(-\mathrm{i}\omega t)$. Actually, our attention will be focused on the determination of the time-reduced field ϕ^0 and this is going to be the case for the vector $\mathbf{A}$ and the dyadic $\widetilde{G}$ as well. To simplify things, we omit the superscript '0', having in mind that all the produced space fields have to be multiplied with the time factor $\exp(-\mathrm{i}\omega t)$, if the full time-space dependence is to be established. However this time factor is reduced in all steps of the scattering problem that follows in last section and so its role is not significant. It would be mentioned here that the specific time reduction is very common and extensively used in many branches of different origin [26]. In the framework of dyadic solutions, we mention here the similarity with Einstein's field equations of linearized theory [27], involving the wave (D'Alembert) operator and describing weak gravitational waves. Although the wave operator is much simpler than the operator appearing in equation (31), the starting point to determine harmonic dyadic waves, in both approaches, is this specific time reduction. Then equation (31) obtains the form

$$\Delta\left[\left(1-\xi_1^2\Delta\right)(\lambda+2\mu)\Delta^2\phi+\rho\omega^2\left(\Delta\phi-h_1^2\Delta^2\phi\right)\right]=0. \tag{32}$$

Excluding space-harmonic (stationary) solutions, since we consider propagating waves, we infer that

$$\left(1-\xi_1^2\Delta\right)(\lambda+2\mu)\Delta\phi+\rho\omega^2\left(\phi-h_1^2\Delta\phi\right)=0$$

or

$$-\xi_1^2(\lambda+2\mu)\Delta^2\phi+\left[(\lambda+2\mu)-\rho h_1^2\omega^2\right]\Delta\phi+\rho\omega^2\phi=0. \tag{33}$$

We write equation (33) in the form

$$(\alpha\Delta+\beta)(\gamma\Delta+\delta)\phi=0 \tag{34}$$

with

$$\alpha\gamma=-\xi_1^2(\lambda+2\mu), \qquad \alpha\delta+\beta\gamma=(\lambda+2\mu)-\rho h_1^2\omega^2, \qquad \beta\delta=\rho\omega^2. \tag{35}$$

At first sight, it seems that equation (34) provides wave solutions with wavenumbers $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$. Actually, solving the system of equations (35), we find that the ratios $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$ are always equal but there are two alternatives for these terms,

$$\frac{\beta}{\alpha}\left(=\frac{\delta}{\gamma}\right)=\frac{2\rho\omega^2}{(\lambda+2\mu)-\rho h_1^2\omega^2\pm\sqrt{D(\omega)}} \tag{36}$$

where

$$D(\omega)=\left(\lambda+2\mu-\rho h_1^2\omega^2\right)^2+4\rho\omega^2\xi_1^2(\lambda+2\mu). \tag{37}$$

Consequently,

$$\phi\in\ker\left(\Delta+k_1^2(\omega)\right)\cup\ker\left(\Delta+k_2^2(\omega)\right), \tag{38}$$

where

$$k_1^2(\omega)=\frac{2\rho\omega^2}{(\lambda+2\mu)-\rho h_1^2\omega^2+\sqrt{D(\omega)}}, \qquad k_2^2(\omega)=\frac{2\rho\omega^2}{(\lambda+2\mu)-\rho h_1^2\omega^2-\sqrt{D(\omega)}}. \tag{39}$$

It is worth noting that $k_2^2(\omega)<0$ and so the wavenumber $k_2(\omega)$ is imaginary corresponding to attenuated waves. In the limiting case $h_1,\xi_1\to 0$ and $\lambda\to\lambda_{\text{mac}}$, $\mu\to\mu_{\text{mac}}$, we easily find that $k_1(\omega)\to\sqrt{\frac{\rho}{\lambda_{\text{mac}}+2\mu_{\text{mac}}}}\,\omega$ and $k_2(\omega)\to \mathrm{i}\infty$ giving place to classical elasticity results where no attenuated waves exist and the real wavenumber refers to the longitudinal elastic waves propagating in the isotropic elastic medium with the Lame constants λ_{mac} and μ_{mac}.

In the essence of this work is the construction of a complete set of eigendyadics in the space of solutions of the involved equations expressed mainly in spherical coordinates (although the construction process can be adapted to Cartesian or cylindrical frames), as well as the exploitation of this set for the solution of scattering problems involving media with microstructure. Equation (38) provides that in spherical coordinates

$$\phi\in\left\{g_n^l(k_ir)Y_n^m(\mathbf{r});\, i=1,2, n=0,1,2,\ldots,m\leqslant|n|, l=1,2,3,4\right\}, \tag{40}$$

where Y_n^m are the spherical harmonics, g_n^l stand for the spherical Bessel functions ($l=1,2$), or alternatively for the spherical Hankel functions ($l=3,4$).

In the following part of this work, we will use extensively the Navier eigenvectors **L**, **M**, **N** expressed in spherical coordinates through the relations

$$\mathbf{L}_n^{m,l}(\mathbf{r};k)=\nabla\chi_n^{m,l}(\mathbf{r};k) \tag{41}$$

$$\mathbf{M}_n^{m,l}(\mathbf{r};k) = \nabla \times (\mathbf{r}\chi_n^{m,l}(\mathbf{r};k)) \tag{42}$$

$$\mathbf{N}_n^{m,l}(\mathbf{r};k) = \frac{1}{k}\nabla \times \mathbf{M}_n^{m,l}(\mathbf{r};k), \tag{43}$$

where $\chi_n^{m,l}(\mathbf{r};k) = g_n^l(kr)Y_n^m(\widehat{\mathbf{r}})$ solves the scalar Helmholtz equation corresponding to the wavenumber k.

Consequently, the first part of the decomposition of the dyadic solution (29)

$$\widetilde{u} = \nabla\nabla\phi + \nabla\nabla \times \mathbf{A} + \nabla \times \nabla \times \widetilde{G}$$

is contained in the large set

$$\nabla\nabla\phi \in \left\{\nabla\mathbf{L}_n^{m,l}(\mathbf{r};k_i);\, i = 1,2,\, n = 0,1,2,\ldots,|m| \leqslant n,\, l = 1,2,3,4\right\}. \tag{44}$$

Remark that the dyadic solution $\nabla\mathbf{L}_n^{m,l}(\mathbf{r};k_2)$ constitutes a field of exponential type and contains actually radial functions which are modified spherical Bessel or Hankel functions.

To construct now the possible vector functions $\mathbf{A}$ appearing in representation (29), we apply on the differential equation (30) the operator $\nabla \times (\nabla\cdot)$ obtaining

$$(\lambda + 2\mu)\left(1 - \xi_1^2\Delta\right)\nabla \times \nabla \times \Delta^2\mathbf{A} = \rho\nabla \times \nabla \times \Delta\ddot{\mathbf{A}} - \rho h_1^2\nabla \times \nabla \times \Delta^2\ddot{\mathbf{A}}.$$

Taking time-harmonic dependence we obtain

$$(\lambda + 2\mu)\left(1 - \xi_1^2\Delta\right)\nabla \times \nabla \times \Delta^2\mathbf{A} = -\rho\omega^2\nabla \times \nabla \times \Delta\mathbf{A} + \rho h_1^2\omega^2\nabla \times \nabla \times \Delta^2\mathbf{A}. \tag{45}$$

Using the well-known property $\nabla \times \nabla\times = \nabla(\nabla\cdot) - \Delta$ and the fact that $\mathbf{A}$ is solenoidal we deduce that

$$\Delta^2\left\{(\lambda + 2\mu)\left(1 - \xi_1^2\Delta\right)\Delta\mathbf{A} - \rho h_1^2\omega^2\Delta\mathbf{A} + \rho\omega^2\mathbf{A}\right\} = 0$$

and searching again exclusively for propagating solutions, we obtain

$$-\xi_1^2(\lambda + 2\mu)\Delta^2\mathbf{A} + \left[(\lambda + 2\mu) - \rho h_1^2\omega^2\right]\Delta\mathbf{A} + \rho\omega^2\mathbf{A} = 0, \tag{46}$$

where we recognize the differential operator met in equation (33) but now applying on the vector field $\mathbf{A}$.

The possible representations of $\mathbf{A}$ obeying the free divergence assumption and assuring the spherical invariance are of the form

$$\mathbf{A} = \nabla\psi \times \mathbf{r} \qquad \text{or} \qquad \mathbf{A} = \nabla \times (\nabla\psi \times \mathbf{r})$$

Substitution of these expressions into equation (46) easily leads to the result that ψ again satisfies equation (33). Consequently $\psi \in \ker\left(\Delta + k_i^2\right)$, $i = 1, 2$, result which in combination with the nomenclature introduced by equations (41)–(43), provides easily the outcome that the second term $\nabla\nabla \times \mathbf{A}$ of the decomposition (29) is characterized as follows:

$$\begin{aligned}\nabla\nabla \times \mathbf{A} \in &\left\{\nabla\mathbf{M}_n^{m,l}(\mathbf{r};k_i);\, i = 1,2,\, n = 0,1,2,\ldots,|m| \leqslant n,\, l = 1,2,3,4\right\}\\ &\cup\left\{\nabla\mathbf{N}_n^{m,l}(\mathbf{r};k_i);\, i = 1,2,\, n = 0,1,2,\ldots,|m| \leqslant n,\, l = 1,2,3,4\right\}.\end{aligned} \tag{47}$$

To characterize the third term of the representation (29), i.e. the possible forms of the dyad $\nabla \times \nabla \times \widetilde{G}$, we remark first that the dyadic $\widetilde{G}$ can be selected to be of the form

$$\widetilde{G} = \nabla \times \widetilde{\mathbf{g}}, \tag{48}$$

where $\widetilde{g}$ is another dyadic. Indeed

$$\widetilde{G} = \mathbf{G}_j\widehat{\mathbf{x}}_j = (\nabla\phi_j + \nabla \times \mathbf{g}_j)\widehat{\mathbf{x}}_j = \nabla(\phi_j\widehat{\mathbf{x}}_j) + \nabla \times (\mathbf{g}_j\widehat{\mathbf{x}}_j) = \nabla\mathbf{f} + \nabla \times \widetilde{g}, \tag{49}$$

where Helmholtz decomposition for vector fields has been adopted. The term $\nabla\mathbf{f}$ can be ignored since substituted in the solution $\nabla \times \nabla \times \widetilde{G}$ offers no contribution.

We apply then the operator $\nabla\times\nabla\times$ on the differential equation (30). Using the solenoidal property of $\widetilde{G}$ and the property $\nabla\times\nabla\times=\nabla(\nabla\cdot)-\Delta$ we find that

$$\left(1-\xi_2^2\Delta\right)\mu\Delta^3\widetilde{G}=\rho\Delta^2\ddot{\widetilde{G}}-\rho h_2^2\Delta^3\ddot{\widetilde{G}}.$$

Working with propagating time-harmonic fields we again obtain that

$$-\mu\xi_2^2\Delta^2\widetilde{G}+\left(\mu-\rho h_2^2\omega^2\right)\Delta\widetilde{G}+\rho\omega^2\widetilde{G}=0. \tag{50}$$

There exists a correspondence between the differential equation above and equation (33) established by the parameter correspondence

$$\mu\longleftrightarrow\lambda+2\mu,\qquad h_2\longleftrightarrow h_1,\qquad \xi_2\longleftrightarrow\xi_1$$

Consequently,

$$\widetilde{G}\in\ker\left(\Delta+k_3^2\right)\cup\ker\left(\Delta+k_4^2\right), \tag{51}$$

where

$$k_3^2(\omega)=\frac{2\rho\omega^2}{\mu-\rho h_2^2\omega^2+\sqrt{D'(\omega)}},\qquad k_4^2(\omega)=\frac{2\rho\omega^2}{\mu-\rho h_2^2\omega^2-\sqrt{D'(\omega)}} \tag{52}$$

with

$$D'(\omega)=\left(\mu-\rho h_2^2\omega^2\right)^2+4\rho\omega^2\xi_2^2\mu.$$

We remark that $k_4^2(\omega)$ is always less than zero and so $k_4(\omega)$ is a purely imaginary number corresponding to attenuated waves. Clearly the limiting values of k_3^2 and k_4^2 as $\xi_2, h_2\rightarrow 0$ are $\frac{\rho}{\mu_{\mathrm{mac}}}\omega^2$ and $-\infty$ correspondingly and so we recover the wavenumber of transverse classical waves and again verify the lack of attenuated waves in classical elasticity.

