Upward Point-Set Embeddability

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Abstract. We study the problem of Upward Point-Set Embeddability,
that is the problem of deciding whether a given upward planar digraph
D has an upward planar embedding into a point set S. We show that any
switch tree admits an upward planar straight-line embedding into any
convex point set. For the class of k-switch trees, that is a generalization of
switch trees (according to this definition a switch tree is a 1-switch tree),
we show that not every k-switch tree admits an upward planar straight-
line embedding into any convex point set, for any $k \geq 2$. Finally we show
that the problem of Upward Point-Set Embeddability is NP-complete.

1 Introduction

A *planar straight-line embedding* of a graph G into a point set S is a mapping of
each vertex of G to a distinct point of S and of each edge of G to the straight-
line segment between the corresponding end-points so that no two edges cross
each other. Gritzmann *et al.* [9] proved that outerplanar graphs is the class
of graphs that admit a planar straight-line embedding into every point set in
general position or in convex position. Efficient algorithms are known to embed
outerplanar graphs [3] and trees [4] into any point set in general or in convex
position. From the negative point of view, Cabello [5] proved that the problem of
deciding whether there exists a planar straight-line embedding of a given graph G
into a point set P is NP-hard even when G is 2-connected and 2-outerplanar. For
upward planar digraphs, the problem of constructing upward planar straight-line
embeddings into point sets was studied by Giordano et al. [8], later on by Binucci
et al. [2] and recently by Angelini et al. [1]. While some positive and negative
results are known for the case of upward planar digraphs, the complexity of
testing upward planar straight-line embeddability into point sets has not been
known.

In this paper we continue the study of the problem of upward planar straight-
line embedding of directed graphs into a given point set. Our results include:

- We extend the positive results given in [1,2] by showing that any directed
  switch tree admits an upward planar straight-line embedding into every point
  set in convex position.

- We study directed k-switch trees, a generalization of switch trees (a 1-switch tree is exactly a switch tree). From the construction given in [2] (Theorem 5), we know that for $k \geq 4$ not every k-switch tree admits an upward planar straight-line embedding into any convex point set. Then we fill the gap for 2 and 3-switch trees, by showing that, for any $k \geq 2$ there is a class of k-switch trees $\mathcal{T}_n^k$, and a point set S in convex position, such that any $T \in \mathcal{T}_n^k$ does not admit an upward planar straight-line embedding into S.
- We study the computational complexity of the upward embeddability problem. More specifically, given a n vertex upward planar digraph G and a set of n points on the plane S, we show that deciding whether there exists an upward planar straight-line embedding of G so that its vertices are mapped to the points of S is NP-Complete. The decision problem remains NP-Complete even when G has a single source and the longest simple cycle of G has length four and, moreover, S is an m-convex point set, for some integer $m > 0$.

Due to space constraints we sketch or omit some of the proofs; for the detailed version see [7].

2   Preliminaries

We mostly follow the terminology of [2]. Next, we give some definitions that are used throughout this paper.

Let l be a line on the plane, which is not parallel to the x-axis. We say that point p *lies to the right of l* (resp., *to the left of l*) if p lies on a semi-line that originates on l, is parallel with the x-axis and is directed towards $+\infty$ (resp., $-\infty$). Similarly, if l is a line on the plane, which is not parallel to the y-axis, we say that point p *lies above l* (resp., *below l*) if p lies on a semi-line that originates on l, is parallel with the y-axis and is directed towards $+\infty$ (resp., $-\infty$).

A *point set in general position*, or *general point set*, is a point set such that no three points lie on the same line and no two points have the same y-coordinate. The *convex hull $H(S)$* of a point set S is the point set that can be obtained as a convex combination of the points of S. A *point set in convex position*, or *convex point set*, is a point set such that no point is in the convex hull of the others. Given a point set S, we denote by $b(S)$ and by $t(S)$ the lowest and the highest point of S, respectively. A *one-sided convex point set S* is a convex point set in which $b(S)$ and $t(S)$ are adjacent in the border of $H(S)$. A convex point set which is not one-sided, is called a *two-sided convex point set*. In a convex point set S, the subset of points that lie to the left (resp. right) of the line through $b(S)$ and $t(S)$ is called the *left (resp. right) part of S*. A one-sided convex point set S is called *left-heavy* (resp., *right-heavy*)convex point set if all the points of S lie to the left (resp., to the right) of the line through $b(S)$ and $t(S)$. Note that, a one-sided convex point set is either a left-heavy or a right-heavy convex point set.

