Crossing-Optimal Acyclic HP-Completion for Outerplanar *st*-Digraphs

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Abstract. Given an embedded planar acyclic digraph G, the *acyclic hamiltonian path completion with crossing minimization (Acyclic-HPCCM)* problem is to determine a *hamiltonian path completion set* of edges such that, when these edges are embedded on G, they create the smallest possible number of edge crossings and turn G to a hamiltonian acyclic digraph. In this paper, we present a linear time algorithm which solves the Acyclic-HPCCM problem on any outerplanar st-digraph G. The algorithm is based on properties of the optimal solution and an *st-polygon decomposition of* G. As a consequence of our result, we can obtain for the class of outerplanar st-digraphs upward topological 2-page book embeddings with minimum number of spine crossings.

1 Introduction

A *hamiltonian path* of G is a path that visits every vertex of G exactly once. Determining whether a graph has a hamiltonian path or circuit is NP-complete [3]. The problem remains NP-complete for cubic planar graphs [3], for maximal planar graphs [9] and for planar digraphs [3]. It can be trivially solved in polynomial time for planar acyclic digraphs.

Given a graph $G = (V, E)$, directed or undirected, a non-negative integer $k \leq |V|$ and two vertices $s,\ t \in V$, *the hamiltonian path completion (HPC)* problem asks whether there exists a superset E' containing E such that $|E' - E| \leq k$ and the graph $G' = (V, E')$ has a hamiltonian path from vertex s to vertex t. We refer to G' and to the set of edges $E' \setminus E$ as the *HP-completed graph* and the *HP-completion set* of graph G, respectively. We assume that all edges of a HP-completion set are part of the Hamiltonian path of G', otherwise they can be removed. The hamiltonian path completion problem is NP-complete [2]. For acyclic digraphs the HPC problem is solved in polynomial time [4]. When G is a directed acyclic graph, we can insist on HP-completion sets which leave the HP-completed digraph also acyclic. We refer to this version of the problem as the *acyclic HP-completion problem (acyclic-HPC)*.

A *drawing* Γ of graph G maps every vertex v of G to a distinct point $p(v)$ on the plane and each edge $e = (u, v)$ of G to a simple open curve joining $p(u)$ with $p(v)$. A drawing in which every edge (u, v) is a a simple open curve monotonically increasing in the vertical direction is an *upward drawing*. A drawing Γ of graph

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G is *planar* if no two distinct edges intersect except at their end-vertices. Graph G is called *planar* if it admits a planar drawing Γ.

An embedding of a planar graph G is the equivalence class of planar drawings of G that define the same set of faces or, equivalently, of face boundaries. A planar graph together with the description of a set of faces F is called an *embedded planar graph*.

Let $G = (V, E)$ be an embedded planar graph, E' be a superset of edges containing E, and $\Gamma(G')$ be a drawing of $G' = (V, E')$. When the deletion from $\Gamma(G')$ of the edges in $E' - E$ induces the embedded planar graph G, we say that $\Gamma(G')$ *preserves the embedded planar graph* G.

Definition 1. *Given an embedded planar graph $G = (V, E)$, directed or undirected, a non-negative integer c, and two vertices s, $t \in V$, the* ***hamiltonian path completion with edge crossing minimization (HPCCM) problem*** *asks whether there exists a superset E' containing E and a drawing $\Gamma(G')$ of graph $G' = (V, E')$ such that (i) G' has a hamiltonian path from vertex s to vertex t, (ii) $\Gamma(G')$ has at most c edge crossings, and (iii) $\Gamma(G')$ preserves the embedded planar graph G.*

We refer to the version of the HPCCM problem where the input is an acyclic digraph and we are interested in HP-completion sets which leave the HP-completed digraph also acyclic as the *Acyclic-HPCCM* problem.

Over the set of all HP-completion sets for an embedded planar graph G, and over all of their different drawings that respect G, the one with a minimum number of edge-crossings is called a *crossing-optimal HP-completion set*.