It is clear that for every specific wavenumber k_j, $j=3,4$ and spectral triple (n,m,l) there exist exactly six independent selections for the dyad $\widetilde{G}$ expressed by equation (48) and satisfying the corresponding Helmholtz equation. Indeed, $\widetilde{G}$ can be decomposed in Cartesian coordinates as $\widetilde{G}=(\nabla\times\mathbf{g}_i)\widehat{\mathbf{x}}_i$ and the possible independent selections of the solenoidal vectors $\nabla\times\mathbf{g}_i$ (for every $i=1,2,3$) are combinations of the two solenoidal Navier eigensolutions corresponding to (n,m,l) and k_j (i.e. the functions $\mathbf{M}_n^{m,l}(\mathbf{r},k_j)$ and $\mathbf{N}_n^{m,l}(\mathbf{r},k_j)$). It is proved in appendix B that the complete set of independent solenoidal dyads $\widetilde{G}$ can be selected to be

$$\widetilde{G}\in\{\nabla\times(\mathbf{L}\times\widetilde{\mathbf{I}}),\nabla\times(\mathbf{M}\times\widetilde{\mathbf{I}}),\nabla\times(\mathbf{N}\times\widetilde{\mathbf{I}}),\nabla\times(\mathbf{r}\mathbf{L}),\nabla\times(\mathbf{r}\mathbf{M}),\nabla\times(\mathbf{r}\mathbf{N})\}, \tag{53}$$

where the abbreviated forms of Navier eigenvectors refer to a specific triples (n,m,l) and wavenumber k_j, $j=3,4$. More precisely, given that $\nabla\times\nabla\times\widetilde{G}$ is equal to $-\Delta\widetilde{G}=k^2\widetilde{G}$ we infer that the third term of the decomposition (29) is proportional to $\widetilde{G}$ and so we conclude to

$$\begin{aligned}\nabla\times(\nabla\times\widetilde{G})\in\Big\{&\nabla\times\left(\mathbf{L}_n^{m,l}(\mathbf{r},k_j)\times\widetilde{\mathbf{I}}\right),\nabla\times\left(\mathbf{M}_n^{m,l}(\mathbf{r},k_j)\times\widetilde{\mathbf{I}}\right),\\ &\nabla\times\left(\mathbf{N}_n^{m,l}(\mathbf{r},k_j)\times\widetilde{\mathbf{I}}\right),\nabla\times\left(\mathbf{r}\mathbf{L}_n^{m,l}(\mathbf{r},k_j)\right),\nabla\times\left(\mathbf{r}\mathbf{M}_n^{m,l}(\mathbf{r},k_j)\right),\\ &\nabla\times\left(\mathbf{r}\mathbf{N}_n^{m,l}(\mathbf{r},k_j)\right),\qquad n=0,1,2,\ldots,|m|\leqslant n, l=1,2,3,4\Big\}.\end{aligned} \tag{54}$$

Through equations (29), (44), (47) and (54), we have determined the fundamental eigendyadics constituting the structural solutions of the dynamic gradient elasticity. For every specific triple (n,m,l) and wavenumber we deal with nine dyadic eigensolutions which constitute the analog of the Navier eigenvectors in the dyadic framework. They can so be nominated Navier eigendyadics indicating their relevance to the vector case. It is very important to express these eigensolutions in spherical coordinates. This is necessary in order to handle boundary value problems described geometrically even approximately by the spherical coordinate system. Even in the study of general geometrical boundary value problems, the evocation of collocation-type methods to 'follow' the boundary conditions is strongly supported by the ability to express the involved fields in terms of spherical point source dyadic eigensolutions.

After extended and very elongated analysis, we have constructed the forthcoming spherical representations for the aforementioned dyadic solutions. Special role to these representations play the spherical vector harmonics

$$\mathbf{P}_n^m(\hat{\mathbf{r}}) = \hat{\mathbf{r}}\mathbf{Y}_n^m, \qquad \mathbf{B}_n^m(\hat{\mathbf{r}}) = \mathbf{D}\mathbf{Y}_n^m(\hat{\mathbf{r}}) = \left(\hat{\theta}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\mathbf{Y}_n^m(\hat{\mathbf{r}}), \qquad \mathbf{C}_n^m(\hat{\mathbf{r}}) = \mathbf{B}_n^m(\hat{\mathbf{r}}) \times \hat{\mathbf{r}} \tag{55}$$

(the usually adopted normalization factor has been omitted here for simplicity). More precisely, we are in position to present the decompositions,

$$\begin{aligned}\frac{\nabla\mathbf{L}}{k} &= \ddot{g}_n(kr)\hat{\mathbf{r}}\mathbf{P}_n^m + \frac{\mathrm{d}}{\mathrm{d}(kr)}\left(\frac{1}{kr}g_n(kr)\right)\left[\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s + \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a\right] + \frac{\dot{g}_n(kr)}{kr}\left[\left(\mathbf{D}\mathbf{P}_n^m\right)_s - \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a\right] \\ &\quad + \frac{g_n(kr)}{(kr)^2}\left[\left(\mathbf{D}\mathbf{B}_n^m\right)_s + \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a\right] = \ddot{g}_n(kr)\hat{\mathbf{r}}\mathbf{P}_n^m + \left[\frac{\dot{g}_n(kr)}{kr} - \frac{g_n(kr)}{(kr)^2}\right]\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s \\ &\quad + \frac{\dot{g}_n(kr)}{kr}\left(\mathbf{D}\mathbf{P}_n^m\right)_s + \frac{g_n(kr)}{(kr)^2}\left(\mathbf{D}\mathbf{B}_n^m\right)_s\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{\nabla\mathbf{M}}{k} &= \frac{g_n(kr)}{kr}\left[\left(\mathbf{D}\mathbf{C}_n^m\right)_s - \frac{n(n+1)}{2}\left(\tilde{\mathbf{I}} \times \mathbf{P}_n^m\right)\right] + \dot{g}_n(kr)\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s \\ &\quad + \left[\dot{g}_n(kr) + \frac{g_n(kr)}{kr}\right]\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\nabla\mathbf{N}}{k} &= n(n+1)\left[\frac{g_n(kr)}{(kr)^2} + \frac{\dot{g}_n(kr)}{kr}\right]\hat{\mathbf{r}}\mathbf{P}_n^m + \left[-\frac{\dot{g}_n(kr)}{kr} + n(n+1)\frac{g_n(kr)}{(kr)^2}\right. \\ &\quad \left. - \left(1 + \frac{1}{(kr)^2}\right)g_n(kr)\right]\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s - g_n(kr)\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a + n(n+1)\frac{g_n(kr)}{(kr)^2}\left(\mathbf{D}\mathbf{P}_n^m\right)_s \\ &\quad + \left[\frac{g_n(kr)}{(kr)^2} + \frac{\dot{g}_n(kr)}{kr}\right]\left(\mathbf{D}\mathbf{B}_n^m\right)_s\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{\nabla \times (\mathbf{L} \times \tilde{\mathbf{I}})}{k} &= \frac{1}{k^2}\nabla\nabla\Phi + \Phi\tilde{\mathbf{I}} = \frac{\nabla\mathbf{L}}{k} + \Phi(\tilde{\mathbf{I}} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \Phi\hat{\mathbf{r}}\hat{\mathbf{r}} \\ &= \left[-2\frac{\dot{g}_n(kr)}{(kr)} + n(n+1)\frac{g_n(kr)}{(kr)^2}\right]\hat{\mathbf{r}}\mathbf{P}_n^m + \left[\frac{\dot{g}_n(kr)}{(kr)} - \frac{g_n(kr)}{(kr)^2} - g_n(kr)\right]\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s \\ &\quad + \left[\frac{\dot{g}_n(kr)}{(kr)} + g_n(kr)\right]\left(\mathbf{D}\mathbf{P}_n^m\right)_s + \frac{g_n(kr)}{(kr)^2}\left(\mathbf{D}\mathbf{B}_n^m\right)_s\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{\nabla \times (\mathbf{M} \times \tilde{\mathbf{I}})}{k} &= \frac{(\nabla\mathbf{M}^T)}{k} \\ &= \frac{g_n(kr)}{(kr)}\left[\left(\mathbf{D}\mathbf{C}_n^m\right)_s + \frac{n(n+1)}{2}\left(\tilde{\mathbf{I}} \times \mathbf{P}_n^m\right)\right] + \dot{g}_n(kr)\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s \\ &\quad - \left[\dot{g}_n(kr) + \frac{g_n(kr)}{(kr)}\right]\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a\end{aligned} \tag{60}$$

$$\frac{\nabla\times(\mathbf{N}\times\tilde{\mathbf{I}})}{k}=\frac{(\nabla\mathbf{N}^T)}{k}$$
$$=n(n+1)\left[\frac{g_n(kr)}{(kr)^2}+\frac{\dot{g}_n(kr)}{(kr)}\right]\hat{\mathbf{r}}\mathbf{P}_n^m+\left[-\frac{\dot{g}_n(kr)}{kr}+n(n+1)\frac{g_n(kr)}{(kr)^2}\right.$$
$$\left.-\left(1+\frac{1}{(kr)^2}\right)g_n(kr)\right]\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s+g_n(kr)\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a+n(n+1)\frac{g_n(kr)}{(kr)^2}\left(\mathbf{D}\mathbf{P}_n^m\right)_s$$
$$+\left[\frac{g_n(kr)}{(kr)^2}+\frac{\dot{g}_n(kr)}{(kr)}\right]\left(\mathbf{D}\mathbf{B}_n^m\right)_s \tag{61}$$

$$\nabla\times(\mathbf{r}\mathbf{L})=\left[\dot{g}_n(kr)-\frac{n(n+1)}{2}\frac{g_n(kr)}{(kr)}\right]\left(\tilde{\mathbf{I}}\times\mathbf{P}_n^m\right)+\dot{g}_n(kr)\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s$$
$$-\left[\dot{g}_n(kr)-\frac{g_n(kr)}{(kr)}\right]\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a+\frac{g_n(kr)}{(kr)}\left(\mathbf{D}\mathbf{C}_n^m\right)_s \tag{62}$$

$$\nabla\times(\mathbf{r}\mathbf{M})=-g_n(kr)n(n+1)\left(\mathbf{D}\mathbf{P}_n^m\right)_s+g_n(kr)[n(n+1)-1]\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s+g_n(kr)\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a \tag{63}$$

$$\nabla\times(\mathbf{r}\mathbf{N})=\frac{n(n+1)}{2}\left[\dot{g}_n(kr)+3\frac{g_n(kr)}{(kr)}\right]\left(\tilde{\mathbf{I}}\times\mathbf{P}_n^m\right)+n(n+1)\frac{g_n(kr)}{(kr)}\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s$$
$$+\left[-n(n+1)\frac{g_n(kr)}{(kr)}+\dot{g}_n(kr)+\frac{g_n(kr)}{(kr)}\right]\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a$$
$$+\left[\dot{g}_n(kr)+\frac{g_n(kr)}{(kr)}\right]\left(\mathbf{D}\mathbf{C}_n^m\right)_s, \tag{64}$$

where the subscript $s(a)$ indicates the symmetric (antisymmetric) part of the involved dyadics. We remark that the dyadic eigensolutions of the dynamic equation of gradient elasticity—after being restricted on a specific sphere—are expanded in terms of a set of dyadic spherical harmonics, which constitutes the extension of the vector spherical harmonics. For every specific parameter pair (n, m), this set contains six symmetric and three antisymmetric elements. These dyadic spherical harmonics are independent and merit concrete orthogonality relations. These relations are presented in appendix A and are indispensable for the investigation of the interrelated boundary value problems.

3. Scattering by a spherical penetrable elastic body with microstructure

In this section, we apply the results of the spectral analysis introduced in the previous section to the solution of a representative scattering problem. We consider a spherical region D of radius a centered at the coordinates origin O, occupied by an elastic macroscopically isotropic material with microstructure, fully characterized by the parameters λ_i and μ_i, the densities ρ_i, ρ_i', the gradient parameters $\alpha_{il}, l = 1, 2, \ldots, 5, h_{i1}, h_{i2}$ and the characteristic micro-dimension d_i. The surrounding background space is considered to be elastically isotropic with microstructure as well, physically determined by the exterior parameters λ_e, μ_e, the mass densities ρ_e, ρ_e' and the gradient parameters $\alpha_{el}, l = 1, 2, \ldots, 5, h_{e1}, h_{e2}$ and d_e.

