Crossing-Free Acyclic Hamiltonian Path Completion for Planar *st*-Digraphs

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Abstract. In this paper we study the problem of existence of a crossing-free acyclic hamiltonian path completion (for short, HP-completion) set for embedded upward planar digraphs. In the context of book embeddings, this question becomes: given an embedded upward planar digraph G, determine whether there exists an upward 2-page book embedding of G preserving the given planar embedding.

Given an embedded *st*-digraph G which has a crossing-free HP-completion set, we show that there always exists a crossing-free HP-completion set with at most two edges per face of G. For an embedded N-free upward planar digraph G, we show that there always exists a crossing-free acyclic HP-completion set for G which, moreover, can be computed in linear time. For a width-k embedded planar *st*-digraph G, we show that it can be efficiently tested whether G admits a crossing-free acyclic HP-completion set.

1 Introduction

A *k-page book* is a structure consisting of a line, referred to as *spine*, and of k half-planes, referred to as *pages*, that have the spine as their common boundary. A *book embedding* of a graph G is a drawing of G on a book such that the vertices are aligned along the spine, each edge is entirely drawn on a single page, and edges do not cross each other. If we are interested only in two-dimensional structures we have to concentrate on 2-page book embeddings and to allow spine crossings. These embeddings are also referred to as 2-page *topological* book embeddings.

For acyclic digraphs, an upward book embedding can be considered to be a book embedding in which the spine is vertical and all edges are drawn monotonically increasing in the upward direction. As a consequence, in an upward book embedding of an acyclic digraph G the vertices of G appear along the spine in topological order. If G is planar upward digraph and an upward embedding of G on the plane is given, we are interested to determine a 2-page upward topological book embedding of G which preserves its plane embedding and has minimum number of spine crossings. Giordano et al. [5] showed that an embedded upward planar digraph always admits an upward topological 2-page book embedding (which preserves its plane embedding) with at most one spine crossing per edge. However, in their work no effort was made to minimize the total number of spine crossings.

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The *acyclic hamiltonian path completion with crossing minimization problem* (*Acyclic-HPCCM*) was inspired by its equivalence with the problem of determining an upward 2-page topological book embedding with a minimum number of spine crossings for an embedded planar *st*-digraph [8].

In the *hamiltonian path completion problem* (*HPC*) we are given a graph[1] G and we are asked to identify a set of edges S (refereed to as an *HP-completion set*) such that, when the edges of S are embedded on G they turn it to a hamiltonian graph, that is, a graph containing a hamiltonian path[2]. The resulting hamiltonian graph G_S is referred to as the *HP-completed graph* of G. When we treat the HP-completion problem as an optimization problem, we are interested in HP-completion sets of minimum size. When the input graph G is an embedded planar digraph, an HP-completion set S for G must be naturally extended to include an embedding of its edges on the plane, yielding to an embedded HP-completed digraph G_S. In general, G_S is not planar, and thus, it is natural to attempt to minimize the number of edge crossings of the embedding of the HP-completed digraph G_S instead of the size of the HP-completion set S. This problem is known as *HP-completion with crossing minimization problem* (*HPCCM*) and was first defined in [8]. When the input digraph G is acyclic, we can insist on HP-completion sets which leave the HP-completed digraph G' also acyclic. We refer to this version of the problem as the *Acyclic-HPC problem.* Analogously, we define the *acyclic-HPCCM* which, as stated above, is equivalent to determining 2-page upward topological book embeddings with minimum number of spine crossings for embedded upward planar digraphs. When dealing with the acyclic-HPCCM problem, it is natural to first examine whether there exists an acyclic HP-completion set for a digraph G of zero crossings, i.e., a *crossing-free acyclic HP-completion set* for G. In terms of an upward 2-page topological book embedding, this question is formulated as follows: given an embedded upward planar digraph G, determine whether there exists an upward 2-page book embedding of G without spine crossings preserving G's embedding.

