Unilateral Orientation of Mixed Graphs

Tamara Mchedlidze and Antonios Symvonis

Dept. of Mathematics, National Technical University of Athens, Athens, Greece
{mchet,symvonis}@math.ntua.gr

Abstract. A digraph D is *unilateral* if for every pair x, y of its vertices there exists a directed path from x to y, or a directed path from y to x, or both. A mixed graph $M = (V, A, E)$ with arc-set A and edge-set E *accepts a unilateral orientation*, if its edges can be oriented so that the resulting digraph is unilateral. In this paper, we present the first linear-time recognition algorithm for unilaterally orientable mixed graphs. Based on this algorithm we derive a polynomial algorithm for testing whether a unilaterally orientable mixed graph has a *unique* unilateral orientation.

1 Introduction

A large body of literature has been devoted to the study of mixed graphs (see [1] and the references therein). A mixed graph is *strongly orientable* when its undirected edges can be oriented in such a way that the resulting directed graph is strongly connected, while, it is *unilaterally orientable* when its undirected edges can be oriented in such a way that for every pair of vertices x, y there exists a path from x to y, or from y to x, or both.

Several problems related to the strong orientation of mixed graphs have been studied. Among them are the problems of "recognition of strongly orientable mixed graphs" [2] and "determining whether a mixed graph admits a unique strong orientation" [3,5].

In this paper we answer the corresponding questions for unilateral orientations of mixed graphs, that is, firstly we develop a linear-time algorithm for recognizing whether a mixed graph is unilaterally orientable and, secondly, we provide a polynomial algorithm for testing whether a mixed graph accepts a unique unilateral orientation.

1.1 Basic Definitions

We mostly follow the terminology of [1]. A *graph* $G = (V, E)$ consists of a non-empty finite set V of elements called *vertices* and a finite set E of unordered pairs of vertices, called *edges*. A *directed graph* or *digraph* $D = (V, A)$ consists of a non-empty set of vertices V and a set A of ordered pairs of vertices called *arcs* (or directed edges). We say that in (di)graph G vertex y is *reachable* from vertex x if there is a (directed) path from vertex x to vertex y.

J. van Leeuwen et al. (Eds.): SOFSEM 2010, LNCS 5901, pp. 588–599, 2010.

A *mixed graph* $M = (V, A, E)$ contains both arcs (ordered pairs of vertices in A) and edges (unordered pairs of vertices in E). A *path in a mixed graph* is a sequence of edges and arcs in which consecutive elements (i.e., edges or arcs) are incident on the same vertex and all arcs are traversed in their forward direction. Note that, since a graph (digraph) is a mixed graph having only edges (resp. only arcs), any definition or property concerning mixed graphs also applies to graphs (resp. digraphs).

A *biorientation* of a mixed graph $M = (V, A, E)$ is obtained from M by replacing every edge $(x, y) \in E$ by either arc (x, y), or arc (y, x), or the pair of arcs (x, y) and (y, x). If every edge is replaced by a single arc, we speak of an *orientation* of a mixed graph M. The *complete biorientation* of a mixed graph $M = (V, A, E)$, denoted by $\overleftrightarrow{M}$, is a biorientation of M such that every edge $(x, y) \in E$ is replaced in $\overleftrightarrow{M}$ by the pair of arcs (x, y) and (y, x).

An *underlying graph* $UG(M)$ of a mixed graph $M = (V, A, E)$ is the unique undirected graph G resulting by "removing" the direction from each arc of M, i.e., by turning each arc of A into an edge. A mixed graph M is *connected* if $UG(M)$ is connected.

A digraph D is *strongly connected* (or, just *strong*) if, for every pair x, y of distinct vertices in D, x and y are mutually reachable from each other. A *strong component* of a digraph D is a maximal subdigraph of D which is strong. The *strong component digraph* $SC(D)$ of D is obtained by contracting strong components of D and by identifying any parallel arcs obtained during this process into a single arc. The digraph $SC(D)$ for any digraph D is acyclic as any cycle is fully contained within a single strongly connected component. A digraph D is *unilateral* if, for every pair $x,\ y$ of vertices of D, either x is reachable from y or y is reachable from x (or both).