We consider an elastic time-harmonic dyadic plane wave of the form $\widetilde{u}^{\rm inc}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}$ where

$$\widetilde{u}^{\rm inc}(\mathbf{r})=A_1\widehat{\mathbf{k}\mathbf{k}}\,\mathrm{e}^{\mathrm{i}k_{e,1}(\omega)\hat{\mathbf{k}}\cdot\mathbf{r}}+A_2(\widetilde{\mathbf{I}}-\widehat{\mathbf{k}\mathbf{k}})\,\mathrm{e}^{\mathrm{i}k_{e,3}(\omega)\hat{\mathbf{k}}\cdot\mathbf{r}} \tag{65}$$

is the time-reduced part of the field, propagating in the matrix space and interfering with the penetrable inclusion, giving birth to secondary fields creation. It is worthwhile to note that the dyadic form of this field permits to represent uniformly all the possible polarizations of the incident field and it is sufficient to 'dot' expression (65) on the specific polarization direction in order to acquire the corresponding vector field. The scalars A_1 and A_2 are just the amplitudes of the incident plane waves. Furthermore, we mention that the concrete form of the appeared dyadics reflects the irrotational and solenoidal type of the involved traveling waves. The wavenumbers $k_{e,j}$, $j = 1, 3$, appearing in the representation (65), are determined via expressions (39) and (52), after substituting there the physical parameters referring to the background space. We mention that the remaining wavenumbers $k_{e,j}$, $j = 2, 4$ are absent in the incident field expression since are imaginary and then cannot be present in a traveling plane wave with sources remote to infinity.

The scattering process results in the creation of the scattered field $\widetilde{u}^{\mathrm{sc}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}$, propagating outward the inclusion region, as well as of the 'trapped' standing wave $\widetilde{u}^{i}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}$, inside the scatterer region. The outcome of the above section allows the representation of the produced fields in terms of the constructed spectral eigendyadics as follows:

$$\tilde{u}^{\mathrm{sc}} = \sum_{j=1}^{2}\sum_{|m|\leqslant n}\left[a_{n,e,j}^{m}\frac{\nabla\mathbf{L}_{n,e,j}^{m,3}}{k_{e,j}} + b_{n,e,j}^{m}\frac{\nabla\mathbf{M}_{n,e,j}^{m,3}}{k_{e,j}} + c_{n,e,j}^{m}\frac{\nabla\mathbf{N}_{n,e,j}^{m,3}}{k_{e,j}}\right] + \sum_{j=3}^{4}\sum_{|m|\leqslant n}\left[\alpha_{n,e,j}^{m}\frac{\nabla\times\left(\mathbf{L}_{n,e,j}^{m,3}\times\tilde{\mathbf{I}}\right)}{k_{e,j}} + \beta_{n,e,j}^{m}\frac{\nabla\times\left(\mathbf{M}_{n,e,j}^{m,3}\times\tilde{\mathbf{I}}\right)}{k_{e,j}} + \gamma_{n,e,j}^{m}\frac{\nabla\times\left(\mathbf{N}_{n,e,j}^{m,3}\times\tilde{\mathbf{I}}\right)}{k_{e,j}} + \delta_{n,e,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{L}_{n,e,j}^{m,3}\right) + \epsilon_{n,e,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{M}_{n,e,j}^{m,3}\right) + \zeta_{n,e,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{N}_{n,e,j}^{m,3}\right)\right] \tag{66}$$

$$\tilde{u}^{i} = \sum_{j=1}^{2}\sum_{|m|\leqslant n}\left[a_{n,i,j}^{m}\frac{\nabla\mathbf{L}_{n,i,j}^{m,1}}{k_{i,j}} + b_{n,i,j}^{m}\frac{\nabla\mathbf{M}_{n,i,j}^{m,1}}{k_{i,j}} + c_{n,i,j}^{m}\frac{\nabla\mathbf{N}_{n,i,j}^{m,1}}{k_{i,j}}\right] + \sum_{j=3}^{4}\sum_{|m|\leqslant n}\left[\alpha_{n,i,j}^{m}\frac{\nabla\times\left(\mathbf{L}_{n,i,j}^{m,1}\times\tilde{\mathbf{I}}\right)}{k_{i,j}} + \beta_{n,i,j}^{m}\frac{\nabla\times\left(\mathbf{M}_{n,i,j}^{m,1}\times\tilde{\mathbf{I}}\right)}{k_{i,j}} + \gamma_{n,i,j}^{m}\frac{\nabla\times\left(\mathbf{N}_{n,i,j}^{m,1}\times\tilde{\mathbf{I}}\right)}{k_{i,j}} + \delta_{n,i,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{L}_{n,i,j}^{m,1}\right) + \epsilon_{n,i,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{M}_{n,i,j}^{m,1}\right) + \zeta_{n,i,j}^{m}\nabla\times\left(\mathbf{r}\mathbf{N}_{n,i,j}^{m,1}\right)\right], \tag{67}$$

where the mixture coefficients undertake to encode all the diversifying information about the represented fields since the functional behavior of them is incorporated in the spectral eigendyadics, which is a common base for all the relevant dyadic fields. The indices appeared in every quantity of the form $(\cdot)_{n,s,j}^{m,l}$ serve specific notation. More precisely, the index l determines the type of the involved spherical Bessel ($l = 1, 2$) or spherical Hankel function ($l = 3, 4$), the index s defines the exterior ($s = e$) or interior ($s = i$) medium imposing the use of the corresponding material parameters wherever these appear, while j refers to the concrete wavenumber. Furthermore, the summation $\sum_{|m|\leqslant n}$ is a simplification of the well-known sum $\sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}$ over the integer separation constants of the spherical geometry. It is worthwhile to mention that the total number of coefficients participating in both fields for every pair (n, m) is equal to 36.

Clearly, the interior field $\widetilde{u}^{i}$ is represented through eigenvectors constructed with spherical Bessel functions and modified spherical Bessel functions of the first kind to ensure regularity in the vicinity of the coordinate system origin, while the exterior scattered field $\widetilde{u}^{\mathrm{sc}}$ involves spherical Hankel functions of the first kind (to ensure radially outward propagating waves) as well as modified spherical Hankel functions of the first kind (to ensure radially attenuated waves).

By construction, the displacement fields introduced by equations (65)–(67) satisfy the time-reduced dyadic equation of gradient elasticity with the convenient physical parameters and obey the necessary regularity and asymptotic behavior. What remains to be accomplished is to force these representations to comply with the boundary conditions of the problem. This plan will result in the formation of linear algebraic systems, whose unknown quantities are just the expansion coefficients of the representations (66) and (67).

The total displacement field in the exterior region $\mathsf{R}^3\backslash D$ is the superposition of the incident and the scattered wave, i.e. $\widetilde{u}^e = \widetilde{u}^{\rm sc} + \widetilde{u}^{\rm inc}$. As we cross the discontinuity surface $r = |\mathbf{r}| = a$, the adjacent fields $\widetilde{u}^e$ and $\widetilde{u}^i$ must obey the continuity conditions stemmed from the form of the boundary fields expressed by equations (15)–(18). More precisely, the conditions describing the interrelation between the corresponding fields across the boundary of the scatterer are the following:

$$\widetilde{P}^e(\mathbf{r}) = \widetilde{P}^i(\mathbf{r}), \qquad |\mathbf{r}| = a \tag{68}$$

$$\widetilde{u}^e(\mathbf{r}) = \widetilde{u}^i(\mathbf{r}), \qquad |\mathbf{r}| = a \tag{69}$$

$$\widetilde{R}^e(\mathbf{r}) = \widetilde{R}^i(\mathbf{r}), \qquad |\mathbf{r}| = a \tag{70}$$

$$\frac{\partial}{\partial r}\widetilde{u}^e(\mathbf{r}) = \frac{\partial}{\partial r}\widetilde{u}^i(\mathbf{r}), \qquad |\mathbf{r}| = a, \tag{71}$$

where the superscripts indicate the region in which the quantities tally with.

We express first the incident field in terms of (65) the dyadic eigensolutions of dynamic gradient elasticity as follows:

$$\begin{aligned}\widetilde{u}^{\rm inc}(\mathbf{r}) &= -\frac{A_1}{k_{e,1}^2}\nabla\nabla \mathrm{e}^{\mathrm{i}k_{e,1}\widehat{\mathbf{k}}\cdot\mathbf{r}} + A_2\left(\widetilde{\mathbf{I}} + \frac{\nabla\nabla}{k_{e,3}^2}\right)\mathrm{e}^{\mathrm{i}k_{e,3}(\omega)\widehat{\mathbf{k}}\cdot\mathbf{r}}\\ &= -A_1\sum_{|m|\leqslant n} Z_{n,1}^m(\widehat{\mathbf{k}})\frac{\nabla\mathbf{L}_{n,e,1}^{m,1}}{k_{e,1}} + A_2\sum_{|m|\leqslant n} Z_{n,3}^m(\widehat{\mathbf{k}})\frac{\nabla\times\left(\mathbf{L}_{n,e,3}^{m,1}\times\widetilde{\mathbf{I}}\right)}{k_{e,3}},\end{aligned} \tag{72}$$

where $Z_{n,j}^m(\widehat{\mathbf{k}}) = (2n+1)i^n\frac{(n-m)!}{(n+m)!}\overline{Y}_n^m(\widehat{\mathbf{k}})$ are involved in the decomposition of plane waves in terms of spherical standing waves [28].

We face the boundary conditions (69) and (71). We use the representations (65)–(67) which, restricted on the sphere $r = a$, become expansions in terms of dyadic spherical harmonics. We project then functionally conditions (69)–(71) on these dyadic spherical harmonics. This projection is equivalent by double doting with the dyadic functions and integrate over the unit sphere. The orthogonality relations concerning the complete set of dyadic harmonics are presented in appendix A and are evoked here to provide fully algebraic systems. It is interesting that these orthogonality relations help to obtain separate algebraic systems for every pair (n, m). In other words the terms corresponding to different spectral pairs (n, m) decouple. We remark that—this is actually the case for the remaining boundary conditions as well—the projections on the dyadics $\hat{\mathbf{r}}\mathbf{P}_n^m$, $\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s$, $\left(\mathbf{D}\mathbf{P}_n^m\right)_s$, $\left(\mathbf{D}\mathbf{B}_n^m\right)_s$, $\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a$ lead, for every specific pair (n, m), to separate equations pertaining to different set of unknowns for these obtained after projecting on the remaining dyadic spherical harmonics $\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s$, $\left(\mathbf{D}\mathbf{C}_n^m\right)_s$, $\left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a$, $\left(\tilde{\mathbf{I}}\times\mathbf{P}_n^m\right)$.

Recapitulating conditions (69) and (71) lead to the following algebraic systems for every specific pair (n, m):

$$\begin{aligned}\sum_{j=1}^{2}\left[A_{pq,e,j}^{(n,m)}a_{n,e,j}^{m}+C_{pq,e,j}^{(n,m)}c_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[A_{pq,e,j}^{(n,m)}\alpha_{n,e,j}^{m}+\Gamma_{pq,e,j}^{(n,m)}\gamma_{n,e,j}^{m}+E_{pq,e,j}^{(n,m)}\varepsilon_{n,e,j}^{m}\right]\\ -\sum_{j=1}^{2}\left[A_{pq,i,j}^{(n,m)}a_{n,i,j}^{m}+C_{pq,i,j}^{(n,m)}c_{n,i,j}^{m}\right]\\ -\sum_{j=3}^{4}\left[A_{pq,i,j}^{(n,m)}\alpha_{n,i,j}^{m}+\Gamma_{pq,i,j}^{(n,m)}\gamma_{n,i,j}^{m}+E_{pq,i,j}^{(n,m)}\varepsilon_{n,i,j}^{m}\right]\\ =\Phi_{pq}^{(n,m)},\qquad \text{for}\quad p=1,2;\, q=1,2,\ldots,5\end{aligned}\tag{73}$$

$$\begin{aligned}\sum_{j=1}^{2}\left[B_{pq,e,j}^{(n,m)}b_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[B_{pq,e,j}^{(n,m)}\beta_{n,e,j}^{m}+\Delta_{pq,e,j}^{(n,m)}\delta_{n,e,j}^{m}+Z_{pq,e,j}^{(n,m)}\zeta_{n,e,j}^{m}\right]\\ -\sum_{j=1}^{2}\left[B_{pq,i,j}^{(n,m)}b_{n,i,j}^{m}\right]-\sum_{j=3}^{4}\left[B_{pq,i,j}^{(n,m)}\beta_{n,i,j}^{m}+\Delta_{pq,i,j}^{(n,m)}\delta_{n,i,j}^{m}+Z_{pq,i,j}^{(n,m)}\zeta_{n,i,j}^{m}\right]\\ =Y_{pq}^{(n,m)},\qquad \text{for}\quad p=1,2;\quad q=1,2,3,4.\end{aligned}\tag{74}$$

The system (73) contains ten equations with 20 unknowns, while the system (74) consists of eight equations involving the remaining 16 unknowns. So the subsystems under discussion are half-built and they are going to be completed after the remaining boundary conditions are handled.

The coefficients of the unknowns in the above system along with the coefficients appearing in the forthcoming equations are determined as an outcome of the projection process described above. Their full derivation is impossible to be presented here since it is based on several and elongated manipulations of the emerging surface integrals. We though mention that their derivation is totally analytic. All the coefficients are presented in appendix C. The non-homogeneous terms of the right-hand side of these systems are provided exclusively from the incident wave and are totally determined.

