Journal of Knot Theory and Its Ramifications
Vol. 21, No. 13 (2012) 1240008 (48 pages)

DOI: 10.1142/S0218216512400081

A CATEGORICAL MODEL FOR THE VIRTUAL BRAID GROUP

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Received 23 April 2012
Accepted 1 August 2012
Published

ABSTRACT

This paper gives a new interpretation of the virtual braid group in terms of a strict monoidal category SC that is freely generated by one object and three morphisms, two of the morphisms corresponding to basic pure virtual braids and one morphism corresponding to a transposition in the symmetric group. The key to this approach is to take pure virtual braids as primary. The generators of the pure virtual braid group are abstract solutions to the algebraic Yang–Baxter equation. This point of view illuminates representations of the virtual braid groups and pure virtual braid groups via solutions to the algebraic Yang–Baxter equation. In this categorical framework, the virtual braid group is a natural group associated with the structure of algebraic braiding. We then point out how the category SC is related to categories associated with quantum algebras and Hopf algebras and with quantum invariants of virtual links.

Keywords: Virtual braid group; pure virtual braid group; string connection; strict monoidal category; Yang–Baxter equation; algebraic Yang–Baxter equation; quantum algebra; Hopf algebra; quantum invariant.

Mathematics Subject Classification 2010: 57M27

1. Introduction

This paper gives a new interpretation of the virtual braid group in terms of a tensor category SC with generating morphisms μ_{ij} where this symbol denotes an abstract connecting string between strands i and j in a diagram that otherwise is an identity braid on n strands. These μ_{ij} satisfy the algebraic Yang–Baxter equation and they generate, in this interpretation, the pure virtual braid group. The other generating morphisms of this category are elements v_i that are depicted as virtual crossings between strings i and $i+1$. The generators v_i have all the relations for transpositions generating the symmetric group. An n-strand diagram that is a product of these generators is regarded as a morphism from $[n]$ to $[n]$ where the symbol $[n]$ is regarded as an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. The virtual braid group on n strands is isomorphic to the group of morphisms in the String Category SC from $[n]$ to $[n]$. Given that one studies the algebraic Yang–Baxter equation, it is natural to study the compositions of algebraic braiding operators placed in two out of the n tensor lines and to let the permutation group of the tensor lines act on this algebra as the group generated by the virtual crossings. This construction is in sharp contrast to the role of the virtual crossings in the original form of the virtual knot theory.

Figure 1 illustrates most of the issues. At the top of the figure we have illustrated the pure virtual braid $\mu = \sigma v$ on two strands. The permutation associated with μ is the identity, as each strand returns to its original position. The braiding element σ has been composed with the virtual crossing v, which acts as a permutation of the two strands. With these conventions in place we find that μ satisfies the *algebraic Yang–Baxter equation*

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}$$

and this is equivalent to the statement that σ satisfies the *braiding relation*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

This relationship is well known and it is fundamental to the construction of representations of the Artin braid group and to the construction of quantum link invariants (see [29] for an account of these matters). In this paper we will detail this relationship once again, and we shall see that it leads to alternative ways to understand the concept of virtual braiding and to generalizations of the formulation of quantum invariants of knots and links to quantum invariants of virtual knots and links (taken up to rotational equivalence described below).

Here a notational issue leads to a mathematical concept. View Fig. 1 and notice how we have diagrammed the algebraic Yang–Baxter relation. An element μ_{ij} is shown as a graphical connection between vertical lines labeled i and j respectively. The vertical lines represent different factors in a tensor product in the usual interpretation where $\mu \in \mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A}$ is an algebra that carries a solution to the algebraic Yang–Baxter equation. We call the graphical edge representing μ_{ij} a *string connection* between the strands i and j. *The string connection is a topological model for a logical connection in the mathematics.* The string going from vertical line 1 to vertical line 3 represents μ_{13}, and it has nothing to do with strand 2 except as in the plane the strand 2 happens to come between strands 1 and 3. This means that in our diagram the graphical edge for μ_{13} intersects the vertical strand 2. This intersection is *virtual* in the sense that it is just an artifact of the planar drawing. There is no conceptual connection between μ_{13} and strand 2.

We see that virtuality in the sense of artifactual coincidence of topological entities will be a necessity in depicting logical connection as topological connection. For this reason, the string diagrammatics that we have adopted for the algebraic Yang–Baxter equation can be taken as a starting point for the development of the virtual braid group. In this paper, we have started with the usual virtual braid group and reformulated it in this algebraic context. The attentive reader will see that one could start with the formalism of the algebraic Yang–Baxter equation, construct the appropriate categories and first arrive at the pure virtual braid group and then at the virtual braid group. All of these constructions come from the concept of making topological models for logical connections in mathematical structures.

The Artin braid group B_n is motivated by a combination of topological considerations and the desire for a group structure that is very close to the structure of the symmetric group S_n. The virtual braid group VB_n is motivated at first by a natural extension of the Artin braid group in the context of virtual knot theory. The virtual crossings appear as artifacts of the presentation of virtual knots in the plane where those knots acquire extra crossings that are not really part of the essential structure of the virtual knot. We add virtual crossings to the Artin braid group and follow the principles of virtual knot theory for handling them. These virtual crossings appear crucially in the virtual braid group, and turn into the generators of the symmetric group embedded in the virtual braid group. Thus we arrive at the action of the symmetric group in either case, but with different motivations. Seen from the categorical view, the virtual crossings are interpreted as generators of the symmetric group whose action is added to the algebraic structure of the pure virtual braid group, and they become part of the embedded symmetry of the structure of the virtual braid group. The pure virtual braid group is seen to be a natural monoidal category generated by formal elements satisfying the algebraic Yang–Baxter equation. The virtual braid group is then an extension of the pure virtual braid group by the symmetric group. It has nothing to do with the plane and nothing to do with virtual crossings. It is a natural group associated with the structure of algebraic braiding. This is our motivation for constructing the category SC.

Here is a quick technical description of our category. We define a strict monoidal category SC that is freely generated by one object $*$ and three morphisms $\mu : * \otimes * \to * \otimes *$, $\mu' : * \otimes * \to * \otimes *$ and $v : * \otimes * \to * \otimes *$. This basic structure, subjected to appropriate relations can be understood via morphisms μ_{ij} defined in terms of the generating morphisms, where the symbol μ_{ij} can be interpreted as a connection between strands i and j in a diagram that otherwise is an identity on n strands. The μ_{ij} satisfy the algebraic Yang–Baxter equation in the sense that for $i < j < k$, $\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$. The other basic morphisms of this category are elements v_i that can be depicted as virtual crossings between strings i and $i+1$. The v_i are obtained from v by tensoring with identity morphisms $* \to *$. The v_i

generate the symmetric group S_n. The μ_{ij} are obtained from μ by the action of the symmetric group that is generated by the v_i. Composition with an individual v_i makes a transposition of indices on the μ_{kl}, generating all of them from the basic μ and μ'. An n-strand diagram that is a product of basic morphisms is a morphism from $[n]$ to $[n]$ where the symbol $[n]$ is an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. Here $[n] = * \otimes * \cdots * \otimes *$ for a tensor product of n $*$'s. In Fig. 1 we illustrate the diagrammatic interpretation of μ and the fundamental relation of μ and v with an elementary braiding element σ. The relation is $\mu = \sigma v$. The virtual braid σv is pure in the sense that its associated permutation is the identity.

The category we describe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang–Baxter equation, and it is directly related to more topological points of view about virtual links and virtual braids. In fact, a closely related category, under different motivation, was constructed in [23]

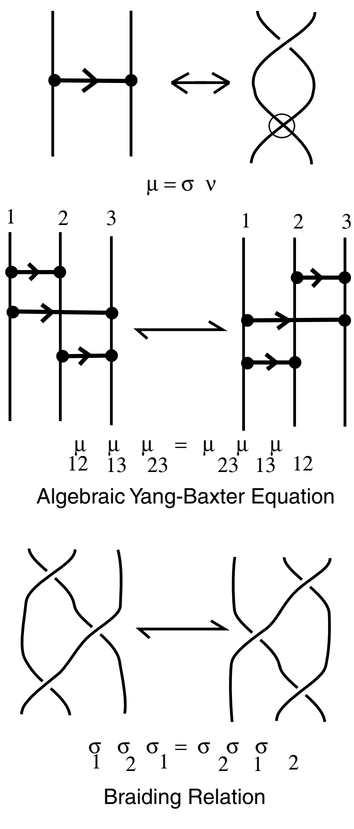

Fig. 1. Algebraic Yang–Baxter equation and braiding relation.

where the intent was to construct a category that would be naturally associated with a Hopf algebra on the one hand, and would receive topological tangles, knots and links under a functor from the tangle category to the Hopf algebra category. The present category, giving the structure of the virtual braid group, is a subcategory of that category associated with a general Hopf algebra. We explain this relationship in detail in Sec. 6 of this paper. See also [2, Remark 10] and references therein for another earlier observation of the relationship of the algebraic Yang–Baxter equation with the pure virtual braid group.

We now describe exactly the structure of the paper. We develop our model for the virtual braid group by first recalling, in Sec. 2, its usual definition motivated by virtual knot theory. We then proceed to reformulate the virtual braid group in terms of the above mentioned generators. By the time we reach Theorem 1, we have reformulated the virtual braid group in terms of the new generators. We then use this approach to give a presentation of the pure virtual braid group in Theorem 3.2. More precisely, in Sec. 2 we give a presentation for the virtual braid group in terms of our stringy model. We start by describing the usual presentation of the virtual braid group in terms of classical braid generators and virtual generators that act as permutations between pairs of adjacent strands in the braid, and relations among them (see Figs. 2–6). Elementary connecting strings (see Fig. 7) are defined as elementary pure virtual braids — products of braid generators and virtual generators as in Fig. 1. We then generalize the notion of connecting string and show that it has the formal diagrammatic property of being stretched and contracted as shown in Fig. 9. This property makes the string a topological model for a logical connection as we have advertised earlier in this introduction. With these constructions we then rewrite presentations for the virtual braid group and, in Sec. 3, show how the connection with strings generates the pure virtual braid group with a set of relations that correspond to the algebraic Yang–Baxter equation. See Theorem 3.2.

In Sec. 4 we construct the String Category discussed in this introduction and we show that the virtual braid group on n strands is isomorphic to the group of morphisms in the String Category SC from $[n]$ to $[n]$ (see Theorem 4.3). In Sec. 5 we detail the relationship with the algebraic Yang–Baxter equation and show how to use solutions of the algebraic Yang–Baxter equation to obtain representations of the pure virtual braid group and virtual braid group. In Sec. 6 we discuss a generalization of the virtual braid group to the virtual tangle category. We show in this section how our work on the structure of the virtual braid group fits into the structure of the virtual tangle category. The virtual tangle category can be used for obtaining invariants of knots and links via Hopf algebras. The invariants we obtain are invariants of *rotational virtual knots and links* where the term rotational means that we do not allow the use of the first virtual Reidemeister move. See Fig. 18. For the virtual tangle category, the rules for regular isotopy of rotational virtuals are shown in Fig. 21. This is a most convenient category for working with virtual knots and links, and every quantum link invariant for classical knots and links extends to an invariant for rotational virtual knots and links. In this section we show how a generalization of the string connectors defined previously in the paper enables the construction of quantum virtual link invariants associated with Hopf algebras. The paper ends with two sections on Hopf algebras. The concept of a quasi-triangular Hopf algebra creates an algebraic context for solutions to the algebraic Yang–Baxter equation. This algebraic context gives rise to categories and relationships with knot theory and virtual knot theory that connect directly with the contents of our investigation.

2. A Stringy Presentation for the Virtual Braid Group

2.1. *The virtual braid group*

Let us begin with a presentation for the virtual braid group. The set of isotopy classes of virtual braids on n strands forms a group, the *virtual braid group* denoted VB_n, that was introduced in [18]. The group operation is the usual braid multiplication (form bb' by attaching the bottom strand ends of b to the top strand ends of b'). VB_n is generated by the usual braid generators $\sigma_1, \ldots, \sigma_{n-1}$ and by the virtual generators $v_1, \ldots, v_{n-1}$, where each virtual crossing v_i has the form of the braid generator σ_i with the crossing replaced by a virtual crossing. See Fig. 2 for illustrations. Recall that in virtual crossings we do not distinguish between under and over crossing. Thus, VB_n is an extension of the classical braid group B_n by the symmetric group S_n, whereby v_i corresponds to the elementary transposition $(i, i+1)$.

Among themselves the braid generators satisfy the usual *braiding relations*:

$$(B1)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$
$$(B2)\ \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } j \neq i \pm 1.$$

Among themselves, the virtual generators are a presentation for the symmetric group S_n, so they satisfy the following *virtual relations*:

$$(S1)\ v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1},$$
$$(S2)\ v_iv_j = v_jv_i, \quad \text{for } j \neq i \pm 1,$$
$$(S3)\ {v_i}^2 = 1.$$

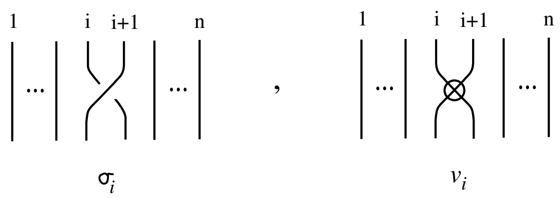

Fig. 2. The generators of VB_n.

The *mixed relations* between virtual generators and braiding generators are as follows:

$$(M1)\ v_i\sigma_{i+1}v_i = v_{i+1}\sigma_i v_{i+1},$$
$$(M2)\ \sigma_i v_j = v_j\sigma_i, \quad \text{for} \quad j \neq i \pm 1.$$

To summarize, the virtual braid group VB_n has the following presentation [18].

$$\mathrm{VB}_n = \left\langle \begin{matrix} \sigma_1, \ldots, \sigma_{n-1}, \\ v_1, \ldots, v_{n-1} \end{matrix} \middle| \begin{matrix} (B1), (B2), \\ (S1), (S2), (S3), \\ (M1), (M2). \end{matrix} \right\rangle \tag{2.1}$$

It is worth noting at this point that the virtual braid group VB_n does not embed in the classical braid group B_n, since the virtual braid group contains torsion elements (the v_i have order two) and it is well known that B_n does not. But the classical braid group embeds in the virtual braid group just as classical knots embed in virtual knots. This fact may be most easily deduced from [26], and can also be seen from [8, 28]. For reference to previous work on virtual knots and braids the reader should consult [4, 6, 11–13, 15, 16, 18–22, 25, 26, 28, 32, 35–37] and references therein. For work on welded braids and welded knots, see [8, 16, 21, 22]. For Markov-type theorems for virtual braids (and welded braids), giving sets of moves on virtual braids that generate the same equivalence classes as the oriented virtual link types of their closures, see [16, 22]. Such theorems are important for understanding the structure and classification of virtual knots and links.

