Computing Upward Topological Book Embeddings of Upward Planar Digraphs*

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Abstract. This paper studies the problem of computing an upward topological book embedding of an upward planar digraph G, i.e. a topological book embedding of G where all edges are monotonically increasing in the upward direction. Besides having its own inherent interest in the theory of upward book embeddability, the question has applications to well studied research topics of computational geometry and of graph drawing. The main results of the paper are as follows.

- Every upward planar digraph G with n vertices admits an upward topological book embedding such that every edge of G crosses the spine of the book at most once.
- Every upward planar digraph G with n vertices admits a point-set embedding on any set of n distinct points in the plane such that the drawing is upward and every edge of G has at most two bends.
- Every pair of upward planar digraphs sharing the same set of n vertices admits an upward simultaneous embedding with at most two bends per edge.

1 Introduction

A *book* consists of a line called *spine* and of k half-planes, called *pages*, having the spine as a boundary. A *book embedding* of a planar graph G is a drawing of G on a book such that the vertices are aligned along the spine, each edge is drawn in a page and shares with the spine only its end-vertices, and no two edges cross. A well-known result is that all planar graphs have a book embedding on four pages and that there exist some planar graphs requiring exactly four pages to be book embedded [28]. Thus, book embeddings of planar graphs are in general three-dimensional representations and if one wants to have a two dimensional drawing of a planar graph where all vertices are collinear, edges must be allowed to cross the spine. Drawings where spine crossings are allowed are known in the literature as *topological book embeddings* [13]. In [10] it is proved that every planar graph admits a topological book embedding in the plane such that every edge crosses the spine at most once.

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Motivated by parallel process scheduling problems, *upward book embeddings* of acyclic digraphs and of posets have also been widely investigated (see e.g., [1,19,20,21,26]). An upward book embedding of an acyclic digraph G is a book embedding of G such that the ordering of the vertices along the spine is a topological ordering of G. Informally, an upward book embedding is a book embedding in which the spine is vertical and the directed edges are drawn as curves monotonically increasing in the upward direction. In contrast to the result in [28] concerning the book embeddability of undirected planar graphs, the minimum number of pages required by an upward book embedding of a planar acyclic digraph is unbounded [19], while the minimum number of pages required by an upward book embedding of an upward planar digraph is not known [1,19,26]. Only some classes of upward planar digraphs requiring a constant number of pages have been established to date (see, e.g. [1,9,21]).

This paper studies the problem of computing an *upward topological book embedding* of an upward planar digraph G, i.e. a topological book embedding of G in 2 pages, where all edges are monotonically increasing in the upward direction. Besides having its own inherent interest in the theory of upward book embeddability, the question has applications to well studied research topics of graph drawing and of computational geometry. The first and more immediate application is in the context of computing drawings of hierarchical structures where it is required to consider not only aesthetic constraints such as the upwardness and the planarity but also semantic constraints expressed in terms of collinearity for a (sub)set of the vertices; for example, in the application domains of knowledge engineering and of project management, PERT diagrams are typically drawn by requiring that critical sequences of tasks be represented as collinear vertices (see, e.g., [8,27]).

Upward topological book embeddings turn out to be also a useful tool to address a classical problem of computational geometry. Let G be a planar graph with n vertices and let S be a set of n distinct points in the plane. A *point-set embedding* of G on S is a planar drawing of G where every vertex of G is mapped to a point of S. The problem of computing point-set embeddings of planar graphs such that the number of bends along the edges be a small constant is the subject of a rich body of literature (including, e.g., [3,4,18,22]). We shall discuss how to use upward topological book embeddings in order to find new results in the context of point-set embeddings of planar acyclic digraphs with the additional constraint that all edges are oriented upward.

Finally, an emerging research direction in graph drawing studies the problem of representing and visually comparing multiple related graphs which typically come from different application domains including software engineering, telecommunications, and computational biology. *Simultaneous embeddings* (see, e.g., [5,6,11,14,16]) aid in visualizing multiple relationships between the same set of objects by keeping common vertices of these graphs in the same positions. An additional contribution of this paper is to apply upward topological book embeddings in the context of simultaneous embeddings of upward planar digraphs.