In this paper we focus on crossing-free hamiltonian path completion sets for embedded upward planar digraphs. Our results include:

1. Given an embedded st-digraph G which has a crossing-free HP-completion set, we show that there always exists a crossing-free HP-completion set with at most two edges per face of G (Theorem 1).

This result finds application to upward 2-page book embeddings. The problem of spine crossing minimization in an upward topological book embedding is defined with a scope to improve the visibility of such drawings. For the class of upward planar digraphs that always admit an upward 2-page book embedding (i.e. a topological book embedding without spine crossings) it make sense to define an additional criterion of visibility. When a graph is embedded in a book, its faces are split by the spine into several adjacent parts. It is clear that the

[1] In this paper, we assume that G is directed.

[2] In the literature, a *hamiltonian graph* is traditionally referred to as a graph containing a hamiltonian cycle. In this paper, we refer to a hamiltonian graph as a graph containing a hamiltonian path.

visibility of a drawing improves if each face is split into as few parts as possible. This result implies that the upward planar digraphs which admit an upward 2-page book embedding also admit one such embedding where each face is divided to at most 3 parts by the spine.

2. Given an embedded N-free upward planar digraph G, we show how to construct a crossing-free HP-completion set for G (Theorem 3). The class of embedded N-free upward planar digraphs is the class of embedded upward planar digraphs that does not contain as a subgraph the embedded N-graph of Figure 1.a. N-free upward planar digraphs have been studied in the context of partially ordered sets (posets) and lattices [1]. The class of N-free upward planar digraphs contains the class of series-parallel digraphs which has been thorough studied in the context of book embeddings [4].

3. Given a width-k embedded planar st-digraph G, we show how to determine whether G admits a crossing-free HP-completion set (Theorem 5). It follows that for fixed-width embedded planar *st*-digraphs, it can be tested in polynomial time whether there exists a crossing-free HP-completion set (and thus, a 2-page upward book embedding). The result is based on a reduction to the *minimum setup scheduling* problem.

For reasons of space, some proofs have been omitted and can be found in [7].

2 Terminology and Notation

Let $G = (V, E)$ be a graph. Throughout the paper, we use the term *"graph"* to refer to both directed and undirected graphs. We use the term *"digraph"* when we want to restrict our attention to directed graphs. We assume familiarity with basic graph theory [6,3]. A *drawing* Γ of graph G maps every vertex v of G to a distinct point $p(v)$ on the plane and each edge $e = (u, v)$ of G to a simple open curve joining $p(u)$ with $p(v)$. A drawing in which every edge (u, v) is a a simple open curve monotonically increasing in the vertical direction is an *upward drawing.* A drawing Γ of graph G is *planar* if no two distinct edges intersect except at their end-vertices. Graph G is called *planar* if it admits a planar drawing Γ. An embedding of a planar graph G is the equivalence class of planar drawings of G that define the same set of faces or, equivalently, of face boundaries. A planar graph together with the description of a set of faces F is called an *embedded planar graph.* Let $G = (V, E)$ be an embedded planar graph, E' be a superset of edges containing E, and $\Gamma(G')$ be a drawing of $G' = (V, E')$. When the deletion from $\Gamma(G')$ of the edges in $E' - E$ induces the embedded planar graph G, we say that $\Gamma(G')$ *preserves the embedded planar graph* G. Let $G = (V, E)$ be a digraph. A vertex of G with in-degree (resp. out-degree) equal to zero (0) is called a *source* (resp., *sink*). An *st-digraph* is an acyclic digraph with exactly one source and exactly one sink. Traditionally, the source and the sink of an *st*-digraph are denoted by s and t, respectively. An *st*-digraph which is planar and, in addition, embedded on the plane so that both of its source and sink appear on the boundary of its external face, is referred to as a *planar st-digraph.* In a planar *st*-digraph G each face f is bounded by two directed paths