The definitions for the connectivity-related terms can be extended for the case of mixed graphs. A mixed graph M is *strongly connected* (or *strong*) if its complete biorientation $\overleftrightarrow{M}$ is strongly connected. A mixed graph M is *unilaterally connected* (or *unilateral*) if its complete biorientation $\overleftrightarrow{M}$ is unilateral.

A mixed graph M is *strongly (unilaterally) orientable* (or, equivalently, *M admits a strong (unilateral) orientation*) if there is an orientation of M which is strongly (resp. unilaterally) connected. A mixed graph M admits a *hamiltonian orientation*, if there is an orientation $\overrightarrow{M}$ of M which is hamiltonian. Note that a graph that admits a hamiltonian orientation also admits a unilateral orientation.

1.2 Problem Definition and Related Work

Given a mixed graph M, it is natural to examine whether M is strongly or unilaterally orientable. The mixed graph M_1 of Figure 1a, is strongly orientable as it is demonstarted by digraph D_1 (Fig. 1b). The directed graphs D_2 and D_3 (Fig. 1c and Fig. 1d) show two unilateral orientations of M, non of which is strong. Robbins [9] proved that an undirected graph is strongly orientable if

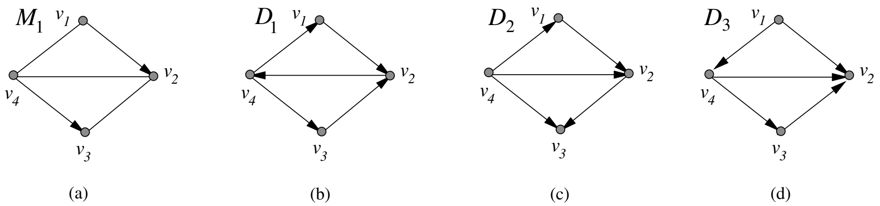

Fig. 1. (a) A mixed graph M_1. (b) A strong orientation of M_1. (c) & (d) Unilateral orientations of M_1 that are not strong, as there is no path from v_3 to v_1.

and only if it is connected and has no bridge[1]. Boesch and Tindel [2] generalized Robbins' result, showing that a mixed multigraph is strongly orientable if and only if it is strongly connected and has no bridges. Given that a digraph with n vertices and m arcs can be tested for strong connectivity and for being bridgeless in $O(m+n)$ time, the characterization given by Boesch and Tindel immediately leads to a polynomial time recognition algorithm of strongly orientable mixed graphs. Chung et al [4] presented an algorithm that computes a strong orientation of a mixed multigraph in linear time. Chartrand et al. [3] provided a characterization of unilaterally orientable graphs by showing that:

Theorem 1 (Chartrand et al., [3]). *A connected graph G has a unilateral orientation if and only if all of the bridges of G lie on a common path.*

As the unilateral orientation presents a different notion of connectivity, it is natural to ask wether a mixed graph admits a unilateral orientation. Even though unilateral orientation is a weaker notion of connectivity, not all mixed graphs admit a unilateral orientation. For example, the mixed graph M_2 in Fig. 2a does not admit a unilateral orientation since there is no directed path between vertices v_2 and v_3 in either direction. In this paper, we present a characterization of unilaterally orientable mixed graphs that leads to a linear-time recognition algorithm. Our characterization can be considered to be a generalization of Theorem 1 for mixed graphs.

Observe that not all mixed graphs admit more than one distinct orientation (strong or unilateral). For example, the mixed graph M_3 of Fig. 2b admits a unique unilateral orientation (given in Fig. 2c). Consider a mixed graph $M = (V, A, E)$ which has a *unique* strong (unilateral) orientation D. Then, we say that D is a *forced strong (resp. unilateral) orientation* for M.

Let $G = (V, E_1 \cup E_2)$ be a graph, let A be an arc-set obtained from an orientation of the edges in E_1, and let $M = (V, A, E_2)$ be the resulting mixed graph. If M has a forced strong (unilateral) orientation then we say that A is a *forcing set* for a strong (resp. unilateral) orientation of G, or simply a *strong (resp. unilateral) forcing set*.