More precisely, the main results in this paper can be listed as follows.

- It is proved that every upward planar digraph G with n vertices admits an upward topological book embedding such that every edge of G crosses the spine of the book at most once. We recall that it is not known how many pages may be required if the edges must be drawn upward but are not allowed to cross the spine [1,19,26]. Our result can be regarded as the upward counterpart of [10], where topological book embeddings of non-oriented planar graphs are studied.
- It is shown that every upward planar digraph G with n vertices admits a point-set embedding on any set of n distinct points in the plane, such that the drawing is upward and every edge of G has at most two bends. Similar results were previously known only for restricted families of upward planar digraphs [9].
- Let G_1 and G_2 be any two upward planar digraphs defined on the same set of n vertices. An *upward simultaneous embedding* of G_1 and G_2 is a pair of upward planar drawings $< \Gamma_1, \Gamma_2 >$ such that Γ_1 is an upward planar drawing of G_1, Γ_2 is an upward planar drawing of G_2, and for each vertex v the point representing v is the same in Γ_1 and in Γ_2.
  It is shown that every pair G_1, G_2 admits an upward simultaneous embedding $< \Gamma_1, \Gamma_2 >$ such that every edge has at most two bends. Non-directed counterparts of this result are in [11,14].

The proofs of the above results are constructive and give rise to polynomial time algorithms. In particular, the drawing algorithm to compute upward topological book embeddings is based on an incremental technique that adds a face at a time by exploiting the interplay between an *st*-numbering of the upward planar digraph given as input and an *st*-numbering of its dual digraph.

The remainder of the paper is organized as follows. Basic definitions are given in Section 2. The problem of computing upward topological book embeddings of upward planar digraphs is studied in Section 3. Upward point-set embeddings and upward simultaneous embeddings are the subject of Sections 4 and 5, respectively. Finally, conclusions and possible directions for future research can be found in Section 6. For reasons of space, proofs have been omitted and can be found in [15].

2 Preliminaries

We assume familiarity with basic graph drawing terminology [2,23,25] and recall in the following only those definitions and properties that will be extensively used in the remainder of the paper.

Let G be a digraph and let u, v be any two vertices of G; (u, v) denotes the directed edge from u to v. An *st*-digraph is a biconnected acyclic digraph with exactly one source s and exactly one sink t, and such that (s, t) is an edge of the digraph. A *planar st-digraph* is an *st*-digraph that is planar and embedded with vertices s and t on the boundary of the external face. The digraph depicted in Figure 1(a) is an example of a planar *st*-digraph.

Property 1. Let v be a vertex of a planar st-digraph G such that $v \neq s$ and $v \neq t$. There exists a path $P \subset G$ such that P is directed from s to t and P includes v.

Property 2. The external face of a planar st-digraph consists of edge (s, t) and of a directed path from s to t.

Let G be a planar st-digraph. For each edge $e = (u, v)$ of G, we denote by *left(e)* (resp. *right(e)*) the face to the left (resp. right) of e in G. Let s^* be the face $right((s,t))$, and let t^* be the face $left((s,t))$. In the rest of the paper we shall always assume that t^* is the external face of G. Faces s^* and t^* are highlighted in Figure 1(a).

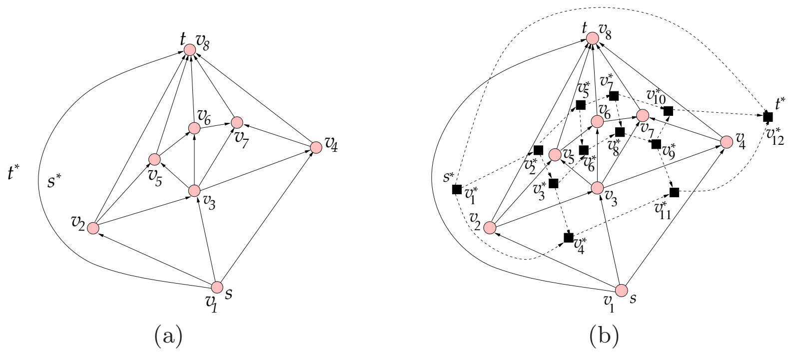

Fig. 1. (a) A planar st-digraph G with an st-numbering of its vertices. Digraph G is a maximal planar st-digraph. (b) st-digraph G (solid) and its dual st-digraph (dashed). The vertices of the dual st-digraph are numbered according to an st-numbering.