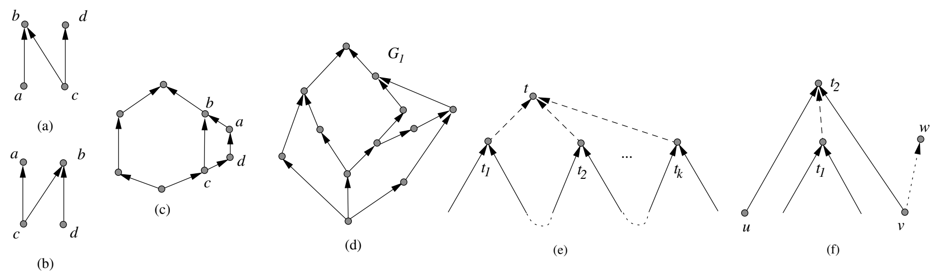

Fig. 1. (a) Embedded N-digraph. (b) Embedded $И$-digraph. (c) Planar digraph that is N-free if treated as an embedded planar digraph, but not N-free as a planar digraph. (d) An embedded N-free planar st-digraph G_1. (e)-(f) The construction for the proof of Theorem 3.

which have two common end-vertices. The common origin (resp., destination) of these paths is called the *source* (resp., *sink*) of f and is denoted by $source(f)$ (resp., $sink(f)$). The leftmost (resp., rightmost) of these two paths is called a *left border* (resp., *right border*) of face f. The *bottom-left* (rest., *bottom-right*) edge of a face f is the first edge on its left(resp., right) border. Similarly we define the *top-left* and the *top-right* edge of a face border. The *right(left) border* of an st-digraph is the rightmost(leftmost) path from its source s to its sink t. A new edge e that is inserted to a face f of a planar st-digraph G, with its origin and destination on the the left and right border of f, respectively, is called a *left-to-right oriented* edge. Analogously, we define a *right-to-left oriented* edge. Following the terminology of posets, the digraph $G_N = (V_N, E_N)$, where $V_N = \{a, b, c, d\}$ and $E_N = \{(a, b), (c, b), (c, d)\}$ is called an N*-digraph*. Then, any digraph that does not contain G_N as a subgraph is called an N*-free* digraph. This definition can be extended to embedded planar digraphs by insisting on a specific embedding. If we adopt the embedding of Figure 1.a. we refer to an *embedded* N*-digraph* while, if we adopt the embedding Figure 1.b. we refer to an *embedded* $И$-digraph. An embedded planar digraph G is then called N-free ($И$-free) if it does not contain any embedded N-digraph ($И$-digraph) as a subgraph. Figure 1.c shows an embedded N-free digraph. However, when its embedding is ignored, the digraph is not N-free since vertices a, b, c, d comprise a N-digraph.

Let $G = (V, E)$ be an embedded planar st-digraph. The external face is split into two faces, s^* and t^*. s^* is the face to the left of the left border of G while t^* is the face to the right of the right border of G. For each $e = (u, v) \in E$, we denote by $left(e)$ (resp. $right(e)$) the face to the left (resp. right) of edge e as we move from u to v. The *dual* of an st-digraph G, denoted by G^*, is a digraph such that: (i) there is a vertex in G^* for each face of G; (ii) for every edge $e \neq (s, t)$ of G, there is an edge $e^* = (f, g)$ in G^*, where $f = left(e)$ and $g = right(e)$. If G^* after this construction contains multiply edges, we substitute them by single edges. It is a well known fact that the dual graph G^* of any planar st-digraph G, is also a planar st-digraph with source s^* and sink t^*. The following definitions were given in [5] for maximal planar st-digraph. Here we extend them for planar st-digraphs. Let $G = (V, E)$ be a planar st-digraph and G^* be the dual digraph of G. Let $v_1^* = s^*, v_2^*, \ldots, v_m^* = t^*$ be the set of vertices of G^* where the indices are given according to an st-numbering of G^*. By the definition of the dual st-digraph, a vertex v_i^* of G^* $(1 \leq i \leq m)$ corresponds to a face of G. In the following we denote by v_i^* both the vertex of the dual digraph G^* and its corresponding face in digraph G. Face v_k^* is called the *k-th face* of G. Let V_k be the subset of the vertices of G that belong to faces $v_1^*, v_2^*, \ldots, v_k^*$. The subgraph of G induced by vertices in V_k is called the *k-facial subgraph* of G and is denoted by G_k. The next lemma describes how, given an st-digraph G and an st-numbering of its dual, G can be incrementally constructed from its faces. The proof is identical to the proof given in [5] for maximal planar st-digraphs.