In this paper, we present a linear time algorithm which solves the Acyclic-HPCCM problem for outerplanar st-digrpahs. A planar graph G is *outerplanar* if there exist a drawing of G such that all of G's vertices appear on the boundary of the same face (which is usually drawn as the external face). Let $G = (V, E)$ be a digraph. A vertex of G with in-degree equal to zero (0) is called a *source*, while, a vertex of G with out-degree equal to zero is called a *sink*. An *st-digraph* is an acyclic digraph with exactly one source and exactly one sink. Traditionally, the source and the sink of an st-digraph are denoted by s and t, respectively. An st-digraph which is planar (resp. outerplanar) and, in addition, it is embedded on the plane so that both of its source and sink appear on the boundary of its external face, is referred to as a ***planar st-digraph*** (resp. an ***outerplanar st-digraph***). It is known that a planar st-digraph admits a planar upward drawing [5,1]. In the rest of the paper, all st-digraphs will be drawn upward.

1.1 Our Results

The Acyclic-HPCCM problem was introduced by Mchedlidze and Symvonis in [8]. They provided a characterization under which a triangulated st-digraph is hamiltonian. For the class of planar st-digraphs, they established an equivalence between the acyclic-HPCCM problem and the problem of determining an upward 2-page topological book embedding with a minimal number of spine crossings.

For any outerplanar *triangulated* st-digraph G they reported a linear-time algorithm that solves the Acyclic-HPCCM problem *with at most one crossing per edge* of G [7].

In this paper, we derive a linear time algorithm that solves the Acyclic-HPCCM problem for any outerplanar st-digraph G. Our algorithm extends the results presented in [7] in two ways: (a) it does not require G to be triangulated, and (b) it takes into account all acyclic HP-completion sets (and not only those which cause at most 1 crossing per edge of G). The algorithm is based on properties of the optimal solution and a decomposition of graph G.

More specifically, we show that (i) for any st-polygon (i.e., an outerplanar st-digraph with no edge connecting its two opposite sides) there is always a crossing-optimal acyclic HP-completion set of size at most 2 (Section 2, Theorem 2), and, (ii) for any outerplanar st-digraph G there exists a crossing optimal acyclic HP-completion set which creates at most 2 crossings per edge of G (Section 4, Theorem 4). Based on these properties and the introduced st-polygon decomposition of an outerplanar st-digraph (Section 3), we derive a linear time algorithm that solves the Acyclic-HPCCM problem for outerplanar st-digraphs. Due to space constraints we present only the main results omitting the proofs of the lemmata and theorems; for the detailed version see [6].

2 Outerplanar *st*-Digraphs, Rhombuses and *st*-Polygons

Let $G = (V^l \cup V^r \cup \{s,t\}, E)$ be an outerplanar st-digraph, where s is its source, t is its sink and the vertices in V_l (resp. V_r) are located on the left (resp. right) part of the boundary of the external face. Let $V^l = \{v_1^l, \ \ldots, v_k^l\}$ and $V^r = \{v_1^r, \ \ldots, v_m^r\}$, where the subscripts indicate the order in which the vertices appear on the left (right) part of the external boundary. By convention, source and the sink are considered to lie on both the left and the right sides of the external boundary. Observe that each face of G is also an outerplanar st-digraph. We refer to an edge that has both of its end-vertices on the same side of G as an *one-sided* edge. All remaining edges are referred to as *two-sided* edges. The following lemma presents an essential property of an acyclic HP-completion set of an outerplanar st-digraph G.

Lemma 1. *The acyclic HP-completion set of an outerplanar st-digraph* $G = (V^l \cup V^r \cup \{s,t\}, E)$ *induces a hamiltonian path that visits the vertices of* V_l *(resp.* V_r*) in the order they appear on the left side (resp. right side) of* G.

The outerplanar st-digraph of Figure 1.a is called a *strong rhombus*. It has at least one vertex at each of its sides and it consists of exactly two faces that have edge (v_s, v_t) in common. The edge (v_s, v_t) of a rhombus is referred to as its *median* and is always drawn in the interior of its drawing. The outerplanar st-digraph resulting from the deletion of the median of a strong rhorbus is referred to as a *weak rhombus* (Figure 1.b). We use the term *rhombus* to refer to either a strong or a weak rhombus.