[1] An edge e of a connected mixed graph M is a *bridge* if $M \setminus \{e\}$ is not connected. A mixed graph containing no bridge is called *bridgeless*.

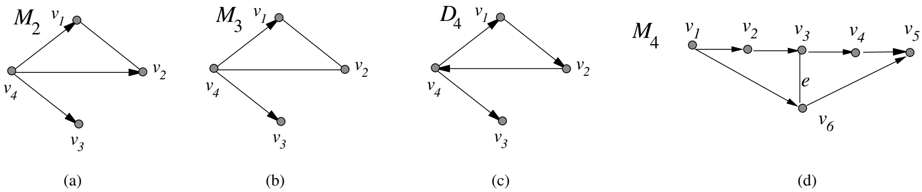

Fig. 2. (a) A mixed graph M_2 which does not admit any unilateral orientation. (b) A mixed graph M_3 which has a forced unilateral orientation. (c) The unique unilateral orientation of M_3. (d) A bridgeless unilateral mixed graph that does not admit a unilateral orientation.

The concept of forced strong (unilateral) orientation of graphs was first introduced by Chartrand et al [3] who defined the *forced strong (resp. unilateral) orientation number* of an undirected graph G to be the cardinality of the minimal forcing set for a strong (resp. unilateral) orientation of G. Strong orientations of graphs were later studied by Farzad et et al [5]. Forced unilateral orientations were studied by Pascovici [8]. In her work, she mentions that finding an efficient algorithm for calculating the forced unilateral orientation number of a graph is an open question which, to the best of our knowledge, has not been answered yet. In this paper, we partially resolve this question. Given a mixed graph $M = (V, A, E)$, we provide an algorithm which tests in polynomial time whether A is a forcing set for a unilateral orientation of M (i.e., it tests whether M has a unique unilateral orientation).

The paper is organized as follows: In Section 2 we develop a linear-time algorithm for recognizing whether a mixed graph accepts a unilateral orientation and, in the case it does, we produce such an orientation. In Section 3, we give a lemma which implies a polynomial algorithm for testing whether a mixed graph accepts a unique unilateral orientation. We conclude in Section 4.

2 Recognition of Unilaterally Orientable Mixed Graphs

2.1 Preliminaries

The following theorem, due to Boesch and Tindell, gives necessary and sufficient conditions for a mixed graph to have a strongly connected orientation.

Theorem 2 (Boesch and Tindell [2]). *A mixed multigraph M admits a strong orientation if and only if M is strong and the underlying multigraph of M is bridgeless.*

Note that, a corresponding theorem for unilaterally orientable graphs (i.e., *"a mixed multigraph M has a unilateral orientation if and only if M is unilateral and the underlying multigraph of M is bridgeless"*) does not hold. This is demonstrated by the mixed graph M_4 in Fig. 2d which is unilateral and bridgeless, but, it does not have a unilateral orientation. To see that, observe that if edge (v_3, v_6) is oriented towards v_6, then vertices v_4 and v_6 are not connected by a directed path in either direction, while, if edge (v_3, v_6) is directed towards v_3 then vertices v_2 and v_6 are not connected by a directed path in either direction.

Lemma 1 ([6], pp. 66). *Digraph D is unilateral if and only if D has a spanning directed walk*[2]*.*

Lemma 2 ([3]). *A tree T admits a unilateral orientation if and only if T is a path.*

A vertex of a directed graph having in-degree (out-degree) equal to zero is referred to as a *source* (resp. *sink*). An *st-digraph* is a directed acyclic digraph having a single source (denoted by s) and a single sink (denoted by t).

Lemma 3 ([7]). *Let D be an st-digraph that does not have a hamiltonian path. Then, there exist two vertices in D that are not connected by a directed path in either direction.*

2.2 A Characterization for Unilaterally Orientable Mixed Graphs

Consider a mixed graph $M = (V, A, E)$ and let $V' \subseteq V$. The mixed subgraph of M *induced* by V', denoted by $M(V')$, is defined as $M(V') = (V', A', E')$ where, $A' = \{(u, v) \mid (u, v) \in A \text{ and } u,\ v \in V'\}$ and $E' = \{(u, v) \mid (u, v) \in E \text{ and } u,\ v \in V'\}$.