Consider a point set S and its convex hull $H(S)$. Let $S_1 = S \setminus H(S)$, ..., $S_m = S_{m-1} \setminus H(S_{m-1})$. If m is the smallest integer such that $S_m = \emptyset$, we say that S is an *m-convex point set*. A subset of points of a convex point set S is

called *consecutive* if its points appear consecutive as we traverse the convex hull of S in the clockwise or counterclockwise direction.

The graphs we study in this paper are directed. By (u, v) we denote an arc directed from u to v. A *switch-tree* is a directed tree T, such that, each vertex of T is either a source of a sink. Note that the longest directed path of a switch-tree has length one[1]. Based on the length of the longest path, the class of switch trees can be generalized to that of k-switch trees. A *k-switch tree* is a directed tree, such that its longest directed path has length k. So, a switch tree is a 1-switch tree. A digraph D is called *path-DAG*, if its underlying graph is a simple path. A *monotone path* $(v_1, v_2, \ldots, v_k)$ is a path-DAG containing arcs (v_i, v_{i+1}), $1 \leq i \leq k - 1$.

An *upward planar directed graph* is a digraph that admits a planar drawing where each edge is represented by a curve monotonically increasing in the y-direction. An *upward straight-line embedding* (*UPSE* for short) of a graph into a point set is a mapping of each vertex to a distinct point and of each arc to a straight-line segment between its end-points such that no two arcs cross and each arc (u, v) has $y(u) < y(v)$. The following results were presented in [2] and are used in this paper.

Lemma 1 (Binucci at al. [2]). *Let T be an n-vertex tree-DAG and let S be any convex point set of size n. Let u be any vertex of T and let $T_1, T_2, \ldots, T_k$ be the subtrees of T obtained by removing u and its incident edges from T. In any UPSE of T into S, the vertices of T_i are mapped into a set of consecutive points of S, for each $i = 1, 2, \ldots, k$.*

Theorem 1 (Binucci at al. [2]). *For every odd integer $n \geq 5$, there exists a $(3n + 1)$-vertex directed tree T and a convex point set S of size $3n + 1$ such that T does not admit an UPSE into S.*

3   Embedding a Switch-Tree into a Point Set in Convex Position

In this section we enrich the positive results presented in [1,2] by proving that, any switch-tree has an UPSE into any point set in convex position. During the execution of the algorithms, presented in the following lemmata, which embed a tree T into a point set S, a *free point* is a point of S to which no vertex of T has been mapped yet. The following two lemmata treat the simple case of a one-sided convex point set and are immediate consequences of a result by Heath et al. [10].

Lemma 2. *Let T be a switch-tree, r be a sink of T, S be a one-sided convex point set so that $|S| = |T|$, and p be S's highest point. Then, T admits an UPSE into S so that vertex r is mapped to point p.*   □

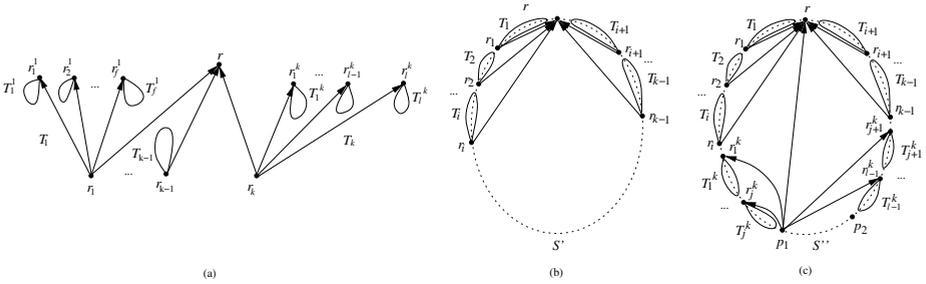[1] The *length* of a directed path is the number of arcs in the path.

Fig. 1. The construction of Lemma 4

Lemma 3. *Let T be a switch-tree, r be a source of T, S be a one-sided convex point set so that $|S| = |T|$, and p be S's lowest point. Then, T admits an UPSE into S so that vertex r is mapped to point p.* □

Now we are ready to proceed to the main result of the section.

Theorem 2. *Let T be a switch-tree and S be a convex point set such that $|S| = |T|$. Then, T admits an UPSE into S.*

The proof of the theorem is based on the following lemma, which extends Lemma 2 from one-sided convex point sets to convex point sets.