Lemma 1. *Assume a planar st-digraph G and let $v_1^* = s^*, v_2^*, \ldots, v_m^* = t^*$ an st-numbering of its dual G^*. Consider the k^{th}-facial subgraph G_k and the $k+1$-th face v_{k+1}^* of G, $(1 \leq k < m)$. Let s_{k+1} be the source of v_{k+1}^*, t_{k+1} be the sink of v_{k+1}^*, s_{k+1}, u_1^l, $u_2^l, \ldots,$ u_i^l, t_{k+1} be its left border, and s_{k+1}, u_1^r, $u_2^r, \ldots,$ u_j^r, t_{k+1} be its right border. Then:*

a. G_k is a planar st-digraph.

b. The vertices $s_{k+1}, u_1^l, u_2^l, \ldots, u_i^l, t_{k+1}$ are vertices of the right border of G_k.

c. G_{k+1} can be built from G_k by an addition of a single directed path $s_{k+1}, u_1^r, u_2^r, \ldots, u_j^r, t_{k+1}$. □

Let $G = (V, E)$ be an embedded planar st-digraph which has an acyclic crossing-free HP-completion set S. By $G_S = (V, E \bigcup S)$ we denote the HP-completed acyclic digraph and by P_{G_S} the resulting hamiltonian path. Note that, as S creates zero crossings with G, each edge of S is drawn within a face of G and, therefore, G_S is a planar st-digraph.

3 Two Edges Per Face Are Enough

In this Section, we prove that an embedded planar st-digraph G which has a crossing-free acyclic HP-completion set, always admits a crossing-free HP-completion set with at most two edges per face of G. This result implies that the upward planar st-digraphs which admit an upward 2-page book embedding also admit one such embedding where each face is divided to at most 3 parts by the spine. This improves the quality of the book embedding drawing.

Theorem 1. *Assume an embedded planar st-digraph G which has an acyclic crossing-free HP-completion set S. Then, there exists another acyclic crossing-free HP-completion set S' for G containing at most two edges per face of G.*

Sketch of proof: The proof is based on the fact that any three consecutive edges of the HP-completion set drawn on the same face can be substituted by a single HP-completion edge (see Figure 2). □

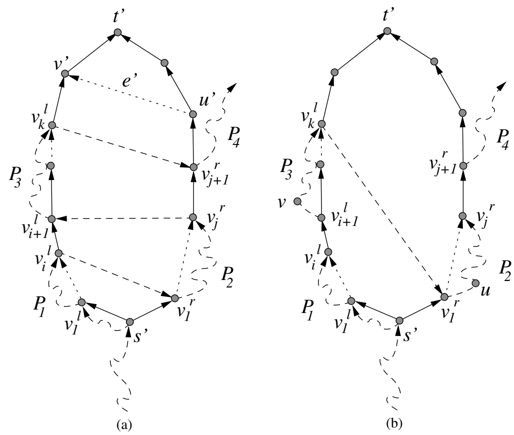

Fig. 2. (a) A crossing-free acyclic HP-completion set S which places at least three edges to a face of an st-digraph. (b) An equivalent crossing-free acyclic HP-completion set S' where the three edges $(v_i^l, v_1^r), (v_j^r, v_{i+1}^l), (v_k^l, v_{j+1}^r)$ were substituted by a single edge (v_k^l, v_1^r).