Let M be a mixed graph, let $D_i = (V_i, E_i),\ 1 \le i \le k$, be the strong components of the complete biorientation $\overleftrightarrow{M}$ of M. The *strong components* M_i, $1 \le i \le k$, of mixed graph M are defined as: $M_i = M(V_i),\ 1 \le i \le k$, that is, M_i is the mixed subgraph of M induced by V_i. Note that each M_i is strong since, by definition, D_i is its complete biorientation.

The *strong component digraph* of a mixed graph M, denoted by $SC(M)$, is obtained by contracting each strong component of M into a single vertex and by identifying all parallel arcs that are created during this process into a single arc. Fig. 3a shows a mixed graph having three strong components and Fig. 3b shows its corresponding strong component digraph. Note that the strong component digraph of any mixed graph is acyclic.

The next lemma gives the first necessary condition for a mixed graph M to have a unilateral orientation.

Lemma 4. *If a mixed graph M admits a unilateral orientation then its strong component digraph $SC(M)$ has a hamiltonian path.*

Proof. For the sake of contradiction, assume that $SC(M)$ has no hamiltonian path. Since $SC(M)$ is an acyclic digraph, it has at least one source and at least

[2] A *spanning directed walk* of a digraph is a directed path that visits all the vertices of the digraph, some possibly more than once.

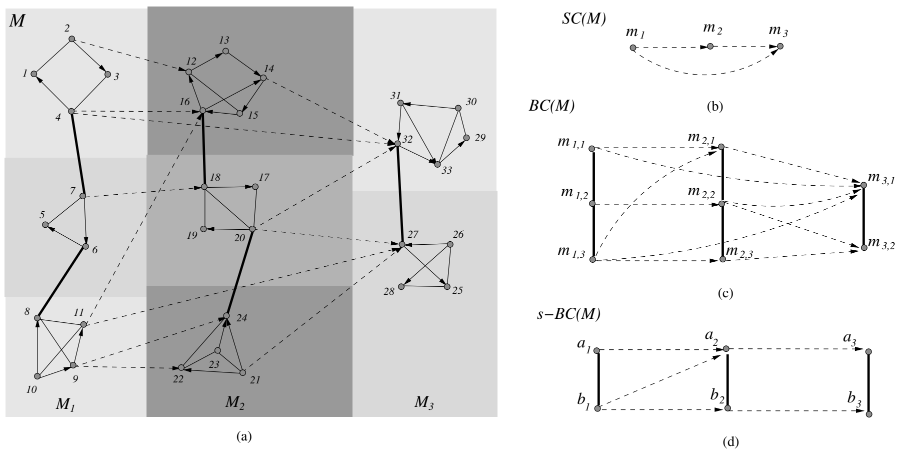

Fig. 3. (a) A mixed graph M. M_1, M_2 and M_3 are the three strong components of M. Dashed edges connect M_i with M_j, $i, j \in \{1, 2, 3\}$, $i \neq j$. The bold edges are the bridges of each M_i. (b) The strong component digraph $SC(M)$ of M. (c) The bridgeless-component mixed graph $BC(M)$ of M. (d) The simplified bridgeless-component mixed graph for mixed graph M, where the vertices a_i, b_i denote the endpoints of a bridge path $B(M_i)$.

one sink. If there are two or more sources (sinks) then it is clear that any two of the sources (sinks) are not connected by a directed path in either direction. If there is exactly one source and exactly one sink in $SC(M)$ then $SC(M)$ is an *st*-digraph and, by Lemma 3, there are two vertices of $SC(M)$ that are not connected by a directed path in either direction. So, in either case, we can identify two vertices of $SC(M)$, call them m_i and m_j, that are not connected by a directed path in either direction.

By the definition of the strong component digraph $SC(M)$, m_i and m_j correspond to contracted strong components of M. Let these strong components be M_i and M_j, respectively. Since m_i and m_j are not connected by a directed path in either direction, then for each vertex u of M_i and for each vertex v of M_j there is no directed path in the complete biorientation digraph $\overleftrightarrow{M}$ connecting u with v. Therefore, there is no path in the mixed graph M connecting u and v in either direction, and thus, there can be no orientation of M that creates a directed path connecting u with v in either direction. This is a clear contradiction of the assumption that M admits a unilateral orientation. □

Let $M = (V, A, E)$ be a strong mixed graph and $B \subseteq E$ be the bridges of M. Note that B might be empty. Then, all components of graph $M \setminus B$ are strong and bridgeless.