Equation (70) expresses the continuity of $\widetilde{R}=\widehat{\mathbf{n}}\cdot\widetilde{\boldsymbol{\mu}}^{1243}\cdot\widehat{\mathbf{n}}$ across the spherical interphase. The complicated form of $\widetilde{\boldsymbol{\mu}}$ given by equation (9) reveals the complexity of the required analysis to apply the projection process on the dyadic spherical harmonics after the substitution of the expansions (65)–(67) is performed. We finally obtain equations of similar structure with (73) and (74) as follows:

$$\begin{aligned}\sum_{j=1}^{2}\left[\hat{A}_{q,e,j}^{(n,m)}a_{n,e,j}^{m}+\hat{C}_{q,e,j}^{(n,m)}c_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[\hat{A}_{q,e,j}^{(n,m)}\alpha_{n,e,j}^{m}+\hat{\Gamma}_{q,e,j}^{(n,m)}\gamma_{n,e,j}^{m}+\hat{E}_{q,e,j}^{(n,m)}\varepsilon_{n,e,j}^{m}\right]\\ -\sum_{j=1}^{2}\left[\hat{A}_{q,i,j}^{(n,m)}a_{n,i,j}^{m}+\hat{C}_{q,i,j}^{(n,m)}c_{n,i,j}^{m}\right]-\sum_{j=3}^{4}\left[\hat{A}_{q,i,j}^{(n,m)}\alpha_{n,i,j}^{m}+\hat{\Gamma}_{q,i,j}^{(n,m)}\gamma_{n,i,j}^{m}+\hat{E}_{q,i,j}^{(n,m)}\varepsilon_{n,i,j}^{m}\right]\\ =\hat{\phi}_{q}^{(n,m)},\qquad \text{for}\quad q=1,2,3,4,5\end{aligned}\tag{75}$$

$$\sum_{j=1}^{2}\left[\hat{B}_{q,e,j}^{(n,m)}b_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[\hat{B}_{q,e,j}^{(n,m)}\beta_{n,e,j}^{m}+\hat{\Delta}_{q,e,j}^{(n,m)}\delta_{n,e,j}^{m}+\hat{Z}_{q,e,j}^{(n,m)}\zeta_{n,e,j}^{m}\right]$$

$$
\begin{aligned}
&-\sum_{j=1}^{2}\left[\hat{B}_{q,i,j}^{(n,m)} b_{n,i,j}^{m}-\sum_{j=3}^{4}\left[\hat{B}_{q,i,j}^{(n,m)} \beta_{n,i,j}^{m}+\hat{\Delta}_{q,i,j}^{(n,m)} \delta_{n,i,j}^{m}+\hat{Z}_{q,i,j}^{(n,m)} \zeta_{n,i,j}^{m}\right]\right. \\
&=\hat{Y}_{q}^{(n,m)}, \qquad \text{for} \quad q=1,2,3,4.
\end{aligned} \tag{76}
$$

The remaining boundary condition involves the continuity of the surface traction $\widetilde{P}$. Using equations (15), (24) and restrict ourselves to the time-harmonic case, we infer that, on the spherical surface,

$$
\begin{aligned}
\widetilde{P}^{s}(\mathbf{r}) = {} & \widehat{\mathbf{r}} \cdot \widetilde{\tau}-\widehat{\mathbf{r}} \widehat{\mathbf{r}}: \frac{\partial \widetilde{\boldsymbol{\mu}}}{\partial r}-\widehat{\mathbf{r}} \cdot\left(\nabla_{S} \cdot \widetilde{\boldsymbol{\mu}}(\mathbf{r})\right)-\widehat{\mathbf{r}} \cdot\left(\nabla_{S} \cdot \widetilde{\boldsymbol{\mu}}^{2134}\right)+\left[\left(\nabla_{S} \cdot \widehat{\mathbf{r}}\right) \widehat{\mathbf{r}} \widehat{\mathbf{r}}-\nabla_{S} \cdot \widehat{\mathbf{r}}\right]: \widetilde{\boldsymbol{\mu}} \\
& -\rho \omega^{2}\left[\left(h_{2}^{2}-\frac{\rho^{\prime} d^{2}}{3 \rho}\right) \widehat{\mathbf{r}} \times(\nabla \times \widetilde{u})+\left(2 h_{2}^{2}-\frac{\rho^{\prime} d^{2}}{3 \rho}\right) \frac{\partial \widetilde{u}}{\partial r}\right. \\
& \left.+\left(h_{1}^{2}-2 h_{2}^{2}+\frac{\rho^{\prime} d^{2}}{3 \rho}\right) \widehat{\mathbf{r}}(\nabla \cdot \widetilde{u}(\mathbf{r}))\right].
\end{aligned}
$$

We remark that only in the surface traction enter the micro-inertia parameters h_1 h_2 and the constants ρ', d. Imposing the continuity of the projection of surface traction on the dyadic spheroidal harmonics on the scatterer's surface leads after extended analysis to the systems

$$
\begin{aligned}
&\sum_{j=1}^{2}\left[\breve{A}_{q,e,j}^{(n,m)} a_{n,e,j}^{m}+\breve{C}_{q,e,j}^{(n,m)} c_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[\breve{A}_{q,e,j}^{(n,m)} \alpha_{n,e,j}^{m}+\breve{\Gamma}_{q,e,j}^{(n,m)} \gamma_{n,e,j}^{m}+\breve{E}_{q,e,j}^{(n,m)} \varepsilon_{n,e,j}^{m}\right] \\
&-\sum_{j=1}^{2}\left[\breve{A}_{q,i,j}^{(n,m)} a_{n,i,j}^{m}+\breve{C}_{q,i,j}^{(n,m)} c_{n,i,j}^{m}\right]-\sum_{j=3}^{4}\left[\breve{A}_{q,i,j}^{(n,m)} \alpha_{n,i,j}^{m}+\breve{\Gamma}_{q,i,j}^{(n,m)} \gamma_{n,i,j}^{m}+\breve{E}_{q,i,j}^{(n,m)} \varepsilon_{n,i,j}^{m}\right] \\
&\qquad =\breve{\Phi}_{q}^{(n,m)}, \qquad \text{for} \quad q=1,2,3,4,5
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
&\sum_{j=1}^{2}\left[\breve{B}_{q,e,j}^{(n,m)} b_{n,e,j}^{m}\right]+\sum_{j=3}^{4}\left[\breve{B}_{q,e,j}^{(n,m)} \beta_{n,e,j}^{m}+\breve{\Delta}_{q,e,j}^{(n,m)} \delta_{n,e,j}^{m}+\breve{Z}_{q,e,j}^{(n,m)} \zeta_{n,e,j}^{m}\right] \\
&\qquad -\sum_{j=1}^{2}\left[\breve{B}_{q,i,j}^{(n,m)} b_{n,i,j}^{m}\right]-\sum_{j=3}^{4}\left[\breve{B}_{q,i,j}^{(n,m)} \beta_{n,i,j}^{m}+\breve{\Delta}_{q,i,j}^{(n,m)} \delta_{n,i,j}^{m}+\breve{Z}_{q,i,j}^{(n,m)} \zeta_{n,i,j}^{m}\right] \\
&\qquad =\breve{Y}_{q}^{(n,m)}, \qquad \text{for} \quad q=1,2,3,4,
\end{aligned} \tag{78}
$$

where all the involved coefficients are presented in appendix C. Collecting the outcome of the above analysis, we remark that the first sub-blocks of the constructed systems (i.e. (73), (75), (77)) constitute a set of 20 equations with 20 unknowns, whilst the second sub-blocks (i.e. (74), (76), (78)) form a set of 16 equations referring to the remaining 16 unknown coefficients of the spectral representation of the dyadic fields. It is very interesting to note that, as presented in appendix C, the non-homogeneous terms $Y_{pq}^{n,m}$, $\widehat{Y}_{q}^{n,m}$, $\breve{Y}_{q}^{n,m}$ vanish for the case we examine, i.e. the plane wave excitation. Consequently only the system (73), (75) and (77) is necessary to be solved providing the pertaining 20 coefficients while the remaining 16 coefficients vanish identically. The numerical investigation of the scattering problem under discussion will be based on the analytic settlement constructed in this paper. It cannot, of course, be included in this paper, since it is an extensive independent and polyparametric task on which the authors already work.

Appendix A. Orthogonality relations of dyadic spherical harmonics

The dyadic spheroidal harmonics are divided in two groups. The first set contains the symmetric dyads

$$\hat{\mathbf{r}}\mathbf{P}_n^m,\ \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s = \frac{\hat{\mathbf{r}}\mathbf{B}_n^m + \mathbf{B}_n^m\hat{\mathbf{r}}}{2}, \qquad \left(\mathbf{D}\mathbf{P}_n^m\right)_s = \frac{\mathbf{D}\mathbf{P}_n^m + \left(\mathbf{D}\mathbf{P}_n^m\right)^T}{2}, \qquad \left(\mathbf{D}\mathbf{B}_n^m\right)_s = \frac{\mathbf{D}\mathbf{B}_n^m + \left(\mathbf{D}\mathbf{B}_n^m\right)^T}{2},$$

$$\left(\mathbf{D}\mathbf{C}_n^m\right)_s = \frac{\mathbf{D}\mathbf{C}_n^m + \left(\mathbf{D}\mathbf{C}_n^m\right)^T}{2}, \qquad \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s = \frac{\hat{\mathbf{r}}\mathbf{C}_n^m + \mathbf{C}_n^m\hat{\mathbf{r}}}{2}$$

and the second one the antisymmetric elements

$$\left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a = \frac{\hat{\mathbf{r}}\mathbf{B}_n^m - \mathbf{B}_n^m\hat{\mathbf{r}}}{2}, \qquad \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a = \frac{\hat{\mathbf{r}}\mathbf{C}_n^m - \mathbf{C}_n^m\hat{\mathbf{r}}}{2}, \qquad \widetilde{\mathbf{I}} \times \mathbf{P}_n^m,$$

all these elements are independent and merit the following orthogonality properties:

$$\int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\hat{\mathbf{r}}\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = \int_{S^2} Y_n^m \overline{Y}_{n'}^{m'}\, \mathrm{d}S = \delta_{nn'}\delta_{mm'} \left\| Y_n^m \right\|_{S^2}^2 \tag{A.1}$$

$$\int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\mathbf{D}\mathbf{P}_{n'}^{m'}}\right)_s \mathrm{d}S = 0 \tag{A.2}$$

$$\int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\mathbf{D}\mathbf{B}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\mathbf{D}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a \mathrm{d}S = 0 \tag{A.3}$$

$$\int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \hat{\mathbf{r}}\mathbf{P}_n^m : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = 0 \tag{A.4}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_s \mathrm{d}S = \frac{\left\|\mathbf{B}_n^m\right\|_{S^2}^2}{2}\delta_{nn'}\delta_{mm'} = \frac{n(n+1)}{2}\delta_{nn'}\delta_{mm'}\left\| Y_n^m \right\|_{S^2}^2 \tag{A.5}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = 0 \tag{A.6}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{P}_{n'}^{m'}}\right)_s \mathrm{d}S = \frac{\left\|\mathbf{B}_n^m\right\|_{S^2}^2}{2}\delta_{nn'}\delta_{mm'} = \frac{n(n+1)}{2}\delta_{nn'}\delta_{mm'}\left\| Y_n^m \right\|_{S^2}^2 \tag{A.7}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{B}_{n'}^{m'}}\right)_s \mathrm{d}S = -\frac{\left\|\mathbf{B}_n^m\right\|_{S^2}^2}{2}\delta_{nn'}\delta_{mm'} = -\frac{n(n+1)}{2}\delta_{nn'}\delta_{mm'}\left\| Y_n^m \right\|_{S^2}^2 \tag{A.8}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = 0 \tag{A.9}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_s : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = 0 \tag{A.10}$$

$$\int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{P}_{n'}^{m'}}\right)_s \mathrm{d}S = \left[2 + \frac{n(n+1)}{2}\right]\delta_{nn'}\delta_{mm'}\left\| Y_n^m \right\|_{S^2}^2 \tag{A.11}$$

$$\int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{B}_{n'}^{m'}}\right)_s \mathrm{d}S = -\frac{3n(n+1)}{2}\delta_{nn'}\delta_{mm'}\left\| Y_n^m \right\|_{S^2}^2 \tag{A.12}$$

$$\int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\mathbf{D}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = 0 \tag{A.13}$$

$$\int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\mathbf{D}\mathbf{P}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = 0 \tag{A.14}$$

$$\int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = 0 \tag{A.15}$$

$$\int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\overline{\mathbf{DB}_{n'}^{m'}}\right)_s \mathrm{d}S = \left[(n(n+1))^2 + \frac{3n(n+1)}{2}\right] \delta_{nn'}\delta_{mm'} \left\| Y_n^m \right\|_{S^2}^2 \tag{A.16}$$