Lemma 4. *Let T be a switch-tree, r be a sink of T, S be a convex point set such that $|S| = |T|$. Then, T admits an UPSE into S so that vertex r is mapped to the highest point of S.*

Proof. Let $T_1, \ldots, T_k$ be the sub-trees of T that are connected to r by an edge (Figure 1.a) and let $r_1, \ldots, r_k$ be the vertices of $T_1, \ldots, T_k$, respectively, that are connected to r. Observe that, since T is a switch tree and r is a sink, vertices $r_1, \ldots, r_k$ are sources.

We draw T on S as follows. We start by placing the trees $T_1, T_2, \ldots$ on the left side of the point set S as long as they fit, using the highest free points first. This can be done in an upward planar fashion by Lemma 3 (Figure 1.b). Assume that T_i is the last placed subtree. Then, we continue placing the trees $T_{i+1}, \ldots, T_{k-1}$ on the right side of the point set S. This can be done due to Lemma 3. Note that the remaining free points are consecutive point of S, denote these points by S'. To complete the embedding we draw T_k on S'. Let $T_1^k, \ldots, T_l^k$ the subtrees of T_k, that are connected to r_k by an arc. Let also $r_1^k, \ldots, r_l^k$ be the vertices of $T_1^k, \ldots, T_l^k$, respectively, that are connected to r_k (Figure 1.a). Note that $r_1^k, \ldots, r_l^k$ are all sinks. We start by drawing $T_1^k, T_2^k, \ldots$ as long as they fit on the left side of point set S', using the highest free points first. This can be done in an upward planar fashion by Lemma 2. Assume that T_j^k is the last placed subtree (Figure 1.c). Then, we continue on the right side of the point set S' with the trees $T_{j+1}^k, \ldots, T_{l-1}^k$. This can be done again by Lemma 2. Note that there are exactly $|T_l^k| + 1$ remaining free points since we have not yet drawn T_l^k and vertex

r_k of T_k. Denote by S'' the remaining free points and note that S'' consists of consecutive points of S. If S'' is a one-sided point set then we can proceed by using the Lemma 2 again and the result follows trivially. Assume now that S'' is a two-sided convex point set and let p_1 and p_2 be the highest points of S'' on the left and on the right, respectively. W.l.o.g., let $y(p_1) < y(p_2)$. Then, we map r_k to p_1. By using the lemma recursively, we can draw T_l^k on $S'' \setminus \{p_1\}$ so that r_l^k is mapped to p_2. The proof is completed by observing that all edges connecting r_k to $r_1^k, \ldots, r_l^k$ and $r_1, \ldots, r_k$ to r are upward and do not cross each other. □

4 K-Switch Trees

Binucci et al. [2] (see also Theorem 1) presented a class of trees and corresponding convex point sets, such that any tree of this class does not admit an UPSE into its corresponding point set.

   The $(3n + 1)$-size tree T constructed in the proof of Theorem 1[2] has the following structure. It consists of: (i) one vertex r of degree three, (ii) three monotone paths of n vertices: $P_u = (u_n, u_{n-1}, \ldots, u_1)$, $P_v = (v_1, v_2, \ldots, v_n)$, $P_w = (w_1, w_2, \ldots, w_n)$, ($iii$) arcs (r, u_1), (v_1, r) and (w_1, r).

   The $(3n + 1)$-convex point set S, used in the proof of Theorem 1[2], consists of two extremal points on the y-direction, $b(S)$ and $t(S)$, the set L of $(3n-1)/2$ points $l_1, l_2, \ldots, l_{(3n-1)/2}$, comprising the left side of S and the set R of $(3n - 1)/2$ points $r_1, r_2, \ldots, r_{(3n-1)/2}$, comprising the right side of S. The points of L and R are located so that $y(b(S)) < y(r_1) < y(l_1) < y(r_2) < y(l_2) < \ldots < y(r_{(3n-1)/2}) < y(l_{(3n-1)/2}) < y(t(S))$.

   Note that the $(3n + 1)$-node tree T described above is a $(n - 1)$-switch tree. Hence a straightforward corollary of Theorem 1[2] is the following statement.

Corollary 1. *For $k \geq 4$, there exists a k-switch tree T and a convex point set S of the same size, such that T does not admit an UPSE into S.*

From Section 3, we know that any switch tree T, i.e. a 1-switch tree, admits an UPSE into any convex point set. The natural question raised by this result and Corollary 1 is whether an arbitrary 2-switch or 3-switch tree has an UPSE into any convex point set. This question is resolved by the following theorem.