The *bridge graph* of a strong mixed graph M, denoted by $B(M)$, is obtained by contracting in M the vertices of each strong component of $M \setminus B$. Note that a bridge graph of any strong mixed graph is a tree. Fig. 4 shows a strong mixed graph and its bridge graph.

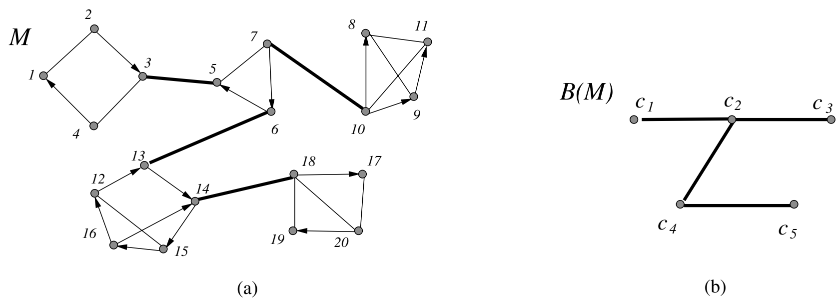

Fig. 4. (a) A strong mixed graph M. (b) The bridge graph $B(M)$ of M.

The following lemma gives the second necessary condition for the mixed graph to have a unilateral orientations.

Lemma 5. *If a mixed graph M admits a unilateral orientation then the bridge graph of each of its strong components is a path.*

Proof. Consider a unilateral orientation D of M. By Lemma 1, D has a spanning walk. This spanning walk of D induces an orientation of the bridges of M and, thus, an orientation of $B(M)$ which is unilateral. Recall that $B(M)$ is a tree. Then, by Lemma 2 we conclude that the bridge graph $B(M)$ of M is a path. □

Let $M = (V, A, E)$ be a mixed graph and let $M_i,\ 1 \leq i \leq k$, be its strong components. Moreover, let B_i be the bridges of $M_i,\ 1 \leq i \leq k$, and let $B = \bigcup_{i=1}^{i=k} B_i$. Then, the set of strong components of the mixed graph $M \setminus B$ is the union of the strong components of each $M_i \setminus B_i,\ 1 \leq i \leq k$. The *bridgeless-component mixed graph* of a mixed graph M, denoted by $BC(M)$, is obtained by contracting in M the vertices of each strong component of $M \setminus B$ into a single vertex and by identifying any parallel arcs created during this process into a single arc. Fig. 3c shows the bridgeless-component mixed graph for the mixed graph of Fig. 3a. Note that the edge set of the bridgeless-component mixed graph $BC(M)$ is exactly set B. Moreover, $BC(M)$ can be considered to consist of a set of (undirected) trees (the bridge graph $B(M_i)$ of each strong component M_i of M) connected by arcs which do not create any cycle. Also observe that the strong component digraph $SC(M)$ of M can be obtained from $BC(M)$ by contracting all bridgeless components of M_i into a single vertex and by identifying all parallel edges created by this process into a single arc.

Observation 1. *Let M be a mixed graph. Then any orientation of $BC(M)$ is acyclic.*

Proof. It follows from the facts that (i) the strong component digraph $SC(M)$ is acyclic and (ii) the bridge graph $B(M')$ of any strong component M' of M is a tree. □

The following theorem provides a characterization of unilaterally orientable mixed graphs.

Theorem 3. *A mixed graph M admits a unilateral orientation if and only if the bridgeless-component mixed graph $BC(M)$ admits a hamiltonian orientation.*

Proof. ($\Rightarrow$) We assume that a mixed graph M admits a unilateral orientation and we show that $BC(M)$ admits a hamiltonian orientation. Consider a unilateral orientation of M. By Lemma 1, the unilateral orientation of M has a spanning walk. That spanning walk induces a spanning walk on $BC(M)$. Since any orientation of $BC(M)$ is acyclic (Observation 1) the induced spanning walk on $BC(M)$ is a hamiltonian path. Thus, $BC(M)$ admits a hamiltonian orientation.