Let G be a planar st-digraph. The *dual* of G is the digraph denoted as G^* such that: (i) there is a vertex in G^* for each face of G; (ii) for every edge $e \neq (s, t)$ of G, G^* has an edge $e^* = (f, g)$ where $f = left(e)$ and $g = right(e)$; (iii) G^* has an edge (s^*, t^*). Figure 1(b) depicts with dashed edges the dual digraph of the digraph of Figure 1(a).

Property 3. Let G be a planar st-digraph and let G^* be the dual digraph of G. Graph G^* is a planar st-digraph with source s^* and sink t^*.

A planar st-digraph is said to be *maximal* if all its faces are triangles, i.e. the boundary of each face has exactly three vertices and three edges. Given any planar st-digraph G, one can always add edges that split faces of G in order to obtain a maximal planar st-digraph that includes G. Figure 1(a) is an example of a maximal planar st-digraph.

Property 4. Let G be a maximal planar st-digraph with more than three vertices. The dual of G is a planar st-digraph without multiple edges.

A planar drawing of a digraph is *upward* if all of its edges are curves monotonically increasing in a common direction which is called the *upward direction* of the drawing. For example, upward directions of an upward planar drawing could be the positive y-direction or the positive x-direction. Figure 1(a) is an example of an upward planar drawing. A digraph that admits an upward planar drawing is said to be *upward planar*. As proved in [7,24], upward planar digraphs are exactly the subgraphs of planar st-digraphs. Also, an upward planar digraph G can always be augmented to become a maximal planar st-digraph. This can be done by adding extra edges that "saturate" the faces of an upward planar drawing of G and by inserting at most two vertices on the external face of such upward planar drawing of G. One of these two extra vertices is the source of the external face of the drawing and the second one is the sink of the external face of the drawing. By using results of [7,12,24] the following can be proved.

Lemma 1. *Let G be an upward planar digraph with n vertices. There exists a maximal planar st-digraph with at most $n+2$ vertices that includes G. Also if an upward planar drawing of G is given, such an st-digraph can be computed in $O(n)$ time.*

An *st-numbering* of an st-digraph G with n vertices, is a numbering of its vertices with the integers $1,\ldots,n$ such that: (i) No two vertices have the same number; (ii) For every edge (u,v), the number of u is less than the number of v. For example, the indices of the vertices in Figure 1(a) are given according to an st-numbering of the depicted st-digraph. The number associated to a vertex v in an st-numbering of an st-digraph is called the *st-number* of v. Let u and v be two vertices of an st-digraph with a given st-numbering; if the st-number of u is less than the st-number of v we say that *u precedes v* and we denote it as $u <_{st} v$.

Lemma 2. [2] *Let G be a planar st-digraph with n vertices. An st-numbering of G can be computed in $O(n)$ time.*

3 Computing Upward Topological Book Embeddings

A 2*-page book* consists of a single vertical line, called *spine*, and of 2 half-planes called *pages* that share the spine as a common boundary. The half-plane on the left-hand side of the spine is the *left page*, the other one is the *right page*. Let p and q be two points of the spine. We say that p is *below* q and denote it as $p < q$ if the y-coordinate of p is smaller than the y-coordinate of q. Let p and q be two points of the spine of a 2-page book such that $p < q$. An *upward arc* (p,q) is a circular arc contained in one of the pages and passing through p, q and r, where r is a point of the perpendicular bisector of segment $\overline{pq}$ at a distance $\frac{d(p,q)}{2}$ from the spine. Points p and q are the *endpoints* of (p,q).