$$\int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\overline{\mathbf{DC}_{n'}^{m'}}\right)_s \mathrm{d}S = \int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a \mathrm{d}S = 0 \tag{A.17}$$

$$\int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\mathbf{DB}_n^m\right)_s : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = 0 \tag{A.18}$$

$$\int_{S^2} \left(\mathbf{DC}_n^m\right)_s : \left(\overline{\mathbf{DC}_{n'}^{m'}}\right)_s \mathrm{d}S = \left[\frac{(n(n+1))^2}{2} + \frac{3n(n+1)}{2}\right] \delta_{nn'}\delta_{mm'} \left\| Y_n^m \right\|_{S^2}^2 \tag{A.19}$$

$$\int_{S^2} \left(\mathbf{DC}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\mathbf{DC}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\mathbf{DC}_n^m\right)_s : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) \mathrm{d}S = 0 \tag{A.20}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a : \left(\overline{\hat{\mathbf{r}}\mathbf{B}_{n'}^{m'}}\right)_a^T \mathrm{d}S = \frac{n(n+1)}{2} \delta_{nn'}\delta_{mm'} \left\| \mathbf{Y}_n^m \right\|^2 \tag{A.21}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a \mathrm{d}S = \int_{S^2} \left(\hat{\mathbf{r}}\mathbf{B}_n^m\right)_a : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right)^T \mathrm{d}S = 0 \tag{A.22}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_a^T \mathrm{d}S = \frac{n(n+1)}{2} \delta_{nn'}\delta_{mm'} \left\| \mathbf{Y}_n^m \right\|^2 \tag{A.23}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_a : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right)^T \mathrm{d}S = 0 \tag{A.24}$$

$$\int_{S^2} \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right) : \left(\tilde{\mathbf{I}} \times \overline{\mathbf{P}_{n'}^{m'}}\right)^T \mathrm{d}S = 2\delta_{nn'}\delta_{mm'} \left\| \mathbf{Y}_n^m \right\|^2 \tag{A.25}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s : \left(\overline{\mathbf{DC}_{n'}^{m'}}\right)_s \mathrm{d}S = -\frac{n(n+1)}{2} \delta_{nn'}\delta_{mm'} \left\| \mathbf{Y}_n^m \right\|^2 \tag{A.26}$$

$$\int_{S^2} \left(\hat{\mathbf{r}}\mathbf{C}_n^m\right)_s : \left(\overline{\hat{\mathbf{r}}\mathbf{C}_{n'}^{m'}}\right)_s \mathrm{d}S = \frac{n(n+1)}{2} \delta_{nn'}\delta_{mm'} \left\| \mathbf{Y}_n^m \right\|^2. \tag{A.27}$$

Appendix B. Investigation of the solenoidal dyadic eigensolutions

Theorem. *The family of solenoidal dyadics*

$$\nabla \times (\mathbf{L} \times \widetilde{\mathbf{I}}), \nabla \times (\mathbf{M} \times \widetilde{\mathbf{I}}), \nabla \times (\mathbf{N} \times \widetilde{\mathbf{I}}), \nabla \times (\mathbf{rL}), \nabla \times (\mathbf{rM}), \nabla \times (\mathbf{rN}),$$

where the abbreviated forms of Navier eigenvalues refer to specific triples (n, m, l) *and wavenumber* k_j, $j = 3, 4$, *form a set of independent dyadics.*

Proof. Let us consider a linear combination of the members of this family to be zero,i.e.,

$$\alpha\nabla \times (\mathbf{L} \times \widetilde{\mathbf{I}}) + \beta\nabla \times (\mathbf{M} \times \widetilde{\mathbf{I}}) + \gamma\nabla \times (\mathbf{N} \times \widetilde{\mathbf{I}}) + \epsilon\nabla \times (\mathbf{rL}) + j\nabla \times (\mathbf{rM}) + n\nabla \times (\mathbf{rN}) = \widetilde{0} \tag{B.1}$$

for every spherical coordinate point (r, θ, ϕ). We intend to prove that the coefficients $\alpha, \beta, \gamma, \epsilon, j, n$ necessarily vanish. Two cases arise:

(i) $n \neq 1$. The linear combination can be written as

$$\alpha[\nabla\mathbf{L} - (\nabla\cdot\mathbf{L})\widetilde{\mathbf{I}}] + \beta(\nabla\mathbf{M})^T + \gamma(\nabla\mathbf{N})^T - \epsilon\mathbf{r}\times\nabla\mathbf{L} - j\mathbf{r}\times\nabla\mathbf{M} - n\mathbf{r}\times\nabla\mathbf{N} = \widetilde{0}. \tag{B.2}$$

Taking the inner product with $\widehat{\mathbf{r}}$ (from the left), we obtain

$$\alpha\left[\frac{\partial\mathbf{L}}{\partial r} - (\nabla\cdot\mathbf{L})\widehat{\mathbf{r}}\right] + \beta\widehat{\mathbf{r}}\cdot(\nabla\mathbf{M})^T + \gamma\widehat{\mathbf{r}}\cdot(\nabla\mathbf{N})^T = \widetilde{0}. \tag{B.3}$$

We need to recall that $\mathbf{L} = \frac{1}{k}\nabla\Phi$ with $\Phi = g(kr)Y(\widehat{\mathbf{r}}) \in ker(\Delta + k^2)$ and that

$$\begin{aligned}
&\mathbf{L} = \dot{g}(kr)\mathbf{P}(\widehat{\mathbf{r}}) + \frac{g(kr)}{kr}\mathbf{P}(\widehat{\mathbf{r}})\\
&\mathbf{M} = \nabla\times(\mathbf{r}\Phi) = g(kr)\mathbf{C}(\widehat{\mathbf{r}})\\
&\mathbf{N} = \frac{1}{k}\nabla\times\mathbf{M} = \frac{n(n+1)g(kr)}{kr}\mathbf{P}(\widehat{\mathbf{r}}) + \left(\dot{g}(kr) + \frac{g(kr)}{kr}\right)\mathbf{B}(\widehat{\mathbf{r}}).
\end{aligned} \tag{B.4}$$

Then

$$\begin{aligned}
\widehat{\mathbf{r}}\cdot(\nabla\mathbf{M})^T &= \frac{1}{r}\mathbf{r}\cdot(\nabla\mathbf{M})^T = \frac{1}{r}\nabla(\mathbf{r}\cdot\mathbf{M}) - \frac{1}{r}\mathbf{M} = -\frac{1}{r}\mathbf{M} = -\frac{1}{r}g(kr)\mathbf{C}(\widehat{\mathbf{r}})\\
\widehat{\mathbf{r}}\cdot(\nabla\mathbf{N})^T &= \frac{1}{r}\nabla(\mathbf{r}\cdot\mathbf{N}) - \frac{\mathbf{N}}{r} = \frac{n(n+1)\dot{g}(kr)}{r}\mathbf{P}(\widehat{\mathbf{r}}) + \left(\frac{n(n+1)g(kr)}{kr^2}\right)\mathbf{B}(\widehat{\mathbf{r}}) - \frac{\mathbf{N}}{r}.
\end{aligned} \tag{B.5}$$

Consequently equation (B.3) becomes

$$\begin{aligned}
&\alpha\left[k\ddot{g}(kr)\mathbf{P}(\widehat{\mathbf{r}}) + \left(\frac{\dot{g}(kr)}{r} - \frac{g(kr)}{kr^2}\right)\mathbf{B}(\widehat{\mathbf{r}}) + kg(kr)\mathbf{P}(\widehat{\mathbf{r}})\right] - \beta\frac{1}{r}g(kr)\mathbf{C}(\widehat{\mathbf{r}})\\
&\quad + \gamma\left[\frac{n(n+1)}{r}\dot{g}(kr)\mathbf{P}(\widehat{\mathbf{r}}) + \frac{n(n+1)}{kr^2}g(kr)\mathbf{B}(\widehat{\mathbf{r}}) - \frac{n(n+1)}{kr^2}g(kr)\mathbf{P}(\widehat{\mathbf{r}})\right.\\
&\quad \left. - \left(\dot{g}(kr) + \frac{g(kr)}{kr}\right)\frac{1}{r}\mathbf{B}(\widehat{\mathbf{r}})\right] = \widetilde{0}.
\end{aligned} \tag{B.6}$$

Exploiting independence of $\mathbf{P}, \mathbf{B}, \mathbf{C}$ we obtain

$$\alpha(kr)^2\ddot{g}(kr) + \gamma n(n+1)(kr)\dot{g}(kr) + [\alpha(kr)^2 - \gamma n(n+1)]g(kr) = 0 \tag{B.7}$$

$$\beta = 0 \tag{B.8}$$

$$\alpha\left(\frac{\dot{g}(kr)}{r} - \frac{g(kr)}{kr^2}\right) + \frac{n(n+1)\gamma}{kr^2}g(kr) - \frac{\gamma}{r}\left(\dot{g}(kr) + \frac{g(kr)}{kr}\right) = 0, \tag{B.9}$$

we recall that $g(kr)$ stands for $g_n(kr)$. As a result, given that $n \neq 1$, equation (B.3) is satisfied only if $\alpha = \gamma = 0$, fact which renders (B.9) automatically valid. (Remark that if $n = 1$, the relation $\alpha = \gamma$ is enough for the satisfaction of equations (B.7) and (B.9).) Vanishing of α, β, γ transforms equation (B.1) to

$$\nabla\times[\epsilon\mathbf{r}\mathbf{L} + j\mathbf{r}\mathbf{M} + n\mathbf{r}\mathbf{N}] = \widetilde{0}. \tag{B.10}$$

So $[\epsilon\mathbf{r}\mathbf{L} + j\mathbf{r}\mathbf{M} + n\mathbf{r}\mathbf{N}] = \nabla\xi$ for some vector function ξ. As a result

$$\epsilon\mathbf{L} + j\mathbf{M} + n\mathbf{N} = \frac{1}{r}\frac{\partial\xi}{\partial r} \tag{B.11}$$

and

$$\frac{\partial \xi}{\partial \phi} = \frac{\partial \xi}{\partial \theta} = 0. \qquad \text{(B.12)}$$

We immediately obtain that $\epsilon = j = n = 0$.

We examine now the second case:

(ii) $n = 1$. As already stated the satisfaction of equation (B.3) is established only if $\alpha = \gamma$ and $\beta = 0$. Taking contraction in the dyadic relation (B.2) we get

$$-2\alpha(\nabla \cdot \mathbf{L}) - j\mathbf{r} \cdot (\nabla \times \mathbf{M}) = 0$$
$$\Rightarrow 2\alpha k\Phi - jk\mathbf{r} \cdot \mathbf{N} = 0 \Rightarrow (2\alpha k - 2j)\Phi = 0 \Rightarrow j = ak. \qquad \text{(B.13)}$$

Taking the exterior product in the place of juxtaposition in relation (B.2) we obtain

$$-\alpha k\mathbf{M} + \epsilon(-k\mathbf{r}\Phi - \nabla(\mathbf{L} \cdot \mathbf{r}) + \mathbf{L}) + \alpha k\mathbf{M} - n\nabla(\mathbf{N} \cdot \mathbf{r}) + n\mathbf{N} = \widetilde{0}. \qquad \text{(B.14)}$$

It can easily be shown that the last equation holds if and only if $\epsilon = n$. Consequently, the combination (B.1) obtains the form

$$\alpha k\left[\frac{1}{k}\nabla \times (\mathbf{L} \times \widetilde{\mathbf{I}}) + \frac{1}{k}\nabla \times (\mathbf{N} \times \widetilde{\mathbf{I}}) + \nabla \times (\mathbf{rM})\right] + \epsilon[\nabla \times (\mathbf{rL}) + \nabla \times (\mathbf{rN})] = \widetilde{0}. \qquad \text{(B.15)}$$

Using the representations of the eigendyadics in equation (B.15) in terms of the spherical dyadic harmonics, projecting on the spherical dyadics $\widehat{\mathbf{r}}\mathbf{P}$ and $(\widehat{\mathbf{r}}\mathbf{C})_a$ and using the orthogonality relations presented in appendix A we obtain that $\epsilon = \alpha = 0$. □

Appendix C. The elements of the linear algebraic subsystems

We introduce the following terminology for all the elements of the linear subsystems under consideration:

$$\mathcal{H}^{(n,m)}_{1q,t,j} = \mathcal{H}^{n,m}_{q,t,j}\big|_{r=a}, \qquad \mathcal{H} \in \{A, C, \Gamma, E, B, \Delta, Z\}, \qquad t = e, i, \; j = 1, 2, 3, 4 \qquad \text{(C.1)}$$

$$\mathcal{H}^{(n,m)}_{2q,t,j} = \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{H}^{n,m}_{q,t,j}\bigg|_{r=a}. \qquad \text{(C.2)}$$