($\Leftarrow$) We assume now that $BC(M)$ admits a hamiltonian orientation and we show that M admits a unilateral orientation. Recall that the vertices of $BC(M)$ correspond to the bridgeless strong components of M. By Theorem 2, it follows that each of these components admits a strong orientation. This strong orientation implies a spanning walk between any pair of vertices of the strong bridgeless component. The hamiltonian orientation of $BC(M)$ implies an orientation of the bridges of the strong components of M. The strong orientation of the strong and bridgeless components of M together with the orientation of the bridges of the strong components of M, result to orientation D of M.

The hamiltonian orientation of $BC(M)$ implies a hamiltonian path and, in turn, an ordering of the strong bridgeless components of M. Based on this ordering, we can easily construct a spanning walk on orientation D. Thus, based on Lemma 1, D is a unilateral orientation. We conclude that M admits a unilateral orientation. □

2.3 The Algorithm

Based on the the characterization of Section 2.2, Algorithm 1 decides whether a mixed graph that is given to its input is unilaterally orientable. In the first step of Algorithm 1, we construct the strong connected digraph $SC(M)$ of M. In order to do so, we have to compute the strongly connected components of the complete biorientation $\overleftrightarrow{M}$ of M. This can be easily accomplished in $O(V + A + E)$ time. In the second step of Algorithm 1, we test whether $SC(M)$ has a hamiltonian path. Note that since $SC(M)$ is an acyclic digraph, it has a hamiltonian path if and only if it has a unique topological ordering. This can be easily tested in linear time to the size of $SC(M)$. In the third step of Algorithm 1, we construct the bridge graph $BM(M_i)$ for each strong component M_i of M. Identifying all bridges can be trivially done by testing whether the removal of each individual edge disconnects the component. By using a depth-first-search based method, the identification of all bridges and the construction of all bridge graphs can be completed in $O(V + A + E)$ time. Testing whether each bridge graph is a path is trivial and can be completed in linear time to the size of the bridge graph.

By utilizing the already constructed bridge graphs of step 3 of the algorithm, we can construct the bridgeless-component mixed graph $BC(M)$ of M in time proportional to its size. Now, it remains to test whether $BC(M)$ is hamiltonian. Note that, since we have reached the fourth step of Algorithm 1, it holds that our graph satisfies two properties. Firstly, all of its bridge graphs are paths (thus, we refer to them as *bridge paths*) and, secondly, its strong component digraph $SC(M)$ is hamiltonian. We will exploit these properties in order to decide whether $BC(M)$ is hamiltonian in linear time.

Let $M_i,\ 1 \le i \le k$, be the strong components of M and assume without loss of generality that they appear on the hamiltonian path of $SC(M)$ in this order. Firstly observe that in a hamiltonian path of $BC(M)$, if one exists, all vertices of the bridge path $B(M_i)$ are visited before the vertices of the bridge path $B(M_j)$, for all $i < j$. Since the graph $SC(M)$ is acyclic, if we leave component M_i before visiting all of its vertices there is no way to return to it and, thus, no hamiltonian path exists. Also observe that, in a hamiltonian path of $BC(M)$ each bridge path is traversed from one of its endpoint to the other and, thus, there are two possible orientation of the bridge path. As a consequence, the hamiltonian path of $BC(M)$, if any, only uses arcs which leave from the endpoints of a bridge path $B(M_i)$ and enter the endpoints of a bridge path $B(M_j),\ i < j$. In addition, these arcs connect consecutive bridge paths. Thus, when testing whether the bridgeless-component mixed graph $BC(M)$ has a hamiltonian path, we can use a simplified leveled mixed graph, denoted by s-$BC(M)$, resulting by eliminating all vertices which are not endpoints of a bridge graph and all arcs that enter or leave them, as well as all arcs connecting vertices on non consecutive bridge pahts. Thus, each level of the graph is either an edge or a single vertex, and the levels which correspond to the strong components of M appear in the order of their corresponding strong component. Fig. 3d shows the simplified bridgeless-component mixed graph for mixed graph M, where the vertices a_i, b_i denote the endpoints of a bridge path $B(M_i)$.