Let G be an upward planar digraph. An *upward topological book embedding* of G is an upward planar drawing Γ of G in a 2-page book such that: (i) All vertices of G are represented as points of the spine (the spine will also be called *spine of* Γ); (ii) Each edge (u, v) of G is represented in Γ as either an upward arc or it consists of two upward arcs (u, z) and (z, v) such that (u, z) is in the left page and (z, v) is in the right page. Let $e = (u, v)$ be an edge of G represented in Γ by two upward arcs (u, z) and (z, v); we say that z is the *spine crossing* of e in Γ. Figure 2(a) shows an upward topological book embedding of the *st*-digraph depicted in Figure 1(a). We remark that, by definition, in an upward topological book embedding every edge can cross the spine at most once.

In the next subsections we study the problem of computing an upward topological book embedding of an upward planar digraph G. Based on Lemma 1, we will describe the drawing procedure by assuming that the input digraph is a maximal planar *st*-digraph. Subsection 3.1 introduces the notion of *k-facial subgraph* of an *st*-digraph, which is used as a guideline for the drawing procedure described in Subsection 3.2.

3.1 The *k*-Facial Subgraph

Let G be a maximal planar *st*-digraph with more than three vertices and let G^* be the dual digraph of G. By Property 4, G^* is a planar *st*-digraph without multiple edges; by Lemma 2, its vertices can be numbered according to an *st*-numbering. Hence, let $\{v_1^* = s^*, v_2^*, \dots, v_m^* = t^*\}$ be the set of vertices of G^* where the indices are given according to an *st*-numbering of G^*. See, for example, Figure 1(b), where the vertices of the dual are numbered according to an *st*-numbering. By definition of dual *st*-digraph, a vertex v_i^* of G^* $(1 \leq i \leq m)$ corresponds to a face of G; in the remainder of the paper we shall denote as v_i^*

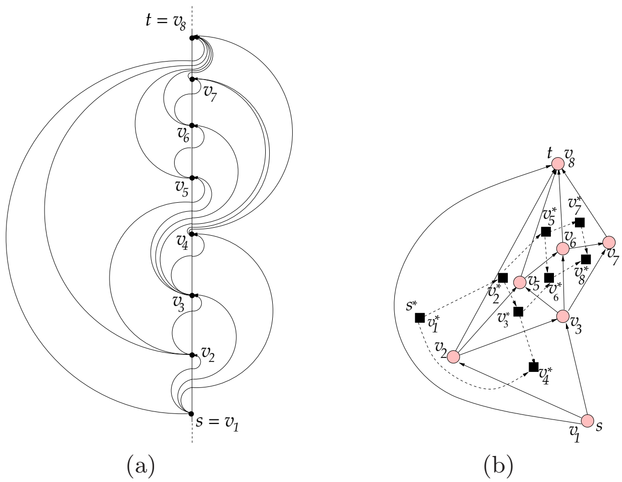

Fig. 2. (a) An upward topological book embedding of the maximal planar *st*-digraph of Figure 1(a). The drawing is computed by using Algorithm `Upward Spine Drawer` of Section 3.2. (b) The 8-facial subgraph of the maximal planar *st*-digraph of Figure 1(a).

both the vertex of the dual digraph G^* and its corresponding face in the primal digraph G.

Let V_k be the subset of the vertices of G that belong to faces $v_1^*, v_2^*, \ldots, v_k^*$. The subgraph of G induced by the vertices in V_k is called the *k-facial subgraph* of G and is denoted as G_k. Face v_k^* is called the *k-th face* of G. Observe that the topology of a k-facial subgraph of G depends on the particular st-numbering chosen for G^*. The drawing algorithm of the next section considers a sequence of k-facial subgraphs of G all defined on a same st-numbering of G^*. Hence, from now on we shall assume that G^* is given together with an st-numbering. As an example, Figure 2(b) shows the 8-facial subgraph of the maximal planar st-digraph depicted in Figure 1(a) assuming that the st-numbering of its dual is the one shown in Figure 1(b). The proof of Lemma 3 relies on properties of the st-numbering of G and of its dual.

