ON COLSON'S THEOREM

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In 1991, Colson [1][1] proved a remarkable theorem about the limitations of *primitive recursive algorithms*, which has the following, even more remarkable consequence:

Colson's Corollary. *If a primitive recursive derivation of* $\min(x, y)$ *is expressed faithfully in a call-by-value programming language* (like C or Lisp), *then one of the two computations* $\min(1, 1000)$ *and* $\min(1000, 1)$ *will take at least* 1000 *steps.*

The point is that the natural algorithm which computes $\min(x, y)$ in $O(\min(x, y))$ steps cannot be matched in efficiency by a primitive recursive program, even though $\min(x, y)$ is a primitive recursive function; and so, as a practical and (especially) a foundational matter, we need to consider "recursion schemes" more general than primitive recursion, even if (ultimately) we are only interested in primitive recursive functions.

In this paper we consider extensions of Colson's Theorem which allow (some) strong schemes of recursion and also a rich variety of "given" functions. Sample (easy to state) result:

Theorem. *Consider primitive-recursive-like derivations, which in addition to composition and primitive recursion allow nested recursion, double recursion and calls to the following functions and* (characteristic functions of) *relations:*

$$x + y, \quad x \dot{-} y, \quad x \div 2, \quad \text{Parity}(x), \quad x = y, \quad x < y$$

For each such derivation of the greatest common divisor function $\gcd(x, y)$ *with time complexity* $c(x, y)$, *there is a sequence of pairs* $\{(x_i, y_i)\}_i$ *and a positive* (rational) *constant* r, *such that* $\lim_i(x_i + y_i) = \infty$, *and for all* i,

$$c(x_i, y_i) \geq r(x_i + y_i).$$

It follows that *Stein's algorithm* which computes $\gcd(x, y)$ with time complexity $O(\log_2(x) + \log_2(y))$, using the givens listed in the theorem and a very simple (but not primitive) recursion scheme, cannot be matched in efficiency using only "primitive-recursive-like" recursive definitions.

[1] Colson proved a general result about call-by-name primitive recursion, which implies this Corollary, and David [2] extended Colson's results using a new method. A direct proof of the simpler version for call-by-value primitive recursion was first given in Fredholm's Thesis [3].

Open problems. I would conjecture positive answers to both of the following questions.

(1) Can we strengthen the theorem to allow among the givens the remainder function $x \bmod y$? This would show that the *Euclidean algorithm* (which calls $x \bmod y$) cannot be expressed by primitive-recursive-like schemes.

(2) Does the theorem hold for μ-recursive derivations?

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References

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