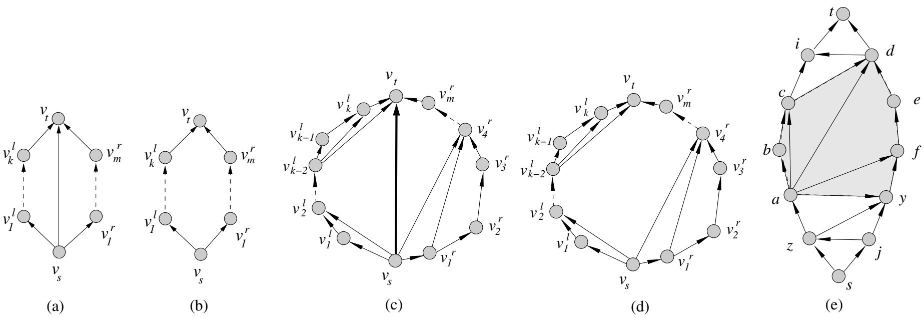

Fig. 1. (a) A strong rhombus. (b) A weak rhombus. (c) A strong *st*-polygon. (d) A weak *st*-polygon. (e) A maximal *st*-polygon.

The following theorem provides a characterization of *st*-digraphs that have a hamiltonian path. Its proof is a trivial extension of the proof given in [8] for triangulated planar *st*-digraphs.

Theorem 1. *Let G be a planar st-digraph. G has a hamiltonian path if and only if G does not contain any rhombus (strong or weak) as a subgraph.*

A *strong st-polygon* is an outerplanar *st*-digraph that contains edge (v_s, v_t) connecting its source v_s to its sink v_t) (Figure 1.c). Edge (v_s, v_t) is referred to as its *median* and it always lies in the interior of its drawing. As a consequence , in a strong *st*-polygon no edge connects a vertex on its left side to a vertex on its right side. The outerplanar *st*-digraph that results from the deletion of the median of a strong *st*-polygon is referred to as a *weak st-polygon* (Figure 1.d). We use the term *st-polygon* to refer to both a strong and a weak *st*-polygon.

Consider an outerplanar *st*-digraph G and one of its embedded subgraphs G_p that is an *st*-polygon (strong or weak). G_p is called a *maximal st-polygon* if it cannot be extended (and still remain an *st*-polygon) by the addition of more vertices to its external boundary. In Figure 1.e, the *st*-polygon $G_{a,d}$ with vertices a (source), b, c, d (sink), e, and f on its boundary is not maximal since the subgraph $G'_{a,d}$ obtained by adding vertex y to it is still an *st*-polygon. However, the *st*-polygon $G'_{a,d}$ is maximal since the addition of either vertex i or z to it does not yield another *st*-polygon.

Observe that an *st*-polygon that is a subgraph of an outerplanar *st*-digraph G fully occupies a *"strip"* of it that is limited by two edges (one adjacent to its source and one to its sink), each having its endpoints at different sides of G. We refer to these two edges as the *limiting edges* of the *st*-polygon. Note that the limiting edges of an *st*-polygon that is an embeded subgraph of an outerplanar graph are sufficient to define it. In Figure 1.e, the maximal *st*-polygon with vertex a as its source and vertex d as its sink is limited by edges (a, y) and (c, d). The next two lemmata describe properties of *st*-polygons.

Lemma 2. *An st-polygon contains exactly one rhombus.*

Lemma 3. *The maximal st-polygons contained in an outerplanar st-digraph G are mutually area-disjoint.*

The following lemmata are concerned with a crossing-optimal acyclic HP-completion set for a single *st*-polygon. They state that there exist crossing optimal acyclic HP-completion sets containing at most two edges and there proofs are based on the construction of a new HP-completion set.

Lemma 4. *Let $R = (V^l \cup V^r \cup \{s,t\}, E)$ be an st-polygon. Let P be an acyclic HP-completion set for R such that $|P| = 2\mu + 1$, $\mu \geq 1$. Then, there exists another acyclic HP-completion set P' for R such that $|P'| = 1$ and the edges of P' create at most as many crossings with the edges of R as the edges of P do. In addition, the hamiltonian paths induced by P and P' have in common their first and last edges.*

Lemma 5. *Let $R = (V^l \cup V^r \cup \{s,t\}, E)$ be an st-polygon. Let P be an acyclic HP-completion set for R such that $|P| = 2\mu$, $\mu \geq 1$. Then, there exists another acyclic HP-completion set P' for R such that $|P'| = 2$ and the edges of P' create at most as many crossings with the edges of R as the edges of P do. In addition, the hamiltonian paths induced by P and P' have in common their first and last edges.*

By combining Lemma 4 with Lemma 5, we get:

Theorem 2. *Any st-polygon has a crossing-optimal acyclic HP-completion set of size at most 2.*

3 *st*-Polygon Decomposition of Outerplanar *st*-Digraphs

The following lemmata state that we can test effectively whether an edge is the median of an *st*-polygon and whether a face is a weak-rhombus. Their proofs are supported by trivial data structures (i.e., for each vertex, a list of neighbors in circular order).