Lemma 3. *Let G be a maximal planar st-digraph with m faces, let G_{k-1} be the $(k-1)$-facial subgraph of G $(2 \leq k \leq m)$ and let G_k be the k-facial subgraph of G. Let v_k^* be the k-th face of G consisting of edges (w, w'), (w', w''), and (w, w''). One of the following statements holds:*

- **(S_1):** *(w, w'') is an edge of the external face of G_{k-1}; (w, w') and (w', w'') are edges of the external face of G_k.*
- **(S_2):** *(w, w') and (w', w'') are edges of the external face of G_{k-1}; (w, w'') is an edge of the external face of G_k.*

The following lemma can be proved by induction and by means of Lemma 3.

Lemma 4. *Let G be a maximal planar st-digraph with m faces and let G_k be the k-facial subgraph of G $(1 \leq k \leq m)$. G_k is a planar st-digraph.*

3.2 The Upward Spine Drawer Algorithm

Let G be a maximal planar st-digraph with m faces, and let $v_1 = s, \ldots, v_n = t$ be the vertices of G ordered according to an st-numbering of G. Algorithm `Upward Spine Drawer` receives G as input and it computes as output an upward topological book embedding of G. The computed upward topological book embedding respects the given upward planar embedding for G. In order to properly describe the algorithm, we need two additional definitions. Let Γ be an upward topological book embedding and let p be a point on the spine of Γ. We say that p is *visible from the right-hand side* if the horizontal line through p does not intersect any upward arc of Γ in the right page. Let v be a vertex of Γ and let p be a point of the spine such that $v < p$. We say that segment $\overline{pv}$ is a *safe interval of* v if: (i) Every point of $\overline{pv}$ is visible from the right-hand side and (ii) $\overline{pv}$ does not contain endpoints of any upward arcs (either in the left or in the right page). Note that the safe interval of v is assumed to be an open set.

A high-level description of Algorithm `Upward Spine Drawer` is as follows. The algorithm computes an upward topological book embedding of G on a 2-page book in m steps. At Step 1, it computes an upward topological book embedding of the 1-facial subgraph G_1. Let Γ_{k-1} be the drawing computed at the end of Step $(k-1)$ $(2 \leq k \leq m)$. At Step k, a drawing Γ_k of the k-facial subgraph G_k is computed by adding a new face to the drawing Γ_{k-1} of G_{k-1}. At each step the following invariant properties are maintained.

`I1`: Let w and w' be two vertices of the external face of G_k such that $w <_{st} w'$ in the st-numbering of G. Then $w < w'$ in Γ_k.

`I2`: For each vertex w of the external face of G_k, w is visible from the right-hand side and w has a safe interval.

A more detailed description of the steps executed by Algorithm `Upward Spine Drawer` is given below; Λ denotes the spine of the 2-page book.

- Step 1, computation of Γ_1: Let $\{s, t, w\}$ be the vertices of the boundary of face v_1^*. Draw s and t along Λ such that s is below t. Let z be a point of the spine such that $s < z < t$. Let (s, z) be the upward arc from s to z in the left page and let (z, t) be the upward arc from z to t in the right page. Draw edge (s, t) in Γ_1 as the curve formed by (s, z) followed by (z, t). Represent w as point of the spine such that $s < w < z$. Select two points z_s and z_w of the spine such that $s < z_s < w$ and $w < z_w < z$. Edge (s, w) is drawn as two upward arcs (s, z_s), (z_s, w) into the left and right page, respectively. Edge (w, t) is drawn as two upward arcs (w, z_w), (z_w, t), into the left and right page, respectively.
- Step k, computation of Γ_k $(2 \leq k \leq m)$: Let Γ_{k-1} be the drawing of G_{k-1} and let $w_1 = s, w_2, \ldots, w_h = t$ be the counterclockwise sequence of the vertices along the external face of Γ_{k-1}. Add face v_k^* to Γ_{k-1} as follows.
  - Statement S_1 of Lemma 3 holds. The boundary of face v_k^* is a three cycle having two consecutive vertices of the external face of Γ_{k-1}, say w_i and w_{i+1} $(1 \leq i \leq h-1)$, and a vertex v of the external face of G_k. Let p be a point above w_i such that segment $\overline{w_i p}$ is the safe interval of w_i. Draw v as a point in the safe interval of w_i. Let z_{w_i} be a point of Λ such that $w_i < z_{w_i} < v$. Draw edge (w_i, v) as the upward arc (w_i, z_{w_i}) in the left page followed by the upward arc (z_{w_i}, v) in the right page. Let z_v be a point of Λ such that $v < z_v < p$. Draw edge (v, w_{i+1}) as the upward arc (v, z_v) in the left page followed by the upward arc (z_v, w_{i+1}) in the right page.
  - Statement S_2 of Lemma 3 holds. The boundary of face v_k^* is a three cycle having three consecutive vertices of the external face of Γ_{k-1} denoted as w_i, w_{i+1}, and w_{i+2} $(1 \leq i \leq h-2)$. Drawing Γ_k is computed by adding edge (w_i, w_{i+2}) to Γ_{k-1} as follows. Let z_{w_i} be a point in the safe interval of w_i. Draw (w_i, w_{i+2}) as the upward arc (w_i, z_{w_i}) in the left page followed by the upward arc (z_{w_i}, w_{i+2}) in the right page.