We can decide whether the s-$BC(M)$ has a hamiltonian path by using a simple dynamic programming algorithm. Let p_i^a be a boolean variable which takes the value **true** if and only if there is a hamiltonian path that traverses all vertices of the first i levels of s-$BC(M)$ and terminates at vertex $a_i,\ 1 \le i \le k$. Similarly we define p_i^b. It is easy to see that the following recursive relations hold:

$$\begin{aligned} &p_1^a = p_1^b = \mathbf{true} \\ &p_i^a = \big(p_{i-1}^a = \mathbf{true} \wedge \exists(a_{i-1}, b_i) \in A'\big) \vee \big(p_{i-1}^b = \mathbf{true} \wedge \exists(b_{i-1}, b_i) \in A'\big) \\ &p_i^b = \big(p_{i-1}^b = \mathbf{true} \wedge \exists(b_{i-1}, a_i) \in A'\big) \vee \big(p_{i-1}^a = \mathbf{true} \wedge \exists(a_{i-1}, a_i) \in A'\big) \\ &(\text{for } 1 < i \le k) \end{aligned}$$

Based on the above equations, we can decide whether there is a hamiltonian path in s-$BC(M)$ (and, as a consequence in $BC(M)$) in $O(k)$ time.

Observation 2. *The bridgeless-component mixed graph $BC(M)$ of a mixed graph M has exactly one hamiltonian path if and only if $p_i^a \oplus p_i^b =$ **true**, $1 \le i \le k$, where k is the number of strong components of M.*

We also note that, in the case where a unilateral orientation exists, we can compute one in linear time. This can be achieved by orienting the bridges of

Algorithm 1. UNILATERAL-ORIENTATION(M)

input : A Mixed graph $M = (V, A, E)$.
output : "YES" if M has a unilateral orientation, "NO" otherwise.
1. Construct the strong connected digraph $SC(M)$ of M.
   {*Denote the strong components of* $SC(M)$ *by* $M_1, \ldots, M_k$.}
2. **if** $SC(M)$ has no hamiltonian path **then return**("NO")
   else
3. For each strong component M_i of M, $1 \leq i \leq k$,
      Construct the bridge graph $B(M_i)$;
      if B_{M_i} is not a simple path **then return**("NO");
   { *All bridge graphs* B_{M_i} *are paths.* }
4. Construct the bridgeless-component mixed graph $BC(M)$ of M
5. **if** $BC(M)$ has no hamiltonian path **then return**("NO");
6. **return**("YES");

$BC(M)$ according to the hamiltonian path of $BC(M)$ and by using a strong orientation for each bridgeless strong component of the bridge graphs.

From the above description, we can state the following theorem:

Theorem 4. *Given a mixed graph* $M = (V, A, E)$ *, we can decide whether* M *admits a unilateral orientation in* $O(V+A+E)$ *time. Moreover, if* M *is unilaterally orientable, a unilateral orientation can be computed in* $O(V + A + E)$ *time.*

We also note that we can prove an additional characterization for unilaterally orientable mixed graphs that can be considered to be a counterpart of Theorem 1 given by Chartrand et al [3]. The proof of the following theorem is based on the properties of the strong component digraph $SC(M)$, and the bridgeless-component mixed graph $BC(M)$ for a mixed graph M.

Theorem 5. *A mixed graph admits a unilateral orientation if and only if all the bridges of its strong components lie on a common path.*

3 Recognition of Unilateral Forcing Sets

Let $M = (V, E, A)$ be a mixed graph. In this section we present a simple lemma stating whether M has a forced unilateral orientation or, equivalently, whether A is a unilateral forcing set for M. Based on this lemma and Algorithm 1, we infer a polynomial algorithm for testing whether A is a unilateral forcing set for M or, equivalently, G has a unique unilateral orientation.

Forced unilateral orientations were studied by Pascovici in [8], where she gave a general lower bound for the forced unilateral orientation number and showed that the unilateral orientation number of a graph G having edge connectivity 1 is equal to $m-n+2$, where m and n are the numbers of edges and vertices of G, respectively.