4 Embedded *N*-Free Upward Planar Digraphs Always Have Crossing-Free Acyclic HP-Completion Sets

In this Section, we study embedded N-free upward planar digraphs. We establish that any embedded N-free upward planar digraph G has a crossing-free acyclic HP-completion set with at most one edge per face of G. Recall that the class of embedded N-free upward planar digraphs is the class of embedded upward planar digraphs that does not contain as a subgraph the embedded N-graph of Figure 1.a. For the class of N-free upward planar embedded digraphs, which is substantially larger than the class of N-free upward planar digraphs, we show that there is always a crossing-free acyclic HP-completion set that can be computed in linear time, thus improving the results given in [1,4]

Theorem 2. *Any embedded N-free planar st-digraph $G = (V, E)$ has an acyclic crossing-free HP-completion set S which contains exactly one edge per face of G. Moreover, S can be computed in $O(V)$ time.*

Proof. Let G^* be the dual graph of G and let $s^* = v_1^*, \ldots, v_m^* = t^*$ be the vertices of G^* ordered according to an st-numbering of G^*. Let G_{k-1} be the $(k-1)$-facial subgraph of G. By Lemma 1, G_k can be constructed from G_{k-1} by adding to the right border of G_{k-1} the directed path forming the right border of v_k^*.

We prove the following stronger statement than the one in the theorem:

Statement 1. *For any G_k $(1 \le k < m)$ there exists an acyclic crossing-free HP-completion set S_k such that the following holds: Let P_k be the resulting hamiltonian paht and let e be an edge of the right border of G_k that is also the bottom-left edge of a face $f \in \{v_{k+1}^*, \ldots v_m^*\}$. Then, edge e is traversed by P_k.*

Proof of Statement 1. If $k = 1$, G_1 consist of a single path, that is the left border of G (Figure 3.a). We let $S_1 = \{\emptyset\}$ and set P_1 to G_1. As all the edges of G_1 are traversed by P_1 it is clear that, any edge e on the right border of G_1 that is also a bottom-left edge of any other face t is traversed by P_1. Assume now that the statement is true for any G_{k-1}, $k < m$. We will show that it is true for G_k. Denote by S_{k-1} a crossing-free acyclic HP-completion set of G_{k-1} and by P_{k-1} the produced hamiltonian path. Let e be an edge on the right border of G_{k-1} that is also the bottom-left edge of v_k^*. By the induction hypothesis, P_{k-1} passes through $e = (s_k, v)$ (see Figure 3.b). Denote by s_k and t_k the source and the sink of v_k^* respectively, and by $v_1^r, \ldots, v_{m_k}^r$ the vertices of the right border of v_k^*. By Lemma 1, s_k and t_k are vertices of the right border of G_{k-1} and G_k can be built from G_{k-1} by adding the path $s_k, v_1^r, \ldots, v_{m_k}^r, t_k$ to it. Suppose first that $m_k \neq 0$ (i.e., the right border of v_k^* contains at least one vertex). Set $S_k = S_{k-1} \bigcup \{(v_{m_k}^r, v)\}$, and $P_k = P_{k-1}[s \ldots s_k], v_1^r, \ldots, v_{m_k}^r, P_{k-1}[v \ldots t]$ (see Figure 3.c). It is clear that P_k is a hamiltonian path of G_k. This is because P_{k-1} is hamiltonian path of G_{k-1} and P_k traverses all newly added vertices. It is also easy to see that S_k is acyclic: the edge $(v_{m_k}^r, v)$ which was added to S_{k-1} creates a single directed path: from vertex s_k to the vertex v, which were already connected by the directed edge (s_k, v) in G_{k-1}. We now show that the bottom-left edge e of any $f \in \{v_{k+1}^* \ldots v_m^*\}$, where e is also on the right border of G_k, is traversed by P_k. The only edge that was added to G_{k-1} to create G_k and is not traversed by P_k, is $e' = (v_{m_k}^r, t_k)$, that is, e' is the last edge of the right border of v_k^*. If e' is also the left bottom edge of a f then the graph has an embedded N-digraph as a subgraph (see the subgraph induced by the vertices $u, t_k, v_{m_k}^r, w$ in Figure 3.c), a contradiction. Otherwise, if the bottom-left edge of f coincides with any other edge of the right border of v_k^*, then the statement holds. If f has its bottom-left edge on the right border of G_{k-1} then, by the induction, a bottom left edge of f is traversed by P_{k-1} and, thus, by P_k. Consider now the case where $m_k = 0$, that is, the right border of v_k^* is a single, transitive edge (see Figure 3.d). In this case, no new vertex is added to G_k, so we set $S_{k+1} = S_k$ and $P_{k+1} = P_k$. Consider now a face $f \in \{v_{k+1}^* \cdots v_m^*\}$. If the bottom-left edge e of f is on the right border of G_k and coincides with the transitive edge (s_k, t_k), then u, t_k, s_k, w form an embedded N-digraph (see Figure 3.d), a contradiction. So e is not (s_k, t_k) and, hence, it is an edge of the right border of G_{k-1}. So, by the induction hypothesis, e is traversed by P_{k-1} and hence by P_k. This completes the proof of the statement. Having proved Statement 1, the theorem follows from the observation that $G_m = G$. The bound on the time needed to compute the crossing-free HP-completion set easily follows from the incremental nature of the described constructive proof. □