Figure 2(a) is an example of drawing computed by Algorithm `Upward Spine Drawer` when the input is the maximal planar st-digraph of Figure 1(a).

Lemma 5. *Let G be a maximal planar st-digraph. Algorithm* `Upward Spine Drawer` *maintains Invariants* `I1` *and* `I2`.

Lemma 6. *Let G be a maximal planar st-digraph. Algorithm* `Upward Spine Drawer` *computes an upward topological book embedding of G.*

We are now ready to present the main result of this section.

Theorem 1. *Every upward planar digraph G with n vertices admits an upward topological book embedding. Also, if an upward planar drawing of G is given, such upward topological book embedding can be computed in $O(n)$ time.*

In the next two sections we discuss applications of Theorem 1 to problems of graph drawing and computational geometry. Namely, Section 4 is devoted to upward drawings with constraints on the position of the vertices, while Section 5 is concerned with simultaneous embeddings of pairs of upward planar digraphs sharing their vertex set.

4 Upward Point-Set Embeddings

Let S be a set of n distinct points on the plane and let G be an upward planar digraph with n vertices. An *h-bend upward point-set embedding* of G on S is an upward planar drawing of G such that each vertex is mapped to a distinct point of S and every edge has at most h bends (notice that the mapping of the vertices to the points of S is not part of the input). A digraph G is *h-bend upward point-set embeddable* if it has an h-bend upward point-set embedding on *any* set of n points in the plane. In [9] it has been proved that an upward planar digraph is 1-bend upward point-set embeddable if and only if it has an upward topological book embedding such that no edge crosses the spine (i.e. it has an upward book embedding on two pages). It has also been proved that the following classes of digraphs admit this type of drawing: tree dags [21], unicyclic dags [21], and two-terminal series-parallel digraphs [9]. Hence, all graphs in these families are 1-bend upward point-set embeddable. However, not all upward planar digraphs have an upward topological book embedding without spine crossings [21] and therefore at least two bends are necessary in the general case. By using Theorem 1 and techniques from [10,22] we can prove that two bends per edge are actually always sufficient. In the following theorem the area of a drawing is the area of the smallest axis-aligned rectangle enclosing the drawing.

Theorem 2. *Every upward planar digraph G with n vertices admits a 2-bend upward point-set embedding on any set S of n distinct points in the plane. Also, if an upward planar drawing of G is given, such 2-bend upward point-set embedding can be computed in $O(n \log n)$ time and in area $O(W^3)$, where W is the width of the smallest axis-aligned rectangle enclosing S.*

5 Upward Simultaneous Embeddings

Let G_1 and G_2 be two planar graphs with the same vertex set, i.e. $V(G_1) = V(G_2) = V$. A *simultaneous embedding* of G_1 and G_2 is a pair of drawings of

G_1 and G_2 such that each drawing is planar and each vertex is represented by the same point in both drawings. The problem of computing a simultaneous embedding of two undirected planar graphs is a classical subject of investigation in the graph drawing literature (see, e.g. [5,6,11,14,16]). This section considers the upward version of this problem and uses Theorem 1 together with techniques from [11,14] to establish upper bounds on the area and number of bends per edge of the computed drawings.