The second mixed relation in the presentation of the virtual braid group will be called the *local detour move* and it is illustrated in Fig. 3. The following relations are also local detour moves for virtual braids and they are easy consequences of the above.

$$\begin{aligned} v_i v_{i+1} {\sigma_i}^{\pm 1} &= {\sigma_{i+1}}^{\pm 1} v_i v_{i+1}, \\ {\sigma_i}^{\pm 1} v_{i+1} v_i &= v_{i+1} v_i {\sigma_{i+1}}^{\pm 1}. \end{aligned} \tag{2.2}$$

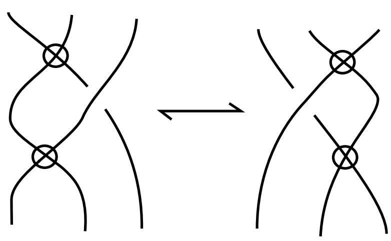

Fig. 3. The local detour.

This set of relations taken together define the basic local isotopies for virtual braids. Each relation is a braided version of a local virtual link isotopy. The local detour move is written equivalently:

$$\sigma_{i+1} = v_i v_{i+1} \sigma_i v_{i+1} v_i. \tag{2.3}$$

Notice that Eq. (2.3) is the braid detour move of the ith strand around the crossing between the $(i+1)$th and the $(i+2)$th strand (see first two illustrations in Fig. 4) and it provides an inductive way of expressing all braiding generators in terms of the first braiding generator σ_1 and the virtual generators $v_1, \ldots, v_{n-1}$ (see first and last illustrations in Fig. 4), that is:

$$\sigma_j = (v_{j-1} \cdots v_2 v_1)(v_j \cdots v_3 v_2)\, \sigma_1 (v_2 v_3 \cdots v_j)(v_1 v_2 \cdots v_{j-1}). \tag{2.4}$$

In [21] we derive the following reduced presentation for VB_n:

$$\mathrm{VB}_n = \left\langle \begin{array}{c|l} & (S1), (S2), (S3) \\ \sigma_1, & \sigma_1 v_j = v_j \sigma_1, \text{ for } j > 2 \\ v_1, \ldots, v_{n-1} & v_1\sigma_1 v_1 v_2 \sigma_1 v_2 v_1 \sigma_1 v_1 = v_2 \sigma_1 v_2 v_1 \sigma_1 v_1 v_2 \sigma_1 v_2 \\ & \sigma_1 v_2 v_3 v_1 v_2 \sigma_1 v_2 v_1 v_3 v_2 = v_2 v_3 v_1 v_2 \sigma_1 v_2 v_1 v_3 v_2 \sigma_1 \end{array} \right\rangle. \tag{2.5}$$

The local detour move gives rise to a generalized *detour move*, by which any box in the braid can be detoured to any position in the braid, see Fig. 5.

Finally, it is worth recalling that in virtual knot theory there are "forbidden moves" involving two real crossings and one virtual. More precisely, there are two types of forbidden moves: One with an over arc, denoted F_1 and another with an under arc, denoted F_2. See [18] for explanations and interpretations. Variants of the forbidden moves are illustrated in Fig. 6. So, relations of the types:

$$\sigma_i v_{i+1} \sigma_i^{-1} = \sigma_{i+1}^{-1} v_i \sigma_{i+1} \ \ (F1) \quad \text{and} \quad \sigma_i^{-1} v_{i+1} \sigma_i = \sigma_{i+1} v_i \sigma_{i+1}^{-1} \ \ (F2) \tag{2.6}$$

are not valid in virtual knot theory.

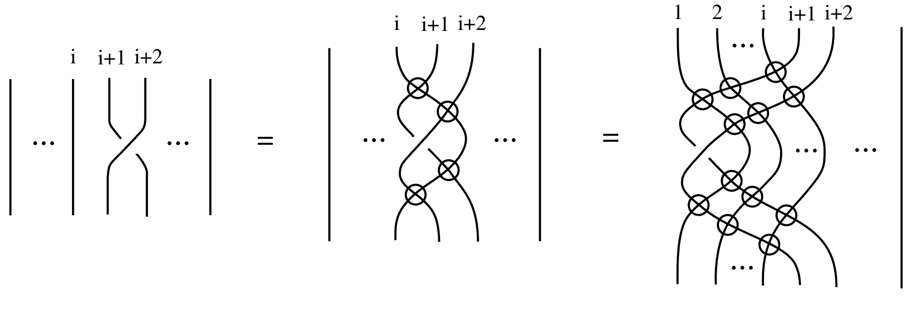

Fig. 4. Detouring the crossing σ_{i+1}.

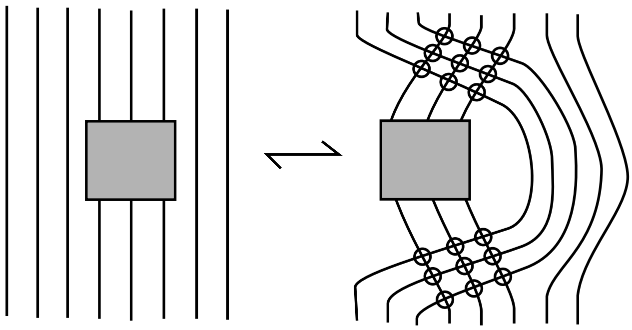

Fig. 5. Detouring a box.

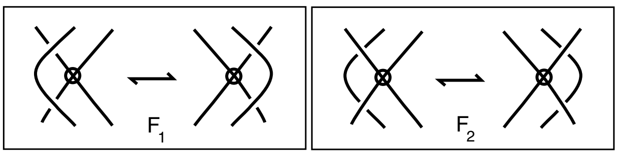

Fig. 6. The forbidden moves.

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2.2.

We now wish to describe a new set of generators and relations for the virtual braid group that makes it particularly easy to describe the pure virtual braid group, VP_n. In order to accomplish this aim, we introduce the following elements of VP_n, for $i = 1, \ldots, n-1$.

$$\mu_{i,i+1} := \sigma_i v_i. \tag{2.7}$$

We indicate $\mu_{i,i+1}$ by a connecting string between the ith and $(i+1)$th strands with a dark vertex on the ith strand, a dark vertex on the $(i+1)$th strand, and an arrow from left to right. View Fig. 7. The inverses $\mu_{i,i+1}^{-1} = v_i\sigma_i^{-1}$ have same directional arrows but are indicated by using white vertices. By detouring it to the leftmost position of the braid, we can write $\mu_{i,i+1}$ in terms of μ_{12} conjugated by a virtual word:

$$\mu_{i,i+1} = (v_{i-1} \cdots v_2 v_1)(v_i \cdots v_3 v_2)\mu_{12}(v_2 v_3 \cdots v_i)(v_1 v_2 \cdots v_{i-1}). \tag{2.8}$$

We also introduce the elements

$$\mu_{i+1,i} := v_i\sigma_i = v_i\mu_{i,i+1}v_i. \tag{2.9}$$

We indicate $\mu_{i+1,i}$ by a connecting string between the ith and $(i+1)$th strands, with a dark vertex on the ith strand, a dark vertex on the $(i+1)$th strand, and an arrow from right to left (reversing the direction from $\mu_{i,i+1}$), view Fig. 7. An illustration of Eq. (2.9) (see top of Fig. 8) explains the reversing of the direction of

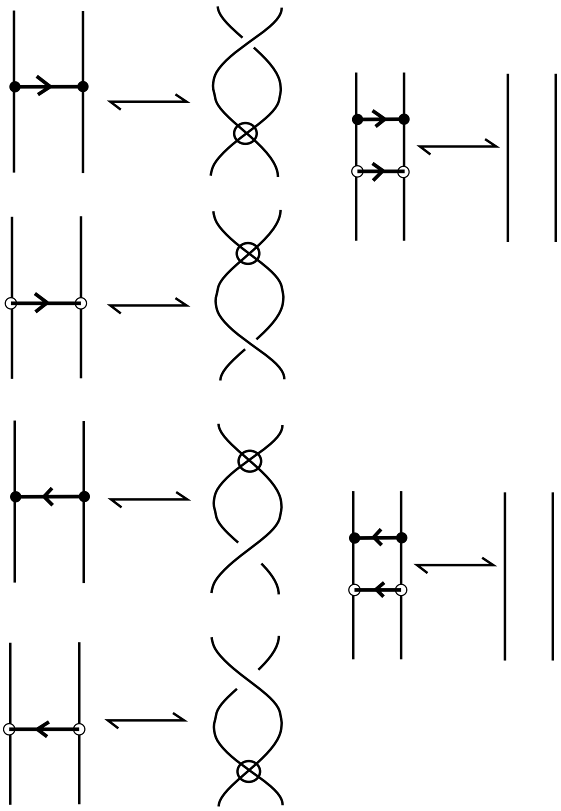


Fig. 7. The elementary connecting strings $\mu_{i,i+1}$, $\mu_{i+1,i}$ and their inverses.

the arrow in the graphical interpretation of $\mu_{i+1,i}$. The inverses $\mu_{i+1,i}^{-1} = \sigma_i^{-1} v_i$ have same directional arrows but are indicated by using white vertices. An analogous equation to Eq. (2.8) holds:

$$\mu_{i+1,i} = (v_{i-1} \cdots v_2 v_1)(v_i \cdots v_3 v_2)\mu_{21}(v_2 v_3 \cdots v_i)(v_1 v_2 \cdots v_{i-1}). \tag{2.10}$$

Definition 2.1. The pure virtual braids $\mu_{i,i+1}, \mu_{i+1,i}$ and their inverses shall be called *elementary connecting strings*.

From Eqs. (2.7) and (2.9) follow directly the relations:

$$v_i \mu_{i+1,i} = \mu_{i,i+1} v_i \quad \text{and} \quad \mu_{i+1,i}^{-1} v_i = v_i \mu_{i,i+1}^{-1}, \tag{2.11}$$

also illustrated in Fig. 8.

Further, we generalize the notion of a connecting string and define, for $i < j$, the element μ_{ij} (a connecting string from strand i to strand j) by the formula

$$\mu_{ij} := v_{j-1} v_{j-2} \cdots v_{i+1} \mu_{i,i+1} v_{i+1} \cdots v_{j-2} v_{j-1}. \tag{2.12}$$

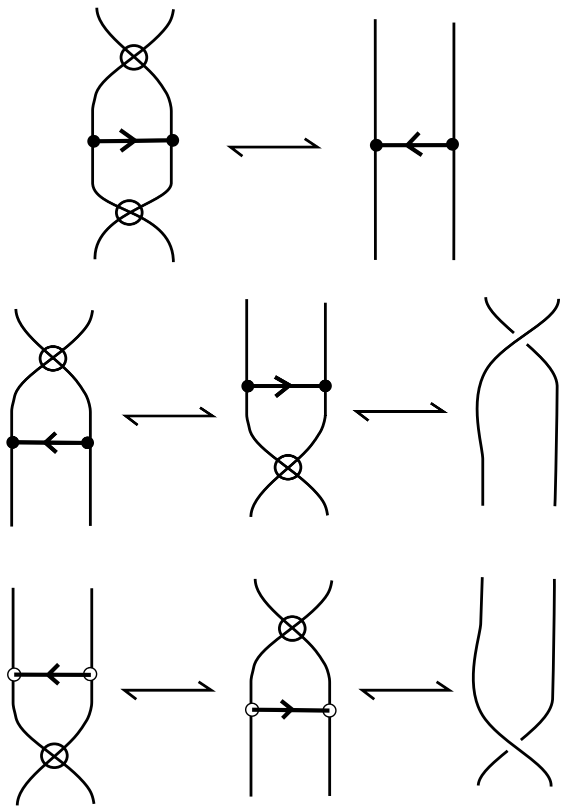


Fig. 8. Relations between the elementary connecting strings.

In a diagram μ_{ij} is denoted by a connecting string from strand i to strand j, with dark vertices on these two strands and an arrow pointing from left to right, view Fig. 9.

We also generalize, for $i < j$, the elements $\mu_{i+1,i}$ to the elements:

$$\mu_{ji} := t_{ij}\mu_{ij}t_{ij}, \tag{2.13}$$

where $t_{ij} = v_i v_{i+1} \cdots v_j \cdots v_{i+1} v_i$ is the element of S_n (generated by the v_i's) that interchanges strands i and j, leaving all other strands fixed. We denote μ_{ji} by a connecting string from strand i to strand j, with dark vertices, and an arrow pointing from right to left. Figure 10 illustrates the example $\mu_{31} = v_2 v_1 v_2 \mu_{13} v_2 v_1 v_2$. It is easily verified that

$$\mu_{ji} = v_{j-1} v_{j-2} \cdots v_{i+1} \mu_{i+1,i} v_{i+1} \cdots v_{j-2} v_{j-1}. \tag{2.14}$$

The inverses of the elements μ_{ij} and μ_{ji} have same directional arrows respectively, but white dotted vertices.

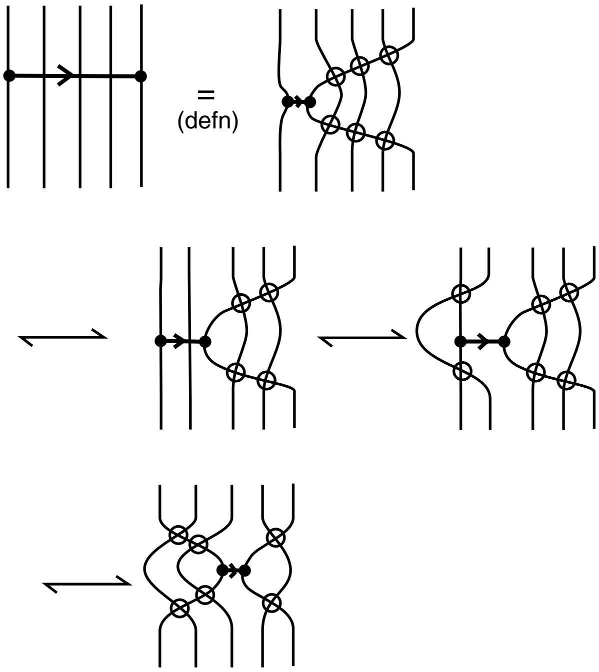


Fig. 9. Connecting strings.