Corollary 1. *Any И-free embedded planar st-digraph $G = (V, E)$ has an acyclic crossing-free HP-completion set S which contains exactly one edge per face of G. Moreover, S can be computed in $O(V)$ time.*

Proof. Reverse the edges of G^* and repeat the proof of Theorem 2.

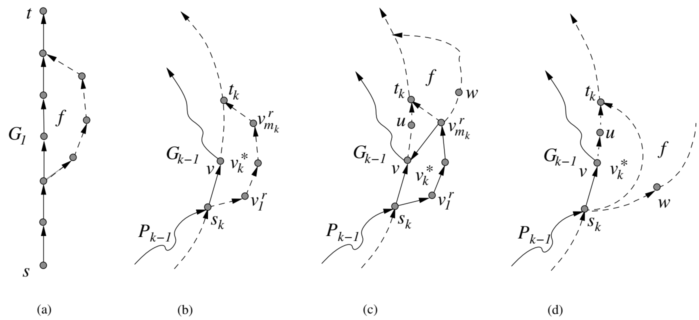

Fig. 3. (a) $G_1 = P_1$ and a face f. The bottom left edge of f is traversed by P_1. (b) G_{k-1} and v^*_k. P_{k-1} is denoted by solid line. (c) A graph G_k for the case that the right border of v^*_k contains at least one vertex. The newly constructed P_k is denoted by solid line. (d) A graph G_k for the case that the right border of v^*_k is a transitive edge.

Theorem 3. *Any embedded N-free upward planar digraph $G = (V, E)$ has an acyclic crossing-free HP-completion set that can be computed in $O(V)$ time.*

Proof. We just prove that, any embedded N-free upward planar digraph G can be transformed to an embedded N-free upward planar st-digraph G' by the addition of few edges. Then, the result follows from Theorem 2. Consider an upward planar embedding Γ of $G = (V, E)$ that is N-free. If the outer face of G contains more than one sink (source), then we add a new super-sink (super-source) vertex. Let $t_1, \ldots, t_k$ be the sinks of G in the outer face. By adding a new vertex t and by joining each t_i to t by an edge, the embedding Γ of G is preserved and remains N-free, because each t_i $1 \leq i \leq k$ has out-degree zero (see Figure 1.e). Let now some sink t_1 be placed in a inner face of G. Let t_2 be the sink of that face. We add edge (t_1, t_2). The addition of the edge (t_1, t_2) creates an embedded N-digraph only if there are edges (v, t_2) and (v, w) in G with (v, w) is the edge following (v, t_2) (in counter clockwise order), out of v. But then, there is already an embedded N-digraph in G (the digraph induced by the vertices u, t_2, v, w in Figure 1.f). A clear contradiction, so (t_1, t_2) can be added to G without creating any embedded N-digraph as a subgraph. The sources are treated similarly. The transformation of G into an st-digraph can be easily completed in linear time. □

