Linear Algebraic Groups and Descriptive Set Theory

A.S. Kechris

This is joint work with Scot Adams.

In mathematics one often deals with problems of classification of objects up to some notion of equivalence by invariants. Frequently these objects can be viewed as elements of a *Polish* (complete separable metric) space X and the equivalence turns out to be a Borel equivalence relation E on X. A *complete classification* of X up to E consists of finding a set of invariants I and a map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$. For this to be of any interest both I and c must be explicit or definable and as simple and concrete as possible. The theory of Borel equivalence relations studies the set theoretic nature of possible invariants and develops a mathematical framework for measuring the complexity of such classification problems.

In organizing this study, the following concept of reducibility is fundamental. Let E, F be equivalence relations on Polish spaces X, Y resp. We say that E is *Borel reducible* to F, in symbols,

$$E \leq_B F,$$

if there is a Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

This simply means that any complete invariants for F work as well for E (after composing with f) and therefore, in some sense, E has a classification problem at most as complicated as that of F.

The structure of this partial order and the corresponding hierarchy of classification problems has been extensively studied over the last decade. In this talk, I will give an introduction to this theory and discuss the recent solution of a long standing problem in this area, in which the ergodic theory of linear algebraic groups, and more particularly the so-called *superrigidity theory* of R. Zimmer, plays a crucial role.