Let G_1 and G_2 be two upward planar digraphs with the same vertex set, i.e. $V(G_1) = V(G_2) = V$. An *upward simultaneous embedding* of G_1 and G_2 is a pair of upward planar drawings Γ_1 of G_1 and Γ_2 of G_2 such that each vertex is represented by the same point in both drawings. An upward simultaneous embedding of G_1 and G_2 will also be denoted as $< \Gamma_1, \Gamma_2 >$. Note that the upward directions of Γ_1 and Γ_2 in $< \Gamma_1, \Gamma_2 >$ are not required to be the same.

Theorem 3. *Every pair of upward planar digraphs G_1 and G_2 such that $V(G_1) = V(G_2) = V$ admits an upward simultaneous embedding with at most two bends per edge. Also, if a pair of upward planar drawings of G_1 and G_2 is given, such upward simultaneous embedding can be computed in $O(n)$ time and in area $O(n^2) \times O(n^2)$, where $n = |V|$.*

6 Conclusions and Open Problems

In this paper we presented a unified approach to studying book-, point-set, and simultaneous embeddability problems of upward planar digraphs. The approach is based on a linear time strategy to compute an upward planar drawing of an upward planar digraph such that all vertices are collinear and each edge has at most two bends. Besides having impact in relevant application domains of graph drawing and computational geometry, the presented results open new research directions in the area of upward planarity with constraints of the positions of the vertices. We therefore conclude this paper by discussing some of the most interesting questions that can be inspired by the presented results.

Upward book embeddability: Theorem 1 shows that an upward topological book embedding of an upward planar digraph can be computed such that every edge crosses the spine at most once. It would be interesting to study the problem of computing upward topological book embeddings with the minimum number of spine crossings.

Upward point-set embeddability: Theorem 2 shows that every upward planar digraph with n vertices has a 2-bend upward point-set embedding on any set on n distinct points in the plane. In [22] it is proved that point-set embeddings of undirected planar graphs may require two bends per edge. This immediately implies that the same lower bound also applies to the upward planar case, and therefore the statement of Theorem 2 is tight in terms of bends per edge. However, it is well-known that every (undirected) outerplanar graph with n vertices has a point-set embedding on any set of n points in general position with straight-line edges and that the outerplanar graphs are the largest family of graphs with this property [18]. It would be interesting to characterize those upward planar digraphs that have an upward point-set embedding with straight-line edges on any set of points in general position.

Upward simultaneous embeddability: Theorem 3 shows that any two upward planar digraphs have an upward simultaneous embedding with at most two bends per edge. It would be interesting to understand whether the number of bends per edge stated in Theorem 3 is also necessary in some cases. We recall that one bend on some of the edges may be required to simultaneously embed pairs of undirected planar graphs [5,14,17] and hence the same lower bound also applies to the problem of computing upward simultaneous embeddings.

A related question asks whether a straight-line upward simultaneous embedding of two upward planar digraphs G_1 and G_2 is always possible in the *no-mapping scenario*. In this scenario, the goal is to compute a pair $< \Gamma_1, \Gamma_2 >$ of straight-line upward planar drawings of G_1 and of G_2 such that the set of points representing the vertices is the same in Γ_1 and in Γ_2, but each vertex can have different coordinates in the two drawings. For example, a straight-forward consequence of the literature is that any number of tree dags and of unicyclic dags can be upward simultaneously embedded without mapping and with straight-line edges. Namely, in [21] it is proved that these graphs admit an upward book embedding with all edges in the same page. Thus, choose a set S of n points in general position such that: (i) the points are in convex position, (ii) all points have distinct y-coordinates, and (iii) the two extreme points in the y-direction are adjacent in the convex hull and all the remaining points are to the left of the upward-directed line they define. Now compute a straight-line upward point-set embedding of each tree or unicyclic dag with n vertices by mapping the vertices to the points of S by increasing y-coordinate and according to the below-to-above order of these vertices along the spine. We find it interesting to study the general question about whether any pair of upward planar digraphs (not just tree dags or unicyclic dags) admit an upward simultaneous embedding without mapping.

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