Lemma 6. *Assume an n vertex outerplanar st-digraph $G = (V^l \cup V^r \cup \{s,t\}, E)$ and an arbitrary edge $e = (s', t') \in E$. If $O(n)$ time is available for the preprocessing of G, we can decide in $O(1)$ time whether e is a median edge of some strong st-polygon. Moreover, the two vertices (in addition to s' and t') that define a maximal strong st-polygon that has edge e as its median can be also computed in $O(1)$ time.*

Lemma 7. *Assume an n vertex outerplanar st-digraph $G = (V^l \cup V^r \cup \{s,t\}, E)$ and a face f with source s' and sink v'. If $O(n)$ time is available for the preprocessing of G, we can decide in $O(1)$ time whether f is a weak rhombus. Moreover, the two vertices (in addition to s' and v') that define a maximal weak st-polygon that contains f can be also computed in $O(1)$ time.*

Denote by $\mathcal{R}(G)$ the set of all maximal st-polygons of an outerplanar st-digraph G, as identified by Lemmata 6 and 7. Observe that not every vertex of G belongs to one of its maximal st-polygons. We refer to the vertices of G that are not part of any maximal st-polygon as *free vertices* and we denote them by $\mathcal{F}(G)$. For example, vertices $s,\ i,\ j,\ z$ and t in the st-digraph of Figure 1.e are free vertices. Also observe that an ordering can be imposed on the maximal st-polygons of an outerplanar st-digraph G based on the ordering of the area disjoint strips occupied by each st-polygon. The vertices which do not belong to some st-polygon are located in the area between the strips occupied by consecutive st-polygons.

The next lemma states that the free vertices, the sources and the sinks of st-polygon are pairwise connected by directed paths.

Lemma 8. *Assume an outerplanar st-digraph G. Let R_1 and R_2 be two of G's consecutive maximal st-polygons and let $V_f \subset \mathcal{F}(G)$ be the set of free vertices lying between R_1 and R_2. Then, the following statements are satisfied:*

a) For any pair of vertices $u,\ v \in V_f$ there is either a path from u to v or from v to u.
b) For any vertex $v \in V_f$ there is a path from the sink of R_1 to v and from v to the source of R_2.
c) If $V_f = \emptyset$, then there is a path from source of R_1 to the source of R_2.

We refer to the source vertex s_i of each maximal st-polygon $R_i \in \mathcal{R}(G),\ 1 \leq i \leq |\mathcal{R}(G)|$ as the *representative* of R_i and we denote it by $r(R_i)$. We also define the representative of a free vertex $v \in \mathcal{F}(G)$ to be v itself, i.e. $r(v) = v$. For any two distinct elements $x,\ y\ \in \mathcal{R}(G) \cup \mathcal{F}(G)$, we define the relation $\angle_p$ as follows: *$x \angle_p y$ iff there exists a path from $r(x)$ to $r(y)$.*

Lemma 9. *Let G be an n node outerplanar st-digraph. Then, relation $\angle_p$ defines a total order on the elements $\mathcal{R}(G) \cup \mathcal{F}(G)$. Moreover, this total order can be computed in $O(n)$ time.*

Definition 2. *Given an outerplanar st-digraph G, the st-*polygon decomposition *$\mathcal{D}(G)$ of G is defined to be the total order of its maximal st-polygons and its free vertices induced by relation $\angle_p$.*

The following theorem follows directly from Lemma 6, Lemma 7 and Lemma 9.

Theorem 3. *An st-polygon decomposition of an n node outerplanar st-digraph G can be computed in $O(n)$ time.*

4 Crossing-Optimal Acyclic HP-Completion Set

In this section, we present some properties of a crossing optimal acyclic HP-completion for an outerplanar st-digraph that will be taken into account in our algorithm. Assume an outerplanar st-digraph $G = (V^l \cup V^r \cup \{s,t\}, E)$ and its st-polygon decomposition $\mathcal{D}(G) = \{o_1,\ \ldots, o_\lambda\}$. By G_i we denote the graph induced by the vertices of elements $o_1, \ldots, o_i,\ i \leq \lambda$. The results of this section are summarized in the following Theorem.