Theorem 3. *For any $n \geq 5$ and for any $k \geq 2$, there exists a class $\mathcal{T}_n^k$ of $3n+1$-vertex k-switch trees and a convex point set S, consisting of $3n+1$ points, such that any $T \in \mathcal{T}_n^k$ does not admit an UPSE into S.*

Proof. For any $n \geq 5$ we construct the following class of trees. Let P_u be an n-vertex path-DAG on the vertex set $\{u_1, u_2, \ldots, u_n\}$, enumerated in the order they are presented in the underlying undirected path of P_u, and such that arcs (u_3, u_2), (u_2, u_1) are present in P_u. Let also P_v and P_w be two n-vertex path-DAGs on the vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, enumerated in the order they are presented in the underlying undirected path of P_v and P_w, and such that arcs $(v_1, v_2), (v_2, v_3)$ and $(w_1, w_2), (w_2, w_3)$ are present

in P_v and P_w, respectively. Let $T(P_u, P_v, P_w)$ be a tree consisting of P_u, P_v, P_w, vertex r and arcs $(r, u_1), (v_1, r), (w_1, r)$.

Let $\mathcal{T}_n^k = \{T(P_u, P_v, P_w) \mid$ the longest directed path in P_u, P_v and P_w has length $k\}$, $k \geq 2$. So, $\mathcal{T}_n^k$ is a class of $3n + 1$-vertex k-switch trees. Let S be a convex point set as described in the beginning of the section. Then any $T \in \mathcal{T}_n^k$ does not admit an UPSE into point set S, see [7] for the detailed proof. $\qquad \square$

5   Upward Planar Straight-Line Point Set Embeddability is NP-Complete

In this section we examine the complexity of testing whether a given n-vertex upward planar digraph G admits an UPSE into a point set S. We show that the problem is NP-complete even for a single source digraph G having longest simple cycle of length at most 4. This result is optimal for the class of cyclic graphs[2], since Angelini et al. [1] showed that every single-source upward planar directed graph with no cycle of length greater than three admits an UPSE into every point set in general position.

Theorem 4. *Given an n-vertex upward planar digraph G and a planar point set S of size n in general position, the decision problem of whether there exists an UPSE of G into S is NP-Complete. The decision problem remains NP-Complete even when G has a single source and the longest simple cycle of G has length at most 4 and, moreover, S is an m-convex point set for some $m > 0$.*

Proof. The problem is trivially in NP. In order to prove the NP-completeness, we construct a reduction from the 3-Partition problem.

> *Problem:* 3-Partition
> *Input:* A bound $B \in \mathbb{Z}^+$, and a set $A = \{a_1, \ldots, a_{3m}\}$ with $a_i \in \mathbb{Z}^+$, $\frac{B}{4} < a_i < \frac{B}{2}$.
> *Output:* m disjoint sets $A_1, \ldots, A_m \subset A$ with $|A_i| = 3$ and $\sum_{a \in A_i} a = B$, $1 \leq i \leq m$.

We use the fact that 3-Partition is a strongly NP-hard problem, i.e. it is NP-hard even if B is bounded by a polynomial in m [6]. Let A and B be the set of the $3m$ positive integers and the bound, respectively, that form the instance (A, B) of the 3-Partition problem. Based on A and B, we show how to construct an upward planar digraph G and a point set S such that G has an UPSE on point set S if and only if the instance (A, B) of the 3-partition problem has a solution.

We first show how to construct G (see Figure 2.a for illustration). We start the construction of G by first adding two vertices s and t. Vertex s is the single source of the whole graph. We then add m disjoint paths from s to t, each of length two. The degree-2 vertices of these paths are denoted by u_i, $i = 1, \ldots, m$. For each $a \in A$, we construct a monotone directed path P_i of length a that has a new vertices and s at its source. Totally, we have $3m$ such paths $P_1, \ldots, P_{3m}$.