In addition wherever the gradient coefficients a_{tj}, $j = 1, 2, \ldots, 5$ appear, the index t is omitted. Then the elements of the subsystems are given by the following relations:

$$A^{(n,m)}_{1,t,j}(r) = \ddot{g}_n(k_{t,j}r), \qquad j = 1, 2 \qquad \text{(C.3)}$$

$$A^{(n,m)}_{1,t,j}(r) = -\frac{2\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{n(n+1)g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 3, 4 \qquad \text{(C.4)}$$

$$A^{(n,m)}_{2,t,j}(r) = 2\left[\frac{\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)^2}\right], \qquad j = 1, 2 \qquad \text{(C.5)}$$

$$A^{(n,m)}_{2,t,j}(r) = 2\left[\frac{\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)^2}\right], \qquad j = 3, 4 \qquad \text{(C.6)}$$

$$A^{(n,m)}_{3,t,j}(r) = \frac{[2 + n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - \frac{2n(n+1)g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 1, 2 \qquad \text{(C.7)}$$

$$A_{3,t,j}^{(n,m)}(r) = \frac{[2+n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + 2g_n(k_{t,j}r) - \frac{2n(n+1)g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 3, 4 \tag{C.8}$$

$$A_{4,t,j}^{(n,m)}(r) = -\frac{[2n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{[n^2(n+1)^2 + 2n(n+1)]g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 1, 2 \tag{C.9}$$

$$A_{4,t,j}^{(n,m)}(r) = -\frac{[2n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - n(n+1)g_n(k_{t,j}r) + \frac{[n^2(n+1)^2 + 2n(n+1)]g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 3, 4 \tag{C.10}$$

$$A_{5,t,j}^{(n,m)}(r) = 0, \; j = 1, 2, 3, 4 \tag{C.11}$$

$$C_{1,t,j}^{(n,m)}(r) = n(n+1)\left[\frac{\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{g_n(k_{t,j}r)}{(k_{t,j}r)^2}\right], \; j = 1, 2 \tag{C.12}$$

$$C_{2,t,j}^{(n,m)}(r) = -\frac{2\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{2n(n+1)g_n(k_{t,j}r)}{(k_{t,j}r)^2} - (1 + \frac{2}{(k_{t,j}r)^2})g_n(k_{t,j}r) j = 1, 2 \tag{C.13}$$

$$C_{3,t,j}^{(n,m)}(r) = -\frac{(2n(n+1))\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - \frac{n(n+1)}{2}g_n(k_{t,j}r) + \frac{n^2(n+1)^2 g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \quad j = 1, 2 \tag{C.14}$$

$$C_{4,t,j}^{(n,m)}(r) = \frac{[n^2(n+1)^2 + 2n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{n(n+1)}{2}g_n(k_{t,j}r) + \frac{[-2n^2(n+1)^2 + 2n(n+1)]g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 1, 2 \tag{C.15}$$

$$C_{5,t,j}^{(n,m)}(r) = -g_n(k_{t,j}r), \qquad j = 1, 2 \tag{C.16}$$

$$\Gamma_{1,t,j}^{(n,m)}(r) = n(n+1)\left[\frac{\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{g_n(k_{t,j}r)}{(k_{t,j}r)^2}\right], \qquad j = 3, 4 \tag{C.17}$$

$$\Gamma_{2,t,j}^{(n,m)}(r) = -\frac{2\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{2n(n+1)g_n(k_{t,j}r)}{(k_{t,j}r)^2} - \left(1 + \frac{2}{(k_{t,j}r)^2}\right)g_n(k_{t,j}r), \quad j = 3, 4 \tag{C.18}$$

$$\Gamma_{3,t,j}^{(n,m)}(r) = -\frac{(2n(n+1))\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} - \frac{n(n+1)}{2}g_n(k_{t,j}r) + \frac{n^2(n+1)^2 g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \quad j = 3, 4 \tag{C.19}$$

$$\Gamma_{4,t,j}^{(n,m)}(r) = \frac{[n^2(n+1)^2 + 2n(n+1)]\dot{g}_n(k_{t,j}r)}{(k_{t,j}r)} + \frac{n(n+1)}{2}g_n(k_{t,j}r) + \frac{[2n(n+1) - n^2(n+1)^2]g_n(k_{t,j}r)}{(k_{t,j}r)^2}, \qquad j = 3, 4 \tag{C.20}$$

$$\Gamma_{5,t,j}^{(n,m)}(r) = g_n(k_{t,j}r), \qquad j = 3, 4 \tag{C.21}$$

$$E_{1,t,j}^{(n,m)}(r) = 0, \qquad j = 3, 4 \tag{C.22}$$

$$E_{2,t,j}^{(n,m)}(r) = -g_n(k_{t,j}r), \qquad j = 3, 4 \tag{C.23}$$

$$E_{3,t,j}^{(n,m)}(r) = -\frac{5}{2}n(n+1)g_n(k_{t,j}r), \qquad j = 3, 4 \tag{C.24}$$

$$E_{4,t,j}^{(n,m)}(r) = \left[n^2(n+1)^2 + \frac{n(n+1}{2}\right]g_n(k_{t,j}r), \qquad j = 3, 4 \tag{C.25}$$

$$E_{5,t,j}^{(n,m)}(r) = g_n(k_{t,j}r), \qquad j = 3, 4 \tag{C.26}$$

$$B_{1,t,j}^{(n,m)}(r) = \dot{g}_n(k_{t,j}r) - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 1, 2, 3, 4 \tag{C.27}$$

$$B_{2,t,j}^{(n,m)}(r) = -\dot{g}_n(k_{t,j}r) + [n(n+1)+3]\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 1, 2, 3, 4 \tag{C.28}$$

$$B_{3,t,j}^{(n,m)}(r) = \dot{g}_n(k_{t,j}r) + \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 1, 2 \tag{C.29}$$

$$B_{3,t,j}^{(n,m)}(r) = -\left(\dot{g}_n(k_{t,j}r) + \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right), \qquad j = 3, 4 \tag{C.30}$$

$$B_{4,t,j}^{(n,m)}(r) = -\frac{n(n+1)}{2}\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}), \qquad j = 1, 2 \tag{C.31}$$

$$B_{4,t,j}^{(n,m)}(r) = \frac{n(n+1)}{2}\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}), \qquad j = 3, 4 \tag{C.32}$$

$$\Delta_{1,t,j}^{(n,m)}(r) = \dot{g}_n(k_{t,j}r) - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.33}$$

$$\Delta_{2,t,j}^{(n,m)}(r) = -\dot{g}_n(k_{t,j}r) + [n(n+1)+3]\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.34}$$

$$\Delta_{3,t,j}^{(n,m)}(r) = -\left(\dot{g}_n(k_{t,j}r) - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right), \qquad j = 3, 4 \tag{C.35}$$

$$\Delta_{4,t,j}^{(n,m)}(r) = \dot{g}_n(k_{t,j}r) - \frac{n(n+1)}{2}\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.36}$$

$$Z_{1,t,j}^{(n,m)}(r) = n(n+1)\frac{g_n(k_{t,j}r)}{(k_{t,j}r)} - \dot{g}_n(k_{t,j}r) - \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.37}$$

$$Z_{2,t,j}^{(n,m)}(r) = [n(n+1)+3]\dot{g}_n(k_{t,j}r) + 3\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.38}$$

$$Z_{3,t,j}^{(n,m)}(r) = -n(n+1)\frac{g_n(k_{t,j}r)}{(k_{t,j}r)} + \dot{g}_n(k_{t,j}r) + \frac{g_n(k_{t,j}r)}{(k_{t,j}r)}, \qquad j = 3, 4 \tag{C.39}$$

$$Z_{4,t,j}^{(n,m)}(r) = \frac{n(n+1)}{2}\left[\dot{g}_n(k_{t,j}r) + 3\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right], \qquad j = 3, 4 \tag{C.40}$$

$$\begin{aligned}\hat{A}_{1,t,j}^{(n,m)} = &-\frac{1}{2}(3a_1 + 2a_2 + 2a_3)k_{t,j}^2\ddot{g}_n(k_{t,j}a) \\ &- \frac{1}{2}(a_1 + 2a_3)A_{1,1,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4 + a_5)k_{t,j}^2\left.\frac{\mathrm{d}^2 A_{1,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a}, \qquad j = 1, 2\end{aligned} \tag{C.41}$$

$$\hat{A}_{2,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2 \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}(a_1+2a_3)A_{1,2,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{2,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.42}$$

$$\hat{A}_{3,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2 \frac{n(n+1)}{2}\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}(a_1+2a_3)A_{1,3,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{3,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.43}$$

$$\hat{A}_{4,t,j}^{(n,m)} = +\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2 \frac{n(n+1)}{2}\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}(a_1+2a_3)A_{1,4,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{4,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.44}$$

$$\hat{A}_{5,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2 \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}, \qquad j=1,2 \tag{C.45}$$

$$\hat{A}_{1,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2[\ddot{g}_n(k_{t,j}a)+g_n(k_{t,j}a)] - \frac{1}{2}a_3A_{1,1,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{1,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=3,4 \tag{C.46}$$

$$\hat{A}_{2,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2 \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}a_3A_{1,2,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{2,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a} + \left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2 \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}, \qquad j=3,4 \tag{C.47}$$

$$\hat{A}_{3,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2 \frac{n(n+1)}{2}\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}a_3A_{1,3,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2 \frac{\mathrm{d}^2 A_{3,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a} + \left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2\left(\frac{n(n+1)}{2}\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - 2\ddot{g}_n(k_{t,j}a)\right), \qquad j=3,4 \tag{C.48}$$

$$\hat{A}_{4,t,j}^{(n,m)} = \frac{1}{2}(a_1+a_3)k_{t,j}^2\left(\frac{n(n+1)}{2}\right)\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}a_3 A_{1,4,t,j}^{(n,m)}k_{t,j}^2$$
$$+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 A_{4,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a} + \left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2\left(-\frac{n(n+1)}{2}\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right.$$
$$\left.+n(n+1)\ddot{g}_n(k_{t,j}a)\right), \qquad j=3,4 \tag{C.49}$$

$$\hat{A}_{5,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}$$
$$+\left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}, \qquad j=3,4 \tag{C.50}$$

$$\hat{C}_{1,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)n(n+1)k_{t,j}^2\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a} - \frac{1}{2}(a_1+2a_3)C_{1,1,t,j}^{(n,m)}k_{t,j}^2$$
$$-\frac{1}{2}(2a_1+2a_2+a_3)n(n+1)k_{t,j}^2\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}$$
$$+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 C_{1,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.51}$$

$$\hat{C}_{2,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2\left(\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)$$
$$-\frac{1}{2}(a_1+2a_3)C_{1,2,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 C_{2,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.52}$$

$$\hat{C}_{3,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2\frac{n(n+1)}{2}\left(\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)$$
$$-\frac{1}{2}(a_1+2a_3)C_{1,3,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 C_{3,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.53}$$

$$\hat{C}_{4,t,j}^{(n,m)} = \frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2\left(\frac{n(n+1)}{2}\right)\left(\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)$$
$$-\frac{1}{2}(a_1+2a_3)C_{1,4,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 C_{4,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.54}$$

$$\hat{C}_{5,t,j}^{(n,m)} = -\frac{1}{2}(3a_1+2a_2+2a_3)k_{t,j}^2\left(\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)$$
$$-\frac{1}{2}(a_1+2a_3)C_{1,5,t,j}^{(n,m)}k_{t,j}^2 + 2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2 C_{5,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j=1,2 \tag{C.55}$$

$$\hat{\Gamma}_{1,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2\left(-k_{t,j}a\dot{g}_n(k_{t,j}a)+n(n+1)\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)$$
$$-\frac{1}{2}a_3\Gamma_{1,1,t,j}^{(n,m)}k_{t,j}^2+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2\Gamma_{1,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}+\left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2$$
$$\times\left[n(n+1)\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}-(k_{t,j}a)\dddot{g}_n(k_{t,j}a)-3\ddot{g}_n(k_{t,j}a)\right.$$
$$\left.-(k_{t,j}a)\dot{g}_n(k_{t,j}a)-g_n(k_{t,j}a)\right],\qquad j=3,4 \tag{C.56}$$

$$\hat{\Gamma}_{2,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)-\frac{1}{2}a_3\Gamma_{1,2,t,j}^{(n,m)}k_{t,j}^2$$
$$+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2\Gamma_{2,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}$$
$$+\left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right),\qquad j=3,4 \tag{C.57}$$