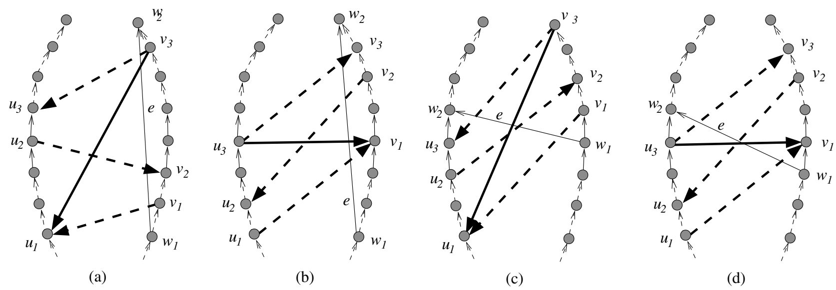

Fig. 2. Configurations of crossing edges used in the proof of Property 1

Theorem 4. *Let $G = (V^l \cup V^r \cup \{s,t\}, E)$ be an outerplanar st-digraph and let $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-polygon decomposition. Then, there exists a crossing optimal acyclic HP-completion set P_{opt} for G such that it satisfies the following properties:*

a) Each edge of E is crossed by at most two edges of P_{opt}.
b) The upper limiting edge e_i of any maximal st-polygon o_i, $i \leq \lambda$, is crossed by at most one edge of P_{opt}. Moreover, the edge crossing e_i, if any, enters G_i.

The proof of Theorem 4 follows immediately from the following properties.

Property 1. *Let $G = (V^l \cup V^r \cup \{s,t\}, E)$ be an outerplanar st-digraph. Then, no edge of E is crossed by more than 2 edges of a crossing-optimal acyclic HP-completion set for G.*

Proof (sketch). For the sake of contradiction, assume that P_{opt} is a crossing-optimal acyclic HP-completion set for G, the edges of which cross some edge $e = (w_1, w_2)$ of G three times. Suppose the edges crossing e are e_1, e_2, e_3. On Figures 2.a-d the edges $e_1 = (v_1, u_1)$, $e_2 = (u_2, v_2)$ and $e_3 = (v_3, u_3)$ are drawn by bold dashed line.

The idea of the proof is to show that we can obtain an acyclic HP-completion set for G that induces a smaller number of crossings than P_{opt}, a clear contradiction. We assume that all edges of P_{opt} participate in the hamiltonian path of G; otherwise they can be discarded.

We distinguish between two cases based on whether edge e is a one-sided or a two-sided edge and for each of these cases we consider the both directions of e_1, from the right to the left and from the left to the right. In all of these cases we substitute the edges e_1, e_2, e_3 of the HP-completion set by a single edge decreasing the total number of crossing created by the HP-completion set (see the bold solid edge on Figures 2.a-d for each case respectively). □

Property 2. *Let $G = (V^l \cup V^r \cup \{s,t\}, E)$ be an outerplanar st-digraph and let $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-polygon decomposition. Then, there exists a crossing optimal acyclic HP-completion set for G such that, for every maximal* *st-polygon* $o_i \in \mathcal{D}(G),\ i \leq \lambda$, *the HP-completion set contains at most one edge that crosses the upper limiting edge of* o_i *and, moreover, this edge enters* G_i. □

5 The Algorithm

The algorithm for obtaining a crossing-optimal acyclic HP-completion set for an outerplanar st-digraph G is a dynamic programming algorithm based on the st-polygon decomposition $\mathcal{D}(G) = \{o_1,\ \ldots, o_\lambda\}$ of G. The following lemmata allow us to compute a crossing-optimal acyclic HP-completion set for an st-polygon and to obtain a crossing-optimal acyclic HP-completion set for G_{i+1} by combining an optimal solution for G_i with an optimal solution for o_{i+1}.

