MARKOV TRACE ON THE ALGEBRA OF BRAIDS AND TIES

FRANCESCA AICARDI AND JESÚS JUYUMAYA

Abstract. We prove that the so–called *algebra of braids and ties* supports a Markov trace. Further, by using this trace in the *Jones recipe* we define invariant polynomials for classical knots and singular knots. Our invariants have three parameters. The invariant for classical knots is an extension of the Homflypt polynomial and the invariant for singular knots is an extension of an invariant of singular knots defined by the second author and S. Lambropoulou.

1. Introduction

The algebra of braids and ties (defined by generators and relations) firstly appeared in [13], having the purpose of constructing new representations of the Braid group. The first author observed that the definition had a redundant relation and provided a graphical interpretation of the generators and relations in terms of braids and ties. In [2] we have investigated this algebra, showing in particular that it is finite dimensional and discussing the representation theory in low dimension.

Let n be a positive integer. The algebra of braids and ties with parameter u is denoted $\mathcal{E}_n(u)$. Its generators can be regarded as elements of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$[14]. Indeed, the defining relations of $\mathcal{E}_n(u)$ come out by imposing the commutation relations of the braid generators of $\mathrm{Y}_{d,n}(u)$ with certain idempotents in $\mathrm{Y}_{d,n}(u)$ appearing in the square of the braid generators, see subsection 3.2.

The algebra $\mathcal{E}_n(u)$ was studied by S. Ryom–Hansen in [21]. He constructs a faithful tensorial representation (Jimbo–type) of this algebra which is used to classify the irreducible representations of $\mathcal{E}_n(u)$. Notably he constructed a basis, showing that the dimension of the algebra is $b_n n!$, where b_n denotes the n–th Bell number. This basis plays a crucial role here to prove that $\mathcal{E}_n(u)$ supports a Markov trace. Also, the algebra was considered by E. Banjo in her Ph. D. thesis, see [3]. She has related $\mathcal{E}_n(u)$ to the ramified partition algebra [19]. More precisely, E. Banjo has shown an explicit isomorphism among the specialized algebra $\mathcal{E}_n(1)$ and a small ramified partition algebra; by using this isomorphism she determines the complex generic representation of $\mathcal{E}_n(u)$.

Looking at the graphical interpretation of the generators $\mathcal{E}_n(u)$ ([2]) it is natural trying to define an invariant of knots through the same mechanism (Jones recipe) defining the famous

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Homflypt polynomial [12]. To do that it is necessary to have a Markov trace on $\mathcal{E}_n(u)$. Since the algebra $\mathcal{E}_n(u)$ was provided with a basis by Ryom–Hansen, a first attempt was to define a trace by the same inductive method used to define the Ocneanu trace on the Hecke algebras, that is, by constructing an isomorphism between the algebra at level n and a direct sum of algebras at lower levels, for details see the proof [12, Theorem 5.1]. Unfortunately, we cannot reproduce this method in our situation because the Ryom–Hansen basis cannot be defined - at least in a simple way - inductively. We have then adopted successfully the method of *relative traces* [6, 20], using as main reference the work of M. Chlouveraki and L. Poulain d'Andecy [6, Section 5], where it is proved that certain affine and cyclotomic Yokonuma–Hecke algebras support a Markov trace. Others works where the method of relative traces appears are [20, 9, 10, 11], but we don't know who was the creator of this method.

In this paper we prove that $\mathcal{E}_n(u)$ supports a Markov trace ρ, that depends on two parameters A and B. Then, by using as ingredient ρ in the Jones recipe [12] and a representation of the braid group (respectively, of the braid monoid) in $\mathcal{E}_n(u)$, we have defined an invariant, $\bar{\Delta}$, for classical knots (respectively, $\bar{\Gamma}$, for singular knots), with parameters u, A and B. Since the definitions of these invariants essentially uses the same formula given by Jones to define the Homflypt polynomial, we can see that the specialization $\bar{\Delta}(u, \mathsf{A}, 1)$ is in fact the Homflypt polynomial. Also, for the same reason it is clear that $\bar{\Delta}(u, \mathsf{A}, 1/m)$ (respectively $\bar{\Gamma}(u, \mathsf{A}, 1/m)$), where m is a positive integer, coincides with the invariant of classical knots (respectively singular knots), defined by the second author and S. Lambropoulou in [16] (respectively [15].)

An immediate question is how strong are the invariants here defined. At this point we want to cite the work in progress [5], where the specialization $\bar{\Delta}(u, \mathsf{A}, 1/m)$ of $\bar{\Delta}$ is studied. The computations show that this invariant have several topological meaning on some families of knots, as the Homflypt polynomial; however, up to now we have no general proof for that. Unfortunately, how much the invariants for singular knots $\bar{\Gamma}$ are useful is an open question.

Finally, we shall note that the invariants defined here can be recovered from an invariant for tied knots, see [1]. The tied knots constitute in fact a new class of knots in the Euclidian space whose definition is motivated by the graphical interpretation of $\mathcal{E}_n(u)$ by braid and ties, given in Section 6.

The structure of the paper is as follows. In Section 2 we give the necessary notations and background. Section 3 is devoted to recall the origin and the definition of the algebra of braids and ties. This section starts with a brief recall of the relations of the algebra of braids and ties with the Yokonuma–Hecke algebra; then, in subsection 3.1 and 3.2 we collect some algebraic properties of the new basis for the algebra $\mathcal{E}_n(u)$, mostly coming from [21]. The Section 4 has two subsections. The first one is devoted to the construction of a family of relative traces (Theorem 2) which are used for the construction of the Markov trace on $\mathcal{E}_n(u)$ (Theorem 3). In Section 5 we construct an invariant of classical links (Theorem 4) and an invariant of singular knots (Theorem 5). These invariants can be interpreted, respectively, as a generalization of the Homplypt polynomial and as a generalization of the invariant defined by the second author and S. Lambropoulou, see Subsection 5.3 for details. Section 6 is devoted to recall the diagrammatic interpretation of the defining generators of $\mathcal{E}_n(u)$ given in [2]. We show that writing the defining monomials relations of the generators in terms of diagrams allows to find out new relations. Furthermore, we show that the computations in terms of diagrams, using the elements of the basis by Ryom–Hansen, become more efficient.

2. Notations and background

2.1. Let u be an indeterminate. We denote by K the field of the rational functions $\mathbb{C}(u)$.

As usual we denote by B_n the braid group on n strands. Thus, B_n has the Artin presentation by generators $\sigma_1, \ldots, \sigma_{n-1}$ and the *braid relations*: $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i \in \{1, \ldots, n-2\}$. We assume the braid generators σ_i have positive crossing, represented by the following diagram:

Let S_n be the symmetric group on symbols and s_i the transposition $(i, i+1)$. Recall that every element $w \in S_n$ can be written (uniquely) in the form

$$w = w_1 w_2 \cdots w_{n-1} \tag{1}$$

where $w_i \in \{1, s_i, s_i s_{i-1}, \ldots, s_i s_{i-1} \cdots s_1\}$

2.2. We denote by $\mathbf{n}$ the set $\{1, \ldots, n\}$ and by $\mathsf{P}(\mathbf{n})$ the set formed by the set–partitions of $\mathbf{n}$. The cardinality of $\mathsf{P}(\mathbf{n})$ is called the n–th Bell number.

The pair $(\mathsf{P}(\mathbf{n}), \preceq)$ is a poset, setting, for any $I := (I_1, \ldots, I_r)$, $J := (J_1, \ldots, J_s) \in \mathsf{P}(\mathbf{n})$

$$I \preceq J \quad \text{if and only if each } J_k \text{ is a union of some } I_m\text{'s.}$$

If $I \preceq J$ we shall say that J *contains* I.

For short we shall omit the subsets of cardinality 1 in the partition. For example, the partition $I = (\{1,3\}, \{2\}, \{4\}, \{5\}, \{6\})$ in $\mathsf{P}(\mathbf{6})$, will be simply written as $I = (\{1,3\})$. So, by writing $6 \notin I$, we will mean that I contains the subset $\{6\}$.

The symmetric group S_n acts naturally on $\mathsf{P}(\mathbf{n})$. More precisely, set $I = (I_1, \ldots, I_m) \in \mathsf{P}(\mathbf{n})$. The action $w(I)$ of $w \in S_n$ on I is given by

$$w(I) := (w(I_1), \ldots, w(I_m)) \tag{2}$$

where $w(I_k)$ is the subset of $\mathbf{n}$ obtained by applying w to the set I_k.

If I and J are two set–partitions in $\mathsf{P}(\mathbf{n})$, we denote $I * J$ the minimal set–partition containing I and J. Let J_k be a subset of $\mathbf{n}$. During the work we will use for short $I \sim J_k$ to indicate $I * (J_k)$. So, $I \sim \{j, m\}$ coincides with I if j and m already belong to the same subset of $\mathbf{n}$ in I, otherwise, $I \sim \{j, m\}$ coincides with I except for the two subsets containing j and m, that merge in a sole set. For short, we shall denote by $I \sim j$ the set–partition $I \sim \{j, j+1\}$. For instance, for the set–partition $I = (\{1,2,4\}, \{3,5,6\})$:

$$I \sim \{1,4\} = I \quad \text{and} \quad I \sim 2 = (\{1,2,3,4,5,6\}).$$

Finally, for $I \in \mathsf{P}(\mathbf{n})$, we denote I/n the element in $\mathsf{P}(\mathbf{n-1})$ that is obtained by removing n from I. For example, for the set–partition I of the example above, $I/6 = (\{1,2,4\}, \{3,5\})$.

3. The algebra of braids and ties

3.1. We recall here the definition of the algebra of braids and ties $\mathcal{E}_n(u)$; by algebra we mean a unital associative algebra over K. For short we shall omit u in $\mathcal{E}_n(u)$.

Definition 1. We set $\mathcal{E}_1 = K$ and for every natural $n > 1$ we define $\mathcal{E}_n$ as the algebra generated by $T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ satisfying the following relations:

$$T_iT_j = T_jT_i \quad \text{for all } i,j \text{ such that } |i-j|>1 \tag{3}$$
$$T_iT_jT_i = T_jT_iT_j \quad \text{for all } i,j \text{ such that } |i-j|=1 \tag{4}$$
$$T_i^2 = 1+(u-1)E_i(1+T_i) \quad \text{for all } i \tag{5}$$
$$E_iE_j = E_jE_i \quad \text{for all } i,j \tag{6}$$
$$E_i^2 = E_i \quad \text{for all } i \tag{7}$$
$$E_iT_i = T_iE_i \quad \text{for all } i \tag{8}$$
$$E_iT_j = T_jE_i \quad \text{for all } i,j \text{ such that } |i-j|>1 \tag{9}$$
$$E_iE_jT_i = T_iE_iE_j = E_jT_iE_j \quad \text{for all } i,j \text{ such that } |i-j|=1 \tag{10}$$
$$E_iT_jT_i = T_jT_iE_j \quad \text{for all } i,j \text{ such that } |i-j|=1. \tag{11}$$

Remark 1. The above definition coincides with the original definition of $\mathcal{E}_n$ under the substitution of u with $1/u$ and of T_i with $-T_i$, see [13].

3.2. Behind the definition of the algebra of braids and ties $\mathcal{E}_n$ there is the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n} = \mathrm{Y}_{d,n}(u)$, where d denotes a positive integer. We refer to [18] for the role of this algebra in knot theory and to [7] for its combinatorial representation theory. The algebra $\mathrm{Y}_{d,n}$ can be regarded as a u–deformation of the wreath product the symmetric group S_n and the cyclic group C_d of order d, in an analogous way as the Hecke algebra is a deformation of S_n. More precisely, the algebra $\mathrm{Y}_{d,n}$ is the algebra generated by the braid generators $g_1, \ldots, g_{n-1}$ together with the framing generators $t_1, \ldots, t_n$ which satisfy the following relations: the braids relation (said of type A) among the g_i's, $t_it_j = t_jt_i$, $g_it_j = t_{s_i(j)}g_i$, $t_i^d = 1$ and

$$g_i^2 = 1+(u-1)e_i(1+g_i) \tag{12}$$

where e_i is defined as

$$e_i := \frac{1}{d}\sum_{s=1}^{d} t_i^s t_{i+1}^{-s}$$

Remark 2. Denote by H_n the Hecke algebra of parameter u, that is, the associative K–algebra defined by generators $h_1, \ldots, h_{n-1}$ subject to the braid relations (of type A) among the h_i's and the Hecke quadratic relations $h_i^2 = u + (u-1)h_i$, for all i. We note that for $d = 1$, the algebra $\mathrm{Y}_{d,n}$ is the Hecke algebra, since the elements t_i are trivial, so $e_i = 1$ for all i, and thus (12) becomes the quadratic Hecke relation. It is now clear that the mappings $g_i \mapsto h_i$ and $t_i \mapsto 1$ define an epimorphism from $\mathrm{Y}_{d,n}$ onto H_n. We denote this epimorphism by ϕ_n.

The definition of the bt–algebra is obtained by considering abstractly the K– algebra generated by the g_i's and the e_i's. Then g_i becomes T_i, e_i becomes E_i and the set of the defining relations of the bt–algebra corresponds to the complete minimal set of relations derived from the commuting relation among the g_i's and the e_i's. Thus, in particular, we have the following proposition.

Proposition 1. *There is a natural algebra morphism $\psi_n : \mathcal{E}_n \to \mathrm{Y}_{d,n}$ defined through the mappings $T_i \mapsto g_i$ and $E_i \mapsto e_i$.*

The proof of this proposition follows from the fact that the defining relations of $\mathcal{E}_n$ are satisfied in $\mathrm{Y}_{d,n}$ by the images of the above mappings (cf. Lemma 2.1[15]).

Remark 3. Notice that the composition $\varphi_n := \phi_n \circ \psi_n$, sending $T_i \mapsto h_i$ and $E_i \mapsto 1$, is an epimorphism from $\mathcal{E}_n$ onto H_n.

3.3. In the present subsection we outline some useful relations among the defining relations and some algebraic properties of the bt–algebra that we will use in the sequel.

In the following proposition we list some relations arising directly from the defining relations of $\mathcal{E}_n$. We shall use these relations along the paper mentioning only this proposition.

Proposition 2. *For all i, j, we have:*

(i) *The elements T_i's are invertible. Moreover,*

$$T_i^{-1} = T_i + (u^{-1} - 1)E_i + (u^{-1} - 1)E_iT_i \tag{13}$$

(ii) $T_iT_jT_i^{-1} = T_j^{-1}T_iT_j$, *for* $|i - j| = 1$

(iii) $T_i^3 - uT_i^2 - T_i + u = 0$.

Now, we extract some useful results from [21]. For $i < j$, we define E_{ij} as

$$E_{ij} = \begin{cases} E_i & \text{for} \quad j = i + 1 \\ T_i \cdots T_{j-2}E_{j-1}T_{j-2}^{-1} \cdots T_i^{-1} & \text{otherwise.} \end{cases} \tag{14}$$

For any nonempty subset J of $\mathbf{n}$ we define $E_J = 1$ if $|J| = 1$ and

$$E_J := \prod_{(i,j)\in J\times J, i<j} E_{ij}$$

Note that $E_{\{i,j\}} = E_{ij}$. Also note that in Lemma 4[21] it is proved that E_J can be computed as

$$E_J = \prod_{j\in J,\, j\neq i_0} E_{i_0 j} \quad \text{where} \quad i_0 = min J. \tag{15}$$

In a similar way one proves that E_J can be computed, writing $J = \{j_0, j_1, \ldots, j_m\}$, with $j_i < j_{i+1}$, as

$$E_J = \prod_{i=1}^{m} E_{j_{i-1}j_i}. \tag{16}$$

Moreover, for $I = \{I_1, \ldots, I_m\} \in \mathsf{P}(\mathbf{n})$, we define E_I as

$$E_I = \prod_k E_{I_k}. \tag{17}$$

The action of S_n on $\mathsf{P}(\mathbf{n})$, transferred to the elements E_I, is given by the following formulae

$$T_w E_I T_w^{-1} = E_{w(I)} \qquad \text{(see [21, Corollary 1])} \tag{18}$$

where $w \in S_n$ and $I \in \mathsf{P}(\mathbf{n})$.

If $w = s_{i_1} \cdots s_{i_k} \in S_n$ is a reduced form for w, we define $T_w := T_{i_1} \cdots T_{i_k}$.

Theorem 1 (Corollary 3[21])**.** *The set $\mathcal{B}_n = \{T_wE_I\,;\, w \in S_n,\, I \in \mathsf{P}(\mathbf{n})\}$ is a linear basis of $\mathcal{E}_n$. Hence the dimension of $\mathcal{E}_n$ is $b_n n!$.*

3.4. Since the T_i's satisfy the braid relations and because of (1), we have that for every $w \in S_n$ the element $T_w \in \mathcal{B}_n$ can be written uniquely as

$$T_w = T_{w_1} T_{w_2} \cdots T_{w_{n-1}}$$

where

$$T_{w_i} \in \{1, T_i, T_i T_{i-1}, \ldots, T_i T_{i-1} \cdots T_1\}.$$

Set $\mathbb{T}_{i,0} = 1$ and for $k \in \{1, \ldots, i\}$, define

$$\mathbb{T}_{i,k} = T_i T_{i-1} \cdots T_k.$$

Thus the element of the basis $\mathcal{B}_n$ can be rewritten as

$$\mathbb{T}_{1,k_1} \mathbb{T}_{2,k_2} \cdots \mathbb{T}_{n-1,k_{n-1}} E_I \tag{19}$$

where $k_j \in \{0, \ldots, j\}$, $j \in \{1, \ldots, n-1\}$ and $I \in \mathsf{P}(\mathbf{n})$.

Notation 1. *It is convenient to denote $\check{\mathbb{T}}_{i,k}$ the element obtained by removing T_i from $\mathbb{T}_{i,k}$, that is, $\check{\mathbb{T}}_{i,k} = T_{i-1} \cdots T_k$. Consequently, $\check{\check{\mathbb{T}}}_{i,k} = T_{i-2} \cdots T_k$.*

Using the defining relations of the algebra $\mathcal{E}_n$ we obtain the following useful relations

$$\mathbb{T}_{i,k} T_j = \begin{cases} \mathbb{T}_{i,k+1} + (u-1)\mathbb{T}_{i,k+1} E_k (1 + T_k) & \text{for} \quad j = k \\ \mathbb{T}_{i,j} & \text{for} \quad j = k-1 \\ T_j \mathbb{T}_{i,k} & \text{for} \quad j \in \{1, \ldots, k-2\} \\ T_{j-1} \mathbb{T}_{i,k} & \text{for} \quad j \in \{k+1, \ldots, i\} \end{cases} \tag{20}$$

Moreover, we will use also the following relations, that are obtained using only the braid relations:

$$T_i \mathbb{T}_{i-1,r} \mathbb{T}_{i,s} = \begin{cases} \mathbb{T}_{i-1,s-1} \mathbb{T}_{i,r} & \text{for} \quad 0 < r < s \\ \mathbb{T}_{i-1,r} \mathbb{T}_{i,r} T_r \cdots T_s & \text{for} \quad r \geq s \end{cases} \tag{21}$$

Notice that:

$$\begin{aligned} \mathbb{T}_{i-1,r} \mathbb{T}_{i,r} T_r \cdots T_s \quad = \quad & \mathbb{T}_{i-1,r} \mathbb{T}_{i,r+1} \mathbb{T}_{r-1,s} + (u-1)\mathbb{T}_{i-1,r} \mathbb{T}_{i,r+1} E_r \mathbb{T}_{r-1,s} \\ & + (u-1)\mathbb{T}_{i-1,r} \mathbb{T}_{i,r+1} E_r \mathbb{T}_{r,s} \end{aligned} \tag{22}$$

Also, from (18) we get

$$\mathbb{T}_{i,j} E_I = E_{\theta_{i,j}(I)} \mathbb{T}_{i,j} \quad \text{and} \quad E_I \mathbb{T}_{i,j} = \mathbb{T}_{i,j} E_{\theta_{i,j}^{-1}(I)} \tag{23}$$

where $\theta_{i,j} := s_i s_{i-1} \cdots s_j$.

Let $I \in \mathsf{P}(\mathbf{n})$, and $k < n$. We shall denote

$$\tau_{n,k}(I) := \theta_{n-2,k}^{-1}(\theta_{n-1,k}(I)/n). \tag{24}$$

By a direct computation we get:

$$\tau_{n,k}(I) = (I \sim \{k, n\})/n \tag{25}$$

Observe that:

- if $n \notin I$, then $\tau_{n,k}(I) = I$.
- If k is an element of the same set of the partition I containing n, then $\tau_{n,k}(I) = I/n$.

Examples 1. *Let* $I = (\{1,2,4,6\},\{3,5\}) \in \mathsf{P}(\mathbf{n})$*, then*

$$\tau_{6,1}(I) = (\{1,2,4\},\{3,5\}) \qquad \textit{and} \qquad \tau_{6,3}(I) = (\{1,2,3,4,5\}).$$

If $I = (\{3,5,6\})$ *then*

$$\tau_{6,3}(I) = (\{3,5\}) \qquad \textit{and} \qquad \tau_{6,2}(I) = (\{2,3,5\}).$$

4. Markov trace

In this section we prove that $\mathcal{E}_n$ supports a Markov trace. To do this, we use the method of relative traces taking as main reference [6], (cf. also [9, 10, 11]). Roughly, the method consists in to defining certain linear maps ϱ_n, called relative traces, from $\mathcal{E}_n$ in $\mathcal{E}_{n-1}$, associated to the tower of the algebras

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$$

Then we prove that the composition of these linear maps is indeed the Markov trace desired, see Theorem 3.

4.1. From now on we fix two parameters A and B in K.

Definition 2. *Let* ϱ_n *be the linear map from* $\mathcal{E}_n$ *to* $\mathcal{E}_{n-1}$ *defined on the basis* $\mathcal{B}_n$ *as follows:*

$$\varrho_n(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-1}}E_I) = \begin{cases} \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_I & \textit{for} \quad k_{n-1}=0 \quad n\notin I \\ \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_{I/n} & \textit{for} \quad k_{n-1}=0 \quad n\in I \\ \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I)} & \textit{for} \quad k_{n-1}\neq 0 \end{cases}$$

Notice that ϱ_n acts as the identity on $\mathcal{E}_{n-1}$, hence $\varrho_n(1)=1$, for all n. Note also that, from the definition of the ϱ_n's, it follows that they satisfy the following:

$$\varrho_n(T_{n-1}) = \varrho_n(E_{n-1}T_{n-1}) = \mathsf{A} \tag{26}$$

$$\varrho_n(E_{n-1}) = \mathsf{B}. \tag{27}$$

Moreover, we have the following theorem.

Theorem 2. *The family* $\{\varrho_n\}_{n>1}$ *satisfies, for all* $X, Z \in \mathcal{E}_{n-1}$ *and* $Y \in \mathcal{E}_n$*:*

$$\varrho_n(XYZ) = X\varrho_n(Y)Z \tag{28}$$

$$\varrho_n(T_{n-1}^{\pm 1}XT_{n-1}^{\mp 1}) = \varrho_{n-1}(X) \tag{29}$$

$$\varrho_{n-1}(\varrho_n(T_{n-1}Y)) = \varrho_{n-1}(\varrho_n(YT_{n-1})) \tag{30}$$

Proof. The theorem is proved verifying separately each statement in the Lemmas 1–3 below. □

Lemma 1. *For all* $X, Z \in \mathcal{E}_{n-1}$ *and* $Y \in \mathcal{E}_n$*, we have:*

(i) $\varrho_n(YZ) = \varrho_n(Y)Z$
(ii) $\varrho_n(XY) = X\varrho_n(Y)$
(iii) $\varrho_n(XYZ) = X\varrho_n(Y)Z$.

Proof. From the linearity of ϱ_n, it follows that it is enough to prove the lemma when $Y \in \mathcal{B}_n$ and X, Z are the generators $T_1,\ldots,T_{n-2}$ and $E_1,\ldots,E_{n-2}$. We set along the proof of the lemma:

$$Y = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-1}}E_I.$$

We start with the case in which Z is one of the generators T_j, with $j \in \{1, \dots, n-2\}$. We have

$$YZ = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}T_jE_{s_j(I)} \tag{31}$$

We shall distinguish now three cases, labeled below as Cases I, II and III.

Case I: $k_{n-1} = 0$.

In the case $n \notin I$, the claim follows since ϱ_n acts as the identity. For the case $n \in I$, we have:

$$\varrho_n(Y)Z = \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_{I/n}T_j = \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}T_jE_{s_j(I/n)}$$

On the other hand, the expression (31) of YZ can be written as a linear combination of elements of the form $WE_{s_j(I)}$ with $W \in \mathcal{B}_{n-1}$. Then, $\varrho_n(YZ) = \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}T_jE_{s_j(I)/n}$. Since s_j does not touch n, it follows that $s_j(I/n) = s_j(I)/n$, hence $\varrho_n(Y)Z = \varrho_n(YZ)$.

Case II: $k_{n-1} \neq 0$ and $n \notin I$.

Now, according to the commutation rules given in (20), we shall distinguish four subcases.

$*$ Subcase $j = k_{n-1} - 1$. We have

$$YZ = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,j}E_{s_j(I)}. \tag{32}$$

Since $n \notin s_j(I)$, according to the definition 2

$$\varrho_n(YZ) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j}E_{s_j(I)},$$

which is equal to $\varrho_n(Y)Z$. Indeed,

$$\varrho_n(Y)Z = (\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_I)T_j = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j}E_{s_j(I)}.$$

$*$ Subcases $j < k_{n-1} - 1$ and $k_{n-1} + 1 \leq j \leq n-1$ are totally analogous to the subcase above.

$*$ Subcase $j = k_{n-1}$. We have $\varrho_n(Y)Z = \varrho_n(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-1}}E_I)T_j$. Then

$$\begin{aligned}\varrho_n(Y)Z &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}T_jE_{s_j(I)}\\ &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}T_j^2E_{s_j(I)}\end{aligned}$$

By splitting T_j^2, we obtain $\varrho_n(Y)Z = W_1 + W_2 + W_3$, where

$$\begin{aligned}W_1 &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_{s_j(I)}\\ W_2 &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_jE_{s_j(I)}\\ W_3 &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}T_jE_jE_{s_j(I)}\end{aligned}$$

On the other hand:

$$\begin{aligned}YZ &= \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}(\mathbb{T}_{n-1,j+1} + (u-1)\mathbb{T}_{n-1,j+1}E_j(1+T_j))E_{s_j(I)}\\ &= W_1' + W_2' + W_3'\end{aligned}$$

where

$$\begin{aligned}W_1' &:= \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,j+1}E_{s_j(I)}\\ W_2' &:= (u-1)\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,j+1}E_jE_{s_j(I)}\\ W_3' &:= (u-1)\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,j+1}T_jE_jE_{s_j(I)}\end{aligned}$$

Now we observe that $W_i = \varrho_n(W_i')$. Therefore $\varrho_n(Y)Z = \varrho_n(YZ)$.

Case III: $k_{n-1} \neq 0$ and $n \in I$.

Again, we will prove the claim using formulae (20). Suppose $j = k_{n-1}-1$. Using the definition 2, we get

$$\varrho_n(YZ) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j}E_{\tau_{n,j}(s_j(I))}$$

where $\tau_{n,j}(s_j(I)) = (s_j(I) \sim \{j,n\})/n$, and

$$\varrho_n(Y) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j+1}E_{\tau_{n,j+1}(I)}$$

where

$$\tau_{n,j+1}(I) = (I \sim \{j+1,n\})/n. \tag{33}$$

Observe that, since $j < n-1$, $\tau_{n,j+1}(I) = s_{n-1}\tau_{n,j}(s_j(I))$. Therefore we have

$$\begin{aligned}
\varrho_n(YZ) &= \mathsf{A}(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j}E_{\tau_{n,j}(s_j(I))}) \\
&= \mathsf{A}(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j+1}T_jE_{\tau_{n,j}(s_j(I))}) \\
&= \mathsf{A}(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,j+1}E_{\tau_{n,j+1}(I)})T_j \\
&= \varrho_n(Y)Z.
\end{aligned}$$

The cases $j < k_{n-1}-1$ and $k_{n-1}+1 \le j \le n-1$ are verified in analogous way.

Suppose now $j = k_{n-1}$. We have

$$YZ = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,j}T_jE_{s_j(I)}$$

and

$$\varrho_n(YZ) = \varrho_n(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,j+1}T_j^2E_{s_j(I)}) = V_1+V_2+V_3$$

being

$$\begin{aligned}
V_1 &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_{\tau_{n,j+1}(s_j(I))} \\
V_2 &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_{\tau_{n,j+1}(s_j(I))}E_j \\
V_3 &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}T_jE_{\tau_{n,j}(s_j(I))}E_j
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varrho_n(Y)Z &= \mathsf{A}(\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j}E_{\tau_{n,j}(I)})T_j \\
&= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}T_j^2E_{s_j(\tau_{n,j}(I))}.
\end{aligned}$$

Splitting T_j^2, we obtain $\varrho_n(Y)Z = V_1'+V_2'+V_3'$

$$\begin{aligned}
V_1' &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_{s_j(\tau_{n,j}(I))} \\
V_2' &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}E_jE_{s_j(\tau_{n,j}(I))} \\
V_3' &= (u-1)\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,j+1}T_jE_jE_{s_j(\tau_{n,j}(I))}.
\end{aligned}$$

We have therefore to verify that $V_i = V_i'$, $i = 1,2,3$. $V_1' = V_1$ and $V_2' = V_2$ since

$$s_j(\tau_{n,j}(I)) = \tau_{n,j+1}(s_j(I)).$$

As for V_3', we have

$$E_jE_{s_j(\tau_{n,j}(I))} = E_{s_j(\tau_{n,j}(I))\sim\{j,j+1\}},$$

and

$$s_j(\tau_{n,j}(I)) \sim \{j,j+1\} = s_j((I \sim \{j,n\})/n) \sim \{j,j+1\}.$$

This partition is the same as that in the expression of V_3, namely

$$\tau_{n,j}(s_j(I)) \sim \{j, j+1\} = ((s_j(I) \sim \{j, n\})/n) \sim \{j, j+1\},$$

since $j < n-1$. Thus we have also $V_3 = V_3'$.

To finish the proof of (i) it remains only to consider the case when $Z = E_j$. We have

$$YE_j = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_{I\sim j}. \tag{34}$$

Observe that $(I/n) \sim j = (I \sim j)/n$, because $j < n$. Applying the Definition 2, we get in all cases $\varrho_n(Y)Z = \varrho_n(YZ)$ since at the end of the left and right sides we have respectively $E_{(I\sim j)/n}$ and $E_{(I/n)\sim j}$.

Now we prove the claim (ii) of the lemma. In the case $k_{n-1} = 0$ and $n \notin I$ the claim is evident, since $Y \in \mathcal{E}_{n-1}$ and ϱ_n acts as the identity on $\mathcal{E}_{n-1}$.

In the case $k_{n-1} = 0$ and $n \in I$, we have $Y = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_I$. Then

$$X\varrho_n(Y) = X\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_{I/n}$$

Now, to compute $\varrho_n(XY)$, we need to express XY as linear combination of elements of the basis $\mathcal{B}_n$, but in the case we are it is enough to express $X' := X\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}$ as linear combination of elements of $\mathcal{B}_{n-1}$, and then to put the element E_I on the right of each terms of this linear combination. Thus, $\varrho_n(XY)$ is the linear combination obtained from the linear combination expressing X', by putting on the right of each term the factor $E_{I/n}$. Hence, we deduce that $X\varrho_n(Y) = \varrho_n(XY)$.

Suppose now that $k_{n-1} \neq 0$. We check firstly the claim for $X = T_m$, where $m \in \{1, \ldots, n-2\}$.

We have $X\varrho_n(Y) = \mathsf{A}T_m\mathbb{T}_{1,k_1}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I)}$. We rewrite it as

$$X\varrho_n(Y) = \mathsf{A}\mathbb{A}(T_m\mathbb{T}_{m-1,r}\mathbb{T}_{m,s})\mathbb{B} \tag{35}$$

where

$$\begin{aligned}\mathbb{A} &= \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{m-2,k_{m-2}}\\ \mathbb{B} &= \mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I)}\end{aligned}$$

On the other hand, we have

$$XY = \mathbb{A}(T_m\mathbb{T}_{m-1,r}\mathbb{T}_{m,s})\mathbb{B}' \tag{36}$$

where $0 \le r \le m-1$, $0 < s \le m$ and

$$\mathbb{B}' = \mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_I.$$

We will compare now $\varrho_n(XY)$ with $X\varrho_n(Y)$, distinguishing the cases $r = 0$ and $r \neq 0$.

Case $r \neq 0$. By using (21) and later (22) we deduce:

$$X\varrho_n(Y) = \begin{cases} \mathsf{A}\mathbb{A}R\mathbb{B} & \text{for} \quad 0 < r \le s \\ \mathsf{A}\mathbb{A}S_1\mathbb{B} + (u-1)\mathsf{A}\mathbb{A}S_2\mathbb{B} + (u-1)\mathsf{A}\mathbb{A}S_3\mathbb{B} & \text{for} \quad s < r \end{cases}$$

where

$$\begin{aligned} R &:= \mathbb{T}_{m-1,s-1}\mathbb{T}_{m,r} \\ S_1 &:= \mathbb{T}_{m-1,r}\mathbb{T}_{m,r+1}\mathbb{T}_{r-1,s} = \mathbb{T}_{m-1,s}\mathbb{T}_{m,r+1} \\ S_2 &:= \mathbb{T}_{m-1,r}\mathbb{T}_{m,r+1}E_r\mathbb{T}_{r-1,s} = \mathbb{T}_{m-1,s}\mathbb{T}_{m,r+1}E_{\{a,b\}} \\ S_3 &:= \mathbb{T}_{m-1,r}\mathbb{T}_{m,r+1}T_rE_r\mathbb{T}_{r-1,s} = \mathbb{T}_{m-1,r}\mathbb{T}_{m,s}E_{\{a,b\}} \end{aligned}$$

being $\{a,b\} = \theta^{-1}_{r-1,s}(\{r,r+1\})$.

Now, by using again (21) and later (22), we get:

$$XY = \begin{cases} \mathbb{A}R\mathbb{B}' & \text{for} \quad 0 < r \leq s \\ \mathbb{A}S_1\mathbb{B}' + (u-1)\mathbb{A}S_2\mathbb{B}' + (u-1)\mathbb{A}S_3\mathbb{B}' & \text{for} \quad s < r \end{cases}$$

Then

$$\varrho_n(XY) = \begin{cases} \mathsf{A}\mathbb{A}R\mathbb{B} & \text{for} \quad 0 < r \leq s \\ \varrho_n(\mathbb{A}S_1\mathbb{B}') + (u-1)\varrho_n(\mathbb{A}S_2\mathbb{B}') + (u-1)\varrho_n(\mathbb{A}S_3\mathbb{B}') & \text{for} \quad s < r \end{cases}$$

Clearly $\mathsf{A}\mathbb{A}S_1\mathbb{B} = \varrho_n(\mathbb{A}S_1\mathbb{B}')$. Now, using (23), we obtain

$$\mathbb{A}S_2\mathbb{B}' \quad = \quad \mathbb{A}(\mathbb{T}_{m-1,s}\mathbb{T}_{m,r+1})\mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_{\{a',b'\}}E_I$$

where $\{a',b'\} := \theta^{-1}_{n-1,k_{n-1}}\cdots\theta^{-1}_{m+1,k_{m+1}}(\{a,b\})$. Now, we have,

$$\varrho_n(\mathbb{A}S_2\mathbb{B}') = \mathsf{A}\mathbb{A}(\mathbb{T}_{m-1,s}\mathbb{T}_{m,r+1})\mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I\sim\{a'b'\})},$$

that is equal to $\mathsf{A}\mathbb{A}S_2\mathbb{B}$ if

$$E_{\{a,b\}}\mathbb{B} = \mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I\sim\{a'b'\})}.$$

But

$$E_{\{a,b\}}\mathbb{B} = \mathbb{B}E_{\{a'',b''\}} = \mathbb{T}_{m+1,k_{m+1}}\cdots\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{(\tau_{n,k_{n-1}}(I))\sim\{a'',b''\}},$$

where $\{a'',b''\} = \theta^{-1}_{n-2,k_{n-1}}\cdots\theta^{-1}_{m+1,k_{m+1}}(\{a,b\})$. Therefore we have to check that

$$(\tau_{n,k_{n-1}}(I)) \sim \{a'',b''\} = \tau_{n,k_{n-1}}(I \sim \{a',b'\}).$$

Observe that $\{a'',b''\} = \tau_{n,k_{n-1}}\{a',b'\}$ since

$$\{a'',b''\} = \theta^{-1}_{n-2,k_{n-1}}\theta^{-1}_{n-1,k_{n-1}}\{a',b'\}.$$

Therefore we can rewrite the set–partition

$$(\tau_{n,k_{n-1}}(I)) \sim \{a'',b''\} = ((I \sim \{n,k_{n-1}\})/n) \sim ((\{a',b'\} \sim \{n,k_{n-1}\})/n)$$

that is clearly the same set–partition as

$$\tau_{n,k_{n-1}}(I \sim \{a',b'\}) = ((I \sim \{a',b'\}) \sim \{n,k_{n-1}\})/n.$$

In a similar way we check that $\varrho_n(\mathbb{A}S_3\mathbb{B}') = \mathsf{A}\mathbb{A}S_3\mathbb{B}$. Therefore, $X\varrho_n(Y) = \varrho(XY)$ whenever $r \neq 0$.

Case $r = 0$. We have that (35) becomes $\mathsf{A}\mathbb{A}T_m\mathbb{T}_{m,s}\mathbb{B} = \mathsf{A}\mathbb{A}T_m^2\mathbb{T}_{m-1,s}\mathbb{B}$. So,

$$T_m\varrho_n(Y) = \mathsf{A}\mathbb{A}\mathbb{T}_{m-1,s}\mathbb{B} + (u-1)\mathsf{A}\mathbb{A}E_m\mathbb{T}_{m-1,s}\mathbb{B} + (u-1)\mathsf{A}\mathbb{A}E_m\mathbb{T}_{m,s}\mathbb{B}$$

Now, (36) becomes $\mathbb{A}(T_m\mathbb{T}_{m,s})\mathbb{B}' = \mathbb{A}(T_m^2\mathbb{T}_{m-1,s})\mathbb{B}'$. Then

$$T_mY = \mathbb{A}\mathbb{T}_{m-1,s}\mathbb{B}' + (u-1)\mathbb{A}(E_m\mathbb{T}_{m-1,s})\mathbb{B}' + (u-1)\mathbb{A}(E_m\mathbb{T}_{m,s})\mathbb{B}'$$

The equality $\varrho_n(T_mY) = T_m\varrho_n(Y)$ is thus obtained as in the previous case comparing the three terms in both members of the equality.

Finally we check the case (ii) when $X = E_m$, with $1 \le m \le n-2$. Let

$$Y = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-1}}E_I.$$

First case: $k_{n-1} = 0$. Since $Y \in \mathcal{E}_n$, $I/n \neq I$. We have

$$XY = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_IE_{\{a,b\}}$$

where $\{a,b\} = (\theta_{1,k_1}\theta_{2,k_2}\cdots\theta_{n-2,k_{n-2}})^{-1}(\{m,m+1\})$. Moreover $E_IE_{\{a,b\}} = E_{I\sim\{a,b\}}$.

Therefore,

$$\varrho_n(XY) = \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_{(I\sim\{a,b\})/n}$$

On the other hand, we have

$$\begin{aligned} X\varrho_n(Y) &= E_m\mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-2,k_{n-2}}E_{I/n} \\ &= \mathsf{B}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-2,k_{n-1}}E_{(I/n)\sim\{a,b\}} \end{aligned}$$

Since $m \le n-2$, a and b cannot be higher than $n-1$, therefore $(I\sim\{a,b\})/n = (I/n)\sim\{a,b\}$, so that we get $\varrho_n(XY) = X\varrho_n(Y)$.

Second case: $k_{n-1} \neq 0$. We have

$$XY = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-1}}E_{I\sim\{a,b\}}$$

where $\{a,b\} = (\theta_{1,k_1}\theta_{2,k_2}\cdots\theta_{n-1,k_{n-1}})^{-1}(\{m,m+1\})$ and $E_{I\sim\{a,b\}} = E_IE_{\{a,b\}}$.

Therefore,

$$\varrho_n(XY) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I\sim\{a,b\})}.$$

On the other hand, we have

$$\begin{aligned} X\varrho_n(Y) &= E_m\mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I)} \\ &= \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(I)\sim\{c,d\}} \end{aligned}$$

where $\{c,d\} = (\theta_{1,k_1}\theta_{2,k_2}\cdots\theta_{n-2,k_{n-1}})^{-1}(\{m,m+1\})$. Now, $\varrho_n(XY) = X\varrho_n(Y)$ if the two partitions $\tau_{n,k_{n-1}}(I\sim\{a,b\})$ and $\tau_{n,k_{n-1}}(I)\sim\{c,d\}$ are equal, i.e., if

$$((I\sim\{a,b\})\sim\{k_{n-1},n\})/n = ((I\sim\{k_{n-1},n\})/n)\sim\{c,d\}. \tag{37}$$

But we have

$$\{c,d\} = \theta_{n-1,k_{n-2}}^{-1}(\theta_{n-1,k_{n-1}}\{a,b\})$$

Then $\{c,d\} = \tau_{n,k_{n-1}}(\{a,b\})$, i.e.,

$$\{c,d\} = (\{a,b\}\sim\{k_{n-1},n\})/n.$$

and therefore the two partitions in (37) coincide.

The proof of (iii) follows immediately after that we have proved (i) and (ii). □

Lemma 2. *For all $X \in \mathcal{E}_{n-1}$, we have:*

(i) $\varrho_n(T_{n-1}^{-1} X T_{n-1}) = \varrho_{n-1}(X)$
(ii) $\varrho_n(T_{n-1} X T_{n-1}^{-1}) = \varrho_{n-1}(X)$

Proof. From the linearity of ϱ_n, it is enough to consider X in the basis $\mathcal{B}_{n-1}$. Let

$$X = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_J \in \mathcal{B}_{n-1}$$

where $k_{n-2} \neq 0$ otherwise the statements are trivial.

We prove now (i). To simplify the calculations, let's write X as

$$X = (\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}})T_{n-2}\check{\mathbb{T}}_{n-2,k_{n-2}}E_J = X' T_{n-2}\check{\mathbb{T}}_{n-2,k_{n-2}}E_J$$

where $X' := \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}}$. Then we write the set–partition J as $\check{J} \sim \{m, n-1\}$, so that $E_J = E_{m,n-1}E_{\check{J}}$. Then we rewrite

$$X = X'\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}E_{\check{J}}$$

so that, by Lemma 1,

$$\varrho_{n-1}(X) = X'\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1})E_{\check{J}}$$

We have to compare it with

$$\varrho_n(T_{n-1}^{-1}XT_{n-1}) = X'\varrho_n(T_{n-1}^{-1}\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}T_{n-1})E_{\check{J}}.$$

Therefore, we have to prove the equality

$$\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}) = \varrho_n(T_{n-1}^{-1}\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}T_{n-1}) \tag{38}$$

The left member of (38) is equal to

$$\mathsf{A}\check{\mathbb{T}}_{n-2,k_{n-2}}E_{\tau_{n-1,k_{n-2}}\{m,n-1\}} = \mathsf{A}\check{\mathbb{T}}_{n-2,k_{n-2}}E_{m,k_{n-2}}$$

The right member of (38) can be calculated, using Lemma 1. Firstly, we write

$$T_{n-1}^{-1}\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}T_{n-1} = T_{n-1}^{-1}T_{n-2}T_{n-1}\,\check{\mathbb{T}}_{n-2,k_{n-2}}E_{m,n},$$

then we have:

$$\begin{aligned}
\varrho_n(T_{n-1}^{-1}\mathbb{T}_{n-2,k_{n-2}}E_{m,n-1}T_{n-1}) &= \varrho_n(T_{n-1}^{-1}T_{n-2}T_{n-1})\,\check{\mathbb{T}}_{n-2,k_{n-2}}E_{m,n}\\
&= \varrho_n(T_{n-2}T_{n-1}T_{n-2}^{-1}E_{m,n})\check{\mathbb{T}}_{n-2,k_{n-2}}\\
&= T_{n-2}\,\varrho_n(T_{n-1}E_{m,n})\,T_{n-2}^{-1}\check{\mathbb{T}}_{n-2,k_{n-2}}\\
&= T_{n-2}\,\mathsf{A}\,E_{m,n-1}T_{n-2}^{-1}\check{\mathbb{T}}_{n-2,k_{n-2}}\\
&= \mathsf{A}\,E_{m,n-2}\check{\mathbb{T}}_{n-2,k_{n-2}} = \mathsf{A}\check{\mathbb{T}}_{n-2,k_{n-2}}E_{m,k_{n-2}}.
\end{aligned}$$

since $\theta_{n-3,k_{n-2}}^{-1}(\{m,n-2\}) = \{m,k_{n-2}.\}$. The proof is concluded

The proof of (ii) is essentially the same.

□

Lemma 3. *For all $X \in \mathcal{E}_n$, we have:*

(i) $\varrho_{n-1}(\varrho_n(E_{n-1}X)) = \varrho_{n-1}(\varrho_n(XE_{n-1}))$.
(ii) $\varrho_{n-1}(\varrho_n(T_{n-1}X)) = \varrho_{n-1}(\varrho_n(XT_{n-1}))$

Proof. Without loss of generality, we can suppose $X \in \mathcal{B}_n$. Set

$$X = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_J$$

We prove first the claim (i). Invoking Lemma 1, we get

$$\begin{aligned}\varrho_{n-1}(\varrho_n(E_{n-1}X)) &= \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}}\varrho_{n-1}(\varrho_n(E_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_J))\\ \varrho_{n-1}(\varrho_n(XE_{n-1})) &= \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}}\varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_{n-1}E_J))\end{aligned}$$

Thus, it is enough to prove that $E = F$, where

$$\begin{aligned}E &:= \varrho_{n-1}(\varrho_n(E_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_J))\\ F &:= \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}E_{n-1}E_J)).\end{aligned}$$

To do that, we consider four cases, distinguishing if k_{n-1} and k_{n-2} vanish or not. In the case $k_{n-1} = k_{n-2} = 0$ it is evident that $E = F$.

Case $k_{n-1} = 0$ **and** $k_{n-2} \neq 0$**.** We have

$$F = \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}E_{n-1}E_J)) = \mathsf{B}\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{(J\sim(n-1))/n}))$$

On the other part

$$\begin{aligned}E = \varrho_{n-1}(\varrho_n(E_{n-1}\mathbb{T}_{n-2,k_{n-2}}E_J)) &= \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}E_{\theta^{-1}_{n-2,k_{n-2}}(\{n-1,n\})}E_J))\\ &= \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}\varrho_n(E_{\theta^{-1}_{n-2,k_{n-2}}(\{n-1,n\})}E_J))\end{aligned}$$

Now, we have $\theta^{-1}_{n-2,k_{n-2}}(\{n-1,n\}) = \{k_{n-2},n\}$. So, we get

$$\varrho_n(E_{\{k_{n-2},n\}}E_J) = \mathsf{B}E_{(J\sim\{k_{n-2},n\})/n}$$

In the case in which none of the sets of the partition J contain n, evidently:

$$(J\sim\{n-1,n\})/n = (J\sim\{k_{n-2},n\})/n = J,$$

so that $E = F$. In the case in which J contains a set $\{a,\dots,n\}$, i.e. $J = (\check{J},\{a,\dots,n\})$,

$$\begin{aligned}(J\sim\{n-1,n\})/n &= \{\check{J}\sim\{a,\dots,n-1\}\} := J_1\\ (J\sim\{k_{n-2},n\})/n &= \{\check{J}\sim\{a,\dots,k_{n-2}\}\} := J_2\end{aligned}$$

Now:

$$F = B(\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{J_1})$$

and

$$E = B(\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{J_2}).$$

We have, for $i = 1, 2$.

$$\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}E_{J_i}) = A\check{\mathbb{T}}_{n-2,k_{n-2}}E_{J_i'},$$

where $J_i' = \tau_{n-1,k_{n-2}}(J_i) = (J_i\sim\{n-1,k_{n-2}\})/(n-1)$. Evidently $J_1' = J_2'$, since J_1 and J_2 are identical up to the transposition of $(n-1)$ with k_{n-1}. Therefore $F = E$.

Case $k_{n-1} \neq 0$ **and** $k_{n-2} = 0$**.** We have

$$F = \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-1,k_{n-1}}E_{n-1}E_J)) \quad\text{and}\quad E = \varrho_{n-1}(\varrho_n(E_{n-1}\mathbb{T}_{n-1,k_{n-1}}E_J))$$

But, $E_{n-1}\mathbb{T}_{n-1,k_{n-1}}E_J = \mathbb{T}_{n-1,k_{n-1}}E_{\{k_{n-1},n\}}E_J$, since $\theta^{-1}_{n-1,k_{n-1}}(\{n-1,n\}) = \{k_{n-1},n\}$. Then

$$E = \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-1,k_{n-1}}E_{J\sim\{k_{n-1},n\}})) = \varrho_{n-1}(\mathsf{A}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{J_1})$$

and

$$F = \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-1,k_{n-1}} E_{J\sim\{(n-1),n\}})) = \varrho_{n-1}(\mathsf{A}\,\check{\mathbb{T}}_{n-1,k_{n-1}} E_{J_2}),$$

where $J_1 = (J \sim \{k_{n-1}, n\})/n$ and $J_2 = ((J \sim \{(n-1), n\}) \sim \{k_{n-1}, n\})/n$.

Thus E and F can be written as follows (for $i = 1$ and $i = 2$ respectively)

$$\mathsf{A}\,\varrho_{n-1}(\check{\mathbb{T}}_{n-1,k_{n-1}} E_{J_i}) = \mathsf{A}^2\,\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} E_{J_i'},$$

where $J_i' = (J_i \sim \{k_{n-1}, (n-1)\})/(n-1)$.

Thus the equality $E = F$ follows, as in the preceding case, from the fact that $J_1' = J_2'$.

Case $k_{n-1} \neq 0$ and $k_{n-2} \neq 0$. From Lemma 1, we get $F = \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \varrho_n(\mathbb{T}_{n-1,k_{n-1}} E_{n-1} E_J))$. Then

$$F = \mathsf{A}\,\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \check{\mathbb{T}}_{n-1,k_{n-1}} E_{J_1})$$

where $J_1 = \tau_{n,k_{n-1}}(J \sim (n-1)) = (J \sim \{k_{n-1}, (n-1), n\}))/n$.

On the other side, $E_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}} E_J = \mathbb{T}_{n-2,k_{n-2}} E_{\{k_{n-2},n\}} \mathbb{T}_{n-1,k_{n-1}} E_J$.

Call $\{a, b\} = \theta^{-1}_{n-1,k_{n-1}}(\{k_{n-2}, n\})$. Observe that $\{a, b\} = \{k_{n-2}, k_{n-1}\}$ if $k_{n-2} < k_{n-1}$, whereas $\{a, b\} = \{k_{n-1}, k_{n-2} + 1\}$ if $k_{n-2} \geq k_{n-1}$.

Using Lemma 1, we obtain

$$\begin{aligned} E &= \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \varrho_n(E_{\{k_{n-2},n\}} \mathbb{T}_{n-1,k_{n-1}} E_J)) = \\ &= \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \varrho_n(\mathbb{T}_{n-1,k_{n-1}} E_{\{a,b\}} E_J)) = \\ &= \mathsf{A}\,\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \check{\mathbb{T}}_{n-1,k_{n-1}} E_{J_2}), \end{aligned}$$

where $J_2 = \tau_{n,k_{n-1}}(J \sim \{a, b\}) = (J \sim \{a, k_{n-1}, n\})/n$, where $a = k_{n-2}$ if $k_{n-2} < k_{n-1}$ and $a = k_{n-2} + 1$ otherwise.

Now, $J_1 \neq J_2$, so, we have to compare $\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \mathbb{T}_{n-2,k_{n-1}} E_{J_i})$, for $i = 1, 2$. To calculate ϱ_{n-1}, it is convenient to write $\mathbb{T}_{n-2,k_{n-2}} \mathbb{T}_{n-2,k_{n-1}}$ as

$$\mathbb{T}_{n-2,k_{n-2}} \mathbb{T}_{n-2,k_{n-1}} = T_{n-2} T_{n-3} T_{n-2} \check{\check{\mathbb{T}}}_{n-2,k_{n-2}} \check{\mathbb{T}}_{n-2,k_{n-1}}.$$

Then, using the relation $T_{n-2}T_{n-3}T_{n-2} = T_{n-3}T_{n-2}T_{n-3}$, and Lemma 1, we get

$$\varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}} \mathbb{T}_{n-2,k_{n-1}} E_{J_i}) = T_{n-3}\ \varrho_{n-1}(T_{n-2} E_{J_i'})\ T_{n-3} \check{\check{\mathbb{T}}}_{n-2,k_{n-2}} \check{\mathbb{T}}_{n-2,k_{n-1}},$$

where $J_i' = \Theta(J_i')$, being $\Theta = \theta_{n-3,k_{n-2}} \theta_{n-3,k_{n-1}}$. Let $\{m, \ldots, n\}$ be the set of the partition J containing n, and denote $\check{J}$ the set–partition obtained from J by removing the set $\{m, \ldots, n\}$. Then

$$\begin{aligned} J_1 = (J \sim \{k_{n-1}, n-1, n\})/n = \check{J} \sim \{m, \ldots, k_{n-1}, n-1\}, \\ J_2 = (J \sim \{a, k_{n-1}, n\})/n = \check{J} \sim \{m, \ldots, a, k_{n-1}\}. \end{aligned}$$

We can write therefore

$$\begin{aligned} \Theta(J_1) = \Theta(\check{J}) \sim \Theta(\{m, \ldots, k_{n-1}, n-1\}), \\ \Theta(J_2) = \Theta(\check{J}) \sim \Theta(\{m, \ldots, a, k_{n-1}.\}) \end{aligned}$$

Now, we obtain $\Theta(k_{n-1}) = n - 3$, and $\Theta(a) = n - 2$ in both cases. Therefore, since Θ does not touch $n - 1$,

$$\begin{aligned} \Theta(J_1) = \Theta(\check{J}) \sim \{\Theta(m), \ldots, n-3, n-1\}, \\ \Theta(J_2) = \Theta(\check{J}) \sim \{\Theta(m), \ldots, n-3, n-2.\} \end{aligned}$$

Now, we have

$$\varrho_{n-1}(T_{n-2}E_{J_i'}) = \mathsf{A}\, E_{\tau_{n-1,n-2}(J_i')}$$

where, for both $i = 1$ and $i = 2$, we have

$$\tau_{n-1,n-2}(J_i') = (\Theta(J_i) \sim \{(n-1),(n-2)\})/(n-1) = (\Theta(\check{J}) \sim \{\Theta(m),\dots,n-3,n-2,n-1\})/(n-1).$$

Therefore, $\varrho_{n-1}(T_{n-2}E_{J_1'}) = \varrho_{n-1}(T_{n-2}E_{J_2'})$.

We will prove now (ii). We study firstly the cases when $k_{n-1} = 0$ or $k_{n-2} = 0$. In the case $k_{n-1} = 0$, we have:

$$T_{n-1}X = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-1,k_{n-2}}E_J \quad \text{and} \quad XT_{n-1} = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}T_{n-1}E_{s_{n-1}(J)}$$

Then

$$\varrho_n(T_{n-1}X) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-1,k_{n-2}}E_{\tau_{n-1,k_{n-2}}(J)}$$

and

$$\varrho_n(XT_{n-1}) = \mathsf{A}\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-2,k_{n-2}}E_{\tau_{n,n-1}(J)}$$

Now

$$\varrho_{n-1}(\varrho_n(T_{n-1}X)) = \mathsf{A}^2\,\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-2,k_{n-2}}E_{\tau_{n-1,k_{n-2}}(\tau_{n,k_{n-2}}(J))}$$

and

$$\varrho_{n-1}(\varrho_n(XT_{n-1})) = \mathsf{A}^2\,\mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\check{\mathbb{T}}_{n-2,k_{n-2}}E_{\tau_{n-1,k_{n-2}}(\tau_{n-1,k_{n-2}}(J))}.$$

But

$$\tau_{n-1,k_{n-2}}(\tau_{n,k_{n-2}}(J)) = ((J \sim \{k_{n-2},n\})/n) \sim \{k_{n-2},n-1\})/(n-1)$$

and

$$\tau_{n-1,k_{n-2}}(\tau_{n,n-1}(J)) = ((J \sim \{n-1,n\})/n) \sim \{k_{n-2},n-1\})/(n-1).$$

The right members of the preceding two equalities are both equal to

$$((J \sim \{k_{n-2},n-1,n\})/n)/(n-1),$$

so that the proof is completed.

For the case $k_{n-2} = 0$, we have:

$$T_{n-1}X = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}}T_{n-1}\mathbb{T}_{n-1,k_{n-1}}E_J$$

and

$$XT_{n-1} = \mathbb{T}_{1,k_1}\mathbb{T}_{2,k_2}\cdots\mathbb{T}_{n-3,k_{n-3}}\mathbb{T}_{n-1,k_{n-1}}T_{n-1}E_{s_{n-1}(J)}$$

By using Lemma 1 we get that $\varrho_{n-1}(\varrho_n(T_{n-1}X))$ and $\varrho_{n-1}(\varrho_n(XT_{n-1}))$ are different, respectively, in

$$R := \varrho_{n-1}(\varrho_n(T_{n-1}\mathbb{T}_{n-1,k_{n-1}}E_J)) \text{ and } S := \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-1,k_{n-1}}T_{n-1}E_{s_{n-1}(J)}).$$

It is a routine to check that these two last expression are equal for $k_{n-1} = 0, n-1$. Thus, we need to check only that $R = S$ for $0 < k_{n-1} < n-1$. Now,

$$\begin{aligned} T_{n-1}\mathbb{T}_{n-1,k_{n-1}}E_J &= T_{n-1}^2T_{n-2}\mathbb{T}_{n-3,k_{n-1}}E_J \\ &= (T_{n-2} + (u-1)E_{n-1}T_{n-2} + (u-1)E_{n-1}T_{n-1}T_{n-2})\mathbb{T}_{n-3,k_{n-1}}E_J \\ &= (T_{n-2} + (u-1)E_{n-1}T_{n-2} + (u-1)E_{n-1}T_{n-1}T_{n-2})E_{\theta_{n-3,k_{n-1}}(J)}\mathbb{T}_{n-3,k_{n-1}} \end{aligned}$$

Let's us call $\Theta := \theta_{n-3,k_{n-1}}$ Then, by using again Lemma 1, we obtain

$$R = (R_1 + (u-1)R_2 + (u-1)R_3)\mathbb{T}_{n-3,k_{n-1}}$$

where

$$\begin{aligned} R_1 &= \varrho_{n-1}(\varrho_n(T_{n-2}E_{\Theta(J)})) \\ R_2 &= \varrho_{n-1}(\varrho_n(E_{n-1}T_{n-2}E_{\Theta(J)})) \\ R_3 &= \varrho_{n-1}(\varrho_n(E_{n-1}T_{n-1}T_{n-2}E_{\Theta(J)})) \end{aligned}$$

i.e.,

$$\begin{aligned} R_1 &= \mathsf{B}\varrho_{n-1}(T_{n-2}E_{\Theta(J)/n}) = \mathsf{AB}E_{J_1^R} \\ R_2 &= \mathsf{B}\varrho_{n-1}(T_{n-2}E_{(\{n-2,n\}\sim\Theta(J))/n}) = \mathsf{AB}E_{J_2^R} \\ R_3 &= \mathsf{A}\varrho_{n-1}(T_{n-2}E_{\tau_{n,n-2}(\{n-2,n\}\sim\Theta(J))}) = \mathsf{A}^2E_{J_3^R} \end{aligned}$$

where

$$\begin{aligned} J_1^R &= ((\Theta(J)/n)\sim\{n-2,n-1\})/(n-1), \\ J_2^R &= (((\{n-2,n\}\sim\Theta(J))/n)\sim\{n-2,n-1\})/(n-1) \\ J_3^R &= \tau_{n-1,n-2}(\tau_{n,n-2}(\{n-2,n\}\sim\Theta(J)) = J_2^R. \end{aligned}$$

On the other part,

$$\begin{aligned} \mathbb{T}_{n-1,k_{n-1}}T_{n-1}E_{s_{n-1}(J)} &= T_{n-1}T_{n-2}T_{n-1}\mathbb{T}_{n-3,k_{n-1}}E_{s_{n-1}(J)} \\ &= T_{n-2}T_{n-1}T_{n-2}E_{\Theta(s_{n-1}J)}\mathbb{T}_{n-3,k_{n-1}} \end{aligned}$$

Then, again from Lemma 1, we get

$$\begin{aligned} S &= \varrho_{n-1}(\varrho_n(T_{n-2}T_{n-1}T_{n-2}E_{\Theta(s_{n-1}J)}))\mathbb{T}_{n-3,k_{n-1}} \\ &= \mathsf{A}\varrho_{n-1}(T_{n-2}^2E_{\tau_{n,n-2}(\Theta(s_{n-1}J)))})\mathbb{T}_{n-3,k_{n-1}} \\ &= (S_1+(u-1)S_2+(u-1)S_3)\mathbb{T}_{n-3,k_{n-1}} \end{aligned}$$

where

$$\begin{aligned} S_1 &= \mathsf{A}\varrho_{n-1}(E_{J^S}) = \mathsf{AB}E_{J_1^S} \\ S_2 &= \mathsf{A}\varrho_{n-1}(E_{n-2}E_{J^S}) = \mathsf{AB}E_{J_2^S} \\ S_3 &= \mathsf{A}\varrho_{n-1}(T_{n-2}E_{n-2}E_{J^S}) = \mathsf{A}^2E_{J_3^S} \end{aligned}$$

being

$$\begin{aligned} J^S &= \tau_{n,n-2}(\Theta(s_{n-1}J)) = (\Theta(s_{n-1}J)\sim\{n-2,n\})/n, \\ J_1^S &= J^S/(n-1) = ((\Theta(s_{n-1}J)\sim\{n-2,n\})/n)/(n-1), \\ J_2^S &= (J^S\sim\{n-2,n-1\})/(n-1) = (((\Theta(s_{n-1}J)\sim\{n-2,n\})/n)\sim\{n-2,n-1\})/(n-1), \\ J_3^S &= \tau_{n-1,n-2}(J^S\sim\{n-2,n-1\}) = (((\Theta(s_{n-1}J)\sim\{n-2,n\})/n)\sim\{n-2,n-1\})/(n-1) = J_2^S. \end{aligned}$$

Now, observe that

$$(\Theta(s_{n-1}J)\sim\{n-2,n\})/n = ((\Theta(J)/n)\sim\{n-2,n-1\}),$$

since Θ does not touch $n-1,n$. Thus, we have that for $i=1,2,3$, $J_i^R = J_i^S$, and therefore also $R_i = S_i$. The proof is concluded.

In order to finish the proof of (ii), we have to check the claim in the cases k_{n-1} and k_{n-2} both different from 0. We will compute first $\varrho_{n-1}(\varrho_n(T_{n-1}X))$:

$$T_{n-1}X = \mathbb{T}_{1,k_1}\cdots\mathbb{T}_{n-3,k_{n-3}}(T_{n-1}\mathbb{T}_{n-2,k_{n-3}}\mathbb{T}_{n-1,k_{n-3}}E_J)$$

Then, from Lemma 1:

$$\varrho_{n-1}(\varrho_n(T_{n-1}X)) = \mathbb{T}_{1,k_1}\cdots\mathbb{T}_{n-3,k_{n-3}}G$$

where $G := \varrho_{n-1}(\varrho_n(T_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-2}}E_J))$.

We compute now $\varrho_{n-1}(\varrho_n(XT_{n-1}))$. From (23) we have

$$XT_{n-1} = \mathbb{T}_{1,k_1}\cdots\mathbb{T}_{n-1,k_{n-1}}T_{n-1}E_{s_{n-1}(J)}$$

Lemma 1 implies that

$$\varrho_{n-1}(\varrho_n(XT_{n-1})) = \mathbb{T}_{1,k_1}\cdots\mathbb{T}_{n-3,k_{n-3}}H$$

where $H := \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}}T_{n-1}E_{s_{n-1}(J)}))$. Thus, it is enough to prove that $G = H$. We will do this by distinguishing four cases, according to the values of k_{n-1} and k_{n-2}.

$*$ Case $k_{n-1} = n-1$ and $k_{n-2} = n-2$. In this case we have: $\mathbb{T}_{n-2,k_{n-2}} = T_{n-2}$ and $\mathbb{T}_{n-1,k_{n-1}} = T_{n-1}$. We have then

$$G = \varrho_{n-1}(\varrho_n(T_{n-1}T_{n-2}T_{n-1}E_J)) \quad \text{and} \quad H = \varrho_{n-1}(\varrho_n(T_{n-2}T_{n-1}T_{n-1}E_{s_{n-1}(J)}))$$

For G we have

$$\begin{aligned} G &= \varrho_{n-1}(\varrho_n(T_{n-2}T_{n-1}T_{n-2}E_J)) \\ &= \mathsf{A}\varrho_{n-1}(T_{n-2}^2E_{J'}) \\ &= \mathsf{A}\varrho_{n-1}((1+(u-1)E_{n-2}+(u-1)E_{n-2}T_{n-2})E_{J'}) \\ &= G_1+(u-1)G_2+(u-1)G_3 \end{aligned}$$

where $J' = (J\sim\{n,n-2\})/n$ and

$$\begin{aligned} G_1 &:= \mathsf{A}\varrho_{n-1}(E_{J'}) = \mathsf{AB}E_{J'/(n-1)} \\ G_2 &:= \mathsf{A}\varrho_{n-1}(E_{n-2}E_J) = \mathsf{AB}E_{J'\sim\{n-2,n-1\}/(n-1)} \\ G_3 &:= \mathsf{A}\varrho_{n-1}(T_{n-2}E_{n-2}E_J) = \mathsf{A}^2E_{J'\sim\{n-2,n-1\}/(n-1)} \end{aligned}$$

In order to compute H, we firstly note that we have

$$T_{n-2}T_{n-1}T_{n-1}E_{s_{n-1}(J)} = T_{n-2}(1+(u-1)E_{n-1}+(u-1)T_{n-1}E_{n-1})E_{s_{n-1}(J)}$$

Then

$$H = H_1+(u-1)H_2+(u-1)H_3$$

where

$$\begin{aligned} H_1 &:= \varrho_{n-1}(\varrho_n(T_{n-2}E_{s_{n-1}(J)})) = \mathsf{B}\varrho_{n-1}(T_{n-2}E_{s_{n-1}(J)/n}) = \mathsf{AB}E_{s_{n-1}(J)/n} \\ H_2 &:= \varrho_{n-1}(\varrho_n(T_{n-2}E_{n-1}E_{s_{n-1}(J)})) = \mathsf{B}\varrho_{n-1}(T_{n-2}E_{s_{n-1}(J)\sim\{n-1,n\}/n}) \\ &= \mathsf{AB}E_{(((J\sim\{n-1,n\})/n)\sim\{n-2,n-1\})/(n-1)} \\ H_3 &:= \varrho_{n-1}(\varrho_n(T_{n-2}T_{n-1}E_{n-1}E_{s_{n-1}(J)})) = \mathsf{A}\varrho_{n-1}(T_{n-2}E_{(J\sim\{n-1,n\})/n}) \\ &= \mathsf{A}^2E_{(((J\sim\{n-1,n\})/n)\sim\{n-2,n-1\})/(n-1)} \end{aligned}$$

Thus the equality $G = H$ is a consequence of the equalities $G_i = H_i$, $i = 1,2,3$.

We will analyze now the remaining cases $0 < k_{n-1} < n-1$ and $0 < k_{n-2} < n-2$.

Observe that

$$\begin{aligned} T_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-1}} &= T_{n-1}(T_{n-2}T_{n-1}T_{n-3}\cdots T_{k_{n-2}})\check{\mathbb{T}}_{n-1,k_{n-1}} \\ &= T_{n-2}T_{n-1}T_{n-2}T_{n-3}\cdots T_{k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}} \end{aligned}$$

Therefore

$$T_{n-1}\mathbb{T}_{n-2,k_{n-2}}\mathbb{T}_{n-1,k_{n-2}}E_J = T_{n-2}T_{n-1}E_{J'}\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}} \tag{39}$$

where $J' := \theta^{-1}_{n-2,k_{n-1}}\theta^{-1}_{n-2,k_{n-2}}(J)$. Thus, by using Lemma 1, we get

$$G = \varrho_{n-1}(T_{n-2}\varrho_n(T_{n-1}E_{J'})\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}) = \mathsf{A}G'$$

where $G' := \varrho_{n-1}(T_{n-2}E_{\tau_{n,n-1}(J')}\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}})$. Now, we have

$$T_{n-2}E_{\tau_{n,n-1}(J')}\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}} = E_{J''}T_{n-2}\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}}$$

where $J'' := s_{n-2}(\tau_{n,n-1}(J'))$. Then

$$\begin{aligned} E_{J''}T_{n-2}\mathbb{T}_{n-2,k_{n-2}}\check{\mathbb{T}}_{n-1,k_{n-1}} &= E_{J''}T_{n-2}(T_{n-2}T_{n-3}T_{n-2}\cdots T_{k_{n-2}})\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \\ &= E_{J''}T_{n-2}T_{n-3}T_{n-2}T_{n-3}\cdots T_{k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \\ &= E_{J''}T_{n-3}T_{n-2}T_{n-3}T_{n-3}\cdots T_{k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \\ &= T_{n-3}(T_{n-2}E_{J'''})T_{n-3}T_{n-3}\cdots T_{k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \end{aligned}$$

where $J''' := s_{n-2}s_{n-3}J'' = (n-3,n-1)(\tau_{n,n-1}(J'))$. Therefore

$$G' = \mathsf{A}\,T_{n-3}E_{\tau_{n-1,n-2}(J''')}T_{n-3}T_{n-3}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}}.$$

In H, we have

$$\begin{aligned} H &= \varrho_{n-1}(\varrho_n(\mathbb{T}_{n-2,k_{n-2}}T_{n-2}\mathbb{T}_{n-1,k_{n-1}}E_{s_{n-1}(J)})) \quad \text{(from (20))} \\ &= \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}T_{n-2}\varrho_n(\mathbb{T}_{n-1,k_{n-1}}E_{s_{n-1}(J)})) \quad \text{(from Lemma 1)} \\ &= \mathsf{A}H' \end{aligned}$$

where $H' := \varrho_{n-1}(\mathbb{T}_{n-2,k_{n-2}}T_{n-2}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(s_{n-1}J)})$. Observe now that

$$\begin{aligned} \mathbb{T}_{n-2,k_{n-2}}T_{n-2}\check{\mathbb{T}}_{n-1,k_{n-1}} &= T_{n-2}T_{n-3}T_{n-2}T_{n-2}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \\ &= T_{n-3}T_{n-3}T_{n-2}T_{n-3}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \end{aligned}$$

Then,

$$\mathbb{T}_{n-2,k_{n-2}}T_{n-2}\check{\mathbb{T}}_{n-1,k_{n-1}}E_{\tau_{n,k_{n-1}}(s_{n-1}J)} = T_{n-3}T_{n-3}T_{n-2}E_IT_{n-3}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}}$$

where $I := \theta_{n-3,k_{n-2}}\theta_{n-3,k_{n-1}}(\tau_{n,k_{n-1}}(s_{n-1}J))$. Thus

$$\begin{aligned} H' &= T_{n-3}T_{n-3}\varrho_{n-1}(T_{n-2}E_I)T_{n-3}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \\ &= \mathsf{A}\,T_{n-3}T_{n-3}E_{\tau_{n-1,n-2}(I)}T_{n-3}\mathbb{T}_{n-4,k_{n-2}}\check{\check{\mathbb{T}}}_{n-1,k_{n-1}} \end{aligned}$$

Therefore, to have $G' = H'$ and then $G = H$, it is enough to prove that

$$T_{n-3}E_{\tau_{n-1,n-2}(J''')}T_{n-3}T_{n-3} = T_{n-3}T_{n-3}E_{\tau_{n-1,n-2}(I)}T_{n-3}$$

i.e. that

$$\tau_{n-1,n-2}(J''') = s_{n-3}\tau_{n-1,n-2}(I).$$

The left member is equal to

$$\tau_{n-1,n-2}((n-3,n-1)\tau_{n,n-1}(\theta^{-1}_{n-2,k_{n-1}}\theta^{-1}_{n-2,k_{n-2}}(J))) \tag{40}$$

while the right member is equal to

$$s_{n-3}\tau_{n-1,n-2}(\theta_{n-3,k_{n-2}}\theta_{n-3,k_{n-1}}(\tau_{n,k_{n-1}}(s_{n-1}J))). \tag{41}$$

Observe that (40)=(41) in the extreme cases when $J = (\ ) = (\{1\},\{2\},\dots,\{n\})$ and $J = (\{1,2,\dots,n\})$. In these cases (40) and (41) are given by $(J/n)/(n-1)$. In the general case we have to distinguish the cases $k_{n-2} < k_{n-1}$ and $k_{n-2} \geq k_{n-1}$, and the proof is done by comparing the set–partitions as for the point (i). We prefer to avoid two further boring pages of calculations, without any new idea. Let's give a non trivial example. Let $n = 7$, $k_{n-1} = 3$ and $k_{n-2} = 1$. Let $J = (\{1,2\},\{3\},\{5,7\},\{4,6\})$. We calculate (40):

$$\begin{aligned} J &= (\{1,2\},\{3\},\{5,7\},\{4,6\}) \\ J' = \theta^{-1}_{5,3}\theta^{-1}_{5,1}(J) &= (\{2,4\},\{5\},\{3,7\},\{6,1\}) \\ J'' = \tau_{7,6}(J') &= (\{2,4\},\{5\},\{3,6,1\}) \\ J''' = (4,6)(J'') &= (\{2,6\},\{5\},\{3,4,1\}) \\ \tau_{6,5}(J''') &= (\{2,5\},\{3,4,1\}). \end{aligned}$$

As for the (41):

$$\begin{aligned} J &= (\{1,2\},\{3\},\{5,7\},\{4,6\}) \\ J' = s_6(J) &= (\{1,2\},\{3\},\{5,6\},\{4,7\}) \\ J'' = \tau_{7,3}(J') &= (\{1,2\},\{5,6\},\{3,4\}) \\ J''' = \theta^{-1}_{4,3}\theta^{-1}_{4,1}(J'') &= (\{5,1\},\{3,6\},\{2,4\}) \\ \tilde{J} = \tau_{6,5}(J''') &= (\{5,1,3\},\{2,4\}) \\ s_4(\tilde{J}) &= (\{4,1,3\},\{2,5\}) \end{aligned}$$

□

4.2. For all $n \geq 1$ define ρ_n the linear map from $\mathcal{E}_n$ to $\mathbb{C}(u,\mathsf{A},\mathsf{B})$ by

$$\rho_n = \varrho_1 \circ \varrho_2 \circ \cdots \varrho_{n-1} \circ \varrho_n$$

Notice that for $k \leq n$ and $X \in \mathcal{E}_k$, we have

$$\rho_n(X) = \rho_k(X) \tag{42}$$

Also, from the definition of ϱ_n, it follows that $\rho_n(1) = 1$. Moreover we have the following theorem.

Theorem 3. *The family $\{\rho_n\}_{n\in\mathbb{N}}$ is a Markov trace. I.e. for all $n \in \mathbb{N}$, ρ_n is a linear map uniquely defined (inductively) by the following rules:*

(i) $\rho_n(1) = 1$
(ii) $\rho_n(XY) = \rho_n(YX)$

(iii) $\rho_{n+1}(XT_n) = \rho_{n+1}(XE_nT_n) = \mathsf{A}\rho_n(X)$
(iv) $\rho_{n+1}(XE_n) = \mathsf{B}\rho_n(X)$
where $X, Y \in \mathcal{E}_n$.

Proof. We will prove (ii). Due to the linearity of ρ_n it is enough to prove for $X \in \mathcal{B}_n$ and when Y is one of the generators T_i and E_i of $\mathcal{E}_n$. We prove it by induction on n. For $n = 2$ clearly the claim is true since $\mathcal{E}_2$ is commutative. Suppose now the claim is true for all k less than n. We are going to prove the claim for n. Let $X \in \mathcal{B}_n$ and $Y = E_j$ or T_j, with $j \in \{T_1, \ldots, T_{n-2}\}$. We have

$$\begin{aligned}
\rho_n(XY) &= (\varrho_1 \circ \ldots \circ \varrho_{n-1})(\varrho_n(XY)) \\
&= (\varrho_1 \circ \ldots \circ \varrho_{n-1})(\varrho_n(X)Y) \quad \text{((i) Lemma 1)} \\
&= (\varrho_1 \circ \ldots \circ \varrho_{n-1})(Y\varrho_n(X)) \quad \text{(induction hypothesis)} \\
&= (\varrho_1 \circ \ldots \circ \varrho_{n-1})(\varrho_n(YX)) \quad \text{((ii) Lemma 1)} \\
&= \rho_n(YX)
\end{aligned}$$

For $Y = T_{n-1}$, E_{n-1} , we have $\rho_n(XY) = \varrho_1 \circ \ldots \circ (\varrho_{n-1}(\varrho_n(XY)))$, then by using respectively part (i) and (ii) of Lemma 3, we obtain $\rho_n(XY) = \varrho_1 \circ \ldots \circ (\varrho_{n-1}(\varrho_n(YX))) = \rho(YX)$.

The proof of the statements (iii) and (iv) are analogous. We shall prove only that $\rho_{n+1}(XT_n) = \mathsf{A}\rho_n(X)$. We have:

$$\begin{aligned}
\rho_{n+1}(XT_n) &= (\varrho_1 \circ \ldots \circ \varrho_n)(\varrho_{n+1}(XT_n)) \\
&= (\varrho_1 \circ \ldots \circ \varrho_n)(X\varrho_{n+1}(T_n)) \quad \text{((ii) Lemma 1)} \\
&= (\varrho_1 \circ \ldots \circ \varrho_n)(X\mathsf{A}) \\
&= \mathsf{A}(\varrho_1 \circ \ldots \circ \varrho_n)(X) \;=\; \mathsf{A}\rho_n(X).
\end{aligned}$$

□

Remark 4. Observe that rule (iv) in the above theorem is the condition on the Markov trace of the Yokonuma–Hecke algebra requested to have the invariant defined by S. Lambropoulou and the second author, see [17, 15, 16]. More precisely, this properties allows to factorize the factor $\rho_n(X)$ in the computation of $\rho_{n+1}(XT_n^{-1})$, where $X \in \mathcal{E}_n$, see (48).

5. Applications to Knot invariants

In this section we will construct an invariant for classical knots and another invariant for singular knots. The constructions follow the Jones recipe, that is, they are obtained from normalization and rescaling of the composition of a representation of a braid group/singular braid monoid in $\mathcal{E}_n$ with the trace ρ_n.

In both invariants we will use the element of normalization $\mathsf{L} = \mathsf{L}(u, \mathsf{A}, \mathsf{B})$, defined as follows

$$\mathsf{L} = \frac{\mathsf{A} + (1-u)\mathsf{B}}{u\mathsf{A}}. \tag{43}$$

5.1. In order to define our invariant for classical knots, we shall recall some classical facts and standard notations. Firstly, remember that from the classical theorems of Alexander and Markov, the set of isotopy classes of links in the Euclidian space is in bijection with the set of equivalence classes obtained from the inductive limit of the tower of braid groups $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$, under the *Markov equivalence relation* $\sim$. That is, for all $\alpha, \beta \in B_n$, we have:

(i) $\alpha\beta \sim \beta\alpha$
(ii) $\alpha \sim \alpha\sigma_n$ and $\alpha \sim \alpha\sigma_n^{-1}$.

Secondly, let us denote $\bar{\pi}_{\mathsf{L}}$ the representation of B_n in $\mathcal{E}_n$, namely $\sigma_i \mapsto \sqrt{\mathsf{L}}T_i$. Then, for $\alpha \in B_n$, we define $\bar{\Delta}(\alpha)$

$$\bar{\Delta}(\alpha) := \left(-\frac{1-\mathsf{L}u}{\sqrt{\mathsf{L}}(1-u)\mathsf{B}}\right)^{n-1} (\rho_n \circ \bar{\pi}_{\mathsf{L}})(\alpha) \in K(\sqrt{\mathsf{L}}, B) \tag{44}$$

It is useful to have an alternative expression for $\bar{\Delta}(\alpha)$, in terms of the exponent $e(\sigma)$ of α, where $e(\alpha)$ is the algebraic sum of the exponents of the elementary braids σ_i used for writing α. Then, we have:

$$\bar{\Delta}(\alpha) = \left(-\frac{1-\mathsf{L}u}{\sqrt{\mathsf{L}}(1-u)\mathsf{B}}\right)^{n-1} (\sqrt{\mathsf{L}})^{e(\alpha)} (\rho_n \circ \bar{\pi})(\alpha) \tag{45}$$

where $\bar{\pi}$ is defined as the mapping $\sigma_i \mapsto h_i$. Now, by simplicity, let us define

$$\bar{D} = -\frac{1-\mathsf{L}u}{\sqrt{\mathsf{L}}(1-u)\mathsf{B}} \tag{46}$$

Then, notice that

$$\sqrt{\mathsf{L}}\,\bar{D}\,\mathsf{A} = 1 \quad \text{or equivalently} \quad \mathsf{A} = -\frac{1-u}{1-\mathsf{L}u}\mathsf{B} \tag{47}$$

and $\bar{\Delta}(\alpha)$ can be rewritten as follows:

$$\bar{\Delta}(\alpha) = \bar{D}^{n-1}(\sqrt{\mathsf{L}})^{e(\alpha)}(\rho_n \circ \bar{\pi})(\alpha)$$

Theorem 4. *Let L be a link obtained by closing the braid $\alpha \in B_n$. Then the map $L \mapsto \bar{\Delta}(\alpha)$ defines an isotopy invariant of links.*

Proof. It is enough to prove that $\bar{\Delta}$ respects the Markov equivalence relations. Due to Theorem 3 (ii) it is evident that $\bar{\Delta}$ respects the first Markov equivalence. We are going now to prove the second Markov equivalence. Again, it is easy to check that $\bar{\Delta}(\alpha) = \bar{\Delta}(\alpha\sigma_n)$. In fact, up to now we have only used the properties of the trace ρ_n in which the elements E_i's do not play any role. But now, to prove that $\bar{\Delta}(\alpha) = \bar{\Delta}(\alpha\sigma_n^{-1})$, the defining conditions of ρ_n involving the elements E_i's are crucial (see Remark 4).

For every $\alpha \in B_n$ we have

$$\begin{aligned}\bar{\Delta}(\alpha\sigma_n^{-1}) &= \bar{D}^n(\sqrt{\mathsf{L}})^{e(\alpha\sigma_n^{-1})}(\rho_n(\bar{\pi}(\alpha\sigma_n^{-1}))\\ &= \bar{D}^n(\sqrt{\mathsf{L}})^{e(\alpha)-1}(\rho_n(\bar{\pi}(\alpha)T_n^{-1})).\end{aligned}$$

By using the formulae of T_n^{-1} (see Proposition 2) and the defining rule of ρ_n, we deduce:

$$\rho_n(\bar{\pi}(\alpha)T_n^{-1}) = \rho_n(\bar{\pi}(\alpha))(\mathsf{A} + (u^{-1}-1)\mathsf{B} + (u^{-1}-1)\mathsf{A}). \tag{48}$$

Then

$$\begin{aligned}\bar{\Delta}(\alpha\sigma_n^{-1}) &= \bar{D}^n(\sqrt{\mathsf{L}})^{e(\alpha)-1}(u^{-1}\mathsf{A} + (u^{-1}-1)\mathsf{B})\rho_n(\bar{\pi}(\alpha))\\ &= (\bar{D}/\sqrt{\mathsf{L}})(u^{-1}\mathsf{A} + (u^{-1}-1)\mathsf{B})\bar{D}^{n-1}(\sqrt{\mathsf{L}})^{e(\alpha)}\rho_n(\bar{\pi}(\alpha))\\ &= (\bar{D}/\sqrt{\mathsf{L}})\mathsf{A}\mathsf{L}\bar{D}^{n-1}(\sqrt{\mathsf{L}})^{e(\alpha)}\rho_n(\bar{\pi}(\alpha))\\ &= \bar{\Delta}(\alpha) \qquad \text{(by (47))}.\end{aligned}$$

□

Example 1. *Let α be the simplest oriented link, formed by two oriented circles with two positive crossings. We obtain*

$$\rho_n(\bar{\pi}(\alpha)) = 1 + (\mathsf{A}+\mathsf{B})(u-1)$$

and

$$\bar{\Delta}(\alpha) = \sqrt{\frac{\mathsf{A}+\mathsf{B}(1-u)}{u\mathsf{A}}}\frac{1+(\mathsf{A}+\mathsf{B})(u-1)}{\mathsf{A}}.$$

Example 2. *Let γ be the trefoil knot. We obtain*

$$\rho_n(\bar{\pi}(\gamma)) = \frac{\mathsf{B}(1-u+u^2-u^3)+A(1-u+u^2)}{u^3}$$

and

$$\bar{\Delta}(\gamma) = \frac{\mathsf{A}(-u^3\mathsf{B}+u^2\mathsf{B}-u\mathsf{B}+\mathsf{B}+u^2\mathsf{A}-u\mathsf{A}+\mathsf{A})}{u(\mathsf{A}+\mathsf{B}-u\mathsf{B})^2}.$$

5.2. For the singular links in the Euclidian space, the singular braid monoid plays an analogous role as that of the braid group for the classical links. The singular braid monoid was introduced independently by three authors, namely J. Baez, J. Birman and L. Smolin (see [14] and the references therein.

Definition 3. *The singular braid monoid SB_n is defined as the monoid generated by the usual braid generators $\sigma_1, \ldots, \sigma_{n-1}$ (invertible) subject to the following relations: the braid relations among the σ_i's together with the following relations:*

$$\begin{array}{rcll} \tau_i\tau_j & = & \tau_j\tau_i & \text{for} \quad |i-j|>1 \\ \sigma_i\tau_j & = & \tau_j\sigma_i & \text{for} \quad |i-j|>1 \\ \sigma_i\tau_i & = & \tau_i\sigma_i & \text{for all} \quad i \\ \sigma_i\sigma_j\tau_i & = & \tau_j\sigma_i\sigma_j & \text{for} \quad |i-j|=1. \end{array}$$

Now, in an analogous way to the classical links, we define the isotopy of the singular links in the euclidian space in purely algebraic terms. More precisely, for the singular links we have the analogous of the classical Alexander theorem which is due to J. Birman [4]. We have also the analogous of the classical Markov theorem which is due to B. Gemein [8]. Thus, the set of the isotopy classes of singular knots is in bijection with the set of equivalence classes defined on the inductive limit associated to the tower of monoids: $SB_1 \subseteq SB_2 \subseteq \cdots \subseteq SB_n \subseteq \cdots$ respect to the equivalence relation $\sim_s$:

(i) $\alpha\beta \sim_s \beta\alpha$
(ii) $\alpha \sim_s \alpha\sigma_n$ and $\alpha \sim_s \alpha\sigma_n^{-1}$

for all $\alpha, \beta \in SB_n$.

Now we have to define a representation of SB_n in the algebra $\mathcal{E}_n$. This representation uses the same expression as in [15] for its definition. More precisely, we define the representation $\bar{\delta}$ by mapping:

$$\sigma_i \mapsto T_i \quad \text{and} \qquad \tau_i \mapsto E_i(1+T_i).$$

Proposition 3. *$\bar{\delta}$ is a representation.*

Proof. It is straightforward to verify that the images of the defining generators of SB_n satisfy the defining relations of SB_n. □

In order to define our invariant for singular knots we need to introduce the exponent for the elements of SB_n. From the definition of SB_n, it follows that every element $\omega \in SB_n$ can be written in the form

$$\omega = \omega_1^{\epsilon_1} \cdots \omega_m^{\epsilon_m}$$

where ω_i are taken from the defining generators of SB_n and $\epsilon_i = 1$ or -1, and assuming moreover that in the case ω_i is a singular braid, its exponent ϵ_i is by definition equal to 1. Then we have the following

Definition 4. [15, Definition 2] *The exponent $\epsilon(\omega)$ of ω is defined as the sum $\epsilon_1 + \cdots + \epsilon_m$.*

For $\omega \in SB_n$, we define $\bar{\Gamma}$ as follows

$$\bar{\Gamma}(\omega) = \left(-\frac{1-\mathsf{L}u}{\sqrt{\mathsf{L}}(1-u)B}\right)^{n-1} (\sqrt{\mathsf{L}})^{\epsilon(\omega)} (\rho_n \circ \bar{\delta})(\omega).$$

We have then the following theorem.

Theorem 5. *Let L be a singular link obtained by closing $\omega \in SB_n$; then the mapping $L \mapsto \bar{\Gamma}(\omega)$ defines an invariant of singular links.*

Proof. The proof is totally analogous to the proof of [15, Theorem 5]. Cf. proof of Theorem 4. □

5.3. Comparisons. In this subsection we shall show as to obtain known invariant polynomial for classical knots from the three–variable invariant $\bar{\Delta}$ defined in this paper.

In [12, Section 6] V. Jones constructed a Homflypt polynomial, denoted X, invariant for classical links, through the composition of the Ocneanu trace τ, of parameter z, on H_n and the representation $\pi_\lambda : B_n \to \mathrm{H}_n$, $\sigma_i \mapsto \sqrt{\lambda} h_i$, where

$$\lambda = \frac{z + (1-u)}{uz}$$

More precisely, for $\alpha \in B_n$, such Homflypt polynomial is defined by

$$\mathrm{X}(\alpha) = \left(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\right)(\tau \circ \pi_\lambda)(\alpha)$$

Thus, setting $\mathsf{A} = z$ and $\mathsf{B} = 1$ in (43), we obtain $\mathsf{L} = \lambda$. Then, for φ_n of Proposition 3, we have $\varphi_n \circ \pi_{\mathsf{L}} = \pi_\lambda$. Also, for these values of A and B we have $\varphi_n \circ \tau_n = \rho_n$. Then

$$\tau_n \circ \pi_\lambda = \tau_n \circ (\varphi_n \circ \pi_{\mathsf{L}}) = \rho_n \circ \pi_{\mathsf{L}}.$$

Therefore it follows that the Hompflypt polynomial X can be obtained from by taking $\mathsf{A} = z$ and specializing $B = 1$.

Now we show how to obtain from $\bar{\Delta}$ the two-parameters invariants of classical links defined in [16].

The Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}$ also supports a Markov trace, denoted tr, of parameters z and $x_1, \ldots, x_{d-1}$, see [14, Theorem 12]. In [17] it is proved that for certain specific values of the parameters trace x_i's it is possible to construct an invariant of classical links Δ. More precisely, these specific values, which are solutions of the so–called E–system, are parametrized by non–empty subsets of the group of integer modulo d. Now, given such a subset S, we shall denote tr_S the trace tr, whenever the parameters x_k's are taken as the solutions of the E–system,

parametrized by S. Now, the mapping $\sigma_i \mapsto \sqrt{\lambda_S} g_i$ defines a representation $\tilde{\pi}_{\lambda_S}$ from B_n to $\mathrm{Y}_{d,n}$, where

$$\lambda_S = \frac{z + (1-u)/|S|}{uz}$$

The two–variable polynomial invariant of classical knots Δ is defined as follows

$$\Delta(\alpha) = \left(-\frac{1-\lambda_S u}{\sqrt{\lambda_S}(1-u)} \right) (\mathrm{tr}_S \circ \tilde{\pi}_{\lambda_S})(\alpha) \qquad (\alpha \in B_n);$$

for details see [16]. By taking the parameter $z = \mathsf{A}$ and specializing B to $1/|S|$, we get that $\lambda_S = \mathsf{L}$. Then, we have $\psi_n \circ \bar{\pi}_{\mathsf{L}} = \tilde{\pi}_{\lambda}$ and $\mathrm{tr}_S \circ \psi_n = \rho_n$. Thus,

$$\mathrm{tr}_S \circ \tilde{\pi}_{\lambda_S} = \mathrm{tr}_S \circ (\psi_n \circ \bar{\pi}_{\mathsf{L}}) = \rho_n \circ \bar{\pi}_{\mathsf{L}}$$

Therefore also the two–variable invariant of classical links Δ can be obtained from the three-variable invariant $\bar{\Delta}$.

6. A diagrammatical interpretation

In this section we recall a diagrammatical interpretation of the defining generators of $\mathcal{E}_n(u)$, given in [2]. Furthermore we introduce a new diagrammatic interpretation of the basis constructed by S. Ryom–Hansen, in which the ties are *elastic*, and can be extended to connect non consecutive threads. This gives a better understanding of the properties of the basis as well as a considerable simplification of the algebraic calculus. This geometrical interpretation also allowed us to define, starting from the trace here defined, an invariant polynomial for tied links, introduced in [1].

6.1. In [2] we have interpreted the generator T_i as the usual braid generator and the generator E_i as a tie between the consecutive strings i and $i+1$.

FIGURE 1. Generators T_i, left, and E_i, right

Indeed, this diagrammatical interpretation reflects coherently, in terms of diagrams, every defining relation of type monomial of $\mathcal{E}_n(u)$ with exception of the monomial relation (7). More precisely, the braid relations (3) and (4) have the well known interpretation in term of diagrams, while the diagrammatical interpretation of relations (6) and (8) – (10) can be seen in Figure 2.

In Figure 2 we see that Relation (11) says that a tie between two threads can move upwards or downwards along a braid as long as *such threads maintain unit distance*. The monomial relation (7) is not natural in terms of diagrams and must be imposed. This relation says that two or more ties between two threads are equivalent to one sole tie (see Figure 3).

Finally, as in the Hecke algebra, the 'quadratic relation'(5) takes account of the splitting of the square of the braid generators in term of the defining generators. This relation is formally shown, in term of diagrams, in Figure 4.

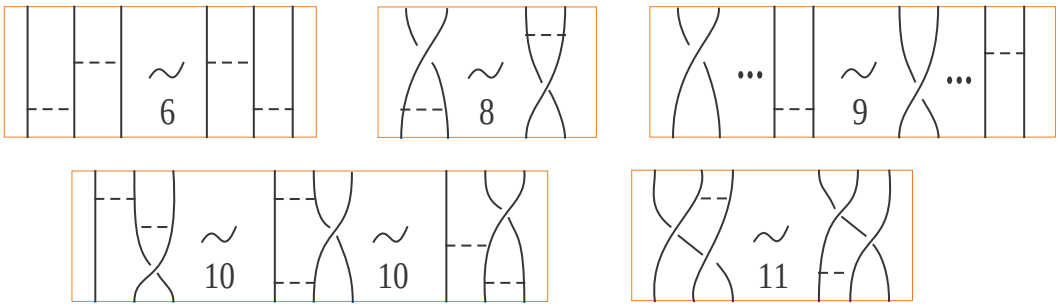


FIGURE 2. Relations (6),and (8)–(11) in diagrams

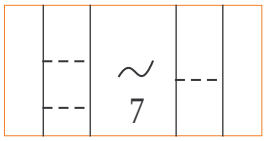


FIGURE 3. Relation (7) in diagrams

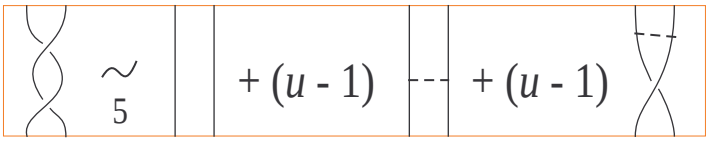


FIGURE 4. Relation (5) in diagrams

Remark 5. From the diagrammatical interpretation it is clear that the defining relations hold substituting the generators T_i by their inverse.

Remark 6. We have already observed, in [2], that *a tie is allowed to bypass a thread*, according to the relation

$$E_{i+1}T_i^{-1}T_{i+1} = \quad T_i^{-1}T_{i+1}E_i$$

which follows directly from the defining relations (11), (13), (7) and (7) of the algebra. In diagrammatical terms, we have the following picture (Figure 5).

6.2. Recall that the linear basis constructed by Ryom–Hansen (Theorem 1) for $\mathcal{E}_n$ consists of elements of the form $E_I T_w$, where $w \in S_n$ and $I \in \mathcal{P}_n$. The diagrammatic interpretations for the elements T_w is standard since the elements T_i's are represented by usual braids. We are going now to describe diagrammatically the elements E_I's, according to the diagrammatic interpretation of the defining generators. This allows to simplify several tedious algebraic computations.

The elements E_I's are defined by means of the $E_{i,j}$'s, where $i < j$, see (14). We introduce now a simple diagrammatic representation of the element $E_{i,j}$, by means of an *elastic tie* (or

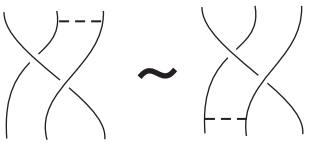


FIGURE 5

spring) connecting the threads i and j, see Figure 6. These new geometrical objects have some properties deduced from the algebra (see [1] for more details and proofs.) For instance, the ties are *transparent* for the threads, i.e., they can be drawn no matter if in front or beyond the threads. We shall say that the spring representing E_{ij} has *length* $j-i$:

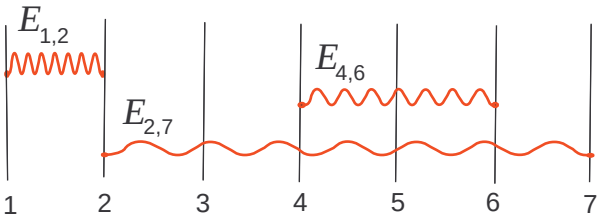


FIGURE 6

Because of the elastic property of the springs, we see immediately the accordance with the original definition of E_{ij}:

$$E_{ij} = T_i^{-1} \cdots T_{j-2}^{-1} E_{j-1} T_{j-2} \cdots T_i.$$

Moreover, in Figure 7 we show how E_{ij} (here, $E_{2,7}$) can be written equivalently by different elements of the algebra.

The elements E_{ij} have another property which allows to rewrite the elements E_I in another form which result more convenient for computations.

We shall show only two particular cases for $n = 7$. This is enough to understand the general case. Set

$$I_1 := \{\{2,3,5,7\},\{1,4,6\}\}, \quad I_2 := \{\{2,3,5,6,7\},\{1\},\{4\}\}$$

Then E_{I_1} and E_{I_2} have the diagrams shown in Figure 8, according to (15).

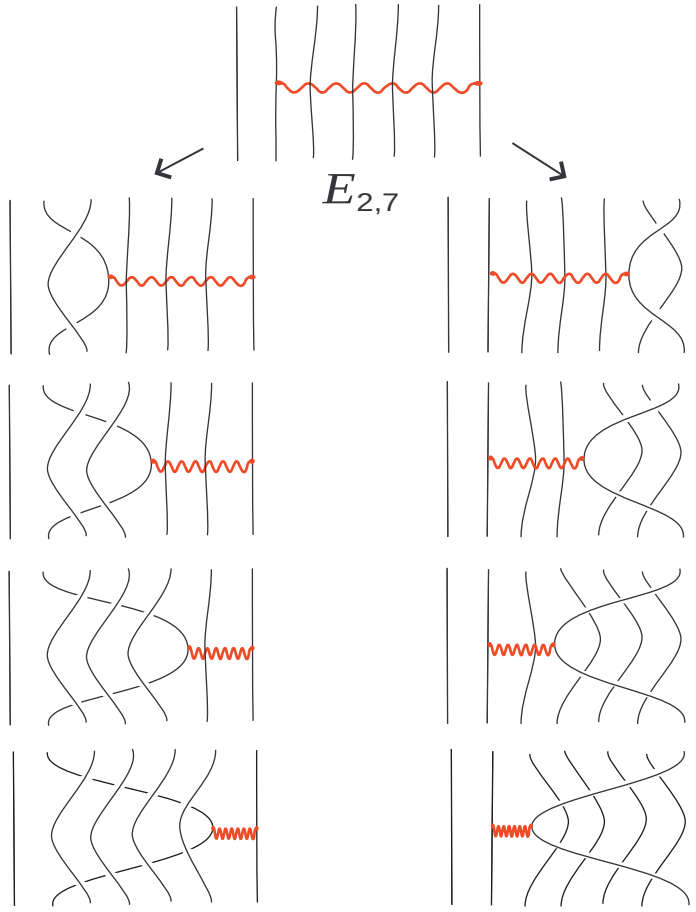


FIGURE 7. $E_{2,7} = T_2T_3T_4T_5E_6T_5^{-1}T_4^{-1}T_3^{-1}T_2^{-1} \quad \sim \quad T_6T_5T_4T_3E_2T_3^{-1}T_4^{-1}T_5^{-1}T_6^{-1}$.

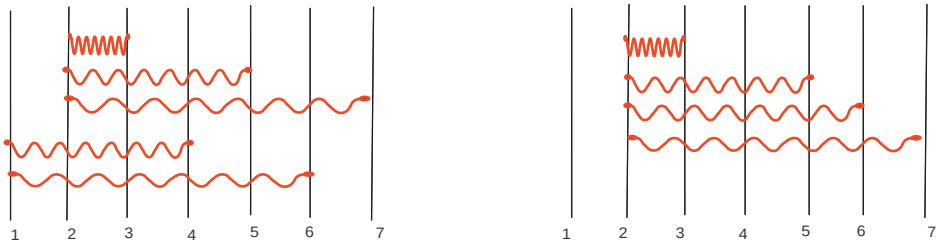


FIGURE 8. $E_{I_1} = (E_{2,3}E_{2,5}E_{2,7})(E_{1,4}E_{1,6})$ $\qquad$ $E_{I_2} = (E_{2,3}E_{2,5}E_{2,6}E_{2,7})$

These elements can be represented by the diagrams pictured in Figure 9, according to the following rule (see formula (16)): If two springs E_{ij} and E_{ik} have in common an end-point

(namely, i), and $i < j < k$, then the product $E_{ij}E_{ik}$ is equivalent to the product $E_{ij}E_{jk}$, i.e., the common part of the springs can be eliminated from the longer spring.

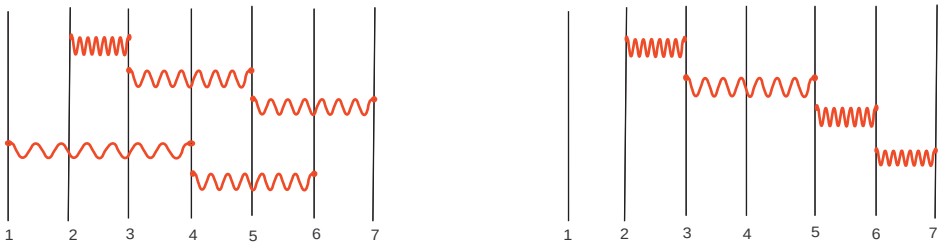
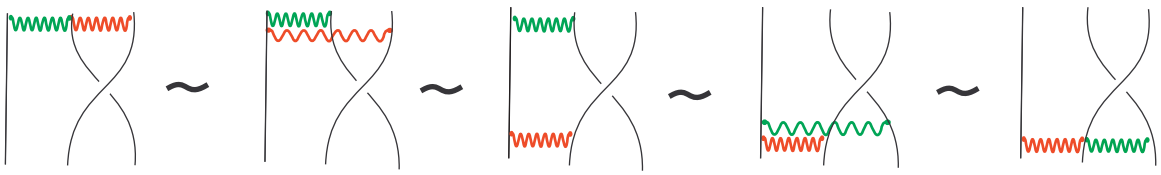


FIGURE 9. $E_{I_1} = (E_{2,3}E_{3,5}E_{5,7})(E_{1,4}E_{4,6})$ $E_{I_2} = (E_{2,3}E_{3,5}E_{5,6}E_{6,7})$

Remark 7. Observe now that the representation of the E_{ij} as a spring allows to simplify considerably the algebra's relations.

For instance, the fact that the generators have the good form of a product of elements E_{ij} time elements of the braids group generators required an elaborated proof in [21]. This becomes evident, having proved that all the springs can be moved upwards along the braid, simply using the property that they can be stretched or shortened, without any operation on the threads.

Moreover, the strange Relation (10) of the algebra (see Figure 2) can be understood in terms of springs as shown below:



Observe also that Relation (11), as well as Remark 6, have a generalization for springs of any length, as shown in the next Figure (case of length equal to 2).

7. Side Comments

We finish with two comments which we think be interesting to be investigated.

7.1. The referee has suggested the following: it would be interesting to know whether there is an integrable model based on the bt–algebra and built with the use of relative traces.

7.2. In Subsection 3.2 was noted that behind the bt–algebra there is the Yokonuma–Hecke algebra. The Yokonuma–Hecke algebra can be regarded as the prototype example of the framization of a knot algebra, see [18] for the concept of framization and knot algebra. More precisely the Yokonuma–Hecke algebra can be considered as the framization of the Hecke algebra. As we mentioned in Section 3.2, the construction of the bt–algebra is obtained by considering abstractly the algebra generated by the braid generators g_i's together with the idempotents e_i's of the Yokonuma–Hecke algebra. In fact, in this new algebra the framing generators are not taken into account. Then the bt–algebra can be considered as a deframization of the Yokonuma–Hecke algebra. Thinking in this way one can define naturally deframizations of all algebras of knots framized in [18]. Moreover, there is a natural deframization associated to certain algebras $\mathrm{Y}(d,m,n)$ defined in [6], where d, n are positive integers and m is either a positive integer or ∞. To be precise, for a positive integer, set u and v_a indeterminates. Set $K_m := \mathbb{C}(u, v_1, \dots v_m)$ for m positive integer and $R_\infty := K$ we could define a deframization of $\mathrm{Y}(d,m,n)$ as the associative algebra over R_m generated by $T_1, \dots, T_{n-1}$, $E_1, \dots E_{n-1}$, $X^{\pm 1}$ subject to the relations (3) to (11) together with the following relations:

$$\begin{aligned} XT_1XT_1 &= T_1XT_1X \\ XT_i &= T_iX \qquad \text{for} \quad i \in \{2, \dots, n-1\} \\ XE_i &= E_iX \qquad \text{for} \quad i \in \{1, \dots, n-1\} \\ (X - v_1)\dots(X - v_m) &= 0 \qquad \text{for} \quad m < \infty \end{aligned}$$

References

[1] F. Aicardi, J. Juyumaya, *Tied Links*, in preparation.

[2] F. Aicardi, J. Juyumaya, *An algebra involving braids and ties*, ictp preprint IC2000179, http://streaming.ictp.trieste.it/preprints/P/00/179.pdf.

[3] E. O. Banjo, *The generic representation theory of the Juyumaya algebra of braids and ties*. Algebr. Represent. Theory 16 (2013), no. 5, 13851395.

[4] J.S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 253–287.

[5] S. Chmutov, S. Jablan, J. Juyumaya, K. Karvounis and S. Lambropoulou *On the knot invariants from the Yokonuma-Hecke algebras*. In progress.

[6] M. Chlouveraki , L. Poulain d'Andecy *Markov traces on affine and cyclotomic Yokonuma-Hecke algebras*.See arXiv:1406.3207.

[7] M. Chlouveraki, L. Poulain d'Andecy *Representation theory of the Yokonuma-Hecke algebras*, Adv. Math. **259** (2014) 134–172.

[8] B. Gemein, *Singular braids and Markov's theorem*, J. Knot Theory Ramifications **6** (1997), no. 4, 441–454.

[9] A.P. Isaev, O. Ogievetsky, *Representations of A-type Hecke algebras*, http://arxiv.org/abs/0912.3701.

[10] A.P. Isaev, O.V. Ogievetsky, *Baxterized Solutions of Reflection Equation and Integrable Chain Models*, http://arxiv.org/abs/math-ph/0510078.

[11] A. P. Isaev, Anatol N. Kirillov, *Bethe subalgebras in Hecke algebra and Gaudin models*, http://arxiv.org/abs/0912.3701.

[12] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. **126** (1987), 335–388.

[13] J. Juyumaya. *Another algebra from the Yokonuma–Hecke algebra*. Preprint ICTP, IC/1999/160.

[14] J. Juyumaya, *Markov trace on the Yokonuma-Hecke Algebra*, J. Knot Theory Ramifications **13** (2004), 25–39.

[15] J. Juyumaya, S. Lambropoulou, *An invariant for singular knots*, J. Knot Theory Ramifications **18**(6) (2009), 825–840.

[16] J. Juyumaya, S. Lambropoulou, *An adelic extension of the Jones polynomial*, in: M. Banagl, D. Vogel (Eds.) The Mathematics of Knots, Theory and Application, in: Contributions in Mathematical and Computational Sciences, **1**, Springer, 2010, 125–142. See also arXiv:0909.2545.
[17] J. Juyumaya and S. Lambropoulou, *p–adic Framed Braids II*, Advances in Mathematics, 234 (2013), 149–191.
[18] J. Juyumaya and S. Lambropoulou, *On the framization of the knot algebras*. To appear in New Ideas in Low–dimensional Topology, L. Kauffman, V. Manturov (eds), Series of Knots and Everything, WS. See: arXiv:1406.6849.
[19] P.P. Martin and A. Elgamal. *Ramified partition algebras.* Mathematische Zeitschrift, 246(3):473500, 2004.
[20] O. Ogievetsky, L. Poulain d'Andecy, *Induced representations and traces for chains of affine and cy clotomic Hecke algebras*, J. Geom. Phys. (2014) http://dx.doi.org/10.1016/j.geomphys.2014.07.005.
[21] S. Ryom–Hansen, *On the Representation Theory of an Algebra of Braids and Ties*, J. Algebr. Comb. **33** (2011), 57–79.

ICTP, Strada Costiera, 11, 34100 Trieste, Italy.
E-mail address: faicardi@ictp.it

Instituto de Matemáticas, Universidad de Valparaíso, Gran Bretaña 1091, Valparaíso, Chile.
E-mail address: juyumaya@uvach.cl