$$\hat{\Gamma}_{3,t,j}^{(n,m)} = -\frac{1}{2}(a_1+a_3)k_{t,j}^2\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)-\frac{1}{2}a_3\Gamma_{1,3,t,j}^{(n,m)}k_{t,j}^2$$
$$+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2\Gamma_{3,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}+\left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2$$
$$\times\left[\frac{n(n+1}{2}\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)-2((k_{t,j}a)\dddot{g}_n(k_{t,j}a)\right.$$
$$\left.+3\ddot{g}_n(k_{t,j}a)+(k_{t,j}a)\dot{g}_n(k_{t,j}a)+g_n(k_{t,j}a))\right],\qquad j=3,4 \tag{C.58}$$

$$\hat{\Gamma}_{4,t,j}^{(n,m)} = \frac{1}{2}(a_1+a_3)k_{t,j}^2\frac{n(n+1)}{2}\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)-\frac{1}{2}a_3\Gamma_{1,4,t,j}^{(n,m)}k_{t,j}^2$$
$$+2(a_4+a_5)k_{t,j}^2\frac{\mathrm{d}^2\Gamma_{4,t,j}^{(n,m)}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}+\left(a_4+\frac{3}{2}a_5\right)k_{t,j}^2$$
$$\times\left[-\frac{n(n+1)}{2}\left(\ddot{g}_n(k_{t,j}a)+\frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right)+n(n+1)((k_{t,j}a)\dddot{g}_n(k_{t,j}a)\right.$$
$$\left.+3\ddot{g}_n(k_{t,j}a)+(k_{t,j}a)\dot{g}_n(k_{t,j}a)+g_n(k_{t,j}a))\right],\qquad j=3,4 \tag{C.59}$$

$$\hat{\Gamma}^{(n,m)}_{5,t,j} = -\frac{1}{2}(a_1 + a_3)k^2_{t,j}\left(\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right) - \frac{1}{2}a_3\Gamma^{(n,m)}_{1,5,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2\Gamma^{(n,m)}_{5,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a} + \left(a_4 + \frac{3}{2}a_5\right)k^2_{t,j}(-1) \times\left[\ddot{g}_n(k_{t,j}a) + \frac{\mathrm{d}}{\mathrm{d}(k_{t,j}r)}\left(\frac{g_n(k_{t,j}r)}{(k_{t,j}r)}\right)\bigg|_{r=a}\right], \qquad j = 3, 4 \tag{C.60}$$

$$\hat{E}^{(n,m)}_{1,t,j} = 0, \qquad j = 3, 4 \tag{C.61}$$

$$\hat{E}^{(n,m)}_{2,t,j} = -\frac{1}{2}a_3E^{(n,m)}_{1,2,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2E^{(n,m)}_{2,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 3, 4 \tag{C.62}$$

$$\hat{E}^{(n,m)}_{3,t,j} = -\frac{1}{2}a_3E^{(n,m)}_{1,3,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2E^{(n,m)}_{3,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 3, 4 \tag{C.63}$$

$$\hat{E}^{(n,m)}_{4,t,j} = -\frac{1}{2}a_3E^{(n,m)}_{1,4,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2E^{(n,m)}_{4,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 3, 4 \tag{C.64}$$

$$\hat{E}^{(n,m)}_{5,t,j} = -\frac{1}{2}a_3E^{(n,m)}_{1,5,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2E^{(n,m)}_{5,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 3, 4 \tag{C.65}$$

$$\hat{B}^{(n,m)}_{1,t,j} = -\frac{1}{2}(3a_1 + 2a_2 + 2a_3)k^2_{t,j}\dot{g}_n(k_{t,j}a) - \frac{1}{2}(a_1 + 2a_3)B^{(n,m)}_{1,1,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2B^{(n,m)}_{1,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 1, 2 \tag{C.66}$$

$$\hat{B}^{(n,m)}_{2,t,j} = \frac{1}{2}(3a_1 + 2a_2 + 2a_3)k^2_{t,j}\dot{g}_n(k_{t,j}a) - \frac{1}{2}(a_1 + 2a_3)B^{(n,m)}_{1,2,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2B^{(n,m)}_{2,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 1, 2 \tag{C.67}$$

$$\hat{B}^{(n,m)}_{3,t,j} = -\frac{1}{2}(3a_1 + 2a_2 + 2a_3)k^2_{t,j}\dot{g}_n(k_{t,j}a) - \frac{1}{2}(a_1 + 2a_3)B^{(n,m)}_{1,3,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2B^{(n,m)}_{3,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 1, 2 \tag{C.68}$$

$$\hat{B}^{(n,m)}_{4,t,j} = -\frac{1}{2}(a_1 + 2a_3)B^{(n,m)}_{1,4,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2B^{(n,m)}_{4,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a}, \qquad j = 1, 2 \tag{C.69}$$

$$\hat{B}^{(n,m)}_{1,t,j} = \frac{1}{2}(a_1 + a_3)k^2_{t,j}\left(\frac{g_n(k_{t,j}a)}{(k_{t,j}a)}\right) - \frac{1}{2}a_3B^{(n,m)}_{1,1,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\frac{\mathrm{d}^2B^{(n,m)}_{1,t,j}}{\mathrm{d}(k_{t,j}r)^2}\bigg|_{r=a} + \left(a_4 + \frac{3}{2}a_5\right)k^2_{t,j}\dot{g}_n(k_{t,j}a), \qquad j = 3, 4 \tag{C.70}$$

$$\hat{B}^{(n,m)}_{2,t,j} = -\frac{1}{2}(a_1 + a_3)k^2_{t,j}\frac{g_n(k_{t,j}a)}{(k_{t,j}a)} - \frac{1}{2}a_3 B^{(n,m)}_{1,2,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\left.\frac{\mathrm{d}^2 B^{(n,m)}_{2,t,j}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a} + \left(a_4 + \frac{3}{2}a_5\right)k^2_{t,j}(-\dot{g}_n(k_{t,j}a)), \qquad j = 3, 4 \tag{C.71}$$

$$\hat{B}^{(n,m)}_{3,t,j} = \frac{1}{2}(a_1 + a_3)k^2_{t,j}\left(\frac{g_n(k_{t,j}a)}{(k_{t,j}a)}\right) - \frac{1}{2}a_3 B^{(n,m)}_{1,3,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\left.\frac{\mathrm{d}^2 B^{(n,m)}_{3,t,j}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a} + \left(a_4 + \frac{3}{2}a_5\right)k^2_{t,j}(-\dot{g}_n(k_{t,j}a)), \qquad j = 3, 4 \tag{C.72}$$

$$\hat{B}^{(n,m)}_{4,t,j} = -\frac{1}{2}a_3 B^{(n,m)}_{1,4,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\left.\frac{\mathrm{d}^2 B^{(n,m)}_{4,t,j}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a}, \qquad j = 3, 4 \tag{C.73}$$

$$\hat{\Delta}^{(n,m)}_{l,t,j} = -\frac{1}{2}a_3 \Delta^{(n,m)}_{1,l,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\left.\frac{\mathrm{d}^2 \Delta^{(n,m)}_{l,t,j}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a}, \qquad l = 1, 2, 3, 4, \qquad j = 3, 4 \tag{C.74}$$

$$\hat{Z}^{(n,m)}_{l,t,j} = -\frac{1}{2}a_3 Z^{(n,m)}_{1,l,t,j}k^2_{t,j} + 2(a_4 + a_5)k^2_{t,j}\left.\frac{\mathrm{d}^2 Z^{(n,m)}_{l,t,j}}{\mathrm{d}(k_{t,j}r)^2}\right|_{r=a}, \qquad l = 1, 2, 3, 4, \qquad j = 3, 4 \tag{C.75}$$

$$\breve{A}^{(n,m)}_{1,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{5}{2}a_1 + 2a_2 + 3a_3 + 4a_4 + 4a_5\right)\right]A^{(n,m)}_{2,1,t,j} - k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]\dot{g}_n(k_{t,j}a) + \frac{k^2_{t,j}}{a}[(3a_1 + 2a_2 + 4a_3 + 2a_4 + 2a_5)]A^{(n,m)}_{1,1,t,j} + \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{1,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 1, 2 \tag{C.76}$$

$$\breve{A}^{(n,m)}_{2,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{5}{2}a_1 + 2a_2 + 3a_3 + 4a_4 + 4a_5\right)\right]A^{(n,m)}_{2,2,t,j} - k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]\frac{g_n(k_{t,j}a)}{k_{t,j}a} + \frac{k^2_{t,j}}{a}[(3a_1 + 2a_2 + 4a_3 + 2a_4 + 2a_5)]A^{(n,m)}_{1,2,t,j} + \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{2,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 1, 2 \tag{C.77}$$

$$\breve{A}^{(n,m)}_{3,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{5}{2}a_1 + 2a_2 + 3a_3 + 4a_4 + 4a_5\right)\right]A^{(n,m)}_{2,3,t,j} - k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]\frac{n(n+1)}{2}\frac{g_n(k_{t,j}a)}{k_{t,j}a} + \frac{k^2_{t,j}}{a}[(3a_1 + 2a_2 + 4a_3 + 2a_4 + 2a_5)]A^{(n,m)}_{1,3,t,j} + \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{3,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 1, 2 \tag{C.78}$$

$$\breve{A}^{(n,m)}_{4,t,j}(r) = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{5}{2}a_1 + 2a_2 + 3a_3 + 4a_4 + 4a_5\right)\right] A^{(n,m)}_{2,4,t,j}$$
$$+ k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]\frac{n(n+1)}{2}\frac{g_n(k_{t,j}a)}{k_{t,j}a}$$
$$+ \frac{k^2_{t,j}}{a}[(3a_1 + 2a_2 + 4a_3 + 2a_4 + 2a_5)]A^{(n,m)}_{1,4,t,j} + \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{4,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 1, 2 \tag{C.79}$$

$$\breve{A}^{(n,m)}_{5,t,j} = -k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]$$
$$\times \frac{g_n(k_{t,j}a)}{k_{t,j}a} + \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{5,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 1, 2 \tag{C.80}$$

$$\breve{A}^{(n,m)}_{1,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right] A^{(n,m)}_{2,1,t,j}$$
$$+ \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{1,t,j}}{\mathrm{d}r}\right|_{r=a} + \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right] A^{(n,m)}_{1,1,t,j}, \qquad j = 3, 4 \tag{C.81}$$

$$\breve{A}^{(n,m)}_{2,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right] A^{(n,m)}_{2,2,t,j}$$
$$+ k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{1}{2}a_3 + a_4 + \frac{3}{2}a_5\right)\right]\left(\frac{g_n(k_{t,j}a)}{(k_{t,j}a)}\right)$$
$$+ \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{2,t,j}}{\mathrm{d}r}\right|_{r=a} + \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right] A^{(n,m)}_{1,2,t,j}, \qquad j = 3, 4 \tag{C.82}$$

$$\breve{A}^{(n,m)}_{3,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right] A^{(n,m)}_{2,3,t,j}$$
$$+ k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{1}{2}a_3 + a_4 + \frac{3}{2}a_5\right)\right]$$
$$\times\left(\frac{n(n+1)}{2}\frac{g_n(k_{t,j}a)}{(k_{t,j}a)} - 2\dot{g}_n(k_{t,j}a)\right)$$
$$+ \left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{3,t,j}}{\mathrm{d}r}\right|_{r=a} + \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right] A^{(n,m)}_{1,3,t,j}, \qquad j = 3, 4 \tag{C.83}$$

$$\breve{A}^{(n,m)}_{4,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right] A^{(n,m)}_{2,4,t,j}$$
$$+ k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h^2_{2,t} - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{1}{2}a_3 + a_4 + \frac{3}{2}a_5\right)\right]$$

$$\times\left(-\frac{n(n+1)}{2}\frac{g_n(k_{t,j}a)}{(k_{t,j}a)}+n(n+1)\dot{g}_n(k_{t,j}a)\right)$$
$$+\left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{4,t,j}}{\mathrm{d}r}\right|_{r=a}+\frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]A^{(n,m)}_{1,4,t,j},\qquad j=3,4 \qquad (C.84)$$

$$\breve{A}^{(n,m)}_{5,t,j}=\left[2\mu_t-\rho_t\omega^2\left(2h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{1}{2}a_1+\frac{3}{2}a_3+3a_4+\frac{5}{2}a_5\right)\right]A^{(n,m)}_{2,5,t,j}$$
$$+k_{t,j}\left[\mu_t-\rho_t\omega^2\left(h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{1}{2}a_1+\frac{1}{2}a_3+a_4+\frac{3}{2}a_5\right)\right]\left(-\frac{g_n(k_{t,j}a)}{(k_{t,j}a)}\right)$$
$$+\left.\frac{\mathrm{d}\hat{A}^{(n,m)}_{5,t,j}}{\mathrm{d}r}\right|_{r=a}+\frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]A^{(n,m)}_{1,5,t,j},\qquad j=3,4 \qquad (C.85)$$