Assume an outerplanar st-digraph G. We denote by $S(G)$ the hamiltonian path on the HP-extended digraph of G that results when a crossing-optimal HP-completion set is added to G. Note that if we are only given $S(G)$ we can infer the size of the HP-completion set and the number of edge crossings. Denote by $c(G)$ the number of edge crossings caused by the HP-completion set inferred by $S(G)$. If we are restricted to Hamiltonian paths that enter the sink of G from a vertex on the left (resp. right) side of G, then we denote the corresponding size of HP-completion set as $c(G, L)$ (resp. $c(G, R)$). Obviously, $c(G) = \min\{c(G, L),\ c(G, R)\}$. Moreover, the notation can be extended to denote by $c^i(G, L)$ ($c^i(G, R)$) the corresponding minimum number of crossings over all HP-completion sets that contain exactly i edges. By Theorem 2, we know that the size of a crossing-optimal acyclic HP-completion set for an st-polygon is at most 2. This notation that restricts the size of the HP-completion set will be used only for st-polygons and thus, only the terms $c^1(G, L),\ c^1(G, R),\ c^2(G, L)$ and $c^2(G, R)$ will be utilized.

We use the operator $\oplus$ to indicate the concatenation of two paths. By convention, the hamiltonian path of a single vertex is the vertex itself.

The next lemma (follows from the Lemmata 4 and 5) states that it is sufficient to examine all HP-completion sets with one or two edges in order to find a crossing-optimal acyclic HP-completion set for an st-polygon.

Lemma 10. *Assume an* n *vertex st-polygon* $o = (V^l \cup V^r \cup \{s, t\}, E)$. *A crossing-optimal acyclic HP-completion set for* o *and the corresponding number of crossings can be computed in* $O(n)$ *time.*

Let $\mathcal{D}(G) = \{o_1,\ \ldots, o_\lambda\}$ be the st-polygon decomposition of G, where element $o_i,\ 1 \leq i \leq \lambda$ is either an st-polygon or a free vertex. Recall that, we denote by $G_i,\ 1 \leq i \leq \lambda$ the graph induced by the vertices of elements $o_1, \ldots, o_i$. Graph G_i is also an outerplanar st-digraph. The same holds for the subgraph of G that is induced by any number of consecutive elements of $\mathcal{D}(G)$.

The next two lemmata describe how to combine the optimal solutions for consecutive st-polygons into an optimal solution for the whole outerplanar st-digraph.

Algorithm 1. ACYCLIC-HPC-CM(G)

input : An Outerplanar st-digraph $G(V^l \cup V^r \cup \{s,t\}, E)$.
output : The minimum number of edge crossing $c(G)$ resulting from the addition of a crossing-optimal acyclic HP-completion set to graph G.

1. Compute the st-polygon decomposition $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ of G;
2. For each element $o_i \in \mathcal{D}(G)$, $1 \le i \le \lambda$,
   compute $c^1(o_i, L)$, $c^1(o_i, R)$ and $c^2(o_i, L)$, $c^2(o_i, R)$:
   if o_i is a free vertex, **then** $c^1(o_i, L) = c^1(o_i, R) = c^2(o_i, L) = c^2(o_i, R) = 0$.
   if o_i is an st-polygon, **then** $c^1(o_i, L)$, $c^1(o_i, R)$, $c^2(o_i, L)$, $c^2(o_i, R)$ are computed
   based on Lemma 10.
3. **if** o_1 is a free vertex, **then** $c(G_1, L) = c(G_1, R) = 0$;
   else $c(G_1, L) = \min\{c^1(o_1, L), c^2(o_1, L)\}$ and
   $c(G_1, R) = \min\{c^1(o_1, R), c^2(o_1, R)\}$;
4. For $i = 1 \ldots \lambda - 1$, compute $c(G_{i+1}, L)$ and $c(G_{i+1}, R)$ as follows:
   if o_{i+1} is a free vertex, **then**
   $c(G_{i+1}, L) = c(G_{i+1}, R) = \min\{c(G_i, L), c(G_i, R)\}$;
   else-if o_{i+1} is an st-polygon sharing **at most** one vertex with G_i, **then**
   $c(G_{i+1}, L) = \min\{c(G_i, L), c(G_i, R)\} + \min\{c^1(o_{i+1}, L), c^2(o_{i+1}, L)\}$;
   $c(G_{i+1}, R) = \min\{c(G_i, L), c(G_i, R)\} + \min\{c^1(o_{i+1}, R), c^2(o_{i+1}, R)\}$;
   else { o_{i+1} is an st-polygon sharing **exactly** two vertices with G_i},
   if $t_i \in V^l$, **then**
   $c(G_{i+1}, L) = \min\{c(G_i, L) + c^1(o_{i+1}, L) + 1, \ c(G_i, R) + c^1(o_{i+1}, L),$
   $c(G_i, L) + c^2(o_{i+1}, L), \ c(G_i, R) + c^2(o_{i+1}, L)\}$
   $c(G_{i+1}, R) = \min\{c(G_i, L) + c^1(o_{i+1}, R), \ c(G_i, R) + c^1(o_{i+1}, R),$
   $c(G_i, L) + c^2(o_{i+1}, R) + 1, \ c(G_i, R) + c^2(o_{i+1}, R)\}$
   else { $t_i \in V^r$ }
   $c(G_{i+1}, L) = \min\{c(G_i, L) + c^1(o_{i+1}, L), \ c(G_i, R) + c^1(o_{i+1}, L),$
   $c(G_i, L) + c^2(o_{i+1}, L), \ c(G_i, R) + c^2(o_{i+1}, L) + 1\}$
   $c(G_{i+1}, R) = \min\{c(G_i, L) + c^1(o_{i+1}, R), \ c(G_i, R) + c^1(o_{i+1}, R) + 1,$
   $c(G_i, L) + c^2(o_{i+1}, R), \ c(G_i, R) + c^2(o_{i+1}, R)\}$
5. **return** $c(G) = \min\{c(G_\lambda, L), c(G_\lambda, R)\}$