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[2] A digraph is *cyclic* if its underling undirected graph contains at least one cycle.
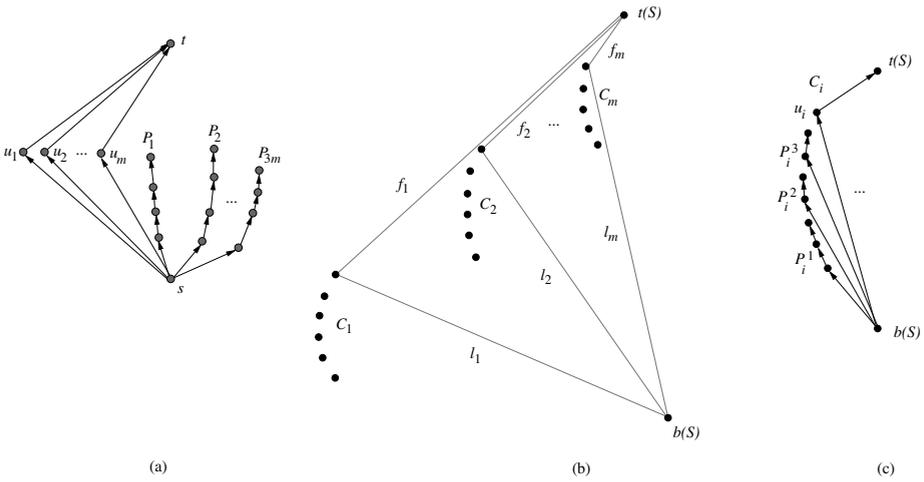
Fig. 2. (*a*) The graph G of the construction used in the proof of NP-completeness. (*b*) The point set S of the construction. (*c*) An UPSE of G on S.

We proceed to the construction of point set S. Let $b(S)$ and $t(S)$ be the lowest and the highest points of S (see Figure 2.b). In addition to $b(S)$ and $t(S)$, S also contains m one-sided convex point sets $C_1, \ldots, C_m$, each of size $B + 1$, so that the points of S satisfy the following properties:

- $C_i \cup \{b(S), t(S)\}$ is a left-heavy convex point set, $i \in \{1, \ldots, m\}$.
- The points of C_{i+1} are higher than the points of C_i, $i \in \{1, \ldots, m-1\}$.
- Let l_i be the line through $b(S)$ and $t(C_i)$, $i \in \{1, \ldots, m\}$. $C_1, \ldots, C_i$ lie to the left of line l_i and $C_{i+1}, \ldots, C_m$ lie to the right of line l_i.
- Let f_i be the line through $t(S)$ and $t(C_i)$, $i \in \{1, \ldots, m\}$. C_j, $j \geq i$, lie to the right of line f_i.
- $\{t(C_i) : i = 1, \ldots, m\}$ is a left-heavy convex point set.

The next statement follows from the properties of point set S.

Statement 1. *Let C_i be one of the left-heavy convex point sets comprising S and let $x \in C_j$, $j > i$. Then, set $C_i \cup \{b(S), x\}$ is also a left-heavy convex point set, with $b(S)$ and x consecutive on its convex hull.* $\qquad \square$

Statement 2. *We can construct a point set S that satisfies all the above requirements so that the area of S is polynomial on B and m.*

Proof of Statement: For each $i \in \{0, \ldots, m-1\}$ we let C_{m-i} to be the set of $B + 1$ points

$$C_{m-i} = \left\{ (-j - i(B+2), j^2 - (i(B+2))^2) \mid j = 1, 2, \ldots, B+1 \right\}$$

Then, we set the lowest point of the set S, called $b(S)$, to be point $(-(B+1)^2 + ((m-1)(B+2))^2, (B+1)^2 - (m(B+2))^2)$ and the highest point of S, called $t(S)$, to be point $(0, (m(B+2))^2)$.

It is easy to verify that all the above requirements hold and that the area of the rectangle bounding the constructed point set is polynomial on B and m. ☐

Statement 3. $|S| = |V(G)| = m(B + 1) + 2.$ ☐

We now proceed to show how from a solution for the 3-Partition problem we can derive a solution for the upward point set embeddability problem. Assume that there exists a solution for the instance of the 3-Partition problem and let it be $A_i = \{a_i^1, a_i^2, a_i^3\}$, $i = 1 \ldots m$. Note that $\sum_{j=1}^{3} a_i^j = B$. We first map s and t to $b(S)$ and $t(S)$, respectively. Then, we map vertex u_i on $t(C_i)$, $i = 1 \ldots m$. Note that the path from s to t through u_i is upward and $C_1, \ldots, C_i$ lie entirely to the left of this path, while $C_{i+1}, \ldots, C_m$ lie to the right of this path. Now each C_i has B free points. We map the vertices of paths P_i^1, P_i^2 and P_i^3 corresponding to a_i^1, a_i^2, a_i^3 to the remaining points of C_i in an upward fashion (see Figure 2.c). It is easy to verify that the whole drawing is upward and planar.

Assume now that there is an UPSE of G into S. We prove that there is a solution for the corresponding 3-Partition problem. The proof is based on the following statements.