5 Crossing-Free Acyclic HP-Completion Sets for Fixed Width *st*-Digraphs

In this section we establish that for any embedded planar st-digraph G of bounded width, there is a polynomial time algorithm determining whether there exists a crossing-free HP-completion set for G. In the case that such an HP-completion set exists, we can easily construct it. A set Q of vertices of G is called *independent* if the graph incident to Q has no edges. Following the terminology of partially ordered sets, we call *width* of G, and denote it by $width(G)$, the maximum integer r such that G has an independent set of cardinality r. In Minimum Setup Scheduling (MSS) we are given a number of jobs that are to be executed in sequence by a single processor. There are constraints which require that certain jobs be completed before another may start; these constraints are given in the form of *precedence dag*. In addition, for each pair i, j of jobs there is a *setup cost* representing the cost of performing job j immediately after job i, denoted by $cost(i, j)$. The objective is to find a one-processor schedule for all jobs which satisfies all the precedence constraints and minimizes the total setup cost incurred. The main idea of the result presented in this section is a simple application of an algorithm solving the *minimum setup scheduling* problem . Given a precedence dag D and a matrix C of costs, $s(D, C)$ denotes the total setup cost of a minimum cost schedule satisfying the constraints given by D. The next theorem follows from the complexity analysis given in [2].

Theorem 4 ([2]). *Given an n-vertex precedence dag D of width k and a matrix C of setup costs, we can compute in $O(n^k k^2)$ time a setup cost $s(P, C)$ of minimum cost schedule, satisfying the constraints given by P.*

In the rest of this section, we show that given a planar *st*-digraph G the problem of determining whether there is a crossing-free acyclic HP-completion set for G can be presented as an instance of MSS. Let $G = (V, E)$ is an embedded planar *st*-digraph. We define the setup cost matrix as follows. Set $C_G[i, j] = 0$ if $(v_i, v_j) \in E$ or v_i and v_j belong to the opposite borders of the same face of G, otherwise set $C_G[i, j] = 1$.

Lemma 2. *Let $G = (V, E)$ be an embedded planar st-digraph. Let also $s(G, C_G)$ be a setup cost of minimum cost schedule satisfying the constraints given by G and setup costs given by C_G. G has an acyclic crossing-free HP-completion set iff $s(G, C_G) = 0$.*

Proof. ($\Rightarrow$) Assume that G has a crossing-free acyclic HP-completion set S and the vertices in the sequence $v_1, v_2, \ldots, v_n$ are enumerated as they appear in the hamiltonian path which is created when S is embedded on G. Then, the sequence $v_1, v_2, \ldots, v_n$ presents a schedule satisfying constraints given by G, otherwise an embedding of S in G would create a cycle. The setup cost for this schedule is $\sum_{i=1}^{n-1} C_G[i, i+1]$. We know that S does not create any crossing with G. Therefore, any two successive vertices v_i and v_{i+1} of the resulting hamiltonian path are either connected by an edge of the graph or belong to the opposite borders of the same face, and thus, $C_G[i, i+1] = 0$. So, we have shown that there is a schedule of setup cost zero and, thus, $s(G, C_G) = 0$.

($\Leftarrow$) Assume now that $s(G, C_G) = 0$, i.e., there exists a one-processor schedule for the jobs represented by the vertices of G which has total setup cost zero and satisfies the precedence constraints given by G. Let $v_1,\ v_2, \ldots,\ v_n$ be the jobs as they appear in this schedule. We construct the set of edges S as follows: Consider any two successive jobs v_i and v_{i+1}. If they are not connected by an edge (v_i, v_{i+1}) of G, then we add this edge to S. All the edges added to S

correspond to two jobs with setup cost zero, and hence represent edges which connect two vertices of the opposite borders of the same face. So we have that S creates in G a hamiltonian path which does not cross any edge of G. Finally we note that (i) there can be no crossings among the edges of S and, (ii) the addition of S to G does not create any cycle. □

Theorem 5. *Let G be a planar st-digraph of width $k \in \mathbb{N}$. Then, in $O(k^2 n^k)$ time we can decide whether G has an crossing-free HP-completion set. In the event that such a set exists, it can be easily computed in the same time bounds.*

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