Lemma 11. *Assume an outerplanar st-digraph G and let $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-polygon decomposition. Consider any two consecutive elements o_i and o_{i+1} of $\mathcal{D}(G)$ that share at most one vertex. Then, the following statements hold:*

a) $S(G_{i+1}) = S(G_i) \oplus S(o_{i+1})$, *and*
b) $c(G_{i+1}) = c(G_i) + c(o_{i+1})$.

Lemma 12. *Assume an outerplanar st-digraph G and let $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-polygon decomposition. Consider any two consecutive elements o_i and o_{i+1} of $\mathcal{D}(G)$ that share an edge. Then, the following statements hold:*

1. $t_i \in V^l \Rightarrow c(G_{i+1}, L) = \min\{ \ c(G_i, L) + c^1(o_{i+1}, L) + 1, \ c(G_i, R) + c^1(o_{i+1}, L),$
$c(G_i, L) + c^2(o_{i+1}, L), \ c(G_i, R) + c^2(o_{i+1}, L)\}$

2. $t_i \in V^l \Rightarrow c(G_{i+1}, R) = \min\{ c(G_i, L) + c^1(o_{i+1}, R),\ c(G_i, R) + c^1(o_{i+1}, R),\ c(G_i, L) + c^2(o_{i+1}, R) + 1,\ c(G_i, R) + c^2(o_{i+1}, R)\}$

3. $t_i \in V^r \Rightarrow c(G_{i+1}, L) = \min\{ c(G_i, L) + c^1(o_{i+1}, L),\ c(G_i, R) + c^1(o_{i+1}, L),\ c(G_i, L) + c^2(o_{i+1}, L),\ c(G_i, R) + c^2(o_{i+1}, L) + 1\}$

4. $t_i \in V^r \Rightarrow c(G_{i+1}, R) = \min\{ c(G_i, L) + c^1(o_{i+1}, R),\ c(G_i, R) + c^1(o_{i+1}, R) + 1,\ c(G_i, L) + c^2(o_{i+1}, R),\ c(G_i, R) + c^2(o_{i+1}, R)\}$

Algorihtm 1 is a dynamic programming algorithm, based on Lemmata 11 and 12, which computes the minimum number of edge crossings $c(G)$ resulting from the addition of a crossing-optimal HP-completion set to an outerplanar st-digraph G. The algorithm can be easily extended to also compute the corresponding hamiltonian path $S(G)$.

Theorem 5. *Given an n vertex outerplanar st-digraph G, a crossing-optimal HP-completion set for G and the corresponding number of edge-crossings can be computed in $O(n)$ time.*

We note that, from Theorem 2 in [8] and our presented algorithm, the following theorem is implied.

Theorem 6. *Given an n vertex outerplanar st-digraph G, an upward 2-page topological book embedding for G with minimum number of spine crossings can be computed in $O(n)$ time.*

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