Statement 4. *In any UPSE of G into S, s is mapped to $b(S)$.* ☐

Statement 5. *In any UPSE of G into S, only one vertex from set $\{u_1, \ldots, u_m\}$ is mapped to point set C_i, $i = 1 \ldots m$.*

Proof of Statement: For the sake of contradiction, assume that there are two distinct vertices u_j and u_k that are mapped to two points of the same point set C_i (see Figures 3). W.l.o.g. assume that u_k is mapped to a point higher than the point u_j is mapped to. We consider three cases based on the placement of the sink vertex t.

Case 1: t is mapped to a point of C_i (Figure 3.a). It is easy to see that arc (s, u_k) crosses arc (u_j, t), a clear contradiction to the planarity of the embedding.

Case 2: t is mapped to $t(S)$ (Figure 3.b). Similar to the previous case since $C_i \cup \{b(S), t(S)\}$ is a one-sided convex point set.

Case 3: t is mapped to a point of C_p, $p > i$, denote it by p_t (Figure 3.c). By Statement 1 $C_i \cup \{b(S), p_t\}$ is a convex point set and points p_t, $b(S)$ are consecutive points of $C_i \cup \{b(S), p_t\}$. Hence, arc (s, u_k) crosses arc (u_j, t), a contradiction. ☐

By Statement 5, we have that each $C_i, i = 1 \ldots m$, contains exactly one vertex from set $\{u_1, \ldots u_m\}$. Without lost of generality, we assume that u_i is mapped to a point of C_i.

Statement 6. *In any UPSE of G into S, vertex t is mapped to either a point of C_m or to $t(S)$.*

Proof of Statement: Vertex t has to be mapped higher than any u_i, $i = 1 \ldots m$, and hence higher than u_m, which is mapped to a point of C_m. ☐
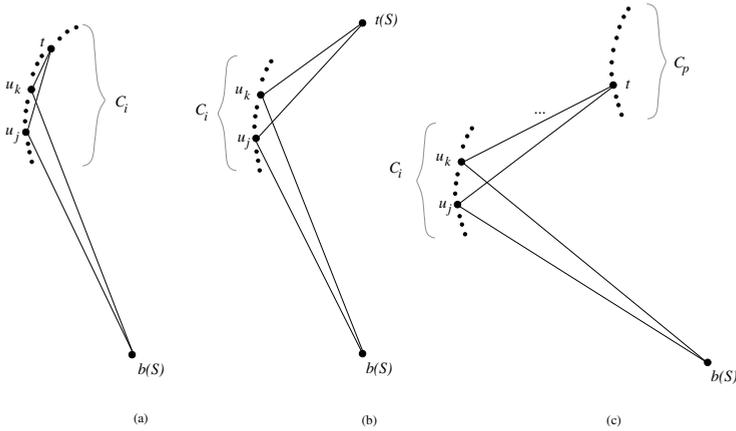
Fig. 3. Mappings used in the proof of Statement 5

Statement 7. *In any UPSE of G into S, vertex u_i is mapped to $t(C_i)$, $1 \leq i \leq m - 1$, moreover, there is no arc (v, w) so that v is mapped to a point of C_i and w is mapped to a point of C_j, $j > i$.*

Proof of Statement: We prove this statement by induction on i, $i = 1 \ldots m - 1$. For the basis, assume that u_1 is mapped to a point p_1 different from $t(C_1)$ (see Figure 4.a). Let p_t be the point where vertex t is mapped. By Statement 6, p_t can be either $t(S)$ or a point of C_m. In both cases, point set $C_1 \cup \{b(S), p_t\}$ is a convex point set, due to the construction of the point set S and the Statement 1. Moreover, the points $b(S)$ and p_t are consecutive on the convex hull of point set $C_1 \cup \{b(S), p_t\}$.

Denote by p the point of C_1 that is exactly above the point p_1. From Statement 5, we know that no u_j, $j \neq 1$ is mapped to the point p. Due to Statement 6, t cannot be mapped to p. Hence there is a path P_k, $1 \leq k \leq 3m$, so that one of its vertices is mapped to p. Call this vertex u. We now consider two cases based on whether u is the first vertex of P_k of not.

Case 1: Assume that there is a vertex v of P_k, such that there is an arc (v, u). Since the drawing of S is upward, v is mapped to a point lower than p and lower than p_1. Since $C_1 \cup \{b(S), p_t\}$ is a convex point set, arc (v, u) crosses arc (u_1, t). A clear contradiction.

Case 2: Let u be the first vertex of P_k. Then, arc (s, u) crosses the arc (u_1, t) since, again, $C_1 \cup \{b(S), p_t\}$ is a convex point set, a contradiction.