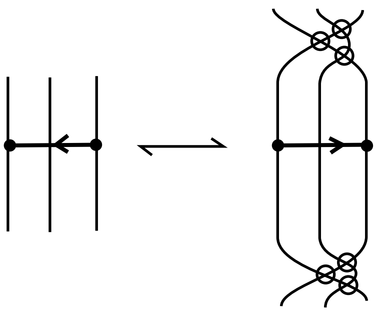


Fig. 10. The exchange of labels between μ_{ij} and μ_{ji}.

Definition 2.2. The elements μ_{ij}, μ_{ji} and their inverses shall be called *connecting strings*.

With the above conventions we can speak of connecting strings μ_{rs} for any r, s. It is important to have the elements μ_{ji} when $j > i$, but in the algebra they are all defined in terms of the μ_{ij}. The importance of having the elements μ_{ji} will become clear when we restrict to the pure virtual braid group.

Remark 2.3. In the definition of μ_{ij} if we consider $\mu_{i,i+1}$ as a virtual box inside the virtual braid we can use the (generalized) detour moves to bring it to any position, as Fig. 9 illustrates. This means that the contraction of μ_{ij} to $\mu_{i,i+1}$ may be pulled anywhere between the ith and the jth strands. By the same reasoning the contraction of μ_{ji} to $\mu_{i+1,i}$ may be also pulled anywhere between the ith and the jth strands.

2.3.

We shall next give some relations satisfied by the connecting strings. Before that we need the following remark.

Remark 2.4. The symmetric group S_n clearly acts on VB_n by conjugation. By their definition [Eqs. (2.7), (2.9), (2.12)–(2.14)], this action on connecting strings translates into permuting their indices, that is, a permutation $\tau \in S_n$ acting on μ_{rs} will change it to $\mu_{\tau(r),\tau(s)}$. This means that S_n acts by conjugation also on the subgroup of VB_n generated by the μ_{ij}'s. Moreover, by Eqs. (2.8) and (2.9), all connecting strings may be obtained by the action of S_n on μ_{12}. For $\sigma \in S_n$ we regard σ both as a product of the elements v_i and as a permutation of the set $\{1, 2, 3, \ldots, n\}$.

Further, any relation in VB_n transforms into a valid relation after acting on it an element of S_n. In particular, a commuting relation between connecting strings will be transformed to a new commuting relation between connecting strings.

Lemma 2.5. *The following relations hold in* VB_n *for all* i.

(1) $v_i\mu_{i,i+1} = \mu_{i+1,i}v_i$, $v_i\mu_{i+1,i} = \mu_{i,i+1}v_i$,
(2) $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1}$, $v_{i+1}\mu_{i+1,i} = \mu_{i+2,i}v_{i+1}$,
(3) $v_{i-1}\mu_{i,i+1} = \mu_{i-1,i+1}v_{i-1}$, $v_{i-1}\mu_{i+1,i} = \mu_{i+1,i-1}v_{i-1}$,
(4) $v_j\mu_{i,i+1} = \mu_{i,i+1}v_j$, $v_j\mu_{i+1,i} = \mu_{i+1,i}v_j$, $j \neq i-1, i, i+1$.

The above local relations generalize to similar ones involving different indices. Relations 1 *are generalized by Eq.* (2.13), *reflecting the mutual reversing of* μ_{ij} *and* μ_{ji}, *recall Figs.* 8 *and* 10. *Relations* 2 *and* 3 *are the local slide moves, as illustrated in Fig.* 11, *and they generalize to the slide moves coming from the defining equations*: $\mu_{i+1,k} = v_i\mu_{ik}v_i$ *for any* $k < i$ *or* $k > i+1$. *Relations* 4 *and their generalizations*: $v_j\mu_{ik} = \mu_{ik}v_j$ *for any* $k \neq i$ *and* $j \neq i-1, i, k-1, k$, *are all commuting relations. All these relations result from the action of any* $\tau \in S_n$ *on* μ_{12}:

$$\boldsymbol{\tau^{-1}\mu_{12}\tau = \mu_{\tau(1),\tau(2)}}. \tag{2.15}$$

Proof. All relations 1, 2 and 3 follow directly from the definitions of the elements μ_{ij} and μ_{ji}. For example, $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1}$ is equivalent to the defining relation $\mu_{i,i+2} = v_{i+1}\mu_{i,i+1}v_{i+1}$. Figure 12 illustrates the proof of a local slide move. Relations 4 follow immediately from the commuting relations $(S2)$ and $(M2)$

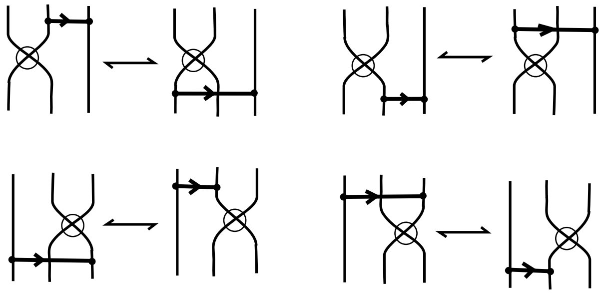


Fig. 11. Slide moves.

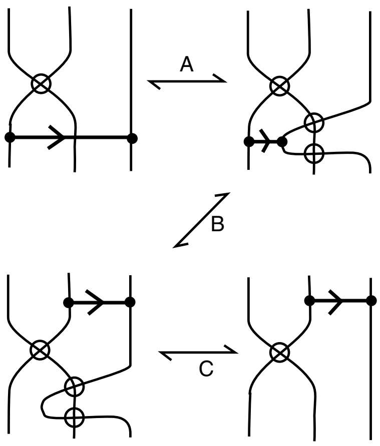


Fig. 12. Proving a local slide move.

of VB_n. The generalizations of all types of moves follow from the local ones after using detour moves. Finally, the derivation of all relations from the action of S_n on μ_{12} is explained in Remark 2.4 and, more precisely, by the Eqs. (2.8), (2.12), (2.10) and (2.14). □

Lemma 2.6. *The following commuting relations among connecting strings hold in* VB_n.

(1) $\boldsymbol{\mu_{12}\mu_{34} = \mu_{34}\mu_{12}}$,
(2) $\mu_{14}\mu_{23} = \mu_{23}\mu_{14}$(*action by* (324)),
(3) $\mu_{13}\mu_{24} = \mu_{24}\mu_{13}$(*action by* (23)).

The above local relations generalize to commuting relations of the form:

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{2.16}$$

All the above commuting relations result from relation 1 *by actions of permutations* (*indicated for relations* 2, 3 *to the right of each relation*). *Moreover, for any choice* *of four strands there are exactly* 24 *such commuting relations that preserve the four strands.*

Proof. Relation 1 clearly rest on the virtual braid commuting relations $(B2)$ and $(M2)$. We shall show how relation 2 reduces to relation 1. In the proof we underline in each step the generators of VB_n on which virtual braid relations are applied.

$$\begin{aligned}
\mu_{i,i+3}\mu_{i+1,i+2} &= v_{i+2}v_{i+1}\mu_{i,i+1}\underline{v_{i+1}v_{i+2}\mu_{i+1,i+2}} \\
&\overset{\text{detour}}{=} v_{i+2}v_{i+1}\underline{\mu_{i,i+1}\mu_{i+2,i+3}}v_{i+1}v_{i+2} \\
&\overset{(1)}{=} \underline{v_{i+2}v_{i+1}\mu_{i+2,i+3}}\mu_{i,i+1}v_{i+1}v_{i+2} \\
&\overset{\text{detour}}{=} \mu_{i+1,i+2}v_{i+2}v_{i+1}\mu_{i,i+1}v_{i+1}v_{i+2} \\
&= \mu_{i+1,i+2}\mu_{i,i+3}.
\end{aligned}$$

Figure 13 illustrates how relation 3 also reduces to relation 1. Notice now that relations 2 and 3 can be derived from relation 1 by conjugation by the permutations (324) and (23) respectively. Let us see how this works specifically for relation 2: the indices of relation 1 against the indices of relation 2 induce the permutation $(324) = v_2v_3$. This means that conjugating relation 1 by the word v_2v_3 will yield relation 2.

Notice also that there are 24 commuting relations in total involving the strands $1, 2, 3, 4$ and indices in any order. Likewise for any choice of four strands. The derivation of all relations from the action of S_n on relation 1 is clear from Remark 2.4. □

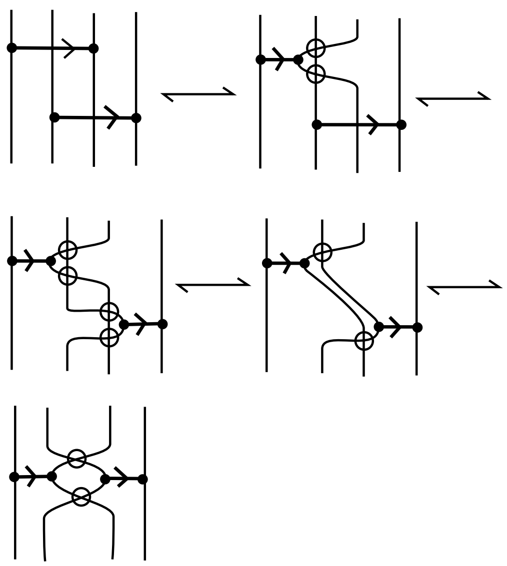


Fig. 13. A local commuting relation.

Lemma 2.7. *The following stringy braid relations hold in* VB_n.

(1) $\boldsymbol{\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}}$,
(2) $\mu_{21}\mu_{23}\mu_{13} = \mu_{13}\mu_{23}\mu_{21}$(*action by* (12)),
(3) $\mu_{13}\mu_{12}\mu_{32} = \mu_{32}\mu_{12}\mu_{13}$(*action by* (23)),
(4) $\mu_{32}\mu_{31}\mu_{21} = \mu_{21}\mu_{31}\mu_{32}$(*action by* (13)),
(5) $\mu_{23}\mu_{21}\mu_{31} = \mu_{31}\mu_{21}\mu_{23}$(*action by* (123)),
(6) $\mu_{31}\mu_{32}\mu_{12} = \mu_{12}\mu_{32}\mu_{31}$(*action by* (132)).

The above relations generalize to three-term relations of the form:

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \textit{for any distinct } i, j, k. \tag{2.17}$$

All six relations stated above result from the action on relation 1 *by permutations of* S_n, *which only permute the indices* $\{1, 2, 3\}$. *These permutations are indicated to the right of each relation. Moreover, for any choice of three strands there are exactly six relations analogous to the above, which all result from relation* 1 *by actions of appropriate permutations that preserve the three strands each time.*

Proof. Figure 14 illustrates relation 1. Relation 1 rests on the braid relations $(B1)$ of VB_n. Indeed, let us prove one relation of this type. See also Fig. 15 for a pictorial proof.

$$\begin{aligned}
\mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,i+1} &= (\sigma_{i+1}\underline{v_{i+1})(v_{i+1}}\sigma_i\underline{v_i v_{i+1})(\sigma_i}v_i) \\
&\overset{(S3,M1)}{=} \underline{\sigma_{i+1}\sigma_i\sigma_{i+1}v_i v_{i+1}v_i} \\
&\overset{(B1,S1)}{=} \sigma_i\sigma_{i+1}\underline{\sigma_i v_{i+1}v_i}v_{i+1} \\
&\overset{(M1,S3)}{=} \sigma_i\sigma_{i+1}\underline{v_{i+1}v_i v_{i+1}}v_{i+1}\sigma_{i+1}v_{i+1} \\
&\overset{(S1)}{=} \sigma_i\underline{\sigma_{i+1}v_i v_{i+1}}v_i v_{i+1}\sigma_{i+1}v_{i+1} \\
&\overset{(M1)}{=} (\sigma_i v_i)(v_{i+1}\sigma_i v_i v_{i+1})(\sigma_{i+1}v_{i+1}) \\
&= \mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2}.
\end{aligned}$$

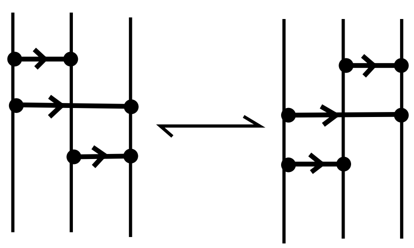


Fig. 14. The stringy braid relation.

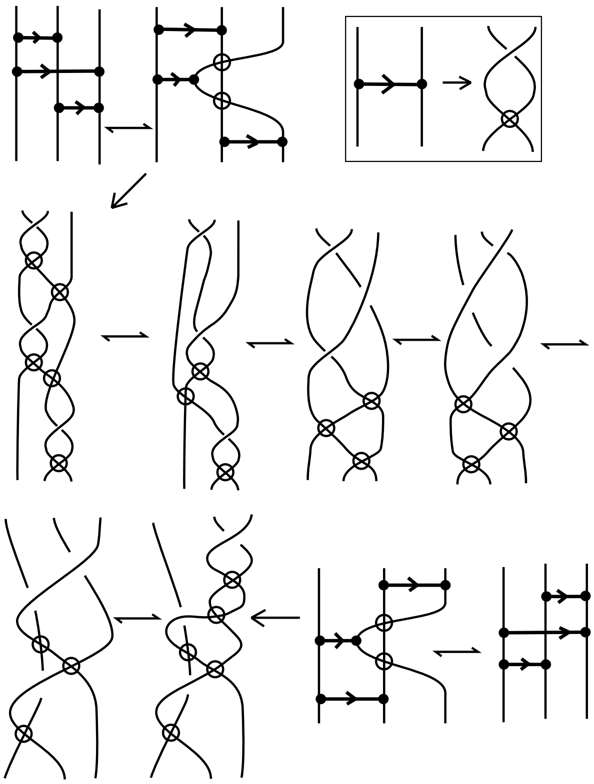


Fig. 15. Proof of the stringy braid relation.

The other five stated relations follow from relation 1. Indeed, substituting the μ_{ji}'s from Eqs. (2.9) and (2.13), and drawing the two sides of a relation we notice that there is always a region where, by the slide relations, all three connecting strings become consecutive without any of them having to be reversed, thus enabling application of the first relation. This diagrammatic argument confirms the fact that all six relations are derived from the first one by the action of appropriate elements of S_n. Let us see how this works specifically for relation 5: the indices of relation 1 against the indices of relation 5 induce the permutation $(123) = v_2v_1$. This means that conjugating relation 1 by the word v_2v_1 will yield relation 5. Finally, the derivation of all stringy braid relations from the action of S_n on relation 1 is clear from Remark 2.4. □

Another remark is now due.

Remark 2.8. The forbidden moves of virtual knot theory are naturally forbidden also in the stringy category. For example, the forbidden relations $(F1), (F2)$

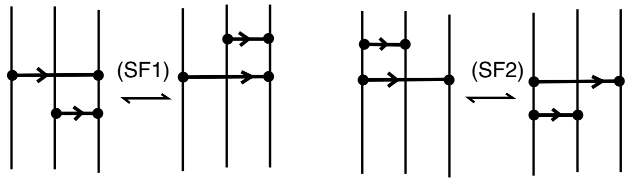


Fig. 16. Stringy forbidden moves.

of Eq. (2.6) translate into the following corresponding *forbidden stringy relations* $(SF1), (SF2)$:

$$\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2} \quad (SF1) \quad \text{and}$$
$$\mu_{i,i+2}\mu_{i,i+1} = \mu_{i,i+1}\mu_{i,i+2} \quad (SF2) \tag{2.18}$$

which, together with all similar relations arising from conjugating the above by permutations, are not valid in the stringy category. See Fig. 16 for illustrations.

2.4. *The stringy presentation*

We will now define an abstract stringy presentation for VB_n that starts from the concept of connecting string and recaptures the virtual braid group. By Eq. (2.7) we have

$$\sigma_i = \mu_{i,i+1}v_i \tag{2.19}$$

so, the connecting strings μ_{ij} can be taken as an alternate set of generators of the virtual braid group, along with the virtual generators v_i. The relations in this new presentation consist the results we proved above in Lemmas 2.5–2.7 describing the interaction of connecting strings with virtual crossings, the commutation properties of connecting strings, the stringy braiding relations and the usual relations $(S1), (S2), (S3)$ in the symmetric group S_n. For the work below, recall that we have defined the element $t_{ij} = v_iv_{i+1}\cdots v_j \cdots v_{i+1}v_i$ that corresponds to the transposition (ij) in S_n.

In any presentation of a group G containing the elements $\{v_1, \ldots, v_{n-1}\}$ and the relations $(S1), (S2), (S3)$ among them, we have an action of the symmetric group S_n on the group G defined by conjugation by an element τ in S_n, expressed in terms of the v_i:

$$g^\tau = \tau g \tau^{-1}$$

for g in G. In particular, we can consider $t_{ij}gt_{ij}$ as the action by the transposition t_{ij} on an element g of G. We will use this action to define a stringy model of the virtual braid group.

Definition 2.9. Let VS_n denote the following stringy group presentation.

$$\mathrm{VS}_n = \left\langle \begin{array}{l} \mu_{ij},\ 1 \le i \ne j \le n, \\ v_1, \ldots, v_{n-1} \end{array} \middle| \begin{array}{l} \tau\mu_{ij}\tau^{-1} = \mu_{\tau(i),\tau(j)},\ \tau \in S_n \\ \mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12} \\ \mu_{12}\mu_{34} = \mu_{34}\mu_{12} \\ (S1), (S2), (S3) \end{array} \right\rangle. \tag{2.20}$$

We can now state the following theorem.

Theorem 2.10. *The stringy group* VS_n *is isomorphic to the virtual braid group* VB_n.

Proof. First we define a homomorphism $F: \mathrm{VB}_n \to \mathrm{VS}_n$ by $F(v_i) = v_i$ and $F(\sigma_i) = \mu_{i,i+1}v_i$, and extend the map to be a homomorphism on words in the generators of the virtual braid group. In order to show that this map is well-defined, we must show that it preserves the relations in the virtual braid group. Since $F(v_i) = v_i$, the relations among the v_i with themselves are preserved identically. The commuting relations in the braid group are $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 2$. Thus we must show that

$$\mu_{i,i+1}v_i\mu_{j,j+1}v_j = \mu_{j,j+1}v_j\mu_{i,i+1}v_i.$$

But this follows immediately from relations 4 of Lemma 2.5 and from Lemma 2.6. The mixed commuting relations $(M2)$ follow also directly from relations $(S2)$ and relations 4 of Lemma 2.5. This completes the verification that the commuting relations in the virtual braid group are compatible with F.

The detour moves $(M2)$ in the virtual braid group go under F to the slide relations of Lemma 2.5. We illustrate this in Fig. 17.

It remains to prove that the braiding relations $(B1)$ carry over to VS_n under F. Indeed:

$$\begin{aligned} F(\sigma_i\sigma_{i+1}\sigma_i) &= \mu_{i,i+1}v_i\mu_{i+1,i+2}v_{i+1}\mu_{i,i+1}v_i \\ &\overset{\text{Lemma 2.5}}{=} \mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2}v_iv_{i+1}v_i, \end{aligned}$$

while

$$\begin{aligned} F(\sigma_{i+1}\sigma_i\sigma_{i+1}) &= \mu_{i+1,i+2}v_{i+1}\mu_{i,i+1}v_i\mu_{i+1,i+2}v_{i+1} \\ &\overset{\text{Lemma 2.5}}{=} \mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,i+1}v_{i+1}v_iv_{i+1}, \end{aligned}$$

and the two expressions are equal from Lemma 2.7 and relations $(S1)$. This completes the proof that the mapping F is a well-defined homomorphism of groups.

We now define an inverse mapping $G: \mathrm{VS}_n \to \mathrm{VB}_n$ by $G(v_i) = v_i$ and $G(\mu_{i,i+1}) = \sigma_iv_i$. At this stage we have two pieces of work to accomplish: We must extend G

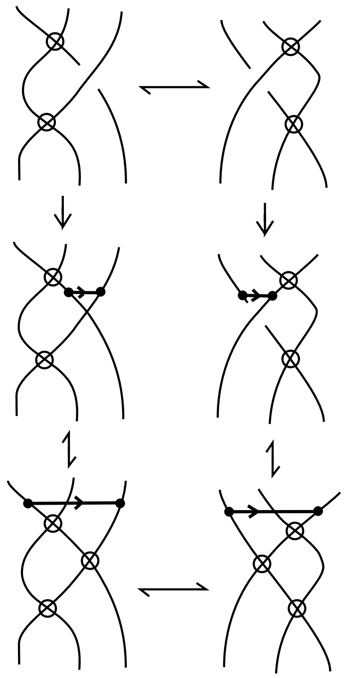


Fig. 17. The detour moves correspond to the slide moves in the stringy category.

to all of VB_n and we must show that G is well-defined and that it preserves the relations in the group presentation. This will be done in the next paragraphs.

First of all, we have the VS_n relations:

$$\tau^{-1}\mu_{ij}\,\tau = \mu_{\tau(i),\tau(j)}$$

for all τ in S_n. In particular, this means that if $\tau(1) = i$ and $\tau(2) = j$, then

$$\mu_{ij} = \tau^{-1}\mu_{12}\tau.$$

Thus we can define

$$G(\mu_{ij}) = \tau^{-1}G(\mu_{12})\tau = \tau^{-1}\sigma_1 v_1 \tau.$$

It is easy to see that this is well-defined by noting that if λ is another permutation such that $\lambda(1) = i$ and $\lambda(2) = j$, then $\lambda = \tau\gamma$ where γ is a permutation that fixes 1 and 2. But such a permutation commutes with $\sigma_1 v_1$ as is easy to see in the virtual braid group. Hence λ can replace τ in the formula for $G(\mu_{ij})$ with no change. We leave it as an exercise for the reader to check that our definition of $G(\mu_{i,i+1})$ in the previous paragraph agrees with the present definition. This completes the definition of the map G. We now need to see that it respects the other relations in VB_n.

We must show that

$$G(\mu_{12}\mu_{34}) = G(\mu_{34}\mu_{12}).$$

Just note that

$$G(\mu_{12}\mu_{34}) = \sigma_1 v_1 \sigma_3 v_3 = \sigma_3 v_3 \sigma_1 v_1 = G(\mu_{34}\mu_{12}),$$

by the commuting relations in the virtual braid group.

Finally, we must prove

$$G(\mu_{12}\mu_{13}\mu_{23}) = G(\mu_{23}\mu_{13}\mu_{12}).$$

Note that $\mu_{13} = v_2\mu_{12}v_2$, so we must prove that in the virtual braid group,

$$\sigma_1 v_1 v_2 \sigma_1 v_1 v_2 \sigma_2 v_2 = \sigma_2 v_2 v_2 \sigma_1 v_1 v_2 \sigma_2 v_2.$$

Figure 15 illustrates how this identity follows via braiding and detour moves.

We have verified that the mapping G is well-defined and, by definition, the compositions $F \circ G$ and $G \circ F$ are the identity on VS_n and VB_n. Therefore VS_n and VB_n are isomorphic groups. This completes the proof of the Theorem. □

Finally, we also give below a reduced presentation for VB_n, which derives immediately from Eq. (2.5).

Proposition 2.11. *The following is a reduced stringy presentation for* VB_n:

$$\mathrm{VB}_n = \left\langle \begin{matrix} \mu_{12}, \\ v_1, \ldots, v_{n-1} \end{matrix} \;\middle|\; \begin{matrix} \mu_{12}v_j = v_j\mu_{12}, \ \text{for } j > 2 \\ \mu_{12}v_2\mu_{12}v_2v_1v_2\mu_{12}v_2v_1 = v_1v_2\mu_{12}v_2v_1v_2\mu_{12}v_2\mu_{12} \\ \mu_{12}v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2 = v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2\mu_{12} \\ (S1), (S2), (S3) \end{matrix} \right\rangle. \tag{2.21}$$

The second relation is the stringy braid relation 1 of Lemma 2.7 and the third relation is the commuting relation 1 of Lemma 2.6.

3. The Pure Virtual Braid Group

3.1. *A presentation for the pure virtual braid group*

From presentation Eq. (2.1) of VB_n we have a surjective homomorphism

$$\pi : \mathrm{VB}_n \to S_n$$

defined by

$$\pi(\sigma_i) = \pi(v_i) = v_i.$$

For a virtual braid b, we refer to $\pi(b)$ as the *permutation associated with the virtual braid* b, and we define the *pure virtual braid group* VP_n to be the kernel of the homomorphism π. Hence, VP_n is a normal subgroup of VB_n of index $n!$. So, $\mathrm{VP}_n \cdot S_n = \mathrm{VB}_n$. Moreover, $\mathrm{VP}_n \cap S_n = \{\mathrm{id}\}$. Hence, $\mathrm{VB}_n = \mathrm{VP}_n \rtimes S_n$. Equivalently, we have the exact sequence

$$1 \to \mathrm{VP}_n \to \mathrm{VB}_n \to S_n \to 1.$$

A presentation for VP_n can be now derived immediately from the stringy presentation of VB_n as an application of the Reidemeister–Schreier process [9, 27, 33]. To see this, we first need the following.

Lemma 3.1. *The subgroup* VP_n *of* VB_n *is generated by the elements* μ_{ij} *for all* $i \neq j$.

Proof. Indeed, by Eqs. (2.7) and (2.9), $\sigma_i = \mu_{i,i+1} v_i = v_i \mu_{i+1,i}$. So, any element $b \in \mathrm{VB}_n$ can be written as a product in the μ_{ij}'s and the v_k's. Furthermore, by the slide relations of Lemma 2.5, all μ_{ij}'s can pass to the top of the braid, leaving at the bottom a word τ in the v_k's, such that $\tau = \pi(b)$. Thus, if $b \in \mathrm{VP}_n$ then τ must be the identity permutation. This completes the proof of the lemma. □

We can now give a stringy presentation of VP_n.

Theorem 3.2. *The following is a presentation for the pure virtual braid group.*

$$\mathrm{VP}_n = \left\langle \mu_{rs},\ r \neq s \;\middle|\; \begin{array}{l} \mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij},\ \textit{for all distinct } i,j,k \\ \mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij},\ \{i,j\} \cap \{k,l\} = \emptyset \end{array} \right\rangle. \tag{3.1}$$

Proof. Having reformulated the presentation of the virtual braid group, the proof is now a direct application of the Reidemeister–Schreier technique. The relations in VP_n arise as conjugations of the relations in VB_n by coset representatives of VP_n in VB_n, which are the elements of S_n. The relations $(S1), (S2), (S3)$ describe S_n and are used for choosing the coset representatives. We now describe the process from the point of view of covering spaces. We have $\mathrm{VP}_n \subset \mathrm{VB}_n$ as a normal subgroup with the subgroup S_n acting on it by conjugation. VP_n is the fundamental group of the covering space E of a cell complex B with fundamental group VB_n, where E has group of deck transformations S_n. Since the elements of the symmetric group lift to paths in the covering space, the relations $\tau\mu_{ij}\tau^{-1} = \mu_{\tau(i),\tau(j)}$ serve to describe the action of the symmetric group on the loops in the covering space (these loops are the lifts of the elements μ_{ij}). We choose basic relations in VP_n to be the lifts at a specific basepoint of the braiding relation $\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}$ and the commuting relation $\mu_{12}\mu_{34} = \mu_{34}\mu_{12}$. All other relations are obtained from these by the action of S_n, and all relations constitute the two orbits of the basic relations under this action. For example the relations

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$$

constitute the orbit under the action of S_n on the single basic braiding relation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}.$$

The same pattern applies to the commuting relations. This gives the statement of the Theorem and completes the proof. □

3.2. *Semidirect product structure*

The virtual braid group and the pure virtual braid group can be described in terms of semidirect products of groups, just as is begun in the paper by Bardakov [1] and continued in [10]. In this section we remark that these decompositions are based on the following algebra: The Yang–Baxter relation has the generic form

$$\mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,+1}$$

which is abstractly in the form

$$ABC = CBA$$

and can be rewritten in the form $B^{-1}ABC = B^{-1}CBA$ or

$$A^B = C^B AC^{-1}.$$

This allows one to rewrite some of the Yang–Baxter relations in terms of the conjugation action of the group on itself, and this is the key to the structural work pioneered by Bardakov.

4. A String Category for the Virtual Braid Group

In this section we summarize our results by pointing out that the string connectors and the virtual crossings can be regarded as generators of a category whose algebraic structure yields the virtual braid group and the pure virtual braid group. There are many relations in the definition of this category. These relations all act to make the string connection a topological model of a logical connection between strands in this tensor category. The specific topological interpretations of all these relations have been discussed in the preceding sections of this paper.

We define a strict monoidal category with generating morphisms μ_{ij} where this symbol is interpreted as an abstract string or connection between strands i and j in a diagram that otherwise is an identity braid on n strands just as defined in the previous sections. The other generators of this category are morphisms v_i that are interpreted as virtual crossings between strings i and $i+1$. The generators v_i have all the relations for transpositions generating the symmetric group. Compositions of these elements generate the morphisms of the category. The relations among these morphisms are exactly the relations described for the v_k and the μ_{ij} in the previous sections. We will now define this category using a minimal number of generators.

Definition 4.1. Consider the strict monoidal category freely generated by one object $*$ and three morphisms

$$\mu : * \otimes * \to * \otimes *,$$

$$\mu' : * \otimes * \to * \otimes *,$$

and

$$v : * \otimes * \to * \otimes *.$$

Let $\mu_{12} = \mu \otimes \mathrm{id}_*$, $\mu_{21} = \mu' \otimes \mathrm{id}_*$, $v_1 = v \otimes \mathrm{id}_*, v_2 = \mathrm{id}_* \otimes v$. Here we express these elements in three strands (tensor factors). For an arbitrary number of tensor factors, we write

$$v_i = \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_* \otimes v \otimes \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_*,$$

where v occurs in the ith place in this tensor product. More generally, it is understood that

$$\mu_{12} = \mu \otimes \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_*$$

and that

$$\mu_{21} = \mu' \otimes \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_*$$

for an arbitrary number of tensor factors.

For each natural number n, the symbols

$$[n] = * \otimes * \otimes \cdots \otimes *$$

with n $*$'s are the objects in the category. One can regard $[n]$ as an ordered row of n points that constitute the top or the bottom of a diagram involving n strands.

Now quotient this category by the following relations (compare with the reduced presentation of the virtual braid group in Proposition 2.11).

(1) $\mu\mu' = \mathrm{id}_{*\otimes*} = \mu'\mu$,
(2) $vv = \mathrm{id}_*$,
(3) $\mu_{12}v_j = v_j\mu_{12}$, for $j > 2$,
(4) $\mu_{12}v_2\mu_{12}v_2v_1v_2\mu_{12}v_2v_1 = v_1v_2\mu_{12}v_2v_1v_2\mu_{12}v_2\mu_{12}$,
(5) $\mu_{12}v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2 = v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2\mu_{12}$,
(6) $v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1}$,
(7) $v_iv_j = v_jv_i$, for $j \neq i \pm 1$.

This quotient is called the *String Category* and denoted SC. The category SC is still strict monoidal.

To recapture the connecting string morphisms μ_{ij} in the String Category context, we follow the formalism of the previous sections. Define

$$\mu_{i,i+1} = \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_* \otimes \mu \otimes \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_*,$$

where μ occurs in the i and $i+1$ places in the tensor product and define

$$\mu_{i+1,i} = \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_* \otimes \mu' \otimes \mathrm{id}_* \otimes \cdots \otimes \mathrm{id}_*,$$

where μ' occurs in the i and $i+1$ places in the tensor product. Define, for $i < j$, the element μ_{ij} by the formula

$$\mu_{ij} = v_{j-1}v_{j-2}\cdots v_{i+1}\mu_{i,i+1}v_{i+1}\cdots v_{j-2}v_{j-1}, \tag{4.1}$$

and define

$$\mu_{ji} = v_{j-1}v_{j-2}\cdots v_{i+1}\mu_{i+1,i}v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{4.2}$$

Remark 4.2. In this notation, relation (4) in Definition 4.1 becomes the algebraic Yang–Baxter equation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12},$$

and relation (5) becomes the commuting relation

$$\mu_{12}\mu_{34} = \mu_{34}\mu_{12}.$$

Then one has, as consequences, the general algebraic Yang–Baxter equation and commuting relations, as we have described them in earlier sections of the paper:

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \text{for all distinct } i, j, k$$

and

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i, j\} \cap \{k, l\} = \emptyset.$$

Diagrammatically, μ_{ij} consists in n parallel strands with a string connector between the ith and jth strands directed from i to j. Similarly, v_i corresponds to a diagram of n strands where there is a virtual crossing between the ith and $(i+1)$th strands. An n-strand diagram that is a product of these generators is regarded as a morphism from $[n]$ to $[n]$ for n any natural number. We interpret μ_{ij} and v_i diagrammatically according to the conventions previously established in this paper.

The morphisms v_i effect the action of the symmetric group and the category models the virtual braid group in the following precise sense.

Theorem 4.3. *The virtual braid group on n strands is isomorphic to the group of morphisms from $[n]$ to $[n]$ in the String Category.*

Proof. By Proposition 2.11, for any positive integer n, the group of endomorphisms of the object $[n] = *^{\otimes n}$ is isomorphic to VB_n. □

The point of this categorical formulation of the virtual braid groups is that we see how these groups form a natural extension of the symmetric groups by formal elements that satisfy the algebraic Yang–Baxter equation. The category we describe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang–Baxter equation. It should be remarked that the relationship between the relations in the virtual pure braid group and the algebraic Yang–Baxter equation was also pointed out in [3]. See also [2, Remark 10]. We have taken this observation further to point out that the virtual braid group is a direct result of forming a convenient category associated with the algebraic Yang–Baxter equation.

For the reader who would like to take the String Category SC as a starting point for the theory of virtual braids, here is a description of how to read our figures for that purpose. Figure 2 illustrates the permutation generators v_i for the String Category. The braiding elements σ_i will be defined in terms of the string generators. Elementary connecting strings are given in Fig. 7. It is implicit in Fig. 7 how to define the braiding elements σ_i by composing string generators with permutations (virtual crossings). See also Fig. 8, which illustrates basic relationships among string generators, permutations and braiding operators. Figure 9 illustrates the general connecting strings and their relations with the permutation operators. In particular, Fig. 9 shows how any string connection can be written in terms of a basic string generator and a product of permutations. Figure 10 illustrates how μ_{ij} and μ_{ji} are related diagrammatically. Figures 11–13 show the basic slide relations between string connections and permutations. Figure 14 illustrates the algebraic Yang–Baxter relation as it occurs for the string connectors.

5. Representations of the Virtual and Pure Virtual Braid Groups

5.1.

Let A be an algebra over a ground ring k. Let $\rho \in A \otimes A$ be an element of the tensor product of A with itself. Then ρ has the form given by the following equation

$$\rho = \sum_{i=1}^{N} e_i \otimes e^i, \tag{5.1}$$

where e_i and e^j are elements of the algebra A. We will write this sum symbolically as

$$\rho = \sum e \otimes e', \tag{5.2}$$

where it is understood that this is short-hand for the above specific summation.

We then define, for $i < j$, $\rho_{ij} \in A^{\otimes n}$ by the equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A, \tag{5.3}$$

where the e occurs in the ith tensor factor and the e' occurs in the jth tensor factor.

With $i < j$ we also define ρ_{ji} by reversing the roles of e and e' as shown in the next equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A, \tag{5.4}$$

where e' occurs in the ith tensor factor and e occurs in the jth tensor factor.

We say that ρ is a *solution to the algebraic Yang–Baxter equation* if it satisfies, in $A^{\otimes n}$ for $n \geq 3$, the equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}. \tag{5.5}$$

It is immediately obvious that if ρ satisfies the algebraic Yang–Baxter equation, then, for any pairwise distinct i, j, k we have

$$\rho_{ij}\rho_{ik}\rho_{jk} = \rho_{jk}\rho_{ik}\rho_{ij}. \tag{5.6}$$

This gives all possible versions of the algebraic Yang–Baxter equation occurring in the tensor product $A^{\otimes n}$.

The following proposition is an immediate consequence of our presentation for the pure virtual braid group.

Proposition 5.1. *Let* VP_n *denote the pure virtual braid group with generators* μ_{ij} *and relations as given in Theorem* 3.2 *of Sec.* 3. *Let A be an algebra with an invertible algebraic solution to the Yang–Baxter equation denoted by $\rho \in A \otimes A$ as described above. Define*

$$\mathrm{rep} : \mathrm{VP}_n \to A^{\otimes n}$$

by the equation

$$\mathrm{rep}(\mu_{ij}) = \rho_{ij}.$$

Then rep *extends to a representation of the virtual braid group to the tensor algebra* $A^{\otimes n}$.

Proof. It follows at once from the definitions of the ρ_{ij} that $\rho_{ij}\rho_{kl} = \rho_{kl}\rho_{ij}$ whenever the sets $\{i, j\}$ and $\{k, l\}$ are disjoint. Thus, we have shown that the ρ_{ij} satisfy all the relations in the pure virtual braid group. This completes the proof of the proposition. □

Next, we show how to obtain representations of the full virtual braid group. To this purpose, consider the algebra $\mathrm{Aut}(A^{\otimes n})$ of linear automorphisms of $A^{\otimes n}$ as a module over A. Assume that we are given an invertible solution to the algebraic Yang–Baxter equation, $\rho \in A \otimes A$, and define $\tilde{\rho}_{ij} : A^{\otimes n} \to A^{\otimes n}$ by the equation $\tilde{\rho}_{ij}(\alpha) = \rho_{ij}\alpha$ where $\alpha \in A^{\otimes n}$. Since ρ is invertible, $\tilde{\rho}_{ij} \in \mathrm{Aut}(A^{\otimes n})$. Let

$P_{ij} : A^{\otimes n} \to A^{\otimes n}$ be the mapping that interchanges the ith and jth tensor factors. Note that $P_{ij} \in \mathrm{Aut}(A^{\otimes n})$. We let P_i denote $P_{i,i+1}$. We now define

$$\mathrm{Rep} : \mathrm{VB}_n \to \mathrm{Aut}(A^{\otimes n})$$

by the equations

$$\mathrm{Rep}(\mu_{ij}) = \tilde{\rho}_{ij} \quad \text{and} \quad \mathrm{Rep}(v_i) = P_i.$$

The next proposition is a consequence of presentation (2.20) for the virtual braid group.

Proposition 5.2. *The mapping* $\mathrm{Rep} : \mathrm{VB}_n \to \mathrm{Aut}(A^{\otimes n})$, *defined above, is a representation of the virtual braid group to a subgroup of* $\mathrm{Aut}(A^{\otimes n})$.

Proof. It is clear that the elements P_i obey all the relations in the symmetric group S_n. By presentation (2.20) it remains to show that letting $\lambda = \mathrm{Rep}(\tau)$ where τ is an element of S_n, the relations

$$\lambda \rho_{ij} \lambda^{-1} = \tilde{\rho}_{\tau(i),\tau(j)}, \quad \tau \in S_n$$

are satisfied in $\mathrm{Aut}(A^{\otimes n})$. Since ρ_{ij} is defined via the placement of the e and e' factors in the summation for ρ on the ith and jth strands, these relations are immediate. This completes the proof of the proposition. □

Remark 5.3. The method we have described for constructing a representation of the virtual braid group from an algebraic solution to the Yang–Baxter equation generalizes the well known construction of a representation of the classical Artin braid group from a solution to the Yang–Baxter equation in braided form. In the usual method for constructing the classical representation, one composes the algebraic solution with a permutation, obtaining a solution to the braiding equation $(B1)$. This composition is the same as our relation

$$\sigma_i = \mu_{i,i+1} v_i$$

between the braiding element σ_i and the stringy generator $\mu_{i,i+1}$ for the pure virtual braid group. Without the concept of virtuality, the direct relationship of the algebraic Yang–Baxter equation with the braid groups would not be apparent. We see that, from an algebraic point of view, the virtual braid group is an entirely natural construction. It is the universal algebraic structure related to viewing solutions to the algebraic Yang–Baxter equation inside tensor products of algebras and endowing these tensor products with the natural permutation action of the symmetric group.

Solutions to the algebraic Yang–Baxter equation are usually thought of as deformations of the identity mapping on a two-fold tensor product $A \otimes A$. We think of a braiding operator as a deformation of a transposition, and so one goes between the algebraic and braided versions of such operators by composition with a transposition.

The Artin braid group B_n is motivated by a combination of topological considerations and the desire for a group structure that is very close to the structure of the symmetric group S_n. We have seen that the virtual braid group VB_n is motivated at first by a natural extension of the Artin braid group in the context of virtual knot theory, but now we see a different motivation for the virtual braid group. Given that one studies the algebraic Yang–Baxter equation in the context of tensor powers of an algebra A, it is thoroughly natural to study the compositions of algebraic braiding operators placed in two out of the n tensor lines (the stringy generators) and to let the permutation group of the tensor lines act on this algebra. As we have seen in (2.20), this is precisely the virtual braid group. Viewed in this way, the virtual braid group has nothing to do with the plane and nothing to do with virtual crossings. It is a natural group associated with the structure of algebraic braiding.

5.2. *A representation category for the string category*

We now give a categorical interpretation of virtual knot theory and the virtual braid group in terms of representation modules associated to an algebra A over a commutative ring k with an algebraic Yang–Baxter element ρ as above. Let $\mathrm{End}(A^{\otimes n})$ denote the linear endomorphisms of $A^{\otimes n}$ as a module over A. View $\mathrm{End}(A^{\otimes n})$ as the set of morphisms in a category Mod_k^n with $A^{\otimes n}$ as the single object. We single out the following morphisms in this category:

(1) $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n \in A^{\otimes n}$ acting on $A^{\otimes n}$ by left multiplication,
(2) the elements of the symmetric group S_n, generated by transpositions of adjacent tensor factors.

In making the representation of VB_n we have used the stringy generators μ_{ij} and mapped them to sums of morphisms of the first type above. The virtual braid group VB_n described via (2.20), can be viewed as a category with one object and generators μ_{ij} and v_k. We let Mod_k denote the category that is obtained by taking all of the categories Mod_k^n together with objects $A^{\otimes n}$ for each natural number n and morphisms from all of the $\mathrm{End}(A^{\otimes n})$.

Remark 5.4. Of course any associative algebra can be seen as a single object category with morphisms the elements of the algebra. But here we have a pictorial representation of the morphisms as stringy braid diagrams. These diagrams, which capture the pure virtual braid group so far, can be generalized by taking the transpositions of the form $P_{i,i+1}$ via a diagram of lines i and $i+1$ crossing through one another to form virtual crossings v_i. Seen from the categorical view that we have developed in these last sections, the virtual crossings are interpreted as generators of the symmetric group whose action is added naturally to the algebraic structure of the pure virtual braid group. By bringing in this action, we expand the pure virtual braid group to the virtual braid group. The virtual crossings have thus become part of the embedded symmetry of the structure of the virtual braid group. This is in sharp contrast to the role of the virtual crossings in the original form of the virtual knot theory. There the virtual crossings appear as artifacts of the presentation of virtual knots in the plane where those knots acquire extra crossings that are not really part of the essential structure of the virtual knot. Nevertheless, these same crossings appear crucially in the virtual braid group, and turn into the generators of the symmetric group embedded in the virtual braid group. With the use of the full set of μ_{ij} in (2.20) the detour moves and other remnants of the virtual crossings as artifacts have completely disappeared into the permutation action. We will continue the categorical discussion for the virtual braid group, after first discussing certain aspects of knot theory and the tangle categories.

We can now state a general representation theorem.

Theorem 5.5. *Any monoidal functor*

$$F : \mathrm{SC} \to \mathrm{Mod}_k$$

gives rise to a representation of VB_n:

$$f \in \mathrm{End}_{\mathrm{SC}}([n]) \simeq \mathrm{VB}_n \longmapsto F(f) \in \mathrm{End}_k(A^{\otimes n}),$$

where $A = F(*)$.

Proof. The proof follows from the previous discussion. □

The representations of VB_n that we have here derived can be interpreted as follows.

Theorem 5.6. *Let* $\rho \in A \otimes A$ *be a solution of the algebraic Yang–Baxter equation, where* A *is an algebra over a commutative ring* k. *One can then define a monoidal functor*

$$F_A : \mathrm{SC} \to \mathrm{Mod}_k$$

by setting $F_A(*) = A$, $F_A(\mu) = \tilde{\rho}$, *and* $F_A(v_i) = P$, *where the endomorphisms* $\tilde{\rho}$ *and* P *of* $A \otimes A$ *are given by*

$$\tilde{\rho}(x \otimes y) = \rho(x \otimes y)$$

and

$$P(x \otimes y) = y \otimes x$$

for all $x, y \in A$.

Proof. The proof follows from the previous discussion. □

5.3. *Virtual Hecke algebra*

From the point of view of the theory of braids the Hecke algebra $H_n(q)$ is a quotient of the group ring $\mathbb{Z}[q, q^{-1}][B_n]$ of the Artin braid group by the ideal generated by the quadratic expressions

$$\sigma_i^2 - z\sigma_i - 1 \tag{5.7}$$

for $i = 1, 2, \ldots, n-1$, where $z = q - q^{-1}$. This corresponds to the identity $\sigma_i - \sigma_i^{-1} = z1$, which is sometimes regarded diagrammatically as a skein identity for calculating knot polynomials. By the same token, we define the *virtual Hecke algebra* $\mathrm{VH}_n(q)$ to be the quotient of the group ring $\mathbb{Z}[q, q^{-1}][\mathrm{VB}_n]$ by the ideal generated by Eqs. (5.7).

There are difficulties in extending structure theorems for the Hecke algebra to corresponding structure theorems for the virtual Hecke algebra, such as finding normal forms, studying the representation theory and constructing Markov traces. Yet, some matters of representations do generalize directly. In particular, let W be a module over $\mathbb{Z}[q, q^{-1}]$ and let $I : W \to W$ be the identity operator. If $R : W \otimes W \to W \otimes W$ is a solution to the Yang–Baxter equation satisfying

$$R^2 = zR + I,$$

then one has a corresponding representation $T : \mathrm{VH}_n(q) \to \mathrm{Aut}(W^{\otimes n})$. This representation is specified as follows.

$$T(\sigma_i) = \sum 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1, \tag{5.8}$$

where R operates on the ith and $(i+1)$th tensor factors, and

$$T(v_i) = \sum 1 \otimes \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots \otimes 1, \tag{5.9}$$

where P acts by permuting the ith and $(i+1)$th tensor factors. It is easy to see that this gives a representation of the virtual Hecke algebra.

One can hope that the presence of such representations would shed light on the existence of a generalization of the Ocneanu trace [14] on the Hecke algebra to a corresponding trace and link invariant using the virtual Hecke algebra. At this point there is an issue about the nature of the generalization. One can aim for a trace on the virtual Hecke algebra that is compatible with the Markov Theorem for virtual knots and links as formulated in [16, 22]. This means that the trace must be compatible with both classical and virtual stabilization. This is a trace that is difficult to achieve. A simpler trace is possible by working in rotational virtual knot theory where virtual stabilization is not allowed [18]. See Sec. 6 for a discussion of unoriented quantum invariants for rotational virtuals. We will report on the relation of this approach with the Markov Theorem for virtual knots and links in a separate paper.

Another line of investigation is suggested by translating the basic Hecke algebra relation into the language of stringy connections. We have $\sigma = \mu v$ for the abstract relation between a braiding generator, a connector and a virtual element. Thus, the virtual Hecke relation $\sigma^2 = z\sigma + 1$ becomes

$$\mu v \mu = z\mu + v,$$

AQ: Please check the citation of Eq. (2.20).

and it is possible to work in the presentation (2.20) of the virtual braid group to find a structure theory for the virtual Hecke algebra.

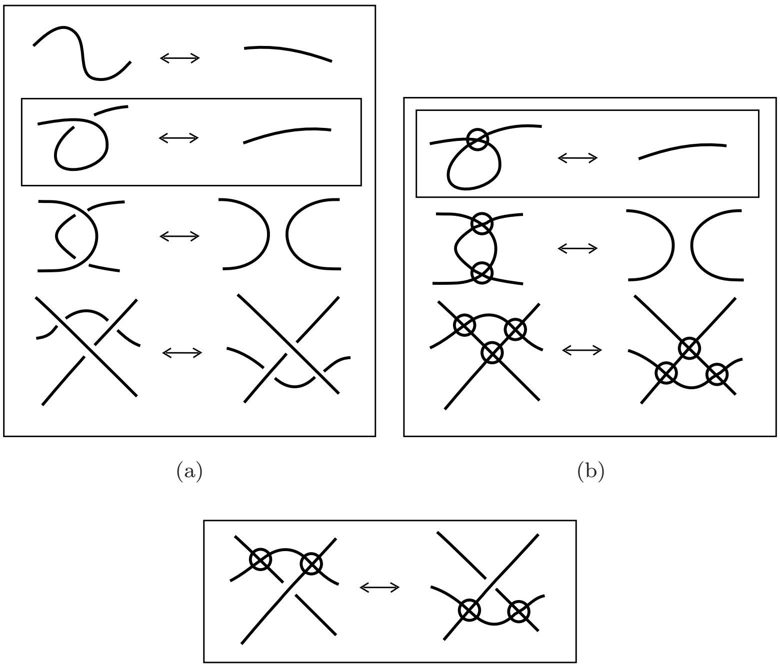
6. Rotational Virtual Links, Quantum Algebras, Hopf Algebras and the Tangle Category

This section will show how the ideas and methods of this paper fit together with representations of quantum algebras (to be defined below) and Hopf algebras and invariants of virtual links. We begin with a quick review of the theory of virtual links (in relation to virtual braids), and we construct the virtual tangle category. This category is a natural generalization of the virtual braid group. A functor from the virtual tangle category to an algebraic category will form a generalization of the representations of virtual braid groups that we have discussed in Sec. 5. This functor is related to (rotational) invariants of virtual knots and links. It is not hard to see that the construction given in this section defines a category (for arbitrary Hopf algebras) that generalizes the String Category given earlier in this paper. The category that we define here contains virtual crossings, special elements that satisfy the algebraic Yang–Baxter equation and also cup and cap operators. The subcategory without the cup and cap operators and without any (symbolic) algebra elements except those involved with the algebraic Yang–Baxter operators is isomorphic to the String Category.

A word to the reader about this section: In one sense this section is a review of known material in the form that Kauffman and Radford [23] have shaped the theory of quantum invariants of knots and three-manifolds via finite-dimensional Hopf algebras. On the other hand, this theory is generalized here to invariants of rotational virtual knots and links. This generalization is new, and it is directly related to the structure of the virtual braid group as described in the earlier part of this paper. We have given a complete sketch of this generalization. The reader should take the word *sketch* seriously and concentrate on the sequence of diagrams that depict the ingredients of the theory. Taking this point of view, the reader can see that the appearance of the algebraic Yang–Baxter element in our diagrams (see Fig. 28) is aided by using a connecting string exactly analogous to the connecting string in the earlier part of the paper. The generalization follows by taking the functorial image of the virtual tangle category defined in this section.

6.1. *Virtual diagrams*

We begin with Fig. 18. This figure illustrates the moves on virtual knot and link diagrams that serve to define the theory of virtual knots and links. Two knot or



(a) (b)

(c)

Fig. 18. Virtual moves.

link diagrams with virtual and classical crossings are said to be *virtually isotopic* if one can be obtained from the other by a finite sequence of these moves. In the figure the moves are divided into type (a), (b) and (c) moves. Moves of type (a) are the classical Reidemeister moves. These are essentially the same as corresponding moves in the Artin braid group except for the boxed move involving a loop in the diagram. The move involving this loop is usually called the *first Reidemeister move.* When we forbid the first Reidemeister move, the equivalence relation is called *regular isotopy.* The moves of type (b) are purely virtual and (except for the move involving a virtual loop) correspond to the properties of virtual crossings in the virtual braid group. We call the equivalence relation that forbids both the virtual loop move and the classical loop move *virtual regular isotopy.* Finally, we have moves of type (c). These are the local detour moves, and they correspond to the mixed moves in the virtual braid group.

In this section we will work with virtual knots and links up to virtual regular isotopy. In addition to the usual kinds of virtual phenomena, we will see some extra features in looking at this equivalence relation. Two virtual knot or link diagrams are said to be *rotationally equivalent* if they are equivalent under virtual regular isotopy. *Rotational virtual knot theory* is the study of the rotational equivalence classes of virtual knot and link diagrams. Studied under this equivalence relation,

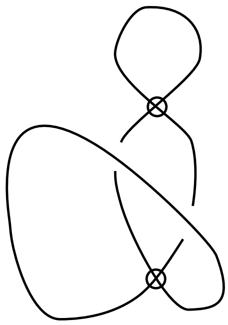


Fig. 19. A rotational virtual knot.

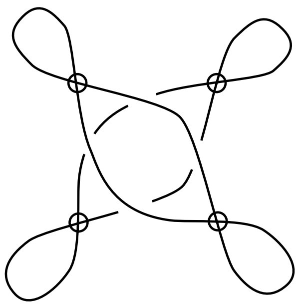


Fig. 20. A rotational virtual link.

virtual knot and link diagrams are called *rotational virtuals.* We shall say that a virtual knot or link is *rotationally knotted* or *rotationally linked* if it is not equivalent to an unknot or an unlink under virtual regular isotopy. View Figs. 19 and 20. In the first figure we illustrate a rotational virtual knot, and in the second we show a rotational virtual link. Both the knot and the link are kept from being trivial by the presence of flat loops as discussed above. There is much more to say about rotational virtuals, and we refer the reader to [18] for some steps in this direction.

6.2. *The virtual tangle category*

The advantage in studying virtual knots up to virtual regular isotopy is that all so-called quantum link invariants generalize to invariants of virtual regular isotopy. This means that virtual regular isotopy is a natural equivalence relation for studying topology associated with solutions to the Yang–Baxter equation.

Here we create a context by defining the *Virtual Tangle Category*, VTC, as indicated in Fig. 21. The tangle category is generated by the morphisms shown in the box at the top of this figure. These generators are: a single identity line,

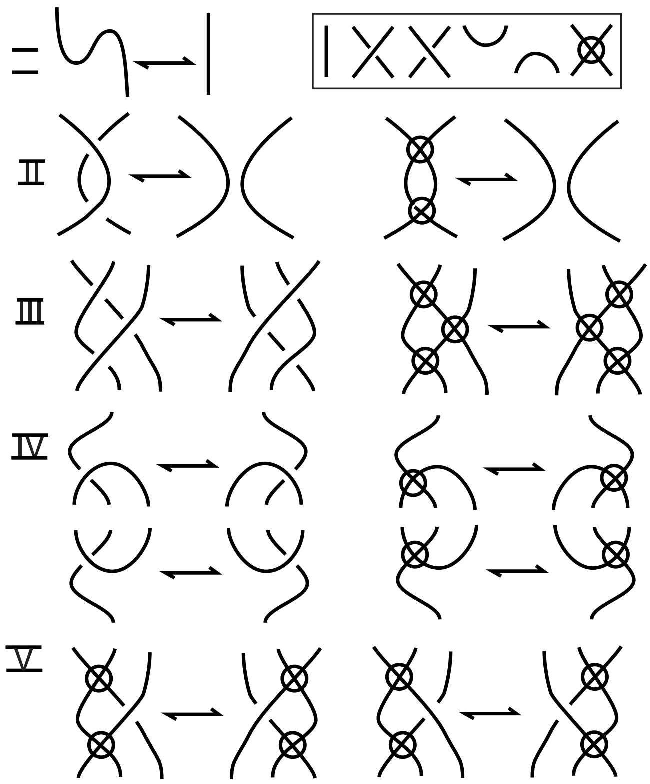


Fig. 21. Regular isotopy with respect to the vertical direction.

right-handed and left-handed crossings, a cap and a cup, a virtual crossing. The objects in the tangle category consist in the set of $[n]$'s where $n = 0, 1, 2, \ldots$. For a morphism $[n] \to [m]$, the numbers n and m denote, respectively, the number of free arcs at the bottom and at the top of the diagram that represents the morphism. The morphisms are like braids except that they can (due to the presence of the cups and caps) have different numbers of free ends at the top and the bottom of their diagrams.

The sense in which the elementary morphisms (line, cup, cap, crossings) generate the tangle category is composition as shown in Fig. 22. For composition, the segments are matched so that the number of lower free ends on each segment is equal to the number of upper free ends on the segment below it. The Fig. 22 shows a virtual trefoil as a morphism from $[0]$ to $[0]$ in the category. The tensor product of morphisms is the horizontal juxtaposition of their diagrams. Each of the seven horizontal segments of the figure represents one of the elementary morphisms tensored with the identity line. Consequently there is a well-defined composition of all of the segments and this composition is a morphism $[0] \to [0]$ that represents the knot.

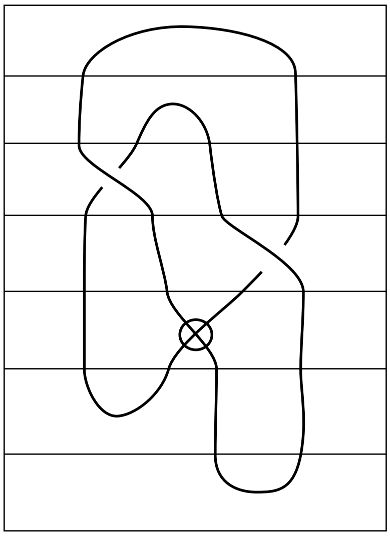


Fig. 22. Virtual trefoil as a morphism in the tangle category.

The basic equivalences of morphisms are shown in Fig. 21. Note that II, III, V are formally equivalent to the rules for unoriented virtual braids. The zeroth move is a cancellation of consecutive maxima and minima, and the move IV is a swing move in both virtual and classical relations of crossings to maxima and minima. It should be clear that the tangle category is a generalization of the virtual braid group with a natural inclusion of unoriented virtual braids as special tangles in the category. Standard braid closure and the plat closure of braids have natural definitions as tangle operations. Any virtual knot or link can be represented in the tangle category as a morphism from [0] to [0], and one can prove that *two virtual links are virtually regularly isotopic if and only if their tangle representatives are equivalent in the tangle category.* None of the rules for equivalence in the tangle category involve either a classical loop or a virtual loop. This means that the virtual tangle category is a natural home for the theory of rotational virtual knots and links.

6.3. *Quantum algebra and category*

Now we shift to a category associated with an algebra that is directly related to our representations of the virtual braid group. We take the following definition [17, 23]: A *quantum algebra* A is an algebra over a commutative ground ring k with an invertible mapping $s: A \to A$ that is an *antipode*, that is $s(ab) = s(b)s(a)$ for all a and b in A, and there is an element $\rho \in A \otimes A$ satisfying the algebraic Yang–Baxter equation as in Eq. (5.5):

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

We further assume that ρ is invertible and that

$$\rho^{-1} = (1_A \otimes s) \circ \rho = (s \otimes 1_A) \circ \rho.$$

The multiplication in the algebra is usually denoted by $m : A \otimes A \to A$ and is assumed to be associative. It is also assumed that the algebra has a multiplicative unit element. The defining properties of a quantum algebra are part of the properties of a Hopf algebra, but a Hopf algebra has a comultiplication $\Delta : A \to A \otimes A$ that is a homomorphism of algebras, plus a list of further relations, including a fundamental relationship between the multiplication, the comultiplication and the antipode. In the interests of simplicity, we shall restrict ourselves to quantum algebras here, but most of the remarks that follow apply to Hopf algebras, and particularly quasi-triangular Hopf algebras. Information on Hopf algebras is included at the end of this section. See [23] for more about these connections.

We construct a category $\mathrm{Cat}(A)$ associated with a quantum algebra A. This category is a very close relative to the virtual tangle category. $\mathrm{Cat}(A)$ differs from the tangle category in that it has only virtual crossings, and there are labeled vertical lines that carry elements of the algebra A. See Fig. 23. Each such labeled line is a morphism in the category. The virtual crossing is a generating morphism as are the cups, caps and labeled lines. The objects in this category are the same entities $[n]$ as in the tangle category. This category is identical in its framework to the tangle category but the crossings are not present and lines labeled with algebra are present. Given $a, b \in A$ we compose the morphisms corresponding to a and b by taking a line labeled ab to be their composition. In other words, if $\langle x \rangle$ denotes the morphism in $\mathrm{Cat}(A)$ associated with $x \in A$, then

$$\langle a \rangle \circ \langle b \rangle = \langle ab \rangle.$$

As for the additive structure in the algebra, we extend the category to an additive category by formally adding the generating morphisms (virtual crossings, cups, caps and algebra line segments). In Fig. 23 we illustrate the composition of such morphisms and we illustrate a number of other defining features of the category $\mathrm{Cat}(A)$.

In the same figure we illustrate how the tensor product of elements $a \otimes b$ is represented by parallel vertical lines with a labeling the left line and b labeling the right line. We indicate that the virtual crossing acts as a permutation in relation to the tensor product of algebra morphisms. That is, we illustrate that

$$\langle a \rangle \otimes \langle b \rangle \circ P = P \circ \langle b \rangle \otimes \langle a \rangle.$$

Here P denotes the virtual crossing of two segments, and is regarded as a morphism $P : V \otimes V \to V \otimes V$ (see remark below). Since the lines interchange, we expect P to behave as the permutation of the two tensor factors.

In Fig. 23 we show the notation V for the object $[1]$ in this category and we use $V \otimes V = [2]$, $V \otimes V \otimes V = [3]$ and so on for all the natural number objects in the category. We write $[0] = k$, identifying the ground ring with the "empty

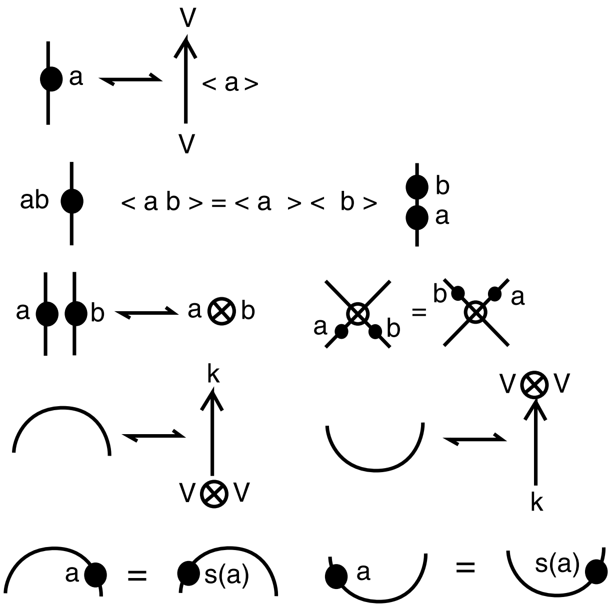


Fig. 23. Morphisms in Cat(A).

object" [0]. It is then axiomatic that $k \otimes V = V \otimes k = V$. Morphisms are indicated both diagrammatically and in terms of arrows and objects in this figure. Finally, the figure indicates the arrow and object forms of the cup and the cap, and crucial axioms relating the antipode with the cup and the cap. A cap is regarded as a morphism from $V \otimes V$ to k, while a cup is regarded as a morphism form k to $V \otimes V$. The basic property of the cup and the cap is the *Antipode Property*: *if one "slides" a decoration across the maximum or minimum in a counterclockwise turn, then the antipode s of the algebra is applied to the decoration.* In categorical terms this property says

$$\mathrm{Cap} \circ (\langle 1 \rangle \otimes a) = \mathrm{Cap} \circ (\langle sa \rangle \otimes 1)$$

and

$$(\langle a \rangle \otimes 1) \circ \mathrm{Cup} = (1 \otimes \langle sa \rangle) \circ \mathrm{Cup}.$$

Here 1 denotes the identity morphism for [0]. These properties and other naturality properties of the cups and the caps are illustrated in Figs. 23 and 24. The naturality properties of the flat diagrams in this category include regular homotopy of immersions (for diagrams without algebra decorations), as illustrated in these figures.

In Fig. 24 we see how the antipode property of the cups and caps leads to a diagrammatic interpretation of the antipode. In the figure we see that the antipode $s(a)$ is represented by composing with a cap and a cup on either side of the morphism

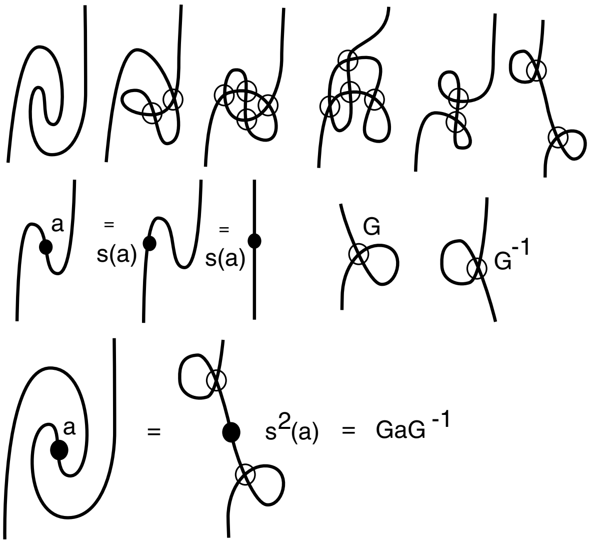


Fig. 24. Diagrammatics of the antipode.

for a. In terms of the composition of morphisms this diagram becomes

$$\langle sa \rangle = (\mathrm{Cap} \otimes 1) \circ (1 \otimes \langle a \rangle \otimes 1) \circ (1 \otimes \mathrm{Cup}).$$

Similarly, we have

$$\langle s^{-1}a \rangle = (1 \otimes \mathrm{Cap}) \circ (1 \otimes \langle a \rangle \otimes 1) \circ (\mathrm{Cup} \otimes 1).$$

This, in turn, leads to the interpretation of the flat curl as an element G in A such that $s^2(a) = GaG^{-1}$ for all a in A. G is a flat curl diagram interpreted as a morphism in the category. We see that, formally, it is natural to interpret G as an element of A. In a so-called *ribbon Hopf algebra* there is such an element already in the algebra. In the general case it is natural to extend the algebra to contain such an element.

6.4. *The basic functor and the rotational trace*

We are now in a position to describe a functor F from the virtual tangle category VTC to Cat(A). (Recall that the virtual tangle category is defined for virtual link diagrams without decorations. It has the same objects as Cat(A).)

$$F : \mathrm{VTC} \to \mathrm{Cat}(A).$$

The functor F decorates each positive crossing of the tangle (with respect to the vertical — see Fig. 26) with the Yang–Baxter element (given by the quantum algebra A) $\rho = \Sigma e \otimes e'$ and each negative crossing (with respect to the vertical) with $\rho^{-1} = \Sigma s(e) \otimes e'$. The form of the decoration is indicated in Fig. 26. Since we have labeled the negative crossing with the inverse Yang–Baxter element, it follows at once that the two crossings are mapped to inverse elements in the category of the algebra. *This association is a direct generalization of our mapping of the virtual braid group to the stringy connector presentation.*

We now point out the structure of the image of a knot, link or tangle under this functor. The key point about this functor is that, because quantum algebra elements can be moved around the diagram, we can concentrate all the image algebra in one place. Because the flat curls are identified with either G or G^{-1}, we can use regular homotopy of immersions to bring the image under F of each component of a virtual link diagram to the form of a circle with a single concentrated decoration (involving a sum over many products) and a reduced pattern of flat curls that can be encoded as a power of the special element G. Once the underlying curve of a link component is converted to a loop with total turn zero, as in Fig. 25, then we can think of such

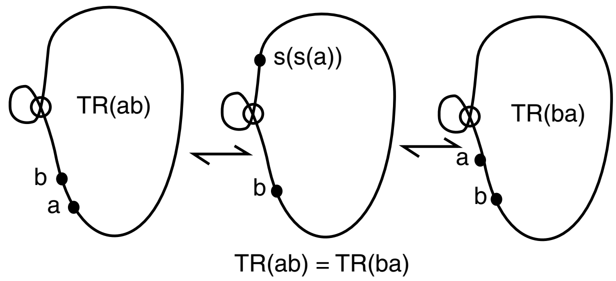


Fig. 25. Formal trace.

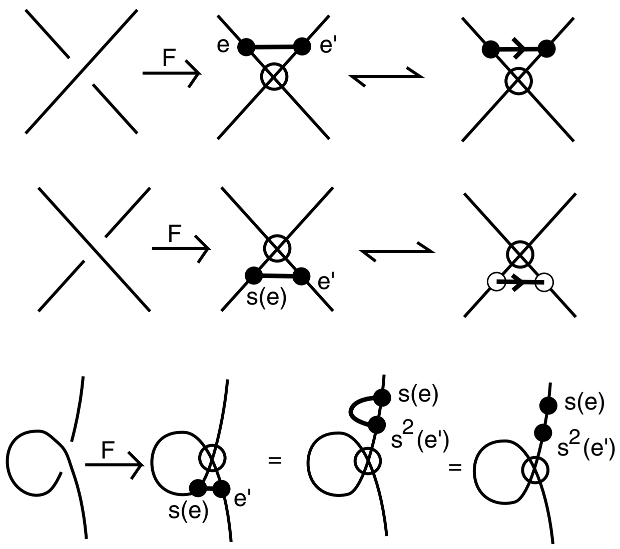


Fig. 26. The functor $F : \mathrm{VTC} \to \mathrm{Cat}(A)$.

a loop, with algebra labeling the loop, as a representative for a formal trace of that algebra and call it $\mathrm{TR}(X)$ as in the figure. In the figure we illustrate that for such a labeling

$$\mathrm{TR}(ab) = \mathrm{TR}(ba),$$

thus one can take a product of algebra elements on a zero-rotation loop up to cyclic order of the product. In situations where we choose a representation of the algebra or in the case of finite-dimensional Hopf algebras where one can use right integrals [23], there are ways to make actual evaluations of such traces. Here we use them formally to indicate the result of concentrating the algebra on the loop.

One further comment is in order about the antipode. In Fig. 27 we show that our axiomatic assumption about the antipode (the sliding rule around maxima and minima) actually demands that the inverse of ρ is $(s \otimes 1_A) \circ \rho = (1_A \otimes s) \circ \rho$. This follows by examining the form of the inverse of the positive crossing in the tangle category by turning that crossing to produce an identity between the positive crossing and the negative crossing twisted with additional maxima and minima. This relationship shows that if we set the functor F on a right-handed crossing as we have done, then the way it maps the inverse crossing is forced and that this inverse corresponds to the inverse of ρ in the quantum algebra. Thus the quantum algebra formula for the inverse of ρ is forced by the topology.

In Fig. 28 we illustrate the entire functorial process for the virtual trefoil of Fig. 22. The virtual trefoil is denoted by K, and we find that $F(K)$ reduces to

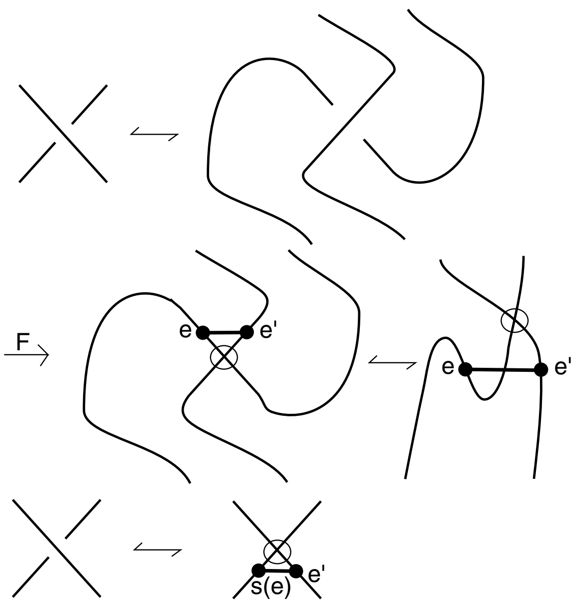


Fig. 27. Inverse and antipode.

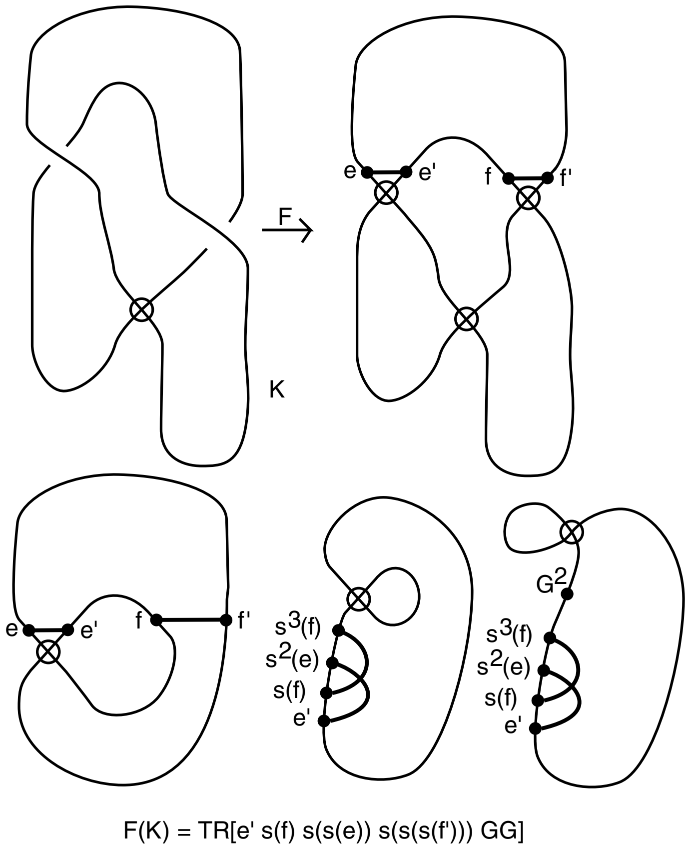


Fig. 28. The functor $F : T \to \mathrm{Cat}(A)$ applied to a virtual trefoil.

a zero-rotation circle with the inscription $e's(f)s^2(e)s^3(f')G^2$. We can, therefore, write the equation

$$F(K) = \mathrm{TR}[e's(f)s^2(e)s^3(f')G^2].$$

Another way to think about this trace expression is to regard it as a Gauss code for the knot that has extra structure. *The chords in the Gauss diagram are the string connectors of the beginning of this paper, generalized to the algebra category* $\mathrm{Cat}(A)$. The powers of the antipode and the power of G keep track of rotational features in the diagram as it lives in the tangle category up to regular isotopy. We now see that the mapping of the virtual braid group to the braid group generated by permutations and string connectors has been generalized to the functor F taking the virtual tangle category to the abstract category of a quantum algebra. We regard this generalization as an appropriate context for thinking about virtual knots, links and braids.

The category Cat(A) of a quantum algebra A can be generalized to an abstract category with labels, virtual crossings, and with stringy connections that satisfy the algebraic Yang–Baxter equation. Each such stringy connection has a left label e or $s(e)$ and a right label e'. We retain the formalism of the antipode as a formal replacement for adjoining a label with a cup and a cap. The resulting *abstract algebra category* will be denoted by $\overline{\text{Cat}(A)}$. Since we take this category with no further relations, the functor $\overline{F}: \text{VTC} \to \overline{\text{Cat}(A)}$ is an equivalence of categories. This functor is the direct analog of our reformulation of the virtual braid group in terms of stringy connectors.

6.5. *Virtual braids and their closures*

The functor $F: \text{VTC} \to \text{Cat}(A)$ defined in the last subsection can be restricted to the category of virtual (unoriented) braids that we will denote here by VB. If the reader then examines the result of this functor he will see that the image of a virtual crossing is a virtual crossing, and the image of a braid generator is a string connection (expressed in terms of Cat(A).). If we use the corresponding functor

$$\overline{F} : \text{VB} \hookrightarrow \text{VTC} \to \overline{\text{Cat}(A)},$$

then the image $\overline{F}$(VB) is an abstract category of string connections and permutations that is (up to orientation) identical with our String Category SC studied throughout this paper. This remark brings us full circle and shows how the String Category fits in the context of the quantum link invariants discussed in this part of the paper. In particular, view the bottom part of Fig. 29 where we have illustrated the image under $\overline{F}$ of a particular virtual braid. Each classical crossing in the braid is replaced by a string connector followed by a virtual crossing. The string connector is interpreted as in the abstract Hopf algebra category, but in the braid image there is no other structure than the connectors and the virtual crossings. This shows how the braid lands in a subcategory that is isomorphic with our main category SC.

Now recall that one can move from virtual braids to virtual knots and links by taking the *braid closure*. The closure $\overline{b}$ of a braid b is obtained by attaching planar disjoint arcs from the outputs of a braid to its inputs as illustrated in Fig. 29. The result of the closure is a virtual knot or link. In particular, this means that we can express rotational quantum link invariants by applying F to the closure of virtual braid and then taking the trace TR described in the last section. Alternatively, one can regard the invariant as A-valued where A is the quantum algebra that supports the functor F. Altogether, this section and the examples in Fig. 29 indicate the close relationship of the different constructions that have been outlined in this paper and how the structure of the virtual braid group is intimately related to quantum link invariants for rotational virtual links.

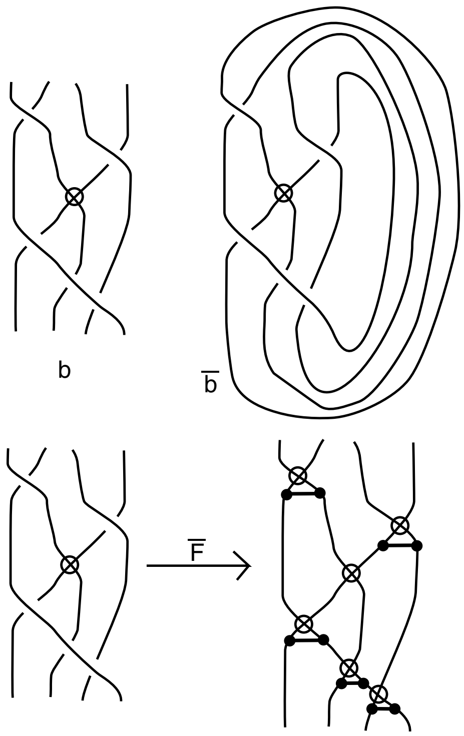


Fig. 29. Virtual braid and closure.

1 ### 6.6. *Hopf algebras and Kirby calculus*

In Fig. 30 we illustrate how one can use this concentration of algebra on the loop in the context of a Hopf algebra that has a right integral. The right integral is a function $\lambda : A \to k$ satisfying

$$\lambda(x)1_A = \Sigma\lambda(x_1)x_2,$$

where the coproduct in the Hopf algebra has formula $\Delta(x) = \Sigma x_1 \otimes x_2$. Here we point out how the use of the coproduct corresponds to doubling the lines in the diagram, and that if one were to associate the function λ with a circle with rotation number one, then the resulting link evaluation will be invariant under the so-called Kirby move. The Kirby move replaces two link components with new ones by doubling one component and connecting one of the components of the double with the other component. Under our functor from the virtual tangle category to the category for the Hopf algebra, a knot goes to a circle with algebra concentrated at x. The doubling of the knot goes to concentric circles labeled with the coproduct $\Delta(x) = \Sigma x_1 \otimes x_2$. Figure 30 shows how invariance under the handle-slide in the tangle category corresponds the integral equation

$$\lambda(x)y = \Sigma\lambda(x_1)x_2y.$$

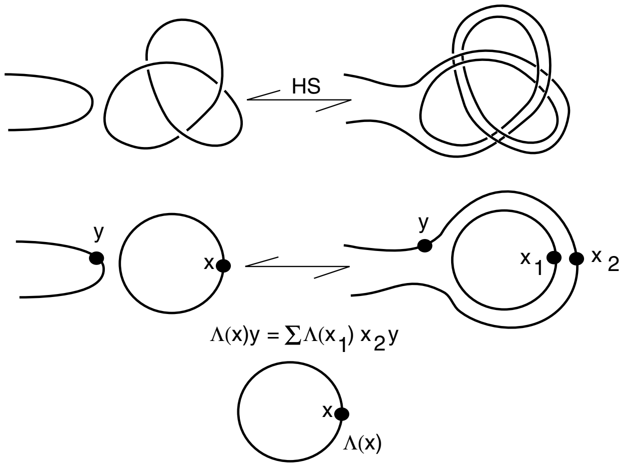


Fig. 30. The Kirby move.

It turns out that classical framed links L have an associated compact oriented three manifold $M(L)$ and that two links related by Kirby moves have homeomorphic three-manifolds. Thus the evaluation of links using the right integral yields invariants of three-manifolds. Generalizations to virtual three-manifolds are under investigation [7]. We only sketch this point of view here, and refer the reader to [23].

6.7. *Hopf algebra*

This section is added for reference about Hopf algebras. Quasitriangular Hopf algebras are an important special case of the quantum algebras discussed in this section.

Recall that a *Hopf algebra* [34] is a bialgebra A over a commutative ring k that has an associative multiplication $m : A \otimes A \to A$, and a coassociative comultiplication, and is equipped with a counit, a unit and an antipode. The ring k is usually taken to be a field. The associative law for the multiplication m is expressed by the equation

$$m(m \otimes 1_A) = m(1_A \otimes m),$$

where 1_A denotes the identity map on A.

The coproduct $\Delta : A \to A \otimes A$ is an algebra homomorphism and is coassociative in the sense that

$$(\Delta \otimes 1_A)\Delta = (1_A \otimes \Delta)\Delta.$$

The *unit* is a mapping from k to A taking 1_k in k to 1_A in A and, thereby, defining an action of k on A. It will be convenient to just identify the 1_k in k and the 1_A in A, and to ignore the name of the map that gives the unit.

The counit is an algebra mapping from A to k denoted by $\epsilon: A \to k$. The following formula for the counit dualize the structure inherent in the unit:

$$(\epsilon \otimes 1_A)\Delta = 1_A = (1_A \otimes \epsilon)\Delta.$$

It is convenient to write formally

$$\Delta(x) = \sum x_1 \otimes x_2 \in A \otimes A$$

to indicate the decomposition of the coproduct of x into a sum of first and second factors in the two-fold tensor product of A with itself. We shall often drop the summation sign and write

$$\Delta(x) = x_1 \otimes x_2.$$

The antipode is a mapping $s: A \to A$ satisfying the equations

$$m(1_A \otimes s)\Delta(x) = \epsilon(x)1_A \quad \text{and} \quad m(s \otimes 1_A)\Delta(x) = \epsilon(x)1_A.$$

It is a consequence of this definition that $s(xy) = s(y)s(x)$ for all x and y in A.

A *quasitriangular Hopf algebra* [5] is a Hopf algebra A with an element $\rho \in A \otimes A$ satisfying the following conditions:

(1) $\rho\Delta = \Delta'\rho$ where Δ' is the composition of Δ with the map on $A \otimes A$ that switches the two factors.

(2)

$$\rho_{13}\rho_{12} = (1_A \otimes \Delta)\rho,$$
$$\rho_{13}\rho_{23} = (\Delta \otimes 1_A)\rho.$$

The symbol ρ_{ij} denotes the placement of the first and second tensor factors of ρ in the i and j places in a triple tensor product. For example, if $\rho = \sum e \otimes e'$ then

$$\rho_{13} = \sum e \otimes 1_A \otimes e'.$$

Conditions (1) and (2) above imply that ρ has an inverse and that

$$\rho^{-1} = (1_A \otimes s^{-1})\rho = (s \otimes 1_A)\rho.$$

It follows easily from the axioms of the quasitriangular Hopf algebra that ρ satisfies the Yang–Baxter equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

A less obvious fact about quasitriangular Hopf algebras is that there exists an element u such that u is invertible and $s^2(x) = uxu^{-1}$ for all x in A. In fact, we may take $u = \sum s(e')e$ where $\rho = \sum e \otimes e'$. This result, originally due to Drinfeld [5], follows from the diagrammatic categorical context of this paper.

An element G in a Hopf algebra is said to be *grouplike* if $\Delta(G) = G \otimes G$ and $\epsilon(G) = 1$ (from which it follows that G is invertible and $s(G) = G^{-1}$). A quasitriangular Hopf algebra is said to be a *ribbon Hopf algebra* [23, 30] if there exists a grouplike element G such that (with u as in the previous paragraph) $v = G^{-1}u$ is in the center of A and $s(u) = G^{-1}uG^{-1}$. We call G a special grouplike element of A.

Since $v = G^{-1}u$ is central, $vx = xv$ for all x in A. Therefore $G^{-1}ux = xG^{-1}u$. We know that $s^2(x) = uxu^{-1}$. Thus $s^2(x) = GxG^{-1}$ for all x in A. Similarly, $s(v) = s(G^{-1}u) = s(u)s(G^{-1}) = G^{-1}uG^{-1}G = G^{-1}u = v$. Thus, the square of the antipode is represented as conjugation by the special grouplike element in a ribbon Hopf algebra, and the central element $v = G^{-1}u$ is invariant under the antipode.

This completes the summary of Hopf algebra properties that are relevant to the last section of the paper.

Acknowledgments

This research has been co-financed by the European Union (European Social Fund — ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) — Research Funding Program: THALIS. Moreover, both authors were partially supported by UIC, NTUA and MFO.

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