$$\breve{C}^{(n,m)}_{1,t,j}=\left[2\mu_t-\rho_t\omega^2\left(2h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{5}{2}a_1+2a_2+3a_3+4a_4+4a_5\right)\right]C^{(n,m)}_{2,1,t,j}$$
$$-k_{t,j}\left[\lambda_t-\rho_t\omega^2\left(h^2_{1,t}-2h^2_{2,t}+\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{3}{2}a_1+2a_2+a_3\right)\right]n(n+1)\frac{g_n(k_{t,j}a)}{k_{t,j}a}$$
$$+\frac{k^2_{t,j}}{a}[(3a_1+2a_2+4a_3+2a_4+2a_5)]C^{(n,m)}_{1,1,t,j}+\left.\frac{\mathrm{d}\hat{C}^{(n,m)}_{1,t,j}}{\mathrm{d}r}\right|_{r=a},\qquad j=1,2 \qquad (C.86)$$

$$\breve{C}^{(n,m)}_{2,t,j}=\left[2\mu_t-\rho_t\omega^2\left(2h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{5}{2}a_1+2a_2+3a_3+4a_4+4a_5\right)\right]C^{(n,m)}_{2,2,t,j}$$
$$-k_{t,j}\left[\lambda_t-\rho_t\omega^2\left(h^2_{1,t}-2h^2_{2,t}+\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{3}{2}a_1+2a_2+a_3\right)\right]\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right)$$
$$+\frac{k^2_{t,j}}{a}[(3a_1+2a_2+4a_3+2a_4+2a_5)]C^{(n,m)}_{1,2,t,j}+\left.\frac{\mathrm{d}\hat{C}^{(n,m)}_{2,t,j}}{\mathrm{d}r}\right|_{r=a},\qquad j=1,2 \qquad (C.87)$$

$$\breve{C}^{(n,m)}_{3,t,j}=\left[2\mu_t-\rho_t\omega^2\left(2h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{5}{2}a_1+2a_2+3a_3+4a_4+4a_5\right)\right]C^{(n,m)}_{2,3,t,j}$$
$$-k_{t,j}\left[\lambda_t-\rho_t\omega^2\left(h^2_{1,t}-2h^2_{2,t}+\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{3}{2}a_1+2a_2+a_3\right)\right]$$
$$\times\frac{n(n+1)}{2}\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right)+\frac{k^2_{t,j}}{a}[(3a_1+2a_2+4a_3+2a_4+2a_5)]C^{(n,m)}_{1,3,t,j}$$
$$+\left.\frac{\mathrm{d}\hat{C}^{(n,m)}_{3,t,j}}{\mathrm{d}r}\right|_{r=a},\qquad j=1,2 \qquad (C.88)$$

$$\breve{C}^{(n,m)}_{4,t,j}=\left[2\mu_t-\rho_t\omega^2\left(2h^2_{2,t}-\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{5}{2}a_1+2a_2+3a_3+4a_4+4a_5\right)\right]C^{(n,m)}_{2,4,t,j}$$
$$+k_{t,j}\left[\lambda_t-\rho_t\omega^2\left(h^2_{1,t}-2h^2_{2,t}+\frac{\rho'_t d^2_t}{3\rho_t}\right)+k^2_{t,j}\left(\frac{3}{2}a_1+2a_2+a_3\right)\right]$$

$$\times \frac{n(n+1)}{2}\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right) + \frac{k_{t,j}^2}{a}[(3a_1+2a_2+4a_3+2a_4+2a_5)]C_{1,4,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{C}_{4,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=1,2 \quad \text{(C.89)}$$

$$\check{C}_{5,t,j}^{(n,m)} = \left[2\mu_t - \rho_t\omega^2\left(2h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{5}{2}a_1+2a_2+3a_3+4a_4+4a_5\right)\right]C_{2,5,t,j}^{(n,m)} - k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h_{1,t}^2 - 2h_{2,t}^2 + \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{3}{2}a_1+2a_2+a_3\right)\right]\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right) + \frac{k_{t,j}^2}{a}[(3a_1+2a_2+4a_3+2a_4+2a_5)]C_{1,5,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{C}_{5,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=1,2 \quad \text{(C.90)}$$

$$\check{\Gamma}_{1,t,j}^{(n,m)} = \left[2\mu_t - \rho_t\omega^2\left(2h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{3}{2}a_3+3a_4+\frac{5}{2}a_5\right)\right]\Gamma_{2,1,t,j}^{(n,m)} + \frac{k_{t,j}^2}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]\Gamma_{1,1,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{\Gamma}_{1,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=3,4 \quad \text{(C.91)}$$

$$\check{\Gamma}_{2,t,j}^{(n,m)}(r) = \left[2\mu_t - \rho_t\omega^2\left(2h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{3}{2}a_3+3a_4+\frac{5}{2}a_5\right)\right]\Gamma_{2,2,t,j}^{(n,m)} + k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{1}{2}a_3+a_4+\frac{3}{2}a_5\right)\right]\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right) + \frac{k_{t,j}^2}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]\Gamma_{1,2,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{\Gamma}_{2,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=3,4 \quad \text{(C.92)}$$

$$\check{\Gamma}_{3,t,j}^{(n,m)} = \left[2\mu_t - \rho_t\omega^2\left(2h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{3}{2}a_3+3a_4+\frac{5}{2}a_5\right)\right]\Gamma_{2,3,t,j}^{(n,m)} + k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{1}{2}a_3+a_4+\frac{3}{2}a_5\right)\right]\frac{n(n+1)}{2} \times\left(\dot{g}_n(k_{t,j}a)-3\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right) + \frac{k_{t,j}^2}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]\Gamma_{1,3,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{\Gamma}_{3,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=3,4 \quad \text{(C.93)}$$

$$\check{\Gamma}_{4,t,j}^{(n,m)} = \left[2\mu_t - \rho_t\omega^2\left(2h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{3}{2}a_3+3a_4+\frac{5}{2}a_5\right)\right]\Gamma_{2,4,t,j}^{(n,m)} + k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h_{2,t}^2 - \frac{\rho_t' d_t^2}{3\rho_t}\right) + k_{t,j}^2\left(\frac{1}{2}a_1+\frac{1}{2}a_3+a_4+\frac{3}{2}a_5\right)\right] \times\left[-\frac{n(n+1)}{2}\left(\dot{g}_n(k_{t,j}a)+\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right)+n^2(n+1)^2\frac{g_n(k_{t,j}a)}{k_{t,j}a}\right] + \frac{k_{t,j}^2}{a}\left[\left(\frac{1}{2}a_1+2a_3+a_4+\frac{1}{2}a_5\right)\right]\Gamma_{1,4,t,j}^{(n,m)} + \left.\frac{\mathrm{d}\hat{\Gamma}_{4,t,j}^{(n,m)}}{\mathrm{d}r}\right|_{r=a}, \qquad j=3,4 \quad \text{(C.94)}$$

$$\breve{\Gamma}^{(n,m)}_{5,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right]\Gamma^{(n,m)}_{2,5,t,j}$$
$$+ k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{1}{2}a_3 + a_4 + \frac{3}{2}a_5\right)\right]$$
$$\times\left(-\dot{g}_n(k_{t,j}a) - \frac{g_n(k_{t,j}a)}{k_{t,j}a}\right)$$
$$+ \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right]\Gamma^{(n,m)}_{1,5,t,j} + \left.\frac{\mathrm{d}\hat{\Gamma}^{(n,m)}_{5,t,j}}{\mathrm{d}r}\right|_{r=a}, \qquad j = 3, 4 \qquad \text{(C.95)}$$

$$\breve{E}^{(n,m)}_{s,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right]E^{(n,m)}_{2,s,t,j}$$
$$+ \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right]E^{(n,m)}_{1,s,t,j} + \left.\frac{\mathrm{d}\hat{E}^{(n,m)}_{s,t,j}}{\mathrm{d}r}\right|_{r=a},$$
$$s = 1, 2, 3, 4, 5, \qquad j = 3, 4 \qquad \text{(C.96)}$$

$$\breve{B}^{(n,m)}_{p,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{5}{2}a_1 + 2a_2 + 3a_3 + 4a_4 + 4a_5\right)\right]B^{(n,m)}_{2,p,t,j}$$
$$- k_{t,j}\left[\lambda_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{3}{2}a_1 + 2a_2 + a_3\right)\right]\Lambda^{(n,m)}_{p,t,j}$$
$$+ \frac{k^2_{t,j}}{a}[(3a_1 + 2a_2 + 4a_3 + 2a_4 + 2a_5)]B^{(n,m)}_{1,p,t,j} + \left.\frac{\mathrm{d}\hat{B}^{(n,m)}_{p,t,j}}{\mathrm{d}r}\right|_{r=a}$$
$$p = 1, 2, 3, 4, \qquad j = 1, 2 \qquad \text{(C.97)}$$

$$\Lambda^{(n,m)}_{1,t,j} = -\Lambda^{(n,m)}_{2,t,j} = \Lambda^{(n,m)}_{3,t,j} = g_n(k_{t,j}a), \qquad j = 1, 2 \qquad \text{(C.98)}$$

$$\Lambda^{(n,m)}_{4,t,j} = 0, \qquad j = 1, 2 \qquad \text{(C.99)}$$

$$\breve{B}^{(n,m)}_{p,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right]B^{(n,m)}_{2,p,t,j}$$
$$+ k_{t,j}\left[\mu_t - \rho_t\omega^2\left(h^2_{1,t} - 2h^2_{2,t} + \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{1}{2}a_3 + a_4 + \frac{3}{2}a_5\right)\right]\Lambda^{(n,m)}_{p,t,j}$$
$$+ \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right]B^{(n,m)}_{1,p,t,j} + \left.\frac{\mathrm{d}\hat{B}^{(n,m)}_{p,t,j}}{\mathrm{d}r}\right|_{r=a},$$
$$p = 1, 2, 3, 4, \qquad j = 3, 4 \qquad \text{(C.100)}$$

$$\Lambda^{(n,m)}_{1,t,j} = -\Lambda^{(n,m)}_{2,t,j} = -\Lambda^{(n,m)}_{3,t,j} = g_n(k_{t,j}a), \qquad j = 3, 4 \qquad \text{(C.101)}$$

$$\Lambda^{(n,m)}_{4,t,j} = 0, \qquad j = 3, 4 \qquad \text{(C.102)}$$

$$\breve{\Delta}^{(n,m)}_{p,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right]\Delta^{(n,m)}_{2,p,t,j} + \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right]\Delta^{(n,m)}_{1,p,t,j} + \left.\frac{\mathrm{d}\hat{\Delta}^{(n,m)}_{p,t,j}}{\mathrm{d}r}\right|_{r=a}, \quad p = 1,2,3,4, \quad j = 3,4 \tag{C.103}$$

$$\breve{Z}^{(n,m)}_{p,t,j} = \left[2\mu_t - \rho_t\omega^2\left(2h^2_{2,t} - \frac{\rho'_t d^2_t}{3\rho_t}\right) + k^2_{t,j}\left(\frac{1}{2}a_1 + \frac{3}{2}a_3 + 3a_4 + \frac{5}{2}a_5\right)\right]Z^{(n,m)}_{2,p,t,j} + \frac{k^2_{t,j}}{a}\left[\left(\frac{1}{2}a_1 + 2a_3 + a_4 + \frac{1}{2}a_5\right)\right]Z^{(n,m)}_{1,p,t,j} + \left.\frac{\mathrm{d}\hat{Z}^{(n,m)}_{p,t,j}}{\mathrm{d}r}\right|_{r=a}, \quad p = 1,2,3,4, \quad j = 3,4 \tag{C.104}$$

$$\Phi^{(n,m)}_{pq} = A_1 Z^m_n(\widehat{\mathbf{k}})A_{pq,e,1} - A_2 Z^m_n(\widehat{\mathbf{k}})A_{pq,e,3} \tag{C.105}$$

$$\widehat{\Phi}^{(n,m)}_{q} = A_1 Z^m_n(\widehat{\mathbf{k}})\widehat{A}_{q,e,1} - A_2 Z^m_n(\widehat{\mathbf{k}})\widehat{A}_{q,e,3} \tag{C.106}$$

$$\breve{\Phi}^{(n,m)}_{q} = A_1 Z^m_n(\widehat{\mathbf{k}})\breve{A}_{q,e,1} - A_2 Z^m_n(\widehat{\mathbf{k}})\breve{A}_{q,e,3} \tag{C.107}$$

$$Y^{(n,m)}_{pq} = \widehat{Y}^{(n,m)}_{pq} = \breve{Y}^{(n,m)}_{q} = 0. \tag{C.108}$$

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