Lemma 6. *A mixed graph $M = (V, A, E)$ admits a unique unilateral orientation if and only if for each edge $e = (u, v) \in E$ either $(V, A \cup \{(u, v)\}, E \setminus \{(u, v)\})$ or $(V, A \cup \{(v, u)\}, E \setminus \{(u, v)\})$ has a unilateral orientation, but not both.*

Proof. ($\Rightarrow$) Let M admit a unique unilateral orientation and assume, for the sake of contradiction, that both $(V, A \cup \{(u, v)\}, E \setminus \{(u, v)\})$ and $(V, A \cup \{(v, u)\}, E \setminus \{(u, v)\})$ have a unilateral orientation. Then these unilateral orientations differ in at least one edge and hence are distinct. A clear contradiction. Otherwise, if we assume that neither $(V, A \cup \{(u, v)\}, E \setminus \{(u, v)\})$ nor $(V, A \cup \{(v, u)\}, E \setminus \{(u, v)\})$ has a unilateral orientation, then we have a contradiction again, as M was supposed to have at least one unilateral orientation.

($\Leftarrow$) Assume now that for *each* edge $e = (u, v) \in E$ either $(V, A \cup \{(u, v)\}, E \setminus \{(u, v)\})$ or $(V, A \cup \{(v, u)\}, E \setminus \{(u, v)\})$ has a unilateral orientation, but not both of them. It is clear that M has at least one unilateral orientation. Assume, for the sake of contradiction, that M has more than one unilateral orientations. Consider any two arbitrary unilateral orientations of M. As these orientations are distinct they differ in at least one edge, say $e' = (u', v') \in E$. So we conclude that both $(V, A \cup \{(u', v')\}, E \setminus \{e'\})$ and $(V, A \cup \{(v', u')\}, E \setminus \{e'\})$ have a unilateral orientation, a clear contradiction. □

Theorem 6. *Given a mixed graph $M = (V, A, E)$, we can decide whether A is a unilateral forcing set for M in $O(E(V + A + E))$ time.*

Proof. Follows directly from Theorem 4 and Lemma 6. □

4 Conclusion

For a mixed graph $M = (V, A, E)$, we presented a linear-time algorithm that recognizes whether M is unilaterally orientable, and in the case where it is, we also presented a characterization leading to a polynomial algorithm for determining whether A is a unilateral forcing set for M. Future research includes the study of the number of unilateral orientations of a mixed graph, as well as the complexity of the problem of finding a unilateral forcing set of minimum size.

References

1. Bang-Jensen, J., Gutin, G.: Digraphs: Theory, Algorithms and Applications. Springer, Heidelberg (2007)
2. Boesch, F., Tindell, R.: Robbins's Theorem for Mixed Multigraphs. American Mathematical Monthly 87(9), 716–719 (1980)
3. Chartrand, G., Harary, F., Schultz, M., Wall, C.E.: Forced Orientation Numbers of a Graph. Congressus Numerantium 100, 183–191 (1994)
4. Chung, F.R.K., Garey, M.R., Tarjan, R.E.: Strongly Connected Orientations of Mixed Multigraphs. Networks 15(4), 477–484 (1985)
5. Farzad, B., Mahdian, M., Mahmoodian, E.S., Saberi, A., Sadri, B.: Forced Orientation of Graphs. Bulletin of Iranian Mathematical Society 32(1), 79–89 (2006)
6. Harary, F., Norman, R.Z., Cartwright, D.: Structural Models: An Introduction to the Theory of Directed Graphs. Addison-Wesley, Reading (1965)

7. Mchedlidze, T., Symvonis, A.: Crossing-Optimal Acyclic Hamiltonian Path Completion and Its Application to Upward Topological Book Embeddings. In: Das, S., Uehara, R. (eds.) WALCOM 2009. LNCS, vol. 5431, pp. 250–261. Springer, Heidelberg (2009)
8. Pascovici, D.: On the Forced Unilateral Orientation Number of a Graph. Discrete Mathematics 187, 171–183 (1997)
9. Robbins, H.E.: A Theorem on Graphs, with an Application to a Problem of Traffic Control. American Mathematical Monthly 46, 218–283 (1939)