So, we have that u_1 is mapped to $t(C_1)$, see Figure 4.b. Observe now that any arc (v, w), such that v is mapped to a point of C_1 and w is mapped to a point $x \in C_2 \cup \ldots \cup C_m \cup \{t(S)\}$ crosses arc (s, u_1), since $C_1 \cup \{b(S), x\}$ is a convex point set. So, the statement is true for $i = 1$.

For the induction step, we assume that the statement is true for C_g and u_g, $g \leq i - 1$, i.e. vertex u_g is mapped to $t(C_g)$ and there is no arc connecting a point of C_g to a point of C_k, $k > g$ and this holds for any $g \leq i - 1$. We now show that
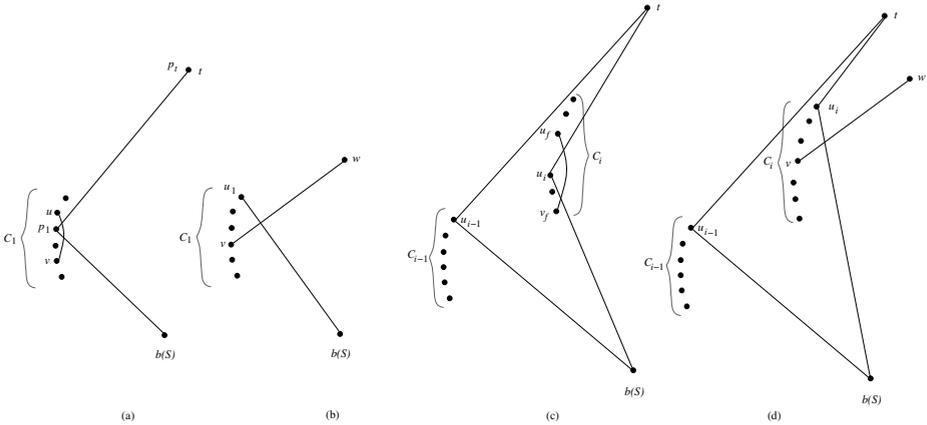
Fig. 4. Mappings used in the proof of Statement 7

it also holds for C_i and u_i. Again, for the sake of contradiction, assume that u_i is mapped to a point p_i different from $t(C_i)$ (see Figure 4.c).

Denote by q the point of C_1 that is exactly above point p_i. From Statement 5, we know that no u_l, $l \neq i$, is mapped to the point q. Due to Statement 6, t can not be mapped to q. Hence, there is a path P_f, so that one of its vertices is mapped to q. Call this vertex u_f. We now consider two cases based on whether u_f is the first vertex of P_f of not.

Case 1: Assume that there is a vertex v_f of P_k such that there is an arc (v_f, u_f). By the induction hypothesis, we know that v_f is not mapped to any C_l, $l < i$. Then, since the drawing of S is upward, v_f is mapped to a point lower than q and lower than p_i. Since $C_i \cup \{b(S), p_t\}$ is a convex point set, arc (v_f, u_f) crosses arc (u_i, t). A clear contradiction.

Case 2: Let u_f be the first vertex of P_k. Then, arc (s, u_f) crosses the arc (u_i, t) since, again, $C_i \cup \{b(S), p_t\}$ is a convex point set, a contradiction.

So, we have shown that u_i is mapped to $t(C_i)$, see Figure 4.d. Observe now that, any arc (v, w), such that v is mapped to a point of C_i and w is mapped to a point $x \in C_{i+1} \cup \ldots \cup C_m \cup \{t(S)\}$ crosses arc (s, u_i), since $C_i \cup \{b(S), x\}$ is a convex point set. So, the statement holds for i. □

A trivial corollary of the previous statement is the following:

Statement 8. *In any UPSE of G into S, any directed path P_j of G originating at s, $j \in \{1, \ldots, 3m\}$, has to be drawn entirely in C_i, for $i \in \{1, \ldots, m\}$.* □

The following statement completes the proof of the theorem.

Statement 9. *In any UPSE of G into S, vertex t is mapped to point $t(S)$.*

Proof of Statement: For the sake of contradiction, assume that t is not mapped to $t(S)$. By Statement 6 we know that t has to be mapped to a point in C_m. Assume first that t is mapped to point $t(C_m)$ (see Figure 5.a). Recall that u_{m-2} and u_{m-1} are mapped to $t(C_{m-2})$ and $t(C_{m-1})$, respectively, and that $\{t(C_i) : i = 1 \ldots m\}$
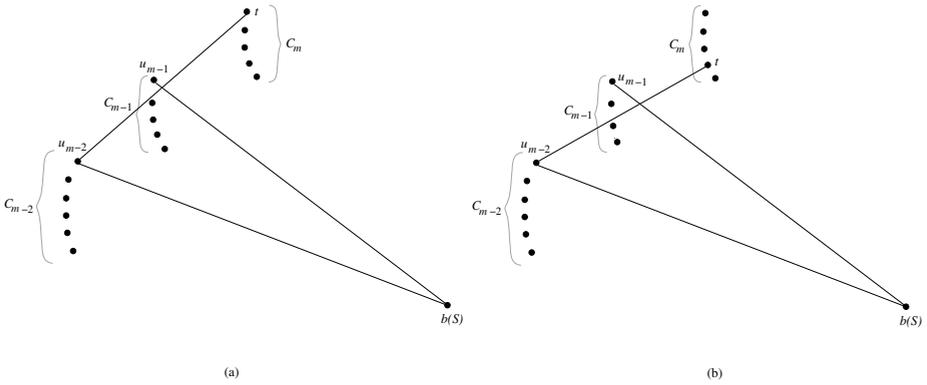
Fig. 5. (a-b) Mappings used in the proof of Statement 9

is a left-heavy convex point set. Hence, points $\{t(C_{m-2}), t(C_{m-1}), t(C_m), b(S)\}$ form a convex point set. It follows that segments $(t(C_{m-2}), t(C_m))$ and $(t(C_{m-1}), b(S))$ cross each other, i.e. edges (s, u_{m-1}) and (u_{m-2}, t) cross, contradicting the planarity of the drawing.

Consider now the case where t is mapped to a point of C_m, say p, different from $t(C_m)$ (see Figure 5.b). Since point p does not lie in triangle $t(C_{m-2})$, $t(C_{m-1}), b(S)$ and point $t(C_{m-1})$ does not lie in triangle $t(C_{m-2}), p, b(S)$, points $\{t(C_{m-2}), t(C_{m-1}), p, b(S)\}$ form a convex point set. Hence, segments $(t(C_{m-2}), p)$ and $(t(C_{m-1}), b(S))$ cross each other, i.e. edges (s, u_{m-1}) and (u_{m-2}, t) cross; a clear contradiction.   □

Let us now combine the above statements in order to derive a solution for the 3-Partition problem when we are given an UPSE of G into S. By Statement 4 and Statement 9, vertices s and t are mapped to $b(S)$ and $t(S)$, respectively. By Statement 5, for each $i = 1 \ldots m$, point set C_i contains exactly one vertex from $\{u_1, \ldots, u_m\}$, say u_i and, hence, the remaining points of C_i are occupied by the vertices of some paths $P_i^1, P_i^2, \ldots, P_i^c$. By Statement 8, $P_i^1, P_i^2, \ldots, P_i^c$ are mapped entirely to the points of C_i. Since C_i has $B + 1$ points, the highest of which is occupied by u_i, we infer that $P_i^1, P_i^2, \ldots, P_i^c$ contain exactly B vertices. We set $A_i = \{a_i^1, a_i^2, \ldots, a_i^c\}$, where a_i^j is the size of path P_i^j, $1 \le j \le c$. Since $\frac{B}{4} < a_i^j < \frac{B}{2}$ we infer that $c = 3$. The subsets A_i are disjoint and their union produces A.

Finally, we note that G has a single source s and the longest simple cycle of G has length 4, moreover the point set S is an m-convex point set for some $m > 1$. This completes the proof.   □

6   Open Problems

In this paper, we continued the study of the upward point-set embeddability problem, initiated in [1,2,8]. We showed that the problem is NP-complete, even if some restrictions are posed on the digraph and the point set. We also extended

the positive and the negative results presented in [1,2] by resolving the problem for the class of k-switch trees, $k \in \mathbb{N}$. The partial results on the directed trees presented in [1,2] and in the present work, may be extended in two ways: (i) by presenting the time complexity of the problem of testing whether a given directed tree admits an upward planar straight-line embedding (UPSE) to a given general/convex point set and (ii) by presenting another classes of trees, that admit/do not admit an UPSE to a given general/convex point set. It would be also interesting to know whether there exists a class of upward planar digraphs $\mathcal{D}$ for which the decision problem whether a digraph $D \in \mathcal{D}$ admits an UPSE into a given point set P remains NP-complete even for